Applications of anamorphosis and mixed reality in a classroom setting

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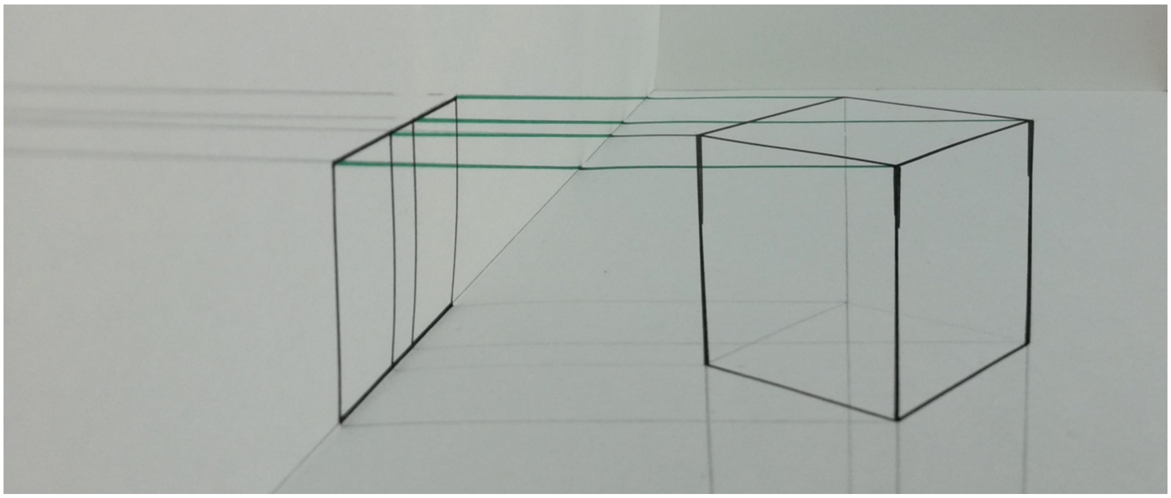

Figure 1: Conical anamorphosis of a cube and its orthogonal projection (work by Andreia (alias), 9th grade student)

ABSTRACT

We report on a novel use of handmade anamorphoses in connection with Mixed Reality in a classroom setting involving Portuguese 9th grade visual education students. This is based on a conceptual reformulation of anamorphosis that makes it intrinsically immersive and connects it naturally with digital visualizations. We propose that this is a useful device for motivating the study of perspective and descriptive geometry for young students.

CCS CONCEPTS

• **Applied computing** → Arts and humanities; • **Computing methodologies** → Computer graphics.

KEYWORDS

Anamorphosis, Education, Drawing, VR, Descriptive Geometry

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1 INTRODUCTION

In this paper we report on a trial of a novel combination of immersive anamorphosis with mixed reality, in a classroom setting involving Portuguese 9th grade elementary school students. We propose that this is a useful instrument for improving spatial visualization and a strategy for preparing and motivating students for the formal study of perspective and descriptive geometry.

We start by describing the definition of anamorphosis proposed by Araújo [1], [2], [3], which serves as the theoretical basis for this applicative work. We then describe the trial performed in an educational setting. Finally, we propose further developments and speculate on the benefits of the proposed approach.

2 ANAMORPHOSIS REFORMED

Anamorphosis has long been a source of wonder for old and young alike. This fascination would make it a powerful didactical device, were it not for the unfortunate misconceptions surrounding it. As pointed out in [3], anamorphosis is seen as the trickster sibling of perspective, charming but not serious. If at all present in visual education curricula, it is relegated to grid deformation tricks, with Holbein's Ambassadors duly trotted out and then put away.

But anamorphosis has suffered a deep review in the age of the prefixed (Virtual, Augmented, Mixed) Realities. Several authors have pointed out that these digital innovations are but the newest iteration of immersive illusions of the past [4], [5]– of Pozzo's church ceilings or Baker's panoramas [6], [7] - but digital technologies bring new flexibility to the table, and their implementation requires a clarity that was often lacking from art historians or philosopher's considerations of the subject (see for instance Mitrović's [8] criticism of Goodman [9]) who often seem to limit the idea of anamorphosis to the simplest sort of "peephole" illusions, a deceptively narrow selection when we consider, say, the full span of Niceron's work [10], and even more unjustified when considering both Andrea Pozzo's techniques [11] and his illusionary church ceilings, which are essentially immersive.

In a recent series of works [2], [3], [12], Araújo has proposed a definition of anamorphosis as a fundamentally immersive concept (spherical anamorphosis) with a view towards a synthesis of rigorous mathematical definition, drawing practice, and technological application. This in a way is putting into words and mathematics what has been the practice of artists such as Pozzo, whose works transcend the blunt definitions that were available to describe them. These works started with a focus on solving problems in spherical perspective drawing [2], but it was soon apparent that this definition turned anamorphosis into the fundamental concept on which all conical perspective is based – both classical and curvilinear, a perspective being defined as an anamorphosis onto a surface followed by a cartographic flattening of the surface, which preserves continuity in some technical sense. In this view the vanishing points are defined already at the level of anamorphosis and perspective is just a convenience of representation. In a twist, the trickster sibling of perspective turns out to be its parent. It is argued in [3] that this view is the conciliation of Euclidean optics with conical perspective after nearly 600 years of polemics.

What is it then, this view of anamorphosis? It is based on a single principle, *radial occlusion*, from which all the rest follows by pure deduction. Radial occlusion is simply the assertion that two spatial points are deemed to "look the same" from a point O if and only if they lie on the same ray from O. This is clear enough: it is the main principle of Euclidean optics, and agreed upon by all perspective textbooks. Yet its centrality and consequences are usually ignored. For it follows from it that the manifold of visual data can be identified with the unit sphere – the *visual sphere*. This defines an equivalence relation (in the technical, mathematical sense) between 3D objects, two objects being equivalent – or anamorphs of each other – if and only if they have the same spherical projection. An infinite class of 3D objects – those that define the same cone from the eye - will look the same when viewed from a give position. This equivalence is between 3D objects, and has no need of projection planes, "oblique" or otherwise, and there is no talk of "deformed objects that look correct", a muddled definition. This eliminates false difficulties like Leonardo's supposed paradox and the confusion of so-called "perspective deformation" (see discussion in [3]). The focus on equivalence puts all anamorphs in an equal footing, be they 3D extensive objects or 2D projections in 3D space: they are anamorphs *of each other*: anamorphosis is a symmetrical relation.

Perhaps the strongest result of this view is that an immersive concept of vanishing points arises naturally, without mention of projection surfaces or perspective. In spherical anamorphosis the notion of vanishing point reaches its most symmetrical definition: each line has exactly two vanishing points, diametrically opposed in the visual sphere (see Figure 2). This is a more natural view than both the awkward asymmetry of linear perspective and the abstract, disembodied view of projective geometry.

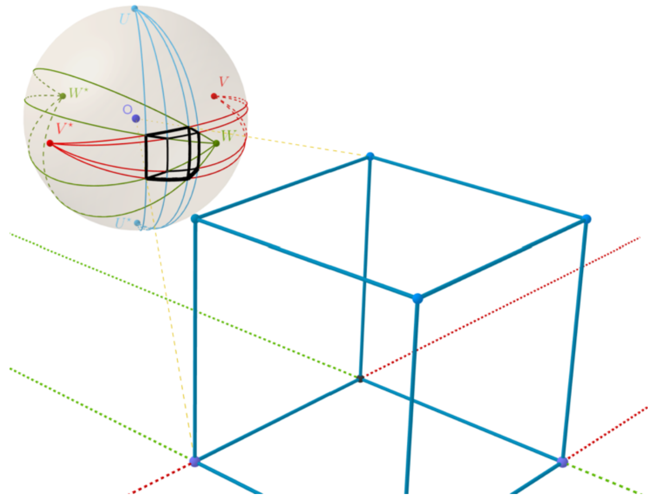

Figure 2: Spherical anamorphosis, following Araújo [3]. The cube has six vanishing points on the visual sphere, sets of parallels converging to pairs of antipodal vanishing points

This theory is however too abstract for young students. Vanishing points are defined in [2] as the topological closure of conical projections. This is a bit much for a 15-year-old. To solve this, it is suggested in [1], [3] that one may take a proposition for a definition: it is a proposition in [2] that the vanishing points of a line l are the points obtained by intersecting the visual sphere with the translation of l to O. This can be taken as a definition and is as understandable as the corresponding definition of vanishing point of linear perspective, given by Taylor [13], [14] in his 1715 treatise and in common use today. You can tell a student: to find the vanishing point of a line, point your index fingers parallel to it; where you are pointing is the vanishing point. This is just the standard advice of classic perspective texts [15] only we do not say "where your finger points in the perspective plane" but "on the imaginary sphere surrounding you". The lack of the plane's limited view immediately implies the existence of two symmetrical vanishing point as soon as the student asks "point in which direction?" and you answer "in both".

These notions have been used for several years [12] for teaching Ph.D. students of a Digital Media Arts Ph.D. program in a way that combines computer graphics algorithms, geometry and drawing. Spherical anamorphosis is made concrete first by the use of an adapted Dürer machine, made to run as an anamorphosis generator; then it is abstracted through descriptive geometry, and finally made immersive through combination with VR and with equirectangular perspective and VR panoramas. It has been suggested [3] that an adaptation of the same method could be used with young students with a dual purpose: to use technologies such as AR to facilitate the teaching of the principles of perspective and descriptive geometry, and to use these principles to open the black boxes that hide the implementations of these technologies. This feedback loop would help students to understand both the theory of representation and the digital technologies they use every day.

3 THE TRIAL

The study put forward several questions, with regards to didactic applications to the 3rd cycle of Portuguese elementary education (grades 7-9, modal age 12-14 years):

- What useful connections can be established between the founding principles of perspective and the possibilities of AR?
- What are the potentialities of using AR applications for the study of representation systems in elementary education?
- What is the full scope of possible uses for Dürer's machine in the study of geometry in the 3rd cycle of elementary education?
- Can the new didactic approach relating anamorphosis, perspective and AR described in [1], [3], [12] be adapted to elementary education?

The study was carried out in two classes of the 3rd cycle in a school in the north of Portugal. The experiences described here are part of a broader formative itinerary, to be developed in future work.

With this educational path, we tried to assess the impact of a new methodology for the study of the perspective in the discipline of Visual Education. This formative itinerary combines the principles of anamorphosis with the new technologies based on augmented reality.

The Visual Education program for the Portuguese 9th grade provides for the development of a didactic unit on representation systems that includes a historical approach to Brunelleschi's and Dürer's contributions to perspective. However, as already mentioned, the theme of anamorphoses is given little relevance in Visual Education textbooks, limited to mentions of deformed images on distorted grids. Within this program we intend to give a new centrality to the notion of anamorphosis as the foundational concept common to perspective, and VR/AR/MR.

In this investigation we follow the *Design-Based Research* (DBR) approach (see Collins [16] and Brown [17]) which is well adapted for the integration of new technologies/educational resources in pedagogical practice. In the educational context, DBR-based research is an approach that involves interactive design processes to develop knowledge and improve educational practices. This methodology arose because some researchers considered that the traditional research models failed to improve educational practices in the context of the classroom. Design-Based Research sees researchers as agents of change, with a focus on deploying interventions to test innovative educational practices and generate effective educational environments. This approach is adequate to the present research objectives due to its dynamic, procedural, contextual and participatory character, this being a theme that allows the application of new digital resources in the classroom. Today, the majority of students own a smartphone, which has enormous potential for educational use and that can be used with benefits both in terms of student motivation and the improvement of their learning, namely in the study of perspective.

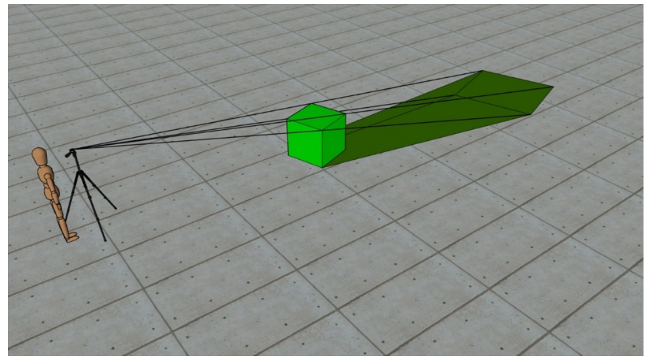

Figure 3: SketchUp image presented to students

3.1 Participants

Thirty-seven students (twenty-four male, thirteen female) from two classes of the third cycle of elementary education aged between 14 and 16 years old from a public school in the north of Portugal participated in this study. Among these, eight students lacked their own smartphone and used tablets provided by the school.

This project adhered to international research ethics standards in education, with respect to good practices in scientific research, ethical principles of investigation, and quality of investigative process, data confidentiality, informed consent, voluntary adherence of participants and assurance that data collected was used for research purposes only. A written document was distributed to students and their guardians describing the purpose of the investigation and informed consent was obtained, as well as the right to use of the images here presented for research purposes. In the presentation of the data, fictitious names are used.

3.2 Technologies

With regard to Augmented Reality technology, we used the HP Reveal application (formerly Aurasma), a free platform for iOS and Android that uses image recognition to combine the real world with interactive virtual content, such as videos and animations, called "Auras". We also used the Augment platform that lets users view their products in 3D in a real environment using tablets or smartphones. This platform also allows a web application to manage 3D content. SketchUp was used for the creation of 3D models since its output is compatible with the Augment platform (Figure 3).

3.3 Data collection and analysis

The data presented here were collected through direct observation and open-ended questioning of the students. We describe student reactions to the various phases of the experience through oral questioning - focus group - conducted throughout the project development. The objective was to examine how the students experienced this path and how they reacted throughout the various phases of their development, giving students a voice to express, in their own words, their experience and the results obtained. The open-ended questions were the object of a content analysis, opting for an emergent categorization based on the semantic criterion which allows

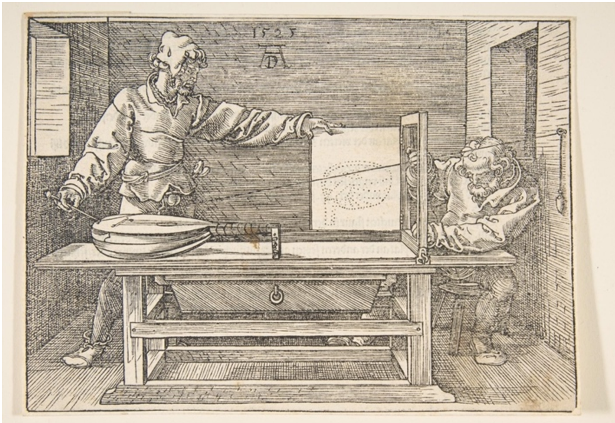

Figure 4: The draughtsman of the lute, Dürer, 1525

"inferences by systematic and objective identification of the specific characteristics of a message" [18].

4 ACTIVITIES AND FINDINGS

Ahead we present the tasks performed by students throughout this intervention, relating their experience through the images they produced together with their collected words and reactions.

4.1 Radial Occlusion and the Dürer machine

Our starting point was an analysis of the well-known engraving "The Draughtsman of the Lute" by Albrecht Dürer (1525) (Figure 4).

Students searched the internet for the historical contextualization of the image, then analyzed it through the interpretation of the content in different dimensions, following the *pedagogy of the education* of the eye proposed by the Integrated Program of Visual Arts.

After this they were assigned the task of using the Dürer machine for representing a cube not only in perspective but in its oblique anamorphic projection in the horizontal plane. With this activity, it was intended for students to identify that the common principle at the core of both perspective and oblique anamorphosis is radial occlusion: two points will look the same to an observer at the viewpoint O if and only if they are on the same ray from O [3]. As described in [3], [12], Dürer's machine can be run backward (from object to vertical canvas) to draw a perspective or forwards (object to table) to make an oblique anamorphosis. These are equivalent to each other and to the object, by radial occlusion. Following Araújo [3] we then opt to define anamorphosis as the equivalence relation entailed by radial occlusion. Equivalent objects may be obtained by any transformation preserving rays, like radial projection towards any surface, curved or not. Perspective is then a special case of anamorphosis.

Just as important as the theoretical explanation is the description of the necessary physical procedures required to actually draw the vertices of the cube and its edges in projection. Anamorphic projection at steep angles to the projection plane can be fiddly and prone to error without proper orientation.

Figure 5: Preparation for the anamorphic construction

Students were asked to use the Dürer machine to project and draw the anamorphosis of a cube onto the table. With a string fixed at a point on a wooden rod, they projected each vertex of the cube on a sheet of paper on the horizontal plane. This work was carried out in groups of four students.

In this phase, the students prepared a work plan (including setup of work groups and task distribution) to be carried out in the construction of a large scale anamorphosis in the school hall.

4.2 Large-scale anamorphosis of a cube

For the large-scale anamorphosis, a camera tripod was used to define the viewpoint, and a cardboard box represented the cube. A cotton rope radiated the vertices onto the floor, to be marked with vinyl adhesive tape. All students took part in the construction (Figure 5, Figure 6).

This physical projection proved crucial for students to internalize the principle of radial occlusion. As claimed in [12] the Dürer machine makes the abstract principle clear by materializing it. After the construction, the students held an evaluation of the main obstacles/difficulties in its execution. All students showed great commitment and enthusiasm in carrying out this task, some taking on leadership roles and distributing tasks to other colleagues.

Students then took time for free expression, eagerly taking multiple shots of themselves interacting with the anamorphosis. With the aid of the application HP Reveal the students performed several Mixed Reality experiences, using the anamorphic cube as a marker to trigger AR images and animations (Figure 7). This made clear that AR itself is an interactive implementation of the same principle of anamorphosis. This use of anamorphosis as trigger, which requires care in finding the exact point of the observer, focuses the attention of the students in the importance of this point, thus stressing in a playful way a fundamental matter for the understanding of the later formal study of conical perspective within the official educational curriculum.

Later, back in the classroom, the students were given a series of images of anamorphic works by street artists Julian Beever, Edgar Mueller and Sérgio Odeith, as QR codes to explore in their phones

Figure 6: Construction of large-scale anamorphosis. Note the images of verticals converging to a vanishing point on the ground plane under the viewpoint *O*

and tablets. We found that these playful and disconcerting images continue to arouse interest in the plastic arts [19], and especially in street art where anamorphic graffiti artworks seem to impress as much as ever.

Due to their relatively small distance to the observation point these illusions are only at their most compelling when seen through the camera, and are therefore already a type of mixed reality object that requires the camera to reach their full effect. The most interesting reaction from the students was not their amazement – common to any observer of these pictures – but their sudden understanding of the common principle in action as revealed by their comments:

> "Ah! ... but this is what we did!" João (9B)
>
> "So we could also make a drawing like this?" Mariana (9B)
>
> "These images look like magic, but now we know the trick." Vasco (9B)
>
> "I really enjoyed doing this work, but now I understand why it matters." Diana (9B)

4.3 Orthographic construction of an anamorphic cube

The next step was to abstract the string of the Dürer machine by replacing it an equivalent orthographic construction drawn with ruler and square (Figure 8). We diagrammed the Dürer setup with a side and top view of a 3.5cm cube. To ensure that all students were successful in carrying out this exercise, even those who normally show more difficulties, we proposed to carry out the drawing on an A3 size sheet of paper, indicating that the cube is 16.5 cm from the observer's point and that the observer is placed 8 cm high in relation to the floor. In this way, all students could validate their progress with the help of the teacher and colleagues.

After completing this exercise, students were asked to place the smartphone's photographic lens in the exact spot of the observer's eye determined by the side and top views, and take a photo to

Figure 7: Augmented reality overlay using the physical anamorphosis as a marker for *HP Reveal*

reveal that the anamorphic illusion had been achieved just as in the construction with the physical thread. Students' reactions were enthusiastic at the success of the illusion.

It is important to realize that as speculated in [3], students were led by the lure of the optical illusion and guided by the physical experience of the Dürer machine to make intuitive, effective and enthusiastic use of the basics of descriptive geometry without any prior training, this being a discipline notorious for having a hard time captivating students, and for giving students a hard time with its requests upon their capacity for spatial visualization and abstraction. And yet they achieved enthusiastically a result whose correct execution is not only visually verifiable but "inherently instagrammable" [3].

As an extra challenge, it was also suggested that students fold the leaf across the ground line. Andreia (not her real name) decided on her own initiative to emphasize the green auxiliary lines and

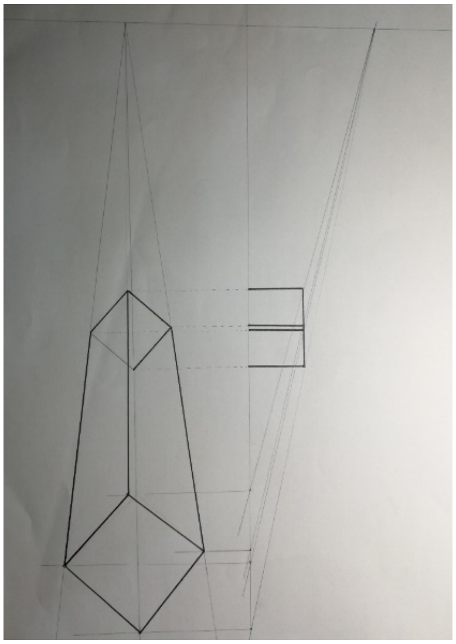

Figure 8: Orthographic views. Drawing by Alexandre 9A

strengthen the edge of the box on the upper surface (to mitigate the thinning effect due to the distance), obtaining the impressive result in Figure 1, shown under the title of this paper. This is a rather *meta*-effect, with an anamorphic view of an orthographic projection, and many levels of abstraction above the trivial exercises involving grid deformations. The fact it could be done by students who never took a formal class on perspective or descriptive geometry is indicative of the validity of the speculations made in [3], [12] regarding the conceptual power and clarity of setting radial occlusion as the foundational concept of both anamorphosis and perspective, and of using the Dürer machine as its physical materialization that makes it intuitive and easy to understand.

Reference to this principle eliminates the visual and conceptual confusion that might arise from the metric deformation of anamorphosis, a conceptual problem that is usually made worse by the ordinary focus on so-called "perspective deformation" or arbitrary "maximal cones of vision" or "Leonardo's paradox", concepts that are often entirely misunderstood. Instead of deformations we have anamorphosis as an equivalence relation (in the mathematical sense of the word), the simple notion that it doesn't matter what the thing looks like outside of point O, because as long as points are in the same ray they are "visually equivalent from O". From this single notion everything arises, even the notion of vanishing point, as an entity in space – a direction in the visual sphere rather than just a point on an arbitrary surface of projection.

This exercise began to show the effectiveness of this approach, as the students already showed some understanding, ability to interpret and critical attitude towards the work developed, registered by the attitudes shown in the classroom and, still, visible in the comments that follow.

> "I found the work very interesting and fun, I don't usually like the course very much, but I really liked this task. I would like to do more tasks like this in the future." David (9B)

> "I found it interesting and fun, as it was a new experience." Margarida (9A)

> "I found it an interesting and simple job to do." Diogo (9B)

> "This job was fantastic, it was cool." Rúben (9B)

> "I liked it a lot, I found it interesting and I learned new things." Carolina (9A)

> "The work is very good, I found it very fun and innovative." Ana Beatriz (9A)

> "I found it interesting, it was easy and I liked the result." Nádia (9B)

These reactions reflect, once again, the adherence and enthusiasm of the students who participated in this intervention, which leads us to affirm its contribution to the development of the skills of visualization, analysis and interpretation of elements of space and representation through drawing. Implicitly, we approach notions of coordinates, vertical plane, horizontal plane, land line, elevation, (visual) angle and distance, fundamental concepts for the study of descriptive geometry and perspective.

4.4 Light, color and shadow in anamorphosis

Using the previous construction, the students continued their work of creative expression with the application of the knowledge already acquired about light and shadow. Thus, each student freely painted their cube, seeking to respect the basic elements of three-dimensionality (Figure 9).

In this work, students had the opportunity to relate new knowledge with content already studied, namely, on the use of shadows for the representation of three-dimensionality, seeking, in this way, to apply knowledge in new learning experiences. All students carried out this task successfully and enthusiastically as can be seen from the comments below.

> "I found this experience interesting and fun." Vasco (9B)

> "I found it interesting, a new point of view that I didn't know." Lucas (9B)

> "I liked the idea of the work. The end result was spectacular." André (9th A)

> "I found a different and interesting job. The best part was seeing the final effect with the smartphone." João Nuno (9A)

> "It was a lot of fun because we learned new things and also explained what an anamorphosis is." Andreia (9A)

> "It was great. . .! I'm going to take a picture to show my brother." Luís (9B)

The students were also invited to use the augmented reality applications HP Reveal and Augment to photograph a mixed reality

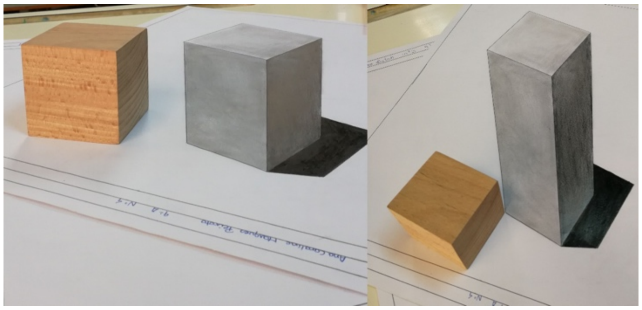

Figure 9: Anamorphosis by Ana Carolina (9A)

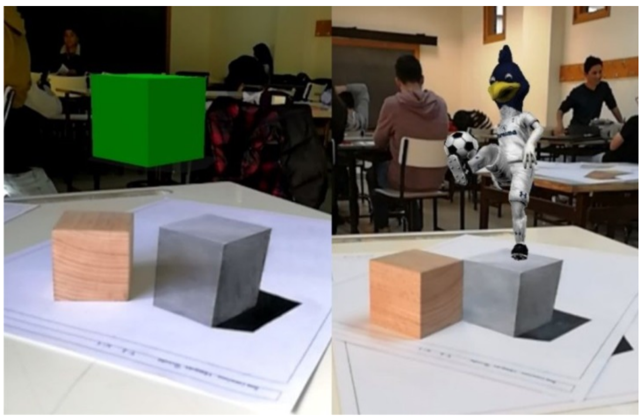

Figure 10: Mixed Reality composition by Ana Carolina (9A)

scenario (Figure 10) that integrated virtual reality (green cube provided via QR code), a real cube, in wood (brown) and a handmade shaded. anamorphosis performed by each of the students (gray).

In this task, the smartphone proved to be an extremely important resource for the accomplishment of the proposed tasks, which was the pleasure of the students, as can be seen from the comments collected during the experience.

> “I had never used my cell phone so much in a class! It was cool and fun.” Andreia (9A)
>
> “After all, reality also deceives us!” Rodrigo (9B)
>
> “What I liked most was the animations.” José Miguel (9B)
>
> “I’m not sure which one I like best? ... all three seem like reality!” Alexandre (9A)

The notion of anamorphosis as equivalence relation between objects, and the equivalence between the various forms it can take, from oblique drawing to AR construct is eloquently expressed by this last student’s remark.

Students made several photographic records exploring the effects of the illusion created by anamorphosis and recognized once again the importance of identifying the viewpoint of the observer, as a spatial point floating above the vanishing point of the verticals.

The graphical representation of the shadows in this exercise provided an opportunity for students to better understand the notion

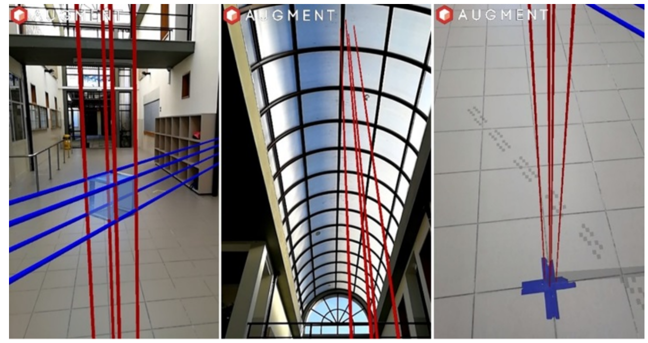

Figure 11: Extension of cube edges to their vanishing points using the Augment application

of depth and simultaneously stimulate their creativity and expressivity. It was also an opportunity for the expressive representation of the concepts of light and shadow, and the understanding of how these vary with the incident angle of light (in diffuse reflections).

4.5 Duality of Vanishing points verified with Augmented Reality

We then considered it appropriate to revisit the first anamorphosis carried out in the school hall in order to challenge students to identify the vanishing points of that representation. After some questioning from the students it was proposed that they follow a QR code link that contained an image of a cube with the extension of its edges and that helped to identify the vanishing points in the anamorphosis built in the school hall (Figure 11).

Due to the natural difficulty of abstraction in students of this age group, it was not clear to us that everyone properly understood the notion of vanishing points applied to this anamorphosis. Nonetheless, some students understood the exercise, as exemplified by Mariana (9b) who commented: "Teacher, I already knew where the vanishing points were". Crucially, it was also clear that each line corresponds to exactly two vanishing points. This concept, involving a great deal of abstraction and not at all corresponding to what is usually taught in perspective arises nonetheless very naturally from the exercises we performed. It is indeed clear that any of the parallel sets of lines of the cube could be extended to meet, in projection, at some point of a surrounding building. This is an immersive notion of anamorphosis, defined for the purposes of spherical perspective work, but very natural when seen in the context of immersive visualizations. The practical difficulty of prolonging the line projections until they meet in a large room (Figure 11) where they would have to meander over a large and intricate space, is resolved by the AR projection, which shows exactly where the lines would have to be drawn and what path they would take to finally meet at the expected points – the points themselves are easily found by just pointing a finger parallel to the spatial line and aligned with the eye and find where that “ray” hits a wall. One may define vanishing points by that construction (bring the line to O and intersect with the visual sphere). This is a theorem in [2] but

Figure 12: Anaglyph. Stereoscopic anamorphosis of a cube

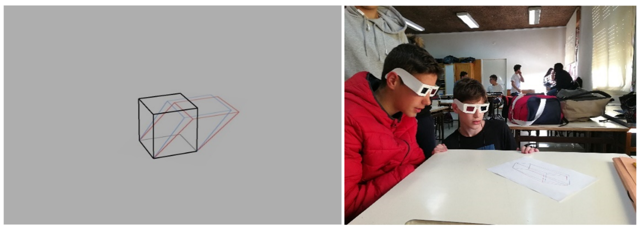

Figure 13: Left: SketchUp representation of the anaglyph. Right: Students observing the anaglyph with 3D glasses

Figure 14: Artifact created to draw a cylindrical perspective

as suggested in [3] it may without loss be taken as a simpler definition for younger students, corresponding neatly to the analogous definition in classical perspective due to Taylor [13].

4.6 Anaglyphs

Again in the classroom, and resuming the work of the orthographic projections of the cube, we moved to a binocular experience by doing an anamorphosis adapted to each eye (in red and blue respectively) and seeing the resulting object with red-blue 3D glasses (Figures 12, 13). This anaglyph creates a true notion of depth and 3D presence. The sensation of this exercise is startling even for adults [3] and is akin to seeing a MR display with no need for electronics. For more on anamorphic anaglyphs see [20], [21].

With this exercise we intend to demonstrate that it is possible to create virtual images using simple technology, again showing how the principle of radial occlusion unites many disciplines. The students were enthusiastic about this experience and made the following comments that clearly demonstrate their positive and surprised reactions to what they were discovering.

> "You can see it really well... There really is a little cube there"
>
> "I see a perfect cube! Oh! Oh my God it's beautiful"
>
> "It's funny (smiling intensely). Ah...! Ok... It's there in middle! It looks like a parallelepiped... Ah! There it is!"
>
> "And if I get up, the cube grows. . .! How strange. . . so cool"
>
> "So the cube grows... now it is here" (he pointed with his finger). "Look at me picking him up."

The last comments refer to the fact that if one moves the eye(s) along a vertical through O, the cube deforms while preserving the vertical's vanishing points, seeming to grow like a 3D animated bar chart. This is a lesson in vanishing points presented to university students in the program described in [12] and yet also perfectly understandable for elementary students, if well guided.

4.7 Curvilinear Perspectives

After this, we transitioned from immersive anamorphoses to curvilinear perspectives. We asked students to observe the room while turning around a vertical axis, and to imagine a way to draw all the walls into a single drawing: "Imagine yourself drawing inside a transparent cylinder". With the purpose-built artifact in Figures 14 and 15, students did just that, intuitively realizing a cylindrical anamorphosis onto transparent acrylic, looking through a hole placed in the cylinder axis. The cylindrical perspective is just the flattening of this drawing onto the plane (Figure 16). In the flattening, lines look curved (sinusoidal) and are no longer anamorphic to the original ones, but still represent the same visual information, in the sense that the anamorphosis could be recovered from the perspective (by reversing the flattening).

Analyzing the result by comparing it to a panoramic photo taken with a smartphone, we moved on to an equirectangular perspective (Figure 17). Unlike the cylindrical perspective, equirectangular perspective captures everything around the observer: it is a total spherical perspective [2], [22], that is, a cartographic flattening of a full spherical anamorphosis, which preserves all its vanishing point pairs. It is not as intuitive, since it doesn't result from a simple (isometric) unfolding like the cylindrical case, but the cylindrical case serves as a stepping stone in the ladder of abstraction. The equirectangular case makes an interesting connection with VR since it is the main format for 360-degree photography. Hence a handmade drawing done in this perspective can be easily viewed as a VR panorama using software readily available even in social platforms like facebook. This simplifies the creation of so-called hybrid models [23], [24], objects straddling the material and the digital field, that are increasingly being used as both artistic media [25], [26], [27] and for technical application in architecture, design, and heritage survey and dissemination [24]. This is an emergent field in which both theoretical methods and computer tool are still being developed [28] and yet it can easily be presented to young students in a very practical way, through the use of perspective grids. In this way students were invited to represent the room in

Figure 15: Artifact created to draw a cylindrical perspective

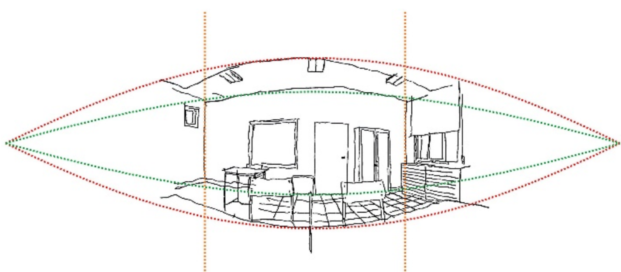

Figure 16: Identification of vanishing points in a cylindrical perspective

Van Gogh's Arles using an equirectangular grid (Figure 17). These perspectives should not be seen so much as an advanced topic, but as an instance of a wider definition of immersive perspective that makes linear perspective more understandable as a special case.

5 CONCLUSION AND DISCUSSION

Overall, the activities of this trial had a strong positive influence over students' reported motivation to study 3D representation. The immersive anamorphosis approach helped to overcome common difficulties in the teaching of graphic representation. It helped to recognize the importance of identifying the viewpoint and the vanishing points in the study of perspective. AR applications proved useful for visualizing the vanishing of parallels towards their vanishing point pairs.

Importantly this view of anamorphic vanishing points – although non-standard – did not conflict with the view required in the standard curricula of linear perspective as it reduces to it in the special case where the projection surface is a plane. But it provides a more powerful definition which led the students in easy succession to tackle problems not only in (classical) anamorphosis and perspective but also in curvilinear perspectives, where dual vanishing points are required (such as in cylindrical and equirectangular perspective). The frankly surprising number and complexity of the concepts acquired (and confidently operated upon) is a demonstration of the promise of this approach. This is manifest in the way the Dürer "anamorphosis machine" motivated the use of basic descriptive geometry in pursuit of an enticing goal of the optical illusion, descriptive geometry being a discipline with notoriously reluctant reception by young students, who struggle with the required abstraction and spatial visualization.

An yet it was with clear motivation, confidence and ease that the students performed these exercises of three-dimensional representation (including the special needs students).

It is therefore supported by this initial trial that the combination of the immersive anamorphosis concept, materialized by the Dürer machine, abstracted by descriptive geometry, and integrated with augmented/mixed reality technologies, has great potential, both in terms of student motivation and in terms of understanding the contents of the various disciplines. It is also worth mentioning its use in the development of the spatial visualization capacity of students in the 3rd cycle of elementary education, thus promoting a better understanding of the concepts related to the teaching of geometry.

Further, it leads to an understanding of the technologies of AR independent of platform-specific implementations.

These results are compatible with the hypothesis of regarding the advantages of using anamorphosis rather than perspective as the fundamental generating concept, and perspective as the derived one, which is argued, is justified both mathematically, historically, and, it is speculated, didactically, the latter seemingly supported by the observations of our trial.

The trial also led to lateral applications, such as the study of light and shadow, namely with regards to the distinction between cast shadows and form shadows, the careful perception and measurement of shadow value/light intensity (in the pursuit of convincing optical illusion), its importance in improving the perception of three-dimensionality.

The results presented here are part of a broader formative itinerary conceived within the scope of the first author's Ph.D. work in the doctoral program of Digital Media-Arts, a joint degree of Univ. Aberta and Univ. of Algarve. This work seeks to analyze the impact of these interventions in the study of perspective in the third cycle of elementary education. Longitudinal studies having been cut short by the current pandemic; we meant to follow students into secondary education to assess the effects of this intervention in their study of descriptive geometry and perspective. This will be an object of future research when conditions allow. These preliminary results allow us nonetheless to conclude for the relevance and adequacy of this approach to 3rd cycle elementary education, as demonstrated by both the students' achievements and their reactions throughout the intervention. We hope that this work may be replicated in other contexts, in order to explore its potential and solidify its findings.

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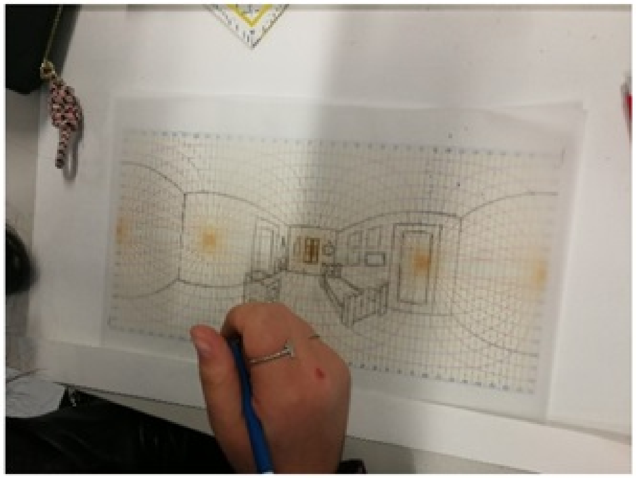

Figure 17: Equirectangular drawing

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