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**Training on Mathematics (TryMaths) Workshop**
**An Introduction to Mathematical Analysis With Real-Life Problems**

# An Introduction to Ordinary Differential Equations

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**Abstract**

This text was prepared as support to a training workshop for teachers of the University of Savannakhet, Laos. Its objective is to provide a very brief introduction to modelling using differential equations, and to present the subject so as to stress basic principles and methods, as well as to point to required background in Mathematical Analysis that further serious study will require.

## Contents



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# 1 Introduction

It is a fact of life, and maybe the greatest insight of Isaac Newton (1642-1726), that humans can better describe nature by measuring and reasoning about very small, "instantaneous" changes. This fact has several far reaching consequences:

**(i)** the importance of *differential equations* in modelling and describing natural and technological phenomena;

**(ii)** the importance of derivatives and of *differential calculus;*

**(iii)** the importance of a reverse process in order to compute the quantities of interest from their rates of change, i.e., an integration procedure and the corresponding *integral calculus;*

**(iv)** the need for a rigorous and coherent theory encompassing all these processes: *Mathematical Analysis.*

In these five lectures we shall give a very brief introduction to ordinary differential equations used as models to some simple real life phenomena and this will allow us to refer to a number of important concepts and results in Mathematical Analysis that underpin the techniques used to study those equations.

# 2 First order linear equations

## 2.1 A simple model for population growth: Malthus (1798)

Consider a population in an isolated location where there is neither emigration nor imigration. In this circunstances, how can the amount of population change with time?

Let $x(t)$ be the population density at time $t$ (= number of inhabitants per unit area). Let $\Delta t$ be a small increment in time and let $\Delta x$ be the corresponding change in population density. Then

$$\Delta x = x(t+\Delta t) - x(t). \tag{1}$$

Now, it is clear from the isolation hypothesis that

$$\Delta x = \Big(\text{births in } ]t, t+\Delta t[\Big) - \Big(\text{deaths in } ]t, t+\Delta t[\Big). \tag{2}$$

It is reasonable to assume that the number of births in the time interval between $t$ and $t+\Delta t$ is proportional to the number of persons at time[3] $t$, $x(t)$, and to the size of the time interval, $\Delta t$. Thus

$$\Big(\text{births in } ]t, t+\Delta t[\Big) = b\,x(t)\,\Delta t, \tag{3}$$

[3] We are assuming the population does not vary significantly and the time interval is not very long, so that the population can be considered almost constant and so the births can be considered proportion to the population at the initial time.

where $b$ is the birth rate, that is: the average number of offspring per person, per unit time.

Similarly, we can assume that the number of deaths in the interval $]t, t+\Delta t[$ can be described by

$$\Big(\text{births in } ]t, t+\Delta t[\Big) = \mu\, x(t)\, \Delta t, \tag{4}$$

where $\mu$ is the death rate (the average number of deaths per person, per unit time).

Thus, using (1)–(4),

$$x(t+\Delta t) - x(t) = b\, x(t)\, \Delta t - \mu\, x(t)\, \Delta t,$$

which can be written as

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} = (b-\mu)x(t),$$

and passing to the limit as $\Delta t \to 0$ in this equality results in the differential equation

$$\frac{dx(t)}{dt} = r\, x(t), \tag{5}$$

where $r := b - \mu$ is the per capita intrinsic rate of change of the population, and where $\frac{dx(t)}{dt} := \lim_{\Delta t \to 0} \frac{x(t+\Delta t)-x(t)}{\Delta t}$ is the derivative of $x(t)$ at $t$.

Equation (5) has as its unknown the function $x(t)$, and this unknown also shows up as the derivative $\frac{dx(t)}{dt}$ computed at the same value of the single independent variable $t$. Because of these, (5) is called an *ordinary differential equation*.

This very simple (and simplistic) example exhibits the characteristic we started to call your attention to in the Introduction: *it is far easier to get a law for the rate of change of a quantity then for the quantity itself:* obtaining the law (5) was a relatively easy task, much easier than to get the expression for $x(t)$ directly from the assumptions about the population (its isolatedness as well as its birth and death laws). However, having obtained the differential law (5) for the evolution of the population, it is necessary to get, from this differential equation, the expression of $x(t)$, i.e., to solve the differential equation. Although it is not exactly difficult in this case, the explicit solution of differential equations can be a dauting task and, in many cases, an impossible one using the usual functions of calculus. We shall see an example of this difficulty later in this course, and shall then discuss what can be done in those cases. Let us, for the moment, go back to (5).

How to solve an equation like (5)? The problem is the one of finding a function $x(t)$ for which both sides of (5) make sense and the equality holds for all values of $t$ in a certain interval (to be determined as part of the answer). Hence, $x(t)$ must be differentiable, because $\frac{dx(t)}{t}$ shows up in the left-hand side of the equation, and for all values of $t$ this derivative has to be equal to the function $x(t)$ multiplied by the constant $r$.

Let us start with an easier problem:

- How to solve equation $\frac{dx(t)}{dt} = f(t)$ for a given function $f(t)$?

If we know that the derivative of a function $x(t)$ is a given known function $f(t)$, then determining the function $x(t)$ is the reverse operation of differentiation, i.e., taking the *anti-derivative,* or the *primitive:*

$$x(t) = \int \frac{dx}{dt} dt = \int f(t) dt.$$

Remember that the anti-derivative is not a well defined operation, in the sense that it never gives a unique answer: there is always an infinite number of anti-derivatives differing from each other by constants, so we really need o write

$$x(t) = \int f(t) dt + C,$$

where $C$ is a constant number. Apart from this kind of non uniqueness, there are no other anti-derivatives. This is the content of the next two results, which pertain to the topic of Mathematical Analysis and that we recall now because of its usefullness to what comes next.

**Proposition 1.** *Let $f$ be a function defined in $I$, an interval of $\mathbb{R}$.*

1. *Let $F$ be an anti-derivative of $f$ in $I$. Then, $\forall c \in \mathbb{R}$, $F + c$ is also an anti-derivative of $f$ in $I$.*

2. *If $F$ and $G$ are two anti-derivatives of $f$ in $I$, then $F - G$ is a constant function in $I$.*

*Proof.*

1. Lef $F$ be an anti-derivative of $f$ in $I$. Then $F$ is differentiable in $I$ and $F' = f$. Let $c \in \mathbb{R}$ be any constant; then $(F + c)' = F' + c' = F' + 0 = F' = f$, and thus $F + c$ is also an anti-derivative of $f$ in $I$.

2. Let $F$ and $G$ be two anti-derivatives of $f$ in $I$. Then $\forall t \in \operatorname{int}(I)$, $(F - G)'(t) = F'(t) - G'(t) = f(t) - f(t) = 0$. But then, by the mean value theorem, this implies that $F - G$ is a constant function in $\operatorname{int}(I)$, and thus, by continuity, it is a also a constant function in $I$. $\square$

**Proposition 2.** *Let $f$ be a function with an anti-derivative in $I$, an interval of $\mathbb{R}$. Let $t_0 \in I, x_0 \in \mathbb{R}$ be arbitrary numbers. Then, there exists a unique function $F$ which is the anti-derivative of $f$ and satisfies $F(t_0) = x_0$.*

*Proof.* Let $\mathcal{F}$ be an anti-derivative of $f$ in $I$, i.e., $\mathcal{F}' = f$. If $\mathcal{F}(t_0) = x_0$, then $F = \mathcal{F}$ concludes the proof, since Proposition 1.-2 implies that every other anti-derivative will differ from $\mathcal{F}$ by a constant, and thus its value at $t_0$ will not be equal to $x_0$. Suppose now that $\mathcal{F}(t_0) \neq x_0$. Consider $F(t) := \mathcal{F}(t) - \mathcal{F}(t_0) + x_0$. Then, since $-\mathcal{F}(t_0) + x_0$ is a constant, and $\mathcal{F}(t)$ is an anti-derivative of $f$ in $I$, Proposition 1.-1 allow us to conclude that $F$ is also an anti-derivative of $f$ in $I$. Clearly $F(t_0) = x_0$, and so the case reduces to the previous one and the proof is concluded. $\square$

We now recall the *Fundamental Theorem of Calculus* which states that, for bounded functions whose integral $F(t) := \int_a^t f(s)ds$ is well defined, and for which $f$ is continuous in $t_0$ (a point in the domain of $F$, $D_F$), then $F$ is differentiable at $x_0$ and $F'(t_0) = f(t_0)$.

In particular, if $f$ is continuous in $D_F$, then $F(t) := \int_a^t f(s)ds$ is differentiable in $D_F$ and it is an anti-derivative of $f$.

An equally important result is *Barrow's formula:* let $f, F$ be two bounded functions defined in $D \subset \mathbb{R} \to \mathbb{R}$ such that $F' = f$. Then, $\forall a, b \in D$, $\int_a^b f(t)dt = F(b) - F(a)$.

For proofs of these results and for further topics on the Riemann integral [2, Chapter 7] should be consulted.

Let us go back to the problem of solving (5). Since we know how to solve an equation of the type

$$\frac{dy(t)}{dt} = f(t) \tag{6}$$

for a given bounded continuous function $f(t)$ (as we saw above, the solution is obtained just by computing an anti-derivative of the right-hand side), it is reasonable that our first approach is to try to transform our equation (5) into an equation like (6). Notice that, in $x(t) \neq 0$ then

$$\frac{dx(t)}{dt} = r\,x(t) \Longleftrightarrow \frac{\frac{dx(t)}{dt}}{x(t)} = r \Longleftrightarrow \frac{d}{dt}\log|x(t)| = r,$$

where log denotes the natural (or Neperian) logarithm. Note that the last term above is of type (6) with $y(t) = \log|x(t)|$. Now, let us anti-differentiate both sides of the last equality with respect to time $t$:

$$\begin{aligned} \int \frac{d}{dt}\log|x(t)|\,dt &= \int r\,dt \\ \log|x(t)| &= rt + C \\ |x(t)| &= |K|\,e^{rt}, \end{aligned}$$

where $|K| = e^C$ is a positive real constant. Thus, since we are looking for differentiable (and hence continuous) functions, we can write

$$x(t) = Ke^{rt} \tag{7}$$

for $K$ a non zero constant. In fact, we observe directly from the differential equation that the identically zero function $x(t) = 0, \forall t \in \mathbb{R}$, is also a solution of (5), and so we can also consider $K = 0$ in (7).

To fix the value of $K$ we need to have extra information: we need to know the value $x(t)$ has at a given point $t = t_0$, say, if $x(t_0) = x_0$ is given, then $x_0 = x(t_0) = Ke^{rt_0} \Leftrightarrow K = x_0e^{-rt_0}$, thus

$$x(t) = x_0e^{r(t-t_0)}. \tag{8}$$

Observe that an alternative way to obtain this result is to integrate $\frac{d}{ds}\log|x(s)| = r$ between $t_0$ and $t$:

$$\begin{aligned}
\frac{d}{ds}\log|x(s)| = r \quad &\Leftrightarrow \quad \int_{t_0}^{t} \frac{d}{ds}\log|x(s)|\,ds = \int_{t_0}^{t} r\,ds \\
&\Leftrightarrow \quad \log|x(t)| - \log|x(t_0)| = r(t-t_0) \\
&\Leftrightarrow \quad \log\left|\frac{x(t)}{x_0}\right| = r(t-t_0) \\
&\Leftrightarrow \quad \left|\frac{x(t)}{x_0}\right| = e^{r(t-t_0)} \\
&\Leftrightarrow \quad x(t) = x_0 e^{r(t-t_0)}
\end{aligned}$$

We concluded that one solution of the differential equation (5) with initial condition $x(t_0) = x_0$ is (8). To prove that (8) is the *only* solution to (5) with that initial condition, observe that, multiplying (5) by $e^{-rt}$ results in

$$e^{-rt}\frac{dx}{dt} - re^{-rt}x = 0 \Longleftrightarrow \frac{d}{dt}\left(e^{-rt}x(t)\right) = 0,$$

which, after integration from $t_0$ to $t$, gives

$$\begin{aligned}
\int_{t_0}^{t} \frac{d}{ds}\left(e^{-rs}x(s)\right)\,ds = \int_{t_0}^{t} 0\,ds \quad &\Leftrightarrow \quad e^{-rt}x(t) - e^{-rt_0}x_0 = 0 \\
&\Leftrightarrow \quad x(t) = x_0 e^{r(t-t_0)},
\end{aligned}$$

thus proving uniqueness.

**Example 1.** *(From [1]). Radioactive substances have unstable atoms that decay by several physical mechanisms. The element Carbon is an indispensable component of life*[4] *the most common isotope is the stable* $^{12}\mathsf{C}$, *and has a naturally occurring radioactive isotope:* $^{14}\mathsf{C}$. *The ratio* $^{14}\mathsf{C}/^{12}\mathsf{C}$ *has been very much constant throughout history. When a living being dies its absorption of carbon ceases and, since the amount of* $^{14}\mathsf{C}$ *decreases due to radioactive decay, the ratio* $^{14}\mathsf{C}/^{12}\mathsf{C}$ *decreases with time. If* $N$ *is the number of radioactive particles in a given sample, then the evolution of* $N$ *with time proceeds according to the differential equation*

$$N' = -kN,$$

*and thus* $N(t) = N(t_0)e^{-k(t-t_0)}$, *for given* $t_0$ *and* $N(t_0)$.

*A common parameter used in radioactive decay studies is the half-life* $T_{1/2}$. *It is defined as the amount of time that has to ellapse for a given initial amount of radioactive atoms to be halfed. Using this definition it is clear that*

$$\frac{1}{2}N(t_0) = N(t_0)e^{-kT_{1/2}} \Longleftrightarrow \log(1/2) = -kT_{1/2} \Longleftrightarrow T_{1/2} = \frac{\log 2}{k} \Longleftrightarrow k = \frac{\log 2}{T_{1/2}}.$$

[4]For instance, it is the second most abundant element in the human body making up about 18.5% in weight —the most common element is Oxygen with 65% of the body weight, which is not surprising given that 50–70% of the body is made of water, $\mathsf{H_2O}$, and in a water molecule about 89% of its mass is due to the Oxygen atom. Even more important than the weight of carbon is its function: all biological mechanisms include carbon compounds and their chemical reactions.

*For $^{14}\mathsf{C}$ the half-life is* 5730 *years. Hence, the solution to the radioactive decay equation is*

$$N(t) = N(t_0)e^{-(t-t_0)(\log 2)/5730},$$

*with time $t$ measured in years.*

*Suppose a carbon sample was found in* 2003 *in a pre-historic cave, and suppose further that after precise laboratory analysis it was found that the ratio $^{14}\mathsf{C}/^{12}\mathsf{C}$ is* 14.5% *of its original value. How old is the sample?*

*From what was presented above, denoting the amount of $^{14}\mathsf{C}$ at time $t$ by $N_{14}(t)$ [and analogously for $^{12}\mathsf{C}$], and remembering that $^{12}\mathsf{C}$ is stable and so the quantity of $^{12}\mathsf{C}$ is constant over time, we can write*

$$0.145 \times \frac{N_{14}(t_0)}{N_{12}(t_0)} = \frac{N_{14}(2003)}{N_{12}(2003)} = \frac{N_{14}(t_0)e^{-(2003-t_0)(\log 2)/5730}}{N_{12}(t_0)},$$

*and thus* $0.145 = e^{-(2003-t_0)(\log 2)/5730}$*, which means that*

$$t_0 = 2003 + \frac{5730 \log 0.145}{\log 2} = -13960 \approx -14000 \textit{ years.}$$

*So, the carbon sample in about fourteen thousand years old, dating from approximately* 12000 *BCE.*

## 2.2 A model from Mathematical Finance and linear non-homogeneous equations

Let us consider the following problem [1]: suppose we have asked for a bank loan of $D_0$ kip at an interest rate of p% a year. Suppose further that you pay the loan at an annual rate of $\rho$ kip. Assume the "conversion period" (i.e., the interval of time between the computation of interests and the repayment instalments) is $\Delta t$, and the repayment is done at the end of each conversion period. With these assumptions the amount of debt you own the bank at time $t + \Delta t$ is given by the following balance

$$\begin{aligned}\left\{\begin{array}{c}\text{debt at time}\\ t+\Delta t\end{array}\right\} &= \\ = \left\{\begin{array}{c}\text{debt at}\\ \text{time } t\end{array}\right\} &+ \left\{\begin{array}{c}\text{interests charged in}\\ \text{the period } [t, t+\Delta t]\end{array}\right\} - \left\{\begin{array}{c}\text{repayment of debt}\\ \text{done at time } t+\Delta t\end{array}\right\},\end{aligned}$$

or, in mathematical notation,

$$D(t+\Delta t) = D(t) + \frac{p}{100}\Delta t D(t) - \rho\Delta t,$$

or also

$$\frac{D(t+\Delta t) - D(t)}{\Delta t} = \frac{p}{100}D(t) - \rho. \tag{9}$$

Assume now that the conversion period $\Delta t$ is let to tend to zero; this means that both the interest rates and repayment instalments are done continuously (or, better: are *computed* as if done continuously, although, due to practical issues, the actual transactions can still be done at some discrete time instants). When in (9) we let $\Delta t \to 0$ we obtain

$$\frac{dD(t)}{dt} = \overline{p}D(t) - \rho \tag{10}$$

where $\overline{p} := \frac{p}{100}$.

Equation (10) is similar to (5) treated before, but not quite the same due to the additive constant term $-\rho$ in the right-hand side. The strategy used to solve (5) by transforming it in an equation of the type $\frac{d}{dt}y(t) = f(t)$ for some known function $f(t)$ and some expression $y(t)$ dependent only on $x(t)$ was rather sucessful. So, we will try to use it again. Remember that to prove uniqueness, in page 6, we multiplied (5) by $e^{-rt}$ and obtained an equation of type $\frac{d}{dt}y(t) = f(t)$ with $f = 0$. Let us now repeat that argument for (10): adding $-\overline{p}D(t)$ to both sides of (10) and multiplying the result by $e^{-\overline{p}t}$ we have

$$e^{-\overline{p}t}\frac{dD(t)}{dt} - \overline{p}e^{-\overline{p}t}D(t) = -\rho e^{-\overline{p}t}$$

which is the same as

$$\frac{d}{dt}\left(e^{-\overline{p}t}D(t)\right) = -\rho e^{-\overline{p}t}.$$

In this problem we know that at time $t = 0$ we asked the bank $D(0) = D_0$ kip, so, integrating between $t = 0$ and $t = T$ we obtain

$$\int_0^T \frac{d}{dt}\left(e^{-\overline{p}t}D(t)\right) dt = -\int_0^T \rho e^{-\overline{p}t}\, dt$$
$$e^{-\overline{p}T}D(T) - D_0 = \frac{\rho}{\overline{p}}\left(e^{-\overline{p}T} - 1\right),$$

from which it follows that

$$D(T) = \frac{1}{\overline{p}}\big(D_0\overline{p} - \rho\big)e^{\overline{p}T} + \frac{\rho}{\overline{p}}. \tag{11}$$

Observe that (11) predicts the following expected behaviour: if the constant annual repayment rate $\rho$ is smaller than the interests charged for the first year, $D_0\overline{p}$, then the loan will never be repayed and, actually, the amound in debt will increase exponentially fast with time $T$. This is certainly a deal no bank will accept! In the other hand, if $\rho > D_0\overline{p}$ then (11) tells us that the time dependent term in the right-hand side is negative and grows exponentially fast in absolute value when time $T$ increases. So, being $D(T)$ a continuous function, the intermediate value theorem guarantees that it will eventually become zero at some $T_* > 0$. This is the time at which the loan will have been fully repayed, and it is easy to conclude from (11) that $D(T_*) = 0$ when

$$T_* = -\overline{p}^{\,-1}\log(\rho - D_0\overline{p}).$$

## 2.3 The general linear first order differential equation

The approach used to solve (10) can, of course, be applied to cases where instead of $-\rho$ in the right-hand side we have a known (continuous) function $f(t)$, and instead of a constant $\overline{p}$ multiplying the unknown function we have a (continuous) function

$r(t)$. Such an equation is called a linear non-homogeneous equation for the unknown function $x = x(t)$:

$$\frac{dx}{dt} = r(t)x + f(t) \tag{12}$$

Subtracting $-r(t)x$ from both sides of (12) we get

$$\frac{dx}{dt} - r(t)x = f(t).$$

We now try to get the left-hand side written down as $\frac{d}{dt}y(t)$ for some function $y$ depending only on $t$ and $x(t)$ (and not on derivatives or integrals of $x(t)$).

The breakthrough is achieved once we note that multiplication of $\frac{dx}{dt} - r(t)x$ by a (yet to be determined) differentiable function $\nu(t)$ results in

$$\nu(t)\frac{dx}{dt} - \nu(t)r(t)x \tag{13}$$

and this quantity is reminescent of the derivative of the product $\frac{d}{dt}(\nu(t)x(t))$ which is equal to

$$\nu(t)\frac{dx}{dt} + \frac{d\nu}{dt}x. \tag{14}$$

In fact, if $\nu(t)$ is chosen such that

$$\frac{d\nu}{dt} = -\nu(t)r(t), \tag{15}$$

then (13) becomes equal to (14). To solve (15) we just apply the same procedure as before to integrate (5), but now $r$ in the right-hand side is a continuous function. A solution is

$$\nu(t) = e^{\int_{t_0}^{t} r(s)ds}, \tag{16}$$

and it is called an *integrating factor* for equation (12).

Thus, multiplying (12) by the integrating factor (16) the left-hand side of the differential equation can be written as the derivative of $y(t) := \nu(t)x(t)$ and we get

$$\frac{d}{dt}\Big(e^{\int_{t_0}^{t} r(s)ds}x(t)\Big) = e^{\int_{t_0}^{t} r(s)ds}f(t).$$

Integrating in $t \in [t_0, T]$ gives

$$e^{\int_{t_0}^{T} r(s)ds}x(T) - x(t_0) = \int_{t_0}^{T} e^{\int_{t_0}^{t} r(s)ds}f(t)dt$$

which, after rearrangement, gives the following expression for the solution of (12) with initial condition $x(t_0)$ at $t = t_0$:

$$x(T) = x(t_0)e^{\int_{t_0}^{T} r(s)ds} + \int_{t_0}^{T} e^{\int_{t}^{T} r(s)ds}f(t)dt. \tag{17}$$

# 3 First order nonlinear equations

## 3.1 The logistic model for population growth: Verhulst (1835)

Let us return to the model for population growth with which we started our study: we obtained the equation (5)

$$\frac{dx}{dt} = rx$$

under the assumption that the net growth rate $r = b - \mu$ is a constant independent of the population effectives $x$. In reality, when the population size $x$ becomes large elements of the same population start to compete with each other for resources (be it food, living space, sexual partners) causing the net growth rate to decrease. Hence, it is reasonable to assume that $r = r(x)$ is a decreasing function of $x$. The simplest such functions are afine functions $r(x) = \lambda - ax$, where $\lambda, a > 0$ are constants.

This was exactly the assumption made in 1838 by the Belgium mathematician Pierre François Verhulst $(1804 - 1849)$, which gave rise to the equation that now bears his name. Under this assumption we have

$$\frac{dx}{dt} = (\lambda - ax)x = \lambda x\Big(1 - \frac{x}{\lambda/a}\Big) = rx\Big(1 - \frac{x}{K}\Big),$$

where $r = \lambda$ and $K = \lambda/a$ are constants.

Observe that, when $x \approx 0$ then $1 - \frac{x}{K} \approx 1$, and the equation is analogous (but not exactly equal!) to Malthus' equation $x' = rx$. So, we call $r > 0$ the *intrinsic growth rate* of the population. However, note that the fact that both equations are close to each other when $x$ is small does not imediately guarantees that their solutions will be close to each other (this is something whose truth has to be proved, but it is clearly outside the scope of these short notes).

When $x > K$ the growth rate becomes negative: the population starts decreasing. So $x = K$ is the maximum size of the population that can be supported by the environment and above this value population starts decreasing: by this reason $K$ is called the *carrying capacity* of the system.

Let us now study Verhulst equation (also called the logistic equation) with initial condition $x(t_0) = x_0$,

$$\begin{cases} \dfrac{dx}{dt} = rx\left(1 - \frac{x}{K}\right), \\ x(t_0) = x_0 \end{cases} \tag{18}$$

The idea is, again, to try separate variables as was done previously for the two linear equations $x' = rx$ and $x' = rx + b$. So, we will try to write the differential equation in (18) in the form $\frac{dy(t)}{dt} = f(t)$ where $f(t)$ is a known function and $y(t)$ is some expression containning the unknown function $x(t)$ but not its derivatives or integrals.

Dividing both termos of Verhult's equation by $x(t)\big(1 - x(t)/K\big)$ we obtain, after integrating in $t$ in the interval $[t_0, T]$,

$$\int_{t_0}^{T} \frac{1}{x(t)\left(1 - \dfrac{x(t)}{K}\right)} \frac{dx}{dt} dt = \int_{t_0}^{T} r dt,$$

which, after a change of variables in the left-hand side and using the initial condition in (18), is the same as

$$\int_{x_0}^{x(T)} \frac{1}{x\left(1-\dfrac{x}{K}\right)} dx = (T-t_0)r. \tag{19}$$

Now to integrate the left-hand side we recall the method of integrating rational functions, which involves the decomposition of the rational function into simple fractions of the type $\frac{1}{x-\alpha}$ and $\frac{1}{x^2+\beta^2}$. Let us see what happens in the present case. As the denominator of the function under the integral is already factorized we see that

$$\frac{1}{x\left(1-\dfrac{x}{K}\right)} = \frac{A}{x} + \frac{B}{\left(1-\dfrac{x}{K}\right)} = \frac{A\left(1-\dfrac{x}{K}\right)+Bx}{rx\left(1-\dfrac{x}{K}\right)} = \frac{A+\left(B-\dfrac{A}{K}\right)x}{rx\left(1-\dfrac{x}{K}\right)}.$$

Consequently, the equality of the polinomials $1 = 1+0x$ and $A+\left(B-\dfrac{A}{K}\right)x$ entails that

$$\begin{cases} A=1 \\ B-\dfrac{A}{K}=0 \end{cases} \Leftrightarrow \begin{cases} A=1 \\ B=\dfrac{1}{K}, \end{cases}$$

and we can conclude that

$$\begin{aligned}
\int_{x_0}^{x(T)} \frac{1}{x\left(1-\dfrac{x}{K}\right)} dx &= \int_{x_0}^{x(T)} \left(\frac{1}{x}+\frac{1}{K-x}\right) dx \\
&= \int_{x_0}^{x(T)} \frac{1}{x} dx + \int_{x_0}^{x(T)} \frac{1}{K-x} dx \\
&= \log\left|\frac{x(T)}{x_0}\right| + \log\left|\frac{K-x_0}{K-x(T)}\right| \\
&= \log\left|\frac{x(T)}{K-x(T)}\frac{K-x_0}{x_0}\right|.
\end{aligned}$$

Substituting into (19), applying the exponential to the resulting equation and multiplying the result by $\frac{x_0}{K-x_0}$, gives

$$\frac{K-x(T)}{x(T)} = \frac{K-x_0}{x_0} e^{-(T-t_0)r},$$

and, after some easy computations, we obtain the following expression for the solution of (18):

$$x(T) = \frac{Kx_0}{x_0+(K-x_0)e^{-(T-t_0)r}} \tag{20}$$

Some conclusions about the behaviour of (20) are noteworthy:

**1.** $x(T) \to K$ as $T \to +\infty$, for all initial conditions $x_0 > 0$.

**2.** The following result is easily computed:

$$\frac{dx}{dt} = \frac{(K - x_0)Kx_0 e^{-(T-t_0)r}}{(x_0 + (K - x_0)e^{-(T-t_0)r})} \quad \begin{cases} > 0 & \text{if } 0 < x_0 < K \\ = 0 & \text{if } x_0 = 0 \vee x_0 = K \\ < 0 & \text{if } x_0 > K \end{cases}$$

and so, the following monotonicity properties are easily concluded: if the initial condition is larger than the carrying capacity the solution is strictly decreasing, and it is strictly increasing if it is smaller (and positive).

Note that conclusion **2** on the monotonicity of solutions can be easily predicted directly from the equation, without the need to computed the precise value of the derivative $\frac{dx}{dt}$ for some particular time instant: in fact, from the differential equation in (18), namely $x' = rx(1-x/K)$, we conclude at once the signs $x' > 0$ if $0 < x < K$, and $x' < 0$ if $x > K$, and, assuming uniqueness of solutions to the Cauchy problem (18) (that will be guarranteed by the Picard-Lindelöf theorem in page 25 of Section 5), this is the same as pointed out before since, because $x = K$ is a solution of the equation, uniqueness implies that solutions that start with $0 < x_0 < K$ [resp. $x_0 > K$] will always stay in this strip [resp. semiplane], i.e., will always have $0 < x < K$ [resp. $x > K$].

**3.** The concavity properties of the solution can also be obtained from differentiating again the explicit result for $\frac{dx}{dt}$ given in **2**, or by using directly (18) instead. Let us now do it this way:

$$\begin{aligned} x' = rx\left(1 - \frac{x}{K}\right) \quad \Longrightarrow \quad x'' &= \left(rx\left(1 - \frac{x}{K}\right)\right)' \\ &= rx'\left(1 - \frac{x}{K}\right) + rx\left(1 - \frac{x}{K}\right)' \\ &= r^2 x\left(1 - \frac{x}{K}\right)^2 - \frac{r}{K} x\, x' \\ &= r^2 x\left(1 - \frac{x}{K}\right)^2 - \frac{r^2}{K} x^2\left(1 - \frac{x}{K}\right) \\ &= r^2 x\left(1 - \frac{x}{K}\right)\left[\left(1 - \frac{x}{K}\right)\frac{x}{K}\right] \\ &= \left(\frac{r}{K}\right)^2 x(K - x)(K - 2x). \end{aligned}$$

So, if $0 < 2x < K$ and $0 < x < K$ (i.e., if $0 < x < K/2$), as well as when $x > K$ we have $x''(t) > 0$ and thus $x(t)$ is convex; while if $K/2 < x < K$ the solution $x(t)$ is concave. See Figure 1 for logistic functions with $0 \leqslant x \leqslant K = 1$. Clearly this same conclusion could have been obtained (at a greater effort) by differentiating once more with respect to time the expression obtained above for $\frac{dx}{dt}$.

## 3.2 Separable equations

Verhulst equation (18), as well the linear equations studied in Section 2, were solved by the same general procedure that resulted in the "separation" of independent from

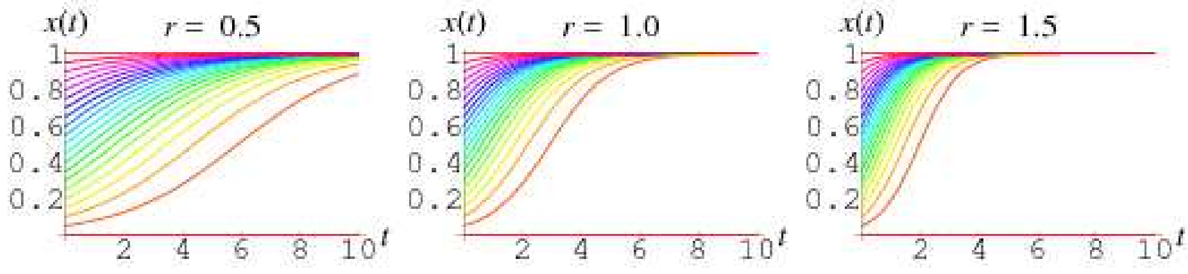


Figure 1: Logistic curves (20)

the dependent variables of the equations. We shall now look at this idea in the general situation.

In general, an equation of the type

$$\frac{dy}{dt} = g(t)h(y) \tag{21}$$

is called a separable equation, because we can proceed as we did before and separate the $t$ from the $y$ dependence: if $h(y)$ is finite and different from zero, we can multiply (21) by $\frac{1}{h(y)}$ to get

$$\frac{1}{h(y)}\frac{dy}{dt} = g(t).$$

Now we can integrate both terms with respect to $t$

$$\int \frac{1}{h(y)}\frac{dy}{dt}dt = \int g(t)dt,$$

and changing variables $t \mapsto y$ (from which $dt = \frac{dt}{dy}dy$) in the first integral we obtain

$$\int \frac{1}{h(y)}dy = \int g(t)dt + c,$$

where $c$ is an arbitrary (real) constant. Call $H$ the function in the left-hand side of this equality, that is:

$$H(y) := \int \frac{1}{h(y)}dy.$$

Then, we can write the solutions of (21) as

$$H(y) = \int g(t)dt + c.$$

Now, since by assumption $H'(y) = \frac{1}{h(y)} \neq 0$, we can apply the inverse function theorem [2, Theorem 6.1.8] to conclude that, for each given $y$, there exists a neighbourhood of that $y$ in which $H$ has a differentiable local inverse, $H^{-1}$, and thus the solution of (21) can be written as

$$y(t) = H^{-1}\left(\int g(t)dt + c\right). \tag{22}$$

All cases studied above were, sooner or later, reduced to a particular case of such a general proceedure.

## 3.3 Change of variables

Some equations, not being separable, can be transformed into separable equations by an appropriate change of variables. An example of these are the so called *non-linear homogeneous equations,* which have the form

$$y' = f\left(\frac{y}{t}\right) \tag{23}$$

where $f$ is a differentiable function of a single real variable $z := y/t$. This form for the right-hand side of the equation suggests we consider the change of variables $y \mapsto z := y/t \Leftrightarrow y = tz$:

$$\frac{dy}{dt} = \frac{d}{dt}(tz) = z + t\frac{dz}{dt} \qquad \text{and} \qquad f\left(\frac{y}{t}\right) = f(z).$$

Thus (23) becomes

$$\begin{aligned} z + t\frac{dz}{dt} &= f(z) \\ t\frac{dz}{dt} &= f(z) - z \\ \frac{dz}{dt} &= (f(z) - z)\frac{1}{t}, \end{aligned} \tag{24}$$

which is a separable equation! Of course after solving (24) we must return to the original variable $z \mapsto y = tz$ in the computed solution.

# 4 Second order linear equations

Until now we studied ordinary differential equations in which the highest derivative was the first. Differential equations of order higher than the first are also very important, both theoretically and in applications. Of these, second order equations are particularly important due to they relevance in Physics: the fundamental relation of dynamics, or Newton's second law, expresses the relation between the force $F$ acting on a body and its acquired aceleration $a$. In one dimensional physical systems the law is $F = ma$. The way second order differential equations shows up in this law is due to the fact that

$$a = \frac{dv}{dt}$$

where $v$ is the body's velocity, which is the derivative of the displacement $x$ of the body with respect to time $t$:

$$v = \frac{dx}{dt}.$$

Thus

$$a = \frac{dv}{dt} = \frac{d}{dt}\frac{dx}{dt} = \frac{d^2x}{dt^2},$$

and Newton's second law can be written as follows:

$$\frac{d^2x}{dt^2} = \frac{1}{m}F. \tag{25}$$

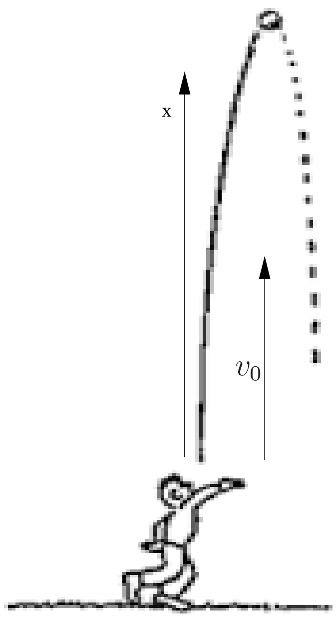


Figure 2: A body thrown from the ground (nearly) vertically upward with an initial velocity $v_0$.

It is clear that, being $m$ a constant (called the inertial mass), if the force $F$ depends only on $t$ (and not on $x$), then the solutions to (25) can be obtained by two integrations. Let us start by considering one such case: suppose a body is thrown from the ground vertically upward with an initial velocity $v_0$. With the coordinate axis shown in Figure 2, the force acting on the body is due to gravity and is $F = -mg$. Thus the differential equation describing the trajectory is

$$\frac{d^2x}{dt^2} = \frac{1}{m}(-mg). \tag{26}$$

with initial condition

$$x(0) = 0, \qquad \frac{dx}{dt}(0) = v_0. \tag{27}$$

Thus, integrating (26) twice we obtain

$$\begin{aligned} \frac{dx}{dt}(t) &= -gt + \alpha \\ x(t) &= -\frac{1}{2}gt^2 + \alpha t + \beta, \end{aligned}$$

for constants $\alpha$ and $\beta$ to be determined using the initial condition (27):

$$\begin{cases} v_0 = -0 + \alpha \\ 0 = -0 + 0 + \beta \end{cases} \Leftrightarrow \begin{cases} \alpha = v_0 \\ \beta = 0. \end{cases} \tag{28}$$

Hence, the trajectory of the body is given by

$$x(t) = v_0 t - \frac{1}{2}gt^2.$$

## 4.1 Free vibrations in a spring-and-mass system

The next simplest equation of second order is actually also an important model from Physics. Consider a spring with a mass $m$ attached to one end and fixed at the other end; suppose the spring can oscillate without any damping.

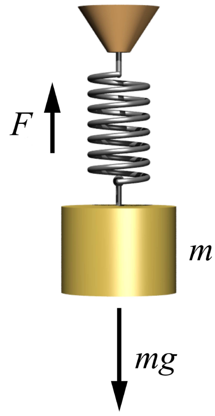


Figure 3: Spring-and-mass oscillator.

If we displace the body from its equilibrium position to a new position at a point at a distance $|x|$ from the equilibrium point, the spring will exert a force that, for small displacements, is proportional to $x$ and acting in opposite direction to the displacement $x$, so

$$F = -kx$$

and the equation describing the vibrations of the spring-and-mass system is

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0.$$

Without loss of generality we can consider[5] $\frac{k}{m} = 1$.

The general way to solve this equation of second order is to transform it into a first order system and apply the theory that can be developed for those systems. The idea is simple: write the second derivative $x''$ as a first derivative of some auxiliary function to be defined. From $x'' = (x')'$ it is natural to consider the auxiliary function $y$ defined by $y := x'$. Thus, our second order equation can be written as

$$x'' + x = 0 \Leftrightarrow \begin{cases} x' = y \\ y' + x = 0. \end{cases} \tag{29}$$

[5]In fact, by changing variables $t \mapsto \tau = t\sqrt{\frac{k}{m}}$ and $X(\tau) = x(t(\tau))$ we have

$$\frac{d^2X}{d\tau^2} = \frac{d}{d\tau}\frac{dX}{d\tau} = \frac{d}{d\tau}\frac{dx}{dt}\frac{dt}{d\tau} = \frac{d^2x}{dt^2}\left(\frac{dt}{dx}\right)^2 = \frac{m}{k}\frac{d^2x}{dt^2} = -x(t) = -X(\tau).$$

Thus, the equation for $X(\tau)$ is $\frac{d^2X}{d\tau^2} + X = 0$, an we conclude that we can always transform the original equation into one for which $k/m = 1$.

In order to solve this problem without having to learn how to solve systems of first order differential equations we shall consider other alternative methods. We start by a very special one that, nevertheless, is interesting since it, somewhat unexpectedly, involves complex functions. The ideia is to build a complex function $z(t) = x(t)+iy(t)$ from the real functions solving (29). What is the differential equation satisfied by $z$? Since $x$ and $y$ satisfy (29) we have

$$z' = (x + iy)' = x' + iy' = y - ix = -i(x + iy) = -iz.$$

So, the differential equation satisfied by $z(t)$ is the exponential model (in $\mathbb{C}$) with "intrinsic growth" coefficient equal to $-i$.

It is clear that what we did before remains valid for complex valued differentiable functions of a single real variable, thus the solution of $z' = -iz$ is $z(t) = z_0e^{-it}$. Now, using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, we can write

$$\begin{aligned}
x(t) + iy(t) &= z(t) \\
&= z_0e^{-it} \\
&= z_0(\cos t - i\sin t) \\
&= (x_0 + iy_0)(\cos t - i\sin t) \\
&= (x_0\cos t + y_0\sin t) + i(y_0\cos t - x_0\sin t)
\end{aligned}$$

and hence

$$x(t) = x_0\cos t + y_0\sin t,$$

which is the solution we where looking for to the spring-and-mass second oreder ordinary differential equation (29), $x'' + x = 0$.

Observe that in this case the "initial condition" must be two quantities: $x_0$, for the value of the initial position of the mass, i.e. $x(t)$ when $t = 0$, and $y_0$ for the value of the initial velocity of the mass, i.e. $x'(t)$ $[= y(t)]$ when $t = 0$.

This is an important and general observation: in differential equations of first order we need to have a single additional value to fix the solution, in second order differential equations we need to be given two additional values, and in differential equations of order $N$ the solutions will depend on $N$ arbitrary constants that must be given in order to fix a single solution.

## 4.2 Damped vibrations in a spring-and-mass system

More realistic (and more complicated) spring-and-mass systems give rise to more general second order equations. For example, if a system is subject to viscous attrition, such as in the case of the spring-and-mass system in the air, under water, or in a container filled with oil or some viscous fluid, there is an attrition force proportional to the velocity,

$$F_{\text{atr}}(t) = -fx'(t),$$

and Newton's law becomes $mx'' = -kx - fx'$, which can also be written as

$$mx'' + fx' + kx = 0 \tag{30}$$

and for this second order linear equation with constant coefficients, if we no not want to transform it into a first order system and use the theory for systems, we need to develop a new approach.

We saw in all previous cases of linear equations (first or second order) that solutions were exponentials (real or complex). So let us start by trying to see if, or when, an exponential function $x(t) = e^{\lambda t}$ is a solution of (30).

The function $x(t) = e^{\lambda t}$ is a solution of (30) if and only if, for all $t \in \mathbb{R}$, we have

$$\begin{aligned} & m\left(e^{\lambda t}\right)'' + f\left(e^{\lambda t}\right)' + k\left(e^{\lambda t}\right) = 0 \\ & m\lambda^2 e^{\lambda t} + f\lambda e^{\lambda t} + k e^{\lambda t} = 0 \\ & \underbrace{\left(m\lambda^2 + f\lambda + k\right)}_{=:P(\lambda)} e^{\lambda t} = 0 \end{aligned}$$

Thus, for $e^{\lambda t}$ to be a solution of (30) $\lambda$ has to be a solution of the polynomial equation $P(\lambda) = 0$, for the second order polynomial defined above, and vice-versa.

What are the possible solutions of $P(\lambda) = 0$? Since $P$ is a second degree polynomial, the answer is easy to obtain from the famous formula for the solution of quadratic equations:

$$\lambda = \frac{-f \pm \sqrt{f^2 - 4mk}}{2m}.$$

So we have three different cases:

**a)** $f^2 - 4mk > 0 \Longrightarrow$ two different *real* solutions $\lambda_1 < \lambda_2 < 0$,

**b)** $f^2 - 4mk < 0 \Longrightarrow$ two different *complex* solutions $\lambda_1 = \overline{\lambda_2} \in \mathbb{C} \setminus \mathbb{R}$,

**c)** $f^2 - 4mk = 0 \Longrightarrow$ just one distinct real solution.

In practical terms, i.e. in real-life situations, only the cases a) and b) are relevant since to have a real-life system for which all the parameters just happen to be so fine tuned that the value of $f^2 - 4mk$ is exactly equal to zero is, in fact, an impossiblility[6]! Consequently, in these introductory notes we will not consider case c). The interested reader is advised to check in any standard book on differential equations (see, e.g., [3, Chap. 3]) what occurs in this case.

In both cases a) and b) we thus have two (linearly independent) solutions $x_1(t) = e^{\lambda_1 t}$ and $x_2(t) = e^{\lambda_2 t}$ and the general solution of (30) is

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \tag{31}$$

where $c_1$ and $c_2$ are constants.

Now, in case a) both $\lambda_1$ and $\lambda_2$ are real numbers and with real arbitrary constants $c_1$ and $c_2$ the expression (31) is the general solution of (30), which are real valued functions. Note that the condition $f^2 - 4mk > 0$ means that the attrition force ($f$) is so strong, compared with the elastic force ($k$) of the spring and to the inertial term ($m$), that the spring does not oscillate at all: the mass goes monotonously to the equilibrium position (eventually after a brief "bump" in the opposite direction: check

[6]Also, no real system is *exactly* modeled by the equation above: it is just a reasonably good approximation, but an approximation nevertheless, as we shall see later.

this statement by analysing the graph of these functions $x(t)$ using the usual tools of Mathematical Analysis [2]. An example of this behaviour is shown in Figure 4.

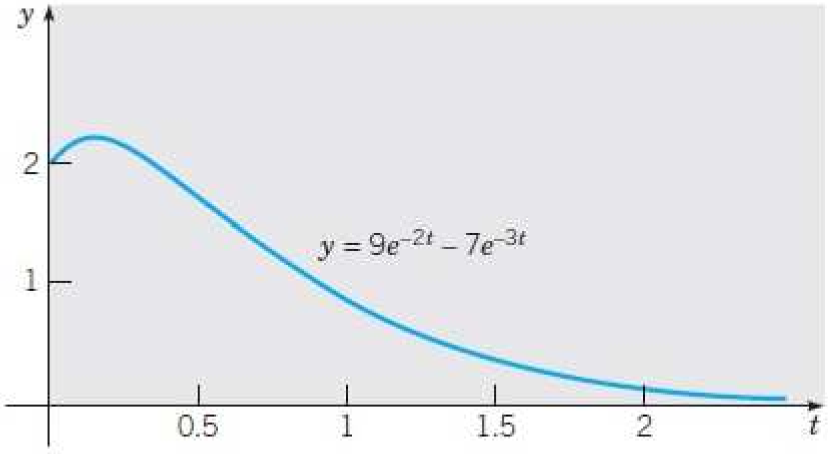


Figure 4: Graph of the function $y(t) = 9e^{-2t} - 7e^{-3t}$, solution to the equation for a spring-and-mass system with initial condition $y(0) = 2, y'(0) = 3$, and parameters $m = 1$, $f = 5$ and $k = 6$; [3, page 142].

In case b), both $\lambda_1$ and $\lambda_2$ are complex numbers, so the functions (31) are complex valued functions. Since we are modelling a real-life situation in which the variable $x(t)$ describe position of the mass relative to the equilibrium position, ant thus is a real valued quantity, we need to approach the problem of getting real valued solutions of (30) out of the complex valued functions $x_j(t) = e^{\lambda_j t}$. The way to do this is to recall to essential factors: (i) equation (30) is linear, and so linear combination of solutions are also solutions, and (ii) Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ allow us to relate the complex exponetial with the real valued sine and cosine functions. So, noting that $\lambda_1 = -u + iv$ and $\lambda_2 = -u - iv$, where $u = f/2m > 0$ and $v = \sqrt{|f^2 - 4mk|}/2m$, we can take the two complex valued functions $x_1(t) = e^{\lambda_1 t}$ and $x_2(t) = e^{\lambda_2 t}$ and form the new functions

$$X_1(t) := \frac{1}{2}\left(e^{\lambda_1 t} + e^{\lambda_2 t}\right) = \frac{1}{2}e^{-ut}\left(e^{vt} + e^{-vt}\right) = e^{-ut}\cos vt,$$

$$X_2(t) := \frac{1}{2i}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right) = \frac{1}{2i}e^{-ut}\left(e^{vt} - e^{-vt}\right) = e^{-ut}\sin vt.$$

Now, we can prove that $X_1$ and $X_2$ are linearly independent functions, and thus the real value general solution of (30) can be written as

$$x(t) = \alpha_1 X_1(t) + \alpha_2 X_2(t) = \alpha_1 e^{-ut}\cos vt + \alpha_2 e^{-ut}\sin vt, \tag{32}$$

for arbitrary real constants $\alpha_1$ and $\alpha_2$, which, of course, can be easily related with the values of the initial position $x(0)$ and initial velocity $x'(0)$ of the spring-and-mass system (do it!).

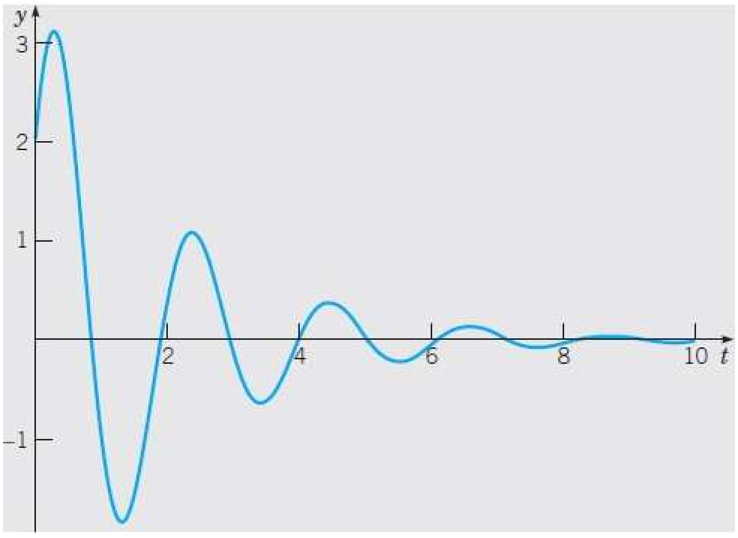


Figure 5: Graph of the function $y(t) = e^{-t/2}(2\cos 3t + 3\sin 3t)$, solution to the equation for a spring-and-mass system with initial condition $y(0) = 2, y'(0) = 8$, and parameters $m = 1$, $f = 1$ and $k = 9.25$; [3, page 161].

As is obvious from (32), the solutions are oscilatory and damped, converging to zero as $t \to \infty$, see Figure 5. This accords with normal experience of damped spring-and-mass systems: the oscillatory behaviour is sustained for some time but their amplitude is progressively diminuished as time increases until the movement all but stops. In our mathematical model movement never dies out completely, which only shows the mathematical model needs improvement to adjust its detailed description to the behaviour or real damped springs, mainly in the very long time behaviour. To this end, however, we would need to go deep into the theory of ordinary differential equations, indeed much deeper than the framework of this introductory course allows.

# 5 Existence of solutions: Picard-Lindelöf theorem

Let us now finally turn to the important problem of how to guarantee that a differential equation has solutions when we do not know how to solve it explicitly.

This is not so strange a problem: in many cases in Mathematics we can prove the existence of some entity without knowing how to explicitly exhibit it, but, after knowing of its existence, we can develop methods to find meaningful approximations; an example is the investigation of zeros of polynomial equations of order $n \geq 1$, $P_n(x) = 0$. If we discover that there are two points $x_1 < x_2$ such that $\mathrm{sgn}(P(x_1))\,\mathrm{sgn}(P(x_2)) = -1$ (i.e., the signals of $P$ at $x_1$ and at $x_2$ are different) then, Bolzano's intermediate value theorem [2, page 138] guarantees the existence of a $\xi \in ]x_1, x_2[$ such that $P(\xi) = 0$, and after this some Numerical Analysis method (bisection [2, page 137], Newton [2, page 194], etc.) can me used to get approximate values of $\xi$ as close to $\xi$ as wanted.

We are going to start with an easy example that, in a situation for which we already know the answer, ilustrates the idea used in the general case. Let us consider the Cauchy (or initial value) problem

$$\begin{cases} \dfrac{dx}{dt} = x \\ x(0) = 1 \end{cases} \tag{33}$$

By now we already know that the solution to (33) is

$$x(T) = e^T, \qquad \forall T \in \mathbb{R}, \tag{34}$$

but suppose for a moment that we do not yet know this, but still want to conclude that (33) has a solution and, if possible, find it.

Up to now we treated a problem such as (33) by looking first at the differential equation, obtaining its general solution, and then use the initial condition to compute the constant showing up in the general solution.

We now take a different point of view. First, instead of dealing with the *differential* equation $\frac{dx(t)}{dt} = x(t)$ we integrate it using the initial time, namely

$$\int_0^T \frac{dx(t)}{dt} dt = \int_0^T x(t) dt.$$

Clearly, the left-hand side can be integrated and we obtain

$$x(T) - x(0) = \int_0^T x(t) dt,$$

and using the initial condition $x(0) = 1$ we obtain

$$x(T) = 1 + \int_0^T x(t) dt. \tag{35}$$

Now, this is **NOT** an answer to our quest to get a solution to (33) since we just changed the Cauchy problem (33) to an *integral equation* that we still have to solve, or at least to prove that it has solutions. So let us continue...

We want to find a solution of (35) which, naturaly, has to have the value 1 when $T = 0$. Not having any idea of what a solution of (35) might be, we do know many functions $x(T)$ that at time $T = 0$ have the value $x(0) = 1$. So, let us ask a simple question: what is the simplest function that satisfies the requirement $x(0) = 1$?

The answer is obvious: the simplest function satisfying $x_0(0) = 1$ is the constant function always equal to 1, namely $x_0(t) = 1$, $\forall t \in \mathbb{R}$: in Figure 5 is the function whose graph is the horizontal straight line drawn in black.

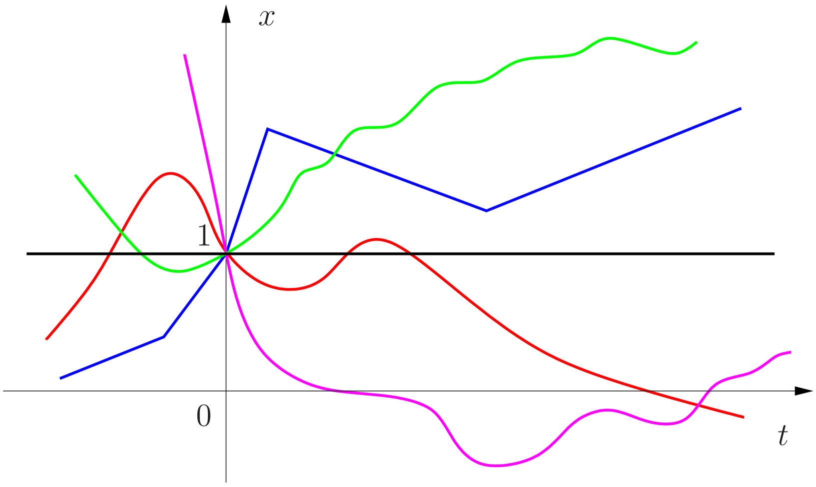


Figure 6: Graphs of several functions satisfying $x(0) = 1$.

Is this function $x_0(t)$ a solution to (35), and hence to (33)? To be a solution to (35) means that if $x_0(t)$ is substituted in the right-hand side of (35) and the computations are performed the result should be equal to the left-hand side for all times $T$. Let us verify what happens in this case: substituting in the right-hand side of (35) $x(t)$ by $x_0(t) \equiv 1$ we obtain

$$1 + \int_0^T 1dt = 1 + T,$$

and this is clearly different from the left-hand side of (35), which is $x_0(T) = 1$.

So $1 + \int_0^T 1dt$ is not equal to $x_0(T)$, and thus is a different function that we must call by a different name, for example $x_1(T)$. So, let us take $x_1(T) = 1 + T$. Note that it is also true that $x_1(0) = 1$, so we can ask the same question we asked before for $x_0$: is $x_1(T)$ a solution to (35)? Let us do the same ckecking as before: substituting $x(t)$ by $x_1(t)$ in the right-hand side of (35) and doing the computations the result should again be $x_1(T)$. We see that substitution $x(t)$ by $x_1(t)$ in the right-hand side of (35) results in

$$\int_0^T (1 + t)dt = 1 + T + \frac{1}{2}T^2,$$

and again this function is certainly different from he left-hand side of (35) with $x$ substituted by $x_1$. So, let us call this new function by a different name, say

$$x_2(t) = 1 + t + \frac{1}{2}t^2.$$

Again the same question: is $x_2(t)$ the solution to (35)? For the same question let us again use the same approach to answer it: compute the right-hand side of (35) with $x_2$ in the place of $x$ and see if the result is equal to the original $x_2$. Performing the computations we have, in the right-hand side,

$$1 + \int_0^T (1 + t + \frac{1}{2}t^2)dt = 1 + T + \frac{1}{2!}T^2 + \frac{1}{3!}T^3,$$

which is clearly diferent from $x_2(T) = 1 + T + \frac{1}{2}T^2$. So the result of the computations is a new function, call it

$$x_3(T) = 1 + T + \frac{1}{2!}T^2 + \frac{1}{3!}T^3.$$

It is possible to prove (and it is not difficult to do so by mathematical induction), that continuing this process indefinitely we obtain a sequence of functions $(x_n(t))_{n\in\mathbb{N}}$, where

$$\begin{aligned}
x_0(t) &= 1 \\
x_1(t) &= 1 + t \\
x_2(t) &= 1 + t + \frac{1}{2!}t^2 \\
x_3(t) &= 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 \\
x_4(t) &= 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 \\
&\vdots \\
x_n(t) &= 1 + t + \frac{1}{2!}t^2 + \ldots + \frac{1}{(n-1)!}t^{n-1} + \frac{1}{n!}t^n \\
&\vdots
\end{aligned}$$

or, recursively,

$$x_{n+1}(T) = 1 + \int_0^T x_n(t)dt, \tag{36}$$

with $x_0(T) \equiv 1$.

So, none of these functions is a solution of the Cauchy problem (33), but the really nice thing now is that, if the sequence $(x_n(t))_{n\in\mathbb{N}}$ converges to some function $x(t)$ when $n \to \infty$, and if in the right-hand side of (36) we can interchange the operations of integration and of passing to the limit in $n$, then we obtain from (36) the equality

$$x(T) = 1 + \int_0^T x(t)dt,$$

which of course means that the limit function $x(T) = \lim_{n\to\infty} x_n(T)$ is, indeed, a solution of (35).

Does this idea work in the present case? Sure it does: just note that

$$x_n(t) = 1 + t + \frac{1}{2!}t^2 + \ldots + \frac{1}{(n-1)!}t^{n-1} + \frac{1}{n!}t^n = \sum_{k=0}^{n} \frac{1}{k!}t^k,$$

and so the limit as $n \to \infty$ is the function

$$x(t) = \sum_{k=0}^{\infty} \frac{1}{k!}t^k, \tag{37}$$

where we know the power series in the right-hand side to be absolutely convergent for all $t \in \mathbb{R}$ and the series is uniformly convergent for $t$ in any compact (= closed and bounded) subset of the real line $\mathbb{R}$.

We should also recognize imediately that, for each $t$ the sum of the series in the right-hand side of (37) is equal to $e^t$, and hence we obtain as a solution of (33) the function

$$x(t) = e^t,$$

which is, of course, the result we already knew.

This iterative process can be applied to much more general Cauchy problems. The idea remains the same:

To prove that the Cauchy problem

$$\begin{cases} \dfrac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases} \tag{38}$$

has a solution, proceed as follows:

1. start by writing (38) as an integral equation

$$x(T) = x_0 + \int_{t_0}^{T} f(t, x(t))dt$$

2. define the sequence of function $(x_n(t))$ iteratively by

$$x_0(T) \equiv x_0$$

$$x_{n+1}(T) = x_0 + \int_{t_0}^{T} f(t, x_n(t))dt$$

3. prove that the sequence $(x_n(t))$ converges to some function as $n \to \infty$:

$$x_*(T) = \lim_{n\to\infty} x_n(T)$$

4. prove that the limit function $x_*$ satisfies the integral equation

$$x_*(T) = x_0 + \int_{t_0}^{T} f(t, x_*(t))dt,$$

   and thus, if it is differentiable, it will be a solution of the original Cauchy problem

5. prove that the limit function $x_*$ is differentiable.

The critical and difficult part of this programme is point 3: to prove that the sequence of iterates $(x_n(t))$ converges to some function. This requires appropriate assumptions about the function $f(t, x)$ in the right-hand side of the differential equation. The precise assumptions turn the above "idea" into the algorithm for the proof of an important theorem in Ordinary Differential Equations: the Picard-Lindelöf theorem (sometimes also known as the Cauchy-Lipschitz theorem) which in a version adapted to a single equation on the real line reads as follows:

**Theorem 1.**
*Let* $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ *be a function defined on an open set* $D$ *of* $\mathbb{R}^2$*, and let* $(t_0, x_0) \in D$*. For* $a, b > 0$ *define the rectangle*

$$\mathcal{R} = \left\{(t, x) \in \mathbb{R}^2 : |t - t_0| \leqslant a \wedge |x - x_0| \leqslant b\right\}$$

*and suppose* $a$ *and* $b$ *are so small that* $\mathcal{R} \subset D$*.*
*Assume that*

1. $f$ *is continuous in* $\mathcal{R}$*.*

2. *there exists a positive constant* $L > 0$ *such that, for all* $(t, y_1)$ *and* $(t, y_2)$ *in* $\mathcal{R}$*, it holds* $|f(t, y_1) - f(t, y_2)| \leqslant L|y_1 - y_2|$*.*

*Then, the Cauchy problem*

$$\begin{cases} \dfrac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

*has exactly one solution* $x(t)$ *defined in a sufficiently small neighborhood of* $t_0$*.*

As referred to above, a rigorous and complete proof of this fundamental result can be done followind the idea presented before. It is, however, beyond the scope of this introductory short course and to present it with the required care will need an amount of time and preparatory work that we clearly do not presently have. Those interested are encouraged to study the proof in a standard text on Ordinary Differential Equations, such as [3, Sec. 2.8] or [4, Chap. 1].

# References


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