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Instantaneous gelation in coagulation dynamics

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Dedicated to Klaus Kirchgässner on the occasion of his sixtieth birthday

1. Introduction

The formation of a distribution of cluster sizes is a common feature in a wide variety of situations. Examples include astrophysics, atmospheric physics and polymer science, [2, 3, 8]. In this paper we discuss some mathematical aspects of the Smoluchowski coagulation equation which is a model for the dynamics of cluster growth. This model involves a coupled infinite system of ordinary differential equations for $c_j \geq 0$, the expected number of clusters consisting of j-particle clusters per unit of volume. The equations are

$$\dot{c}_j(t) = \tfrac{1}{2} \sum_{k=1}^{j-1} a_{j-k,k} c_{j-k}(t) c_k(t) - c_j(t) \sum_{k=1}^{\infty} a_{j,k} c_k(t) \tag{1.1}$$

for $j = 1, 2, \ldots$. The coagulation rates $a_{j,k}$ are nonnegative constants with $a_{j,k} = a_{k,j}$. In the above equation, the first sum represents the rate of change of the j-cluster due to the coalescence of smaller clusters, while the second sum represents the change due to coalescence of the j-cluster with other clusters.

Since particles are neither created nor destroyed in the interactions described by (1.1), we expect the density $\varrho(t) = \sum_{j=1}^{\infty} j c_j$ to be conserved. Certain coagulation kernels however lead to solutions which do not conserve density. For example, take $a_{j,k} = jk$ and initial data $c_1(0) = 1$, $c_j(0) = 0, j = 2, 3, \ldots$. The solution of (1.1) is given by, [6],

$$c_j(t) = \frac{j^{j-3}}{(j-1)!} t^{j-1} \exp(-jt), \quad t \leq 1$$

$$c_j(t) = t^{-1} c_j(1), \quad t \geq 1.$$

In particular, the density $\varrho(t)$ satisfies $\varrho(t) = 1$, $t \leq 1$ and $\varrho(t) = t^{-1}$, $t \geq 1$. The decrease in density for $t \geq 1$ is interpreted as the formation of an infinite cluster or gel. The finite time breakdown of density conservation is known in the physics literature as gelation, and plays an important rôle in the modelling of polymerisation processes by equation (1.1). By gelation time of a solution of (1.1) we mean the time $t_g \geq 0$ such that $\varrho(t) = \varrho(0)$ for all $t \leq t_g$ and $\varrho(t) < \varrho(0)$ otherwise. In the above example $t_g = 1$. A solution of (1.1) with $t_g = 0$ is called an instantaneous gelling solution.

The occurrence of gelation depends not only on the growth rate of $a_{j,k}$ but also on the structure of the kernel. To illustrate this we consider idealised coagulation kernels having the following forms:

$$a_{j,k} = r_j + r_k \tag{1.2}$$

$$a_{j,k} = r_j r_k. \tag{1.3}$$

The additive form (1.2) arises if we assume that interactions of clusters occur randomly with a rate proportional to a measure of the effective surface area of the clusters, while the multiplicative form (1.3) would apply if bond linking was the dominant mechanism. Informally, a j-particle cluster has surface area r_j available for reaction in both cases.

There are existence results for general initial data, for additive kernels when $r_j \leq j$ and for kernels with the multiplicative form when $r_j = o(j)$ as $j \to \infty$ or $r_j = j$, [1, 6]. Also, for additive kernels with (r_i) an arbitrary nonnegative sequence, any solution of equation (1.1) conserves density in its domain of existence [1]. This result contrasts with what happens for kernels of type (1.3), for which an example of a gelling solution when $r_j = j$ was presented above. More generally, for kernels with the multiplicative form (1.3), if $r_j \geq j^\alpha$ with $\alpha > 1/2$, it is conjectured that any non-zero solution has a finite gelation time [5, 7]. This conjecture may be proved for the case $r_j \geq j$ by showing that if c conserves density on $[0, T)$, then for $0 \leq t < T$, $\sum_{k=1}^{\infty} c_j(t) \leq a - bt$ where a and b are constants. If $r_j \geq j^\alpha$ with $\alpha > 1/2$ and $r_j > r_1$ for all $j \geq 2$, Leyvraz, [5] proved the existence of a special instantaneous gelling solution having the form $c_j(t) = A_j(t+1)^{-1}$.

In this paper we prove that if there exist constants α, β with $\beta > \alpha > 1$ such that

$$j^\alpha + k^\alpha \leq a_{j,k} \leq (jk)^\beta$$

then any non-zero solution c of equation (1.1) has zero gelation time. This instantaneous gelation result was demonstrated by van Dongen, [9], for certain coagulation coefficients by means of a formal argument. Our rigorous treatment follows essentially the same development as given in [9].

When applied to systems with additive kernels (1.2) satisfying $r_j \geq j^\alpha$ with $\alpha > 1$, the instantaneous gelation result, together with Theorem 3.6 of [1], implies the nonexistence of solutions for all nontrivial initial data.

It may be argued that coagulation kernels of the above forms with $j^{-1} r_j \to \infty$ as $j \to \infty$ are "unphysical", since the effective surface area of a

j-particle cluster r_j grows faster than the cluster volume. However, examples such as $a_{j,k} = (j^2k + jk^2)^{1/2}(j^{1/3} + k^{1/3})$ have been used as models of physical systems, [10]. Our results prove instantaneous gelation for this case.

In order to simplify the exposition, we impose polynomial growth conditions on the coagulation coefficients. In the final section we indicate how more general cases may be treated.

2. Preliminaries

We first introduce some notation. Let

$$X = \left\{ c = (c_j) : \|c\| = \sum_{j=1}^{\infty} j|c_j| < \infty \right\}.$$

$(X, \|\cdot\|)$ is a Banach space. Set $X^+ = \{c \in X: c_j \geq 0, j = 1, 2, \ldots\}$.

Definition 2.1. Let $T \in (0, \infty]$. A solution $c = (c_j)$ of equation (1.1) on $[0, T)$ is a function c: $[0, T) \to X^+$ such that:

1. each c_j: $[0, T) \to \boldsymbol{R}$ is continuous and $\sup_{t \in [0,T)} \|c(t)\| < \infty$.
2. for all $j = 1, 2, \ldots$ and all $t \in [0, T)$

$$\int_0^t \sum_{k=1}^{\infty} a_{j,k} c_k(s)\, ds < \infty.$$

3. for all $j = 1, 2, \ldots$ and all $t \in [0, T)$

$$c_j(t) = c_j(0) + \int_0^t \left(\tfrac{1}{2} \sum_{k=1}^{j-1} a_{j-k,k} c_{j-k}(s) c_k(s) - c_j(s) \sum_{k=1}^{\infty} a_{j,k} c_k(s) \right) ds.$$

The moments of a solution of (1.1) are defined by

$$S^p(t) = \sum_{j=1}^{\infty} j^p c_j(t) \quad \text{and} \quad S^p_m(t) = \sum_{j=m}^{\infty} j^p c_j(t).$$

The following result, which is proved in [1] is a basic tool in the manipulation of the moments of solutions.

Lemma 2.1. Let c be a solution of (1.1) on $[0, T)$ and let (g_j) be a sequence. Then for $n > 1$ and $0 \leq t_1 \leq t_2 < T$,

$$\sum_{j=1}^{n} g_j [c_j(t_2) - c_j(t_1)]$$
$$= \int_{t_1}^{t_2} \left[\tfrac{1}{2} \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} (g_{j+k} - g_j - g_k) - \sum_{j=1}^{n} \sum_{k=n-j+1}^{\infty} g_j \right] a_{j,k} c_j(s) c_k(s)\, ds.$$

Theorem 2.1. Assume $a_{j,k} > 0$ for all j, k. Let c be a solution of equation (1.1) on $[0, T)$ with initial condition $c_0 \neq 0$. Then for every $t \in (0, T)$ the sequence $(c_j(t))$ has a positive subsequence.

Proof. Assume the contrary, i.e., there exists a $\tau > 0$ and an integer $L > 0$ such that $c_j(\tau) = 0$ for all $j \geq L$. Let

$$\phi_j(t) = \sum_{k=1}^{\infty} a_{j,k} c_k(t), \quad R_j(t) = \tfrac{1}{2} \sum_{k=1}^{j-1} a_{j-k,k} c_{j-k}(t) c_k(t),$$

and

$$E_j(t) = \exp\left(\int_0^t \phi_j(s)\, ds\right).$$

Thus, the equation (1.1) for $j = L$ can be written as

$$0 = c_L(\tau) E_L(\tau) = c_{0L} + \int_0^\tau E_L(s) R_L(s)\, ds.$$

As $\phi_j(t)$, $R_j(t) \geq 0$ and $E_j(t) \geq 1$ for all t, this implies $c_{0L} = 0$ and $R_L(s) = 0$ a.e. $s \in [0, \tau]$. In fact, $R_L \equiv 0$ on $[0, \tau]$ since R_L is a finite sum of continuous functions. Also, $R_L \equiv 0$ on $[0, \tau]$ implies $c_j \equiv 0$ on $[0, \tau]$ for all $j < L$, for suppose there is a J such that $[(L+1)/2] \leq J < L$, and c_J is not identically zero on $[0, \tau]$, where $[x]$ denotes the integer part of x. Since $2J \geq L + 1$, repeating the argument above for $j = 2J$ we obtain $R_{2J} \equiv 0$ on $[0, \tau]$ and, since R_{2J} contains a term $c_J c_J$, c_J must be identically zero. Thus, we have $c_j \equiv 0$ for all $j \geq [(L+1)/2]$. Now applying the same argument with L substituted by $[(L+1)/2]$ and proceeding as above we obtain $c_j(t) = 0$ for all $t \in [0, \tau]$, $j = 1, 2, \ldots$. But then $c_0 = (c_j(0)) = 0$, contrary to the assumption. This proves the result. □

Remark 2.1. This positivity result is optimal in the following sense: It is clear that we should not expect, for $t > 0$, $c_j(t)$ to be positive for all $j \geq 1$ because if the initial data is zero for $j \leq J$, then since there is no fragmentation, no clusters of size less than J can ever be formed. Moreover, the fact that for arbitrary initial data only the positivity of a subsequence can be established, and not that of all $c_j(t)$ for all j sufficiently large, $t > 0$, can be seen by the following example: Let $L \geq 2$ be a fixed integer and consider initial data with $c_j(0) > 0$ if j is an integer multiple of L, and $c_j(0) = 0$ otherwise. Since fragmentation is absent and coagulation of clusters whose size is an integer multiple of L always gives another cluster of the same type, then for all $t > 0$, $c_j(t) = 0$ if j is not an integer multiple of L.

For the proof of Lemma 3.3 we will need the following result

Lemma 2.2. Let (a_j) be a nonnegative sequence with a strictly positive subsequence.

Then, $\lim_{p\to\infty}(\sum_{j=1}^{\infty} j^p a_j)^{1/p} = \infty$.

Proof. Let (a_{j_k}) be a positive subsequence of (a_j). Then, for every k,

$$\left(\sum_{j=1}^{\infty} j^p a_j\right)^{1/p} \geq (j_k^p a_{j_k})^{1/p} = j_k (a_{j_k})^{1/p}$$

so that

$$\liminf_{p\to\infty}\left(\sum_{j=1}^{\infty} j^p a_j\right)^{1/p} \geq \liminf_{p\to\infty} j_k (a_{j_k})^{1/p} = j_k.$$

Since k is arbitrary and $j_k \to \infty$ as $k \to \infty$ we have proved the result. □

3. Non-existence of density conserving solutions

Throughout this section we make the following assumption on the coagulation coefficients:

(H) There exists constants C_L, $C_U > 0$ and $\beta > \alpha > 1$ such that

$C_L(j^\alpha + k^\alpha) \leq a_{j,k} \leq C_U(jk)^\beta$.

Examples of $a_{j,k}$ satisfying (H) are:

(a) $a_{j,k} = j^{\gamma_1}k^{\gamma_2} + k^{\gamma_1}j^{\gamma_2}$ for $\gamma_i \geq 0$ and $\max\{\gamma_1, \gamma_2\} > 1$.

(b) $a_{j,k} = (jk)^\mu (j+k)^{\gamma-\mu}$ for $\mu \geq 0$ and $\gamma > 1$.

In this section we prove the main result of the paper:

Theorem 3.1. Assume (H) holds. Let c be a solution of equation (1.1) on $[0, T)$ with $c_0 \neq 0$. Then c does not conserve density on $[0, t_0)$ for any $t_0 \leq T$.

The strategy of the proof, which is essentially the same as that in [9], is the following: assuming the existence of a density conserving solution of equation (1.1) on $[0, t_0)$ for some $t_0 \in (0, T)$ we prove that all higher moments must be finite in $(0, t_0)$. A contradiction is then obtained by an estimation of the blow-up time of higher moments that allows us to show that for arbitrarily small time $\delta > 0$, there exists a positive number $p_0(\delta)$ such that all p-moments blow-up at a time less then δ for $p > p_0(\delta)$.

Lemma 3.1 (Finite moment property). Let $a_{j,k} \geq C_L(jR_j + kR_k)$, where $C_L > 0$ is a constant and $R_i/\log i \to \infty$ as $i \to \infty$. Let c be a density conserving solution of equation (1.1) on $[0, t_0)$ for some $t_0 > 0$, with initial condition $c_0 \in X^+$, $\|c_0\| = \varrho_0 > 0$. Then, for every p, $S^p(t) < \infty$ for all $0 < t < t_0$.

Proof. As c conserves density on $[0, t_0)$ we have that for all t in $[0, t_0)$, $\|c(t)\| = \varrho_0$. Fix $\tau \in (0, t_0)$ and let t, ε satisfy $0 \le t < \varepsilon \le \tau$. Using density conservation and Lemma 2.1 with $g_j = j$,

$$\begin{aligned} S^1_m(\varepsilon) - S^1_m(t) &= \int_t^\varepsilon \sum_{j=1}^{m-1} \sum_{k=m-j}^{\infty} j a_{j,k} c_j(s) c_k(s)\, ds \\ &\ge C_L \int_t^\varepsilon \sum_{j=1}^{m-1} j c_j(s) \sum_{k=m-j}^{\infty} k R_k c_k(s)\, ds \\ &\ge C_L \int_t^\varepsilon \sum_{j=1}^{m-1} j c_j(s) \sum_{k=m}^{\infty} k R_k c_k(s)\, ds. \end{aligned}$$

Since c conserves density, by Dini's Theorem, the series $\sum_{j=1}^\infty j c_j(s)$ is uniformly convergent for $0 \le s \le \tau$. Hence, there exist a positive m_0 such that for all $m \ge m_0$ and $s \in [0, \tau]$,

$$\sum_{j=1}^{m-1} j c_j(s) \ge \tfrac{1}{2} \sum_{j=1}^{\infty} j c_j(s) = \tfrac{1}{2} \varrho_0 .$$

Thus, defining $\gamma_m = \min_{k \ge m} R_k$, we have, for $m \ge m_0$,

$$S_m(\varepsilon) - S_m(t) \ge \tfrac{1}{2} C_L \varrho_0 \gamma_m \int_t^\varepsilon S_m(s)\, ds.$$

Applying Gronwall's inequality and using $S^1_m(\varepsilon) \le \varrho_0$ we obtain

$$S^1_m(t) \le \varrho_0 \exp[-C_L \varrho_0 \gamma_m (\varepsilon - t)/2].$$

Using $m c_m(t) \le S^1_m(t)$ and the above inequality, we get the bound

$$\sum_{j=m}^{\infty} j^p c_j(t) \le \varrho_0 \sum_{j=m}^{\infty} j^{p-1} e^{-C_L \varrho_0 \gamma_j (\varepsilon - t)/2} \tag{3.1}$$

for every $m \ge m_0$, $0 < t < \varepsilon \le \tau$. Since

$$\lim_{j\to\infty} \frac{\gamma_j}{\log j} = \lim_{j\to\infty} \frac{\min_{k \ge j} R_k}{\log j} \ge \lim_{j\to\infty} \min_{k \ge j} \frac{R_k}{\log k} = \infty,$$

the series on the right-hand side of (3.1) is convergent and the result follows. □

Lemma 3.2. Assume (H) holds. Let c be a density conserving solution of (1.1) on $[0, t_0)$ with initial data $c_0 \ne 0$. Then, if $0 < \delta \le t \le \tau < t_0$,

$$S^p(t) - S^p(\delta) = \tfrac{1}{2} \int_\delta^t \sum_{j,k=1}^{\infty} ((j+k)^p - j^p - k^p) a_{j,k} c_j(s) c_k(s)\, ds.$$

Proof. By Lemma 2.1

$$\sum_{j=1}^{n} j^p[c_j(t) - c_j(\delta)] = \int_\delta^t \tfrac{1}{2} \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} ((j+k)^p - j^p - k^p) a_{j,k} c_j(s) c_k(s)\, ds$$
$$- \int_\delta^t \sum_{j=1}^{n} \sum_{k=n-j+1}^{\infty} j^p a_{j,k} c_j(s) c_k(s)\, ds. \quad (3.2)$$

An easy calculation shows that $(j+k)^p - j^p - k^p \le \mathscr{C}(p)(j^p k + jk^p)(j+k)^{-1}$ for some $\mathscr{C}(p) > 0$. For $\delta \le s \le \tau$ we have, by Lemma 3.1

$$\tfrac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} ((j+k)^p - j^p - k^p) a_{j,k} c_j(s) c_k(s)$$
$$\le \tfrac{1}{2}\mathscr{C}(p) \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{j^p k + jk^p}{j+k} a_{j,k} c_j(s) c_k(s)$$
$$\le \mathscr{C}(p) \sum_{j=1}^{n} \sum_{k=1}^{n} j^{p-1} k a_{j,k} c_j(s) c_k(s)$$
$$\le \mathscr{C}(p) C_U \sum_{j=1}^{n} \sum_{k=1}^{n} j^{p+\beta-1} k^{\beta+1} c_j(s) c_k(s)$$
$$= \mathscr{C}(p) C_U \sum_{j=1}^{n} j^{p+\beta-1} c_j(s) \sum_{k=1}^{n} k^{\beta+1} c_k(s) \le \mathscr{K}$$

where $\mathscr{K}$ is a constant independent of n and s. Also, by Lemma 3.1 the second series in (3.2) converges to zero as $n \to \infty$. The result now follows by letting $n \to \infty$ in (3.2) and using the dominated convergence theorem. □

Lemma 3.3. Assume (H) holds. Let c be a density conserving solution of (1.1) on $[0, t_0)$ with nontrivial initial data $c_0 \in X^+$. Let $\delta, t, \tau \in (0, t_0)$ with δ and τ fixed and $\delta < t \le \tau$. Then, for $p \ge 2$ the p-moment of c has a blow-up time $T^{(p)} \le t_0$ and $\lim_{p\to\infty} T^{(p)} \le \delta$.

Proof. By Lemma 3.2 we have

$$S^p(t) - S^p(\delta) = \tfrac{1}{2} \int_\delta^t \sum_{j,k=1}^{\infty} ((j+k)^p - j^p - k^p) a_{j,k} c_j(s) c_k(s)\, ds$$
$$= \tfrac{1}{2} \int_\delta^t \sum_{j,k=1}^{\infty} \left(\sum_{r=1}^{p-1} \binom{p}{r} j^{p-r} k^r \right) a_{j,k} c_j(s) c_k(s)\, ds$$
$$\ge \tfrac{1}{2} p C_L \int_\delta^t \sum_{j,k=1}^{\infty} (j^{p-1} k + jk^{p-1})(j^\alpha + k^\alpha) c_j(s) c_k(s)\, ds$$
$$\ge p C_L \int_\delta^t \sum_{j,k=1}^{\infty} j^{p+\alpha-1} k c_j(s) c_k(s)\, ds$$
$$= p \varrho_0 C_L \int_\delta^t \sum_{j=1}^{\infty} j^{p+\alpha-1} c_j(s)\, ds. \quad (3.3)$$

Setting $\gamma = \gamma(p) = (\alpha - 1)/(p - 1)$, it follows by Hölder's inequality that

$$\sum_{j=1}^{\infty} j^{p+\alpha-1} c_j \geq \left(\sum_{j=1}^{\infty} j^p c_j \right)^{1+\gamma} \left(\sum_{j=1}^{\infty} j c_j \right)^{-\gamma}.$$

Hence

$$S^p(t) - S^p(\delta) \geq p C_L \varrho_0^{1-\gamma} \int_\delta^t (S^p(s))^{1+\gamma}\, ds. \tag{3.4}$$

By continuity of S^p and standard results in differential inequalities we get

$$S^p(t) \geq \left[(S^p(\delta))^{-\gamma} - C_L \frac{p}{p-1} \varrho_0^{1-\gamma}(t - \delta) \right]^{-\gamma},$$

with the following estimate for the blow-up time, $T^{(p)}$,

$$T^{(p)} \leq \delta + \frac{1}{(\alpha - 1) C_L \varrho_0^{1-\gamma}} (1 - p^{-1}) \left[\frac{1}{S^p(\delta)} \right]^\gamma. \tag{3.5}$$

The sequence $(T^{(p)})$ converges as $p \to \infty$ since it is nonincreasing and bounded below. Using Theorem 2.1 and Lemma 2.2 the limit of the right-hand side of (3.5) as $p \to \infty$ is δ and the result follows. □

Proof of Theorem 3.1. Let c be a solution of equation (1.1) on $[0, T)$, with initial data $c_0 \neq 0$ and assume there exists a $t_0 \in (0, T)$ such that c conserves density in $[0, t_0]$. By Lemma 3.1 all moments are finite in $(0, t_0)$. Let $\delta \in (0, t_0)$ be arbitrary. Then, by Lemma 3.3, for all sufficiently small $\varepsilon > 0$ there exists a $P \geq 2$ such that, for $p \geq P$, the p-moment of c is infinite for $t \geq \delta + \varepsilon$. Since δ can be made arbitrarily small, this contradicts the finite moment property and hence no non-zero solution conserves density in $[0, t_0)$ for any $t_0 > 0$. □

Combining Theorem 3.1 with the results of Ball and Carr (Theorem 3.6 in [1]) we have

Corollary 3.1. Assume (H) holds and that in addition, $a_{j,k} = r_j + r_k + \alpha_{j,k}$ where $r_j \geq C_L j^\alpha$ and $0 \leq \alpha_{j,k} \leq \mathscr{K}(j + k)$, where $\alpha > 1$, $C_L > 0$ and $\mathscr{K} > 0$ are constants. Let $c_0 \neq 0$. Then, equation (1.1) has no solutions with initial data c_0, defined in $[0, T)$, for any $T > 0$.

4. Further remarks

In the previous section we imposed a polynomial growth restriction on the coagulation coefficients. This restriction is not essential and to illustrate the modifications needed for more general cases we consider $\alpha_{j,k} = e^{j+k}$.

By definition of solution,

$$\int_0^t \sum_{k=1}^{\infty} e^k c_k(s)\, ds < \infty,$$

so boundedness of moments $S^p(t)$ would not give extra information. To overcome this difficulty we consider exponential type moments of the form

$$E^p = \sum_{j=1}^{\infty} (e^j)^p c_j. \tag{4.1}$$

Small modifications in the proof of Lemma 3.1 shows that if a solution of (1.1) conserves density in $[0, t_0)$, then all its exponential type moments (4.1) are finite in $(0, t_0)$. This result allows us to prove an analogue of Lemma 3.2 and to get the evolution equation for E^p:

$$E^p(t) - E^p(\delta) = \tfrac{1}{2} \int_\delta^t \sum_{j,k=1}^{\infty} (e^{(j+k)p} - e^{jp} - e^{kp}) a_{j,k} c_j(s) c_k(s)\, ds. \tag{4.2}$$

Since $e^{(j+k)p} - e^{jp} - e^{kp} \geq (1 - 2e^{-1}) e^{(j+k)p}$ we can estimate the right-hand side of (4.2) in a way similar to that in Lemma 3.3 to obtain

$$E^p(t) - E^p(\delta) \geq (\text{const}) \int_\delta^t (E^p(s))^2\, ds. \tag{4.3}$$

Comparing (4.3) with the corresponding result (3.4) for S^p, we see that the right-hand side of (3.4) has a coefficient which is unbounded as $p \to \infty$ while (4.3) does not. This is compensated by $(E^p(s))^2$ in the integrand in (4.3) while the power $1 + \gamma(p)$ in (3.4) converges to 1 as $p \to \infty$. It follows from (4.3) that E^p blows up at times $T^{(p)}$ with

$$T^{(p)} \leq \delta + (\text{const})(E^p(\delta))^{-1},$$

which leads to the same conclusion as Theorem 3.1. For more information on the gelation solution to (1.1) for the exponential case see [4].

More generally, results can be obtained for coagulation kernels satisfying

$$r_j + r_k \leq a_{j,k} \leq C_U (r_j r_k)^\beta$$

with $\beta > 1$. The analogue of the p-moment is

$$R^p = \sum_{j=1}^{\infty} r_j^p c_j.$$

For the method to work, conditions like

$$\mathscr{K}_L r_j^{\alpha_L} r_k^{\beta_L} \leq r_{j+k}^p - r_j^p - r_k^p \leq \mathscr{K}_U r_j^{\alpha_U} r_k^{\beta_U}$$

are needed, where the constants may depend on p. In particular we require extra conditions like $\mathscr{K}_L(p) \sim p$, $\alpha_L(p) \sim p$ (as in the polynomial case) or $\alpha_L(p)$ and $\beta_L(p) \sim p$ (as in the exponential case).

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Abstract

The coagulation equations are a model for the dynamics of cluster growth in which clusters can coagulate via binary interactions to form larger clusters. For a certain class of rate coefficients we prove that the density is not conserved on any time interval.

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