Inhomogenous Marketing Mix Diffusion

Luís G. Pinto, Luís Cavique, Orlando Gomes, and Jorge M. A. Santos

Abstract In this article we extend the Marketing Mix Diffusion (MMD) model to inhomogenous networks (i.e. complex networks of arbitrary topology). The (Homogenous) MMD model is an innovation diffusion model, similar to the Bass model, which includes four decision variables (the 4Ps of Marketing: Product, Price, Place, Promotion). We introduce the Inhomogenous MMD (IMMD) model and we conduct two separate experiments: one based on simulation and another one relying on empirical evidence. The simulation study compares the behavior of the IMMD model with the classic Bass diffusion model. Results suggest that the classic Bass model is able to represent the IMMD curves quite well in most cases. The IMMD is more general and capable of representing extreme scenarios. The empirical study focuses on the geographic diffusion of mobile broadband technology in Japan, combining adoption data with a spatial network of municipalities. The in-sample performance of the model is comparable to the existing methods, which suggests a good explanatory power of the IMMD model.

Keywords Information diffusion · Viral marketing · Marketing science · Network science · Complex networks

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1 Introduction

Classic marketing theory states that **tactical marketing** is about managing the **Marketing Mix (4Ps)**: *Product, Price, Place, and Promotion*. Several empirical studies, however, have suggested that **word-of-mouth (WoM)** plays an important role in the adoption of new products, beyond the marketing mix. While the marketing mix is under the control of the firm, WoM is an effect which the manager can only **influence indirectly**. This idea has given rise to many concepts such as "buzz management", "influencer marketing" and "viral marketing". To manage WoM we need to measure it. It was only with the advent of the internet that it became relatively easy to monitor electronic WoM [6]. In addition we can many times approximate the structure of the underlying **social network** by gathering data online or offline. While these types of activities have become common place, marketing decision support systems have lagged behind. Although we already have diffusion models with all four variables of the marketing mix, and we also have viral marketing and word of mouth models, very little research has been conducted on models that **combine** the **marketing mix** and the **word of mouth effects**. In this article, to the best of our knowledge, we will introduce the **first ever model** that **integrates** these **two types of effects**.

Figure 1 presents the relationship between the **compartmental**, **threshold**, **innovation** and **information diffusion models**. These families can be employed to model word-of-mouth effects (i.e. the spread of information in a social network). We assume that they are all are subsets of the larger information diffusion model family, which is in line with what the related literature suggests [12]. Diffusion processes of these sort can include the **diffusion of new products**, **new diseases**, **rumours**, **news**, **information packets** in computer networks, and many other *analogous processes*. Typically the **diffusion media** is assumed to be a **graph network**, either observed or unobserved. **Compartmental** models originated in the **epidemiological literature** with the *Susceptible-Infected-Recovered model (SIR)* [11], and govern the **spread of diseases** or other processes with similar characteristics, by assuming that we have an **unobserved network** topology which serves as the **diffusion medium**. These

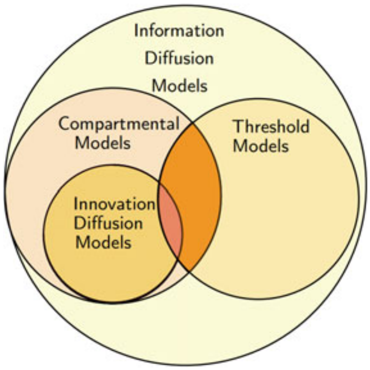

Fig. 1 The relationship between the information diffusion models

models rely on Ordinary Differential Equations (ODEs). **Threshold models**, on the other hand, originated in the **marketing science literature** (see [9] for a survey) to model the problem of **influence maximization** (i.e. choosing the optimal set of seed nodes to maximize the diffusion). These models were created to study the diffusion process at the theoretical level, by assuming that we have an **observed network** structure, and that the diffusion occurs in this network in a **deterministic** manner. The intersection between these two types of models is an open problem. Several frameworks of analysis have attempted to combine aspects of both compartmental and threshold models (see [27], Sect. 3.1.4. for a short survey). **Innovation diffusion models**, a special case of compartmental models, originated in the **marketing science literature**. At the intersection between innovation diffusion models and threshold models some progress has been made, with the discovery of the Niu's Theorem [18], which shows that the Bass model is a limiting case of a threshold type model called Discrete Bass Model when the network size tends to infinity. **Information diffusion models**, along with their marketing applications, have been extensively studied by Kempe [9, 10]. In his work he has studied the **Linear Threshold Models (LTM)** and **Cascade Models (CM)** having derived generalized versions of it. In particular they have shown that their generalized versions are in fact **equivalent**, proving that LTM and CM are particular instances of a more general class of information diffusion models. These simple models can be easily employed to study the theoretical properties of diffusion processes. The focus of much of this literature has been the **Influence Maximization problem**, initially formulated by Domingos and Richardson [24]. The innovation diffusion theory was extensively debated by Rogers in [25]. His theory attempts to explain and predict how **innovations** spread in a **social network**. Bass [2] placed the theory in a formal mathematical setting, incorporating some influence from the **Susceptible-Infected-Recovered (SIR) model** [11]. Most innovation diffusion models, assume that the diffusion process is occurring in a **homogenous network** (i.e. a network where the degree distribution is uniform). Several **inhomogenous** extensions which take into account the network topology have been proposed [4, 8, 23]. Hence, innovation diffusion can be seen as a particular type of information diffusion over complex networks. A recent proposal [20], called **Marketing Mix Diffusion (MMD)** has incorporated all four variables of the traditional **4Ps of marketing** (*Product, Price, Place, Promotion*) in the original Bass framework, building upon the work of [13, 16]. Hereafter, MMD will be referred to as the Homogenous MMD.

The **objective** of this work is to **introduce** a *novel* inhomogenous variant of the Homogenous MMD, which we refer to as **Inhomogenous Marketing Mix Diffusion model** (or **IMMD**), to numerically evaluate some of its properties, and to assess its performance in a real world setting when compared to existing diffusion models. In the following sections we will introduce the IMMD model. Later we introduce a set of simulation results, where we employ an experimental design based on random networks and a sample of the parameter space. Finally we will present the results of an empirical study. In this empirical study we will apply the IMMD model to the **Japanese broadband diffusion dataset**. This dataset includes the *four marketing mix variables* (product quality, price, distribution intensity and advertising investment) for the entire *telecommunications sector* and the mobile broadband adoption over time, as well as a network of *Japanese municipalities*, where edges represent the sociodemographic interaction between cities (adjacency and migration). By applying the IMMD model to this data we obtain the diffusion curve of mobile broadband adoption in Japan between January 2014 and December 2019.

2 Inhomogenous Marketing Mix Diffusion

We introduce in this section an **inhomogenous** version of the original Homogenous MMD model (the IMMD). The proposed model is heavily influenced by the works of [14, 15, 29], since it allows for **continuous threshold dynamics** in **discrete time** and it uses the concept of *aggregation classes*, allowing nodes with similar attributes to share model parameters to reduce the dimensionality of the parameter space. In the following subsections we will use the notation introduced in [7]. Assume that the difference operator $\Delta_t f(t)$ means $\Delta_t f(t) = f(t+1) - f(t)$ (discrete analog of the derivative) and the Riemann sum $\sum_{n:a\to b} f(n) = f(a) + f(a+1) + ... + f(b-2) + f(b-1)$ (discrete analog of the integral). Also, $tanh(x) = \frac{e^{2x}-1}{e^{2x}+1}$ refers to the hyperbolic tangent and that $ReLU$ is the rectifier linear unit ($ReLU(x) = x\ if\ x > 0,\ 0\ otherwise$).

Definition 1 (*Homogenous MMD model*)

$$f_t(X_i, t) = S(U_i, t) + R(P_i, D_i, A_i, t) \tag{1}$$

$$R(P_i, D_i, A_i, t) = pP_i(t)[\bar{m}D_i(t) - f(X_i, t)]A_i(t) \tag{2}$$

$$S(X_i, t) = qU(U_i, t) \tag{3}$$

$$U(U_i, t) = e^{-\frac{t}{\eta}} \int_1^t \frac{e^{\xi/\eta} f(\xi) U_i(\xi)}{\eta} d\xi + \gamma e^{-\frac{t}{\eta}} \tag{4}$$

$$P(P_i, t) = P_i(t) \tag{5}$$

$$D(D_i, t) = D_i(t) \tag{6}$$

$$A(A_i, t) = \sqrt{A_i(t)} \tag{7}$$

$$X_i = (U_i(t), P_i(t), D_i(t), A_i(t)). \tag{8}$$

where $f_t(X_i, t)$ is the combined *radiation-diffusion equation* which gives the **sales rate on moment** t given a *marketing tactic* X_i (which depends on the product U_i, price P_i, distribution D_i and advertising A_i trajectories), S the *diffusion equation* (which captures the word-of-mouth effect, related with the product quality), R the *radiation equation* (which captures the effects of price, advertising and distribution), p is the coefficient of innovation (or coefficient of external influence), q is the coefficient of imitation (or coefficient of internal influence), η is a quality influence decay coefficient, γ the product launch quality coefficient and $\bar{m}$ is the market potential.

We will refer to (4) as the *NGM Integral Equation*. This equation captures the **non-linear effects** of the cumulative product quality on the market in the following manner: as **product quality** varies over time, **new adopters** reinforce the adoption through the **positive word-of-mouth effect** if the quality is **better than expected**. Otherwise, the **new adopters** might generate a **negative word-of-mouth effect**, negatively impacting the product adoption.Hence, the integral, which relates the **continuous and dynamic** cumulative effect of a given change on adoption ($f(\xi)$) and the product quality $U_i(\xi)$ on that instant. Refer to [20] for a deeper explanation and derivation of this function.

Definition 2 (*Inhomogenous MMD model*) To derive the *Inhomogenous MMD* model, we will modify the Homogenous MMD model in the following manner:

1. The **diffusion** will occur on a graph G of **arbitrary topology**.
2. f_t will depend on node level f_j such that $f_t(X_i, t) = \sum_{j \in G} f_j(X_i, t)$.
3. $f_j(X_i, t) = ReLU\left(tanh\left(\bar{m}_j(S_j(U_i, t) + R(P_i, D_i, A_i, t))\right)\right)$.
4. $S_j(X_i, t) = ReLU\left(qU_j(U_i, t)\right)$.
5. The initial values $f_j(X_0, 0)$ will be $-b_j$ which is the **adoption threshold** in a manner similar to the LTM[1] in [9].

A visual representation has been provided on Fig. 2. We can see the resulting model as a special type of **Graph Neural Network (GNN)** (see [28] for a survey), similar to a **Recurrent GNN (RGNN)**, since it uses the previous state and a special Graph node to do the word-of-mouth effect computations (*graph convolution*). In addition we also have a set of "special nodes" (non standard neural network nodes):

1. *Bias* which represents the addition of the initial b_j values.
2. $\int NGM$ is the **NGM Integral Equation** (4).
3. *Im* which is the multiplication by the coefficient of imitation q.
4. G which is the application of the **graph convolution** according to the **topology of** G.
5. *Diff* which is the same as the **difference operator** Δ_t applied to the previous state values of each node.
6. *R+D* which is the addition $S_j(U_i, t) + R(P_i, D_i, A_i, t)$.
7. The **final summation layer** which gives the adoption on t is **equivalent to the function** f, which is the same as the **Riemann sum** of f_t. Likewise $f_t = \Delta_t f(X_i, t)$ should be understood as the **discrete time derivative of** f instead of the continuous time partial derivative defined in the original Homogenous MMD model.

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[1] The use of the hyperbolic tangent ($tanh$) ensures that the outputs are contained between –1 and 1, such that when combined with the $ReLU$ function on Eqs. 3 and 4, only positive outputs will flow into the neighboring nodes. This simulates the adoption dynamic of each individual node.

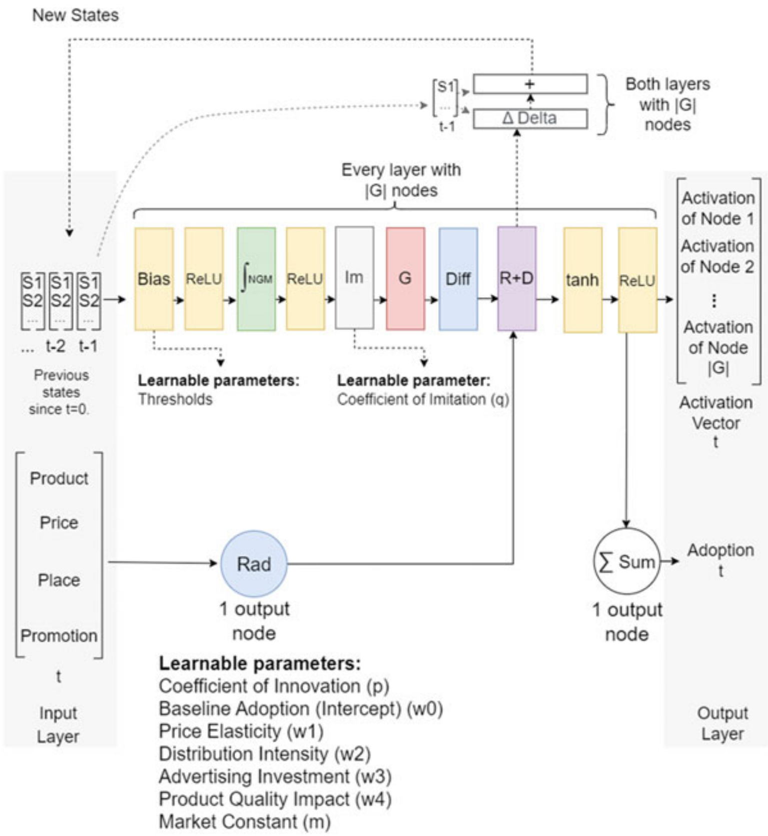
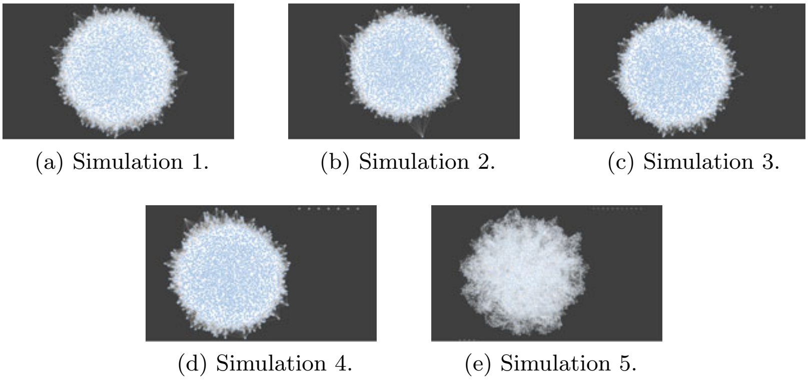


Fig. 2 A visual representation of the model

3 Simulation Results

We conduct a **numerical study** where we compared the curves of the Bass and IMMD models. Our approach employed **Watts-Strogatz** and **Scale-Free** random networks. These two methods allow us to simulate different type of networks which appear in real world applications, including social networks and city networks. We then used an experimental design based on a **latin hypercube sampling** method. Latin hypercubes are a method of **generating random samples** in a efficient manner from a **very large input space**, from which we can only simulate a **limited number of combinations**.

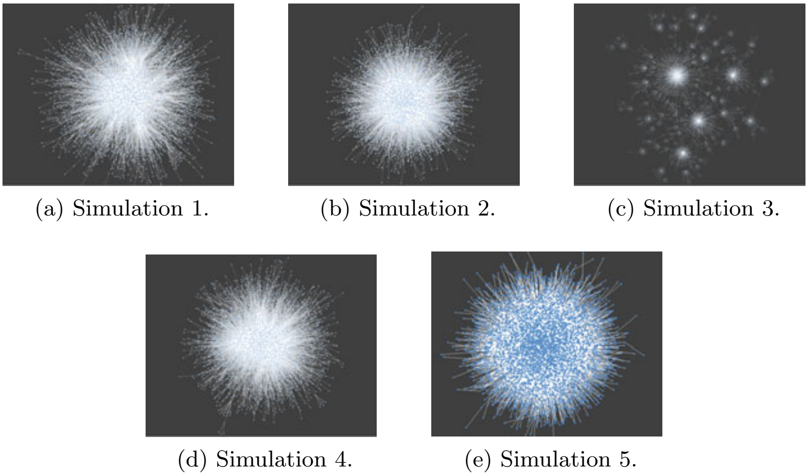
We took 5 samples of 8 variables: ρ,p,q,w_0,w_1,w_2,w_3, and w_4. Where, ρ is either the Watts-Strogatz rewiring probably or the Scale-Free probability of adding a new node connected to an existing node chosen randomly according to the in-degree distribution, p is the coefficient of innovation, q is the coefficient of imitation, w_0 is the baseline adoption, w_1 is the price elasticity, w_2 the distribution intensity, w_3



(a) Simulation 1. (b) Simulation 2. (c) Simulation 3.

(d) Simulation 4. (e) Simulation 5.

Fig. 3 Watts-Strogatz Random graphs used in the simulation



(a) Simulation 1. (b) Simulation 2. (c) Simulation 3.

(d) Simulation 4. (e) Simulation 5.

Fig. 4 Scale-Free Random Graphs used in the simulation

the advertising investment, and w_4 the product quality impact of the IMMD model. The resulting random networks are presented in the Figs. 3 and 4, which allows us to see the effects of the varying ρ parameters. This is especially evident in Fig. 4, but also clear in the network (e) of Fig. 3. With these five samples we simulated 5 paths of length $t \in [1, 50]$. We then tried to adjust the Bass model to these curves using the **least squares method**. As expected the results suggest that IMMD is a **more general model**, capable of generating a **wider variety of curves**, which the Bass model is incapable of representing with absolute accuracy. That said, overall the Bass model fit is still quite good (average fit across all experiments of 98.30%, with minimum of 90.76%), which is consistent with previous results that show that Bass fits empirical data very well in most applications [3] (Figs. 5, 6 and Table 1).

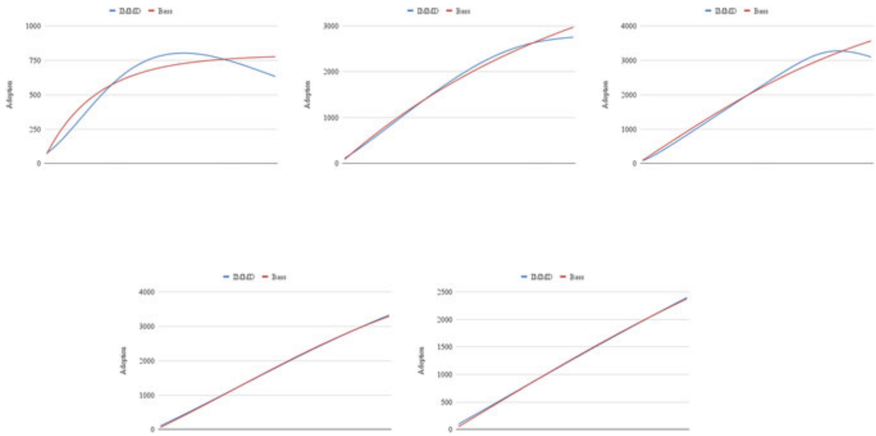


Fig. 5 IMMD simulations versus the Bass model (Watts-Strogatz network)

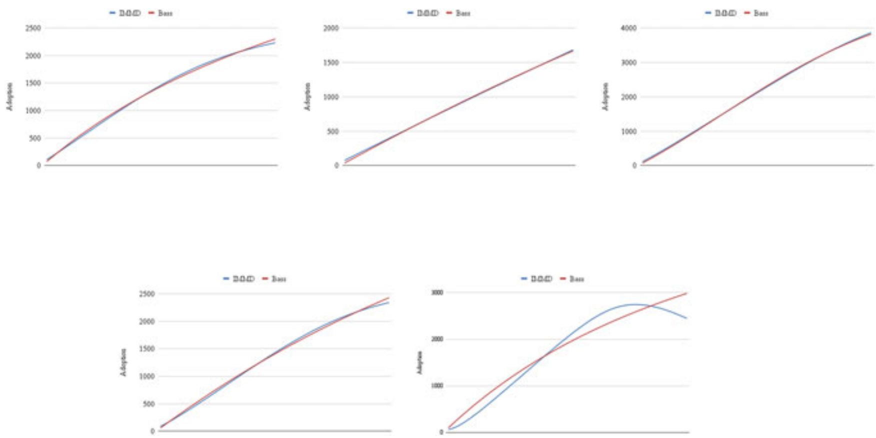


Fig. 6 IMMD simulations versus the Bass model (Scale-Free network)

4 Empirical Results

An **empirical application** was conducted using OECD mobile broadband adoption data from **Japan** (2014–2019) [19]. A marketing mix dataset was constructed combining data from several sources. **Product quality**-related investments came from the **mobile broadband speed** in Japan in Akamai's State of Internet Reports 2014–2019 [1]. **Pricing** data was taken from the Consumer Price Index for Mobile Charges in Japan [22]. **Advertising** investment data for the entire Information/Communications industry sector was used as a proxy for the communications related to mobile broadband adoption [5]. While **point-of-sale investment** data was not available at the industry sector level, *digital distribution efforts* were approximated using the Alexa website rank for the three major telco players in Japan (nttdocomo.co.jp, KDDI.com and softbank.jp) [26].

A latent sector level "digital distribution" factor was extracted from this data using Dynamic Factor Anaysis. Polynomial interpolation (order 4) was applied to all series to adjust the data with different granularity levels. The final dataset included 72 periods (months) for adoption, product, price, place, and promotion data between January 2014 and December 2019.

A network structure was built using **Japanese municipality data**. Edges describe either **municipality adjacency** or significant **inter-prefecture migration**. Migration information from the yearly migration survey [21] was converted to percentages (normalizing by the total migrants in 2014). A cutoff was applied to ignore edges with smaller weights ($w_{ij} = \frac{population_i}{population_j}$ using the 2015 census data). Five models were used to study the mobile broadband adoption curve: **Bass**, **Mesak**, **Homogenous MMD**, and **Inhomogenous MMD** (IMMD). To simplify the estimation of the IMMD, we used a graph segmentation process similar to the work of Morone [14, 15], which studied meso-scale Bass diffusion models that divide the graph into *number of link classes*, according to the degree centrality of each node. We employed this approach to reduce the parameter space. All models were estimated using a **least-squares** approach and the **Nelder-Mead Simplex method** [17] in Python. The IMMD model was implemented using the Tensorflow and NetworkX packages via a local Nvidia Geforce MX150 GPU. The estimation of the IMMD model included a multi-step process (Algorithm 1) with two carefully selected hyperparameters $K = 1385$ (influencer set size) and $L = 20$ (link classes). The results in Table 2 and Fig. 8 (left) suggest that the IMMD model has a **good explanatory power**, comparable to the existing models, (in-sample R^2 statistic of 96%). However it should be noted that the IMMD model is still outperformed by all the other models ($R^2 \geq 99\%$). This might be due to the larger parameter space of the IMMD. The IMMD model significantly underperforms in its predictive ability, as measured by the MAPE in Table 3, and as can be confirmed visually on Fig. 8 (right), with an error of 5% versus 0.81% on the MMD (the most similar model), and 0.16% on the Bass model (Fig. 7).

Algorithm 1 IMMD estimation procedure

```
1: procedure OPTIMIZE(K, L)
2:     Initialize node thresholds set T of size L randomly.
3:     Obtain I influencer set of size K using centrality (eigenvector & degree).
4:     Obtain Θ parameters estimate using IMMD process with T and I parameters.
5: end procedure
```

Table 1 Simulation results

IMDD									Bass	(W-S)	Bass	(S-F)		
Simulation	ρ	p	q	w_0	w_1	w_2	w_3	w_4	p	q	p	q	R^2 (W-S) (%)	R^2 (S-F) (%)
1	0.34	0.19	0.06	0.49	0.06	0.32	0.64	0.49	0.01	0.01	0.02	–0.01	99.96	95.27
2	0.66	0.83	0.27	0.90	0.21	0.78	0.18	0.30	0.01	0.02	0.01	0.00	99.97	99.76
3	0.50	0.40	0.66	0.39	0.79	0.94	0.96	0.11	0.02	0.02	0.01	0.04	98.31	99.96
4	0.95	0.65	0.92	0.72	0.49	0.44	0.39	0.84	0.02	0.00	0.01	0.00	99.22	99.95
5	0.06	0.60	0.57	0.19	0.96	0.14	0.58	0.80	0.01	–0.09	0.01	–0.01	90.76	99.79

Table 2 In-sample results

	Bass diffusion	Mesak diffusion	Homogenous MMD	Inhomogenous MMD
MSE	3.34×10^{10}	$\mathbf{1.44 \times 10^{10}}$	1.15×10^{11}	4.28×10^{11}
RMSE	182626	**119852**	339140	653848
NRMSE	1.06%	**0.70%**	1.97%	3.80%
MAPE	1.183%	**0.762%**	2.189%	4.055%
R^2	**99.82%**	99.75%	99.00%	95.98%

Table 3 Out-of-sample results

	Bass diffusion	Mesak diffusion	Homogenous MMD	Inhomogenous MMD
MSE	$\mathbf{1.04 \times 10^{9}}$	4.83×10^{10}	3.21×10^{10}	1.70×10^{12}
RMSE	**32295**	219684	179156	1304611
NRMSE	**0.18%**	1.22%	1.00%	7.27%
MAPE	**0.16%**	1.16%	0.81%	5.02%

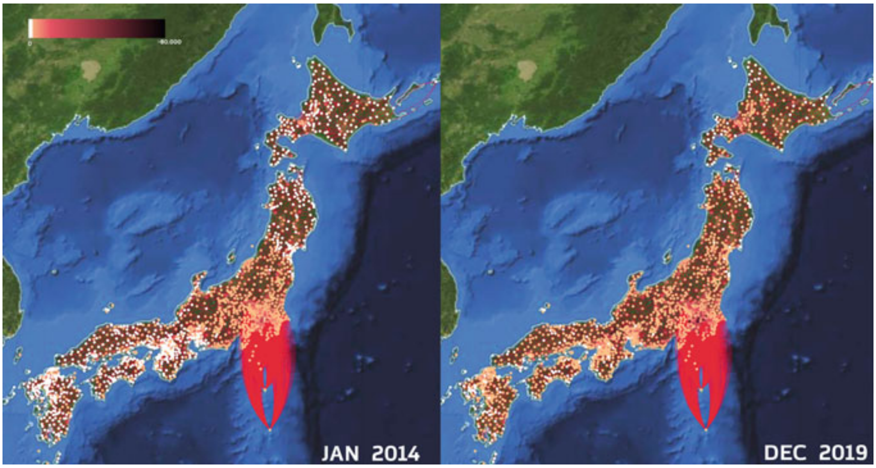


Fig. 7 Initial and final simulated diffusion state of the spatial complex network overlaid on the Japanese geography. The filling color of the nodes represent the number of adopters in each municipality, using the scale from white (0 adopters) to dark red (>80.000 adopters)

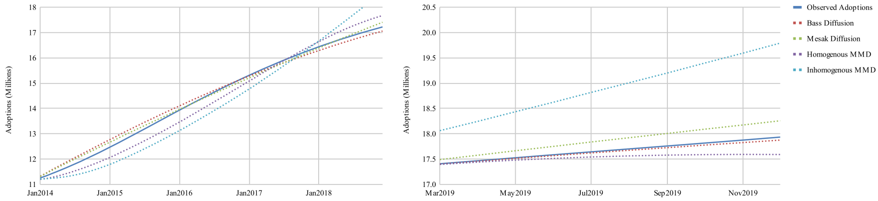


Fig. 8 In-sample (Left) and Out-of-sample (Right) results

5 Conclusions

In this work we have introduced a novel inhomogenous network based diffusion model. This model can be used as a general decision support system for marketing, given that by integrating complex network information, as well as marketing mix controls, we can incorporate all types of data, including social and geographic (spatial) information. Additionally, to the best of our knowledge, we have introduced the first ever inhomogenous diffusion model which **incorporates** all **four** variables of the **marketing mix**. Through our **simulation study** we were able to show that the proposed model **generalizes** the Bass diffusion model, since it is able to represent certain dynamics which occur in inhomogenous networks, that the Bass model doesn't fully capture. In addition the **empirical study** has shown that this model has a **good explanatory power** ($R^2 = 96\%$), comparable to existing models. The study also has **limitations**. Due to **computational constraints**, we could only explore a limited number of cases in our simulation study, from which we draw qualitative conclusions. In addition, the **dataset** we used on the empirical study did not include the adoption at each node over time. Therefore, we had to rely on assumptions about the seed node set and initial adoption in each node, which might not be entirely realistic. There is also no way of confirming if the node activation in the **final state** corresponds to real world mobile broadband adoption at each municipality. The model evaluation and comparison was conducted solely on the ability to explain and predict the **aggregated adoption** in Japan. Finally, we can also note that the predictive ability of our model is significantly worse than the MMD model. This might be related with the large parameter space, or with the low efficiency of the least squares estimators we employed. Further work might focus on obtaining efficient estimators for this model.

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