

OPEN DESCENDANTS OF $U(2N)$ ORBIFOLDS AT RATIONAL RADII

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Abstract

We construct explicitly the open descendants of some exceptional automorphism invariants of $U(2N)$ orbifolds. We focus on the case $N = p_1 \times p_2$, p_1 and p_2 prime, and on the automorphisms of the diagonal and charge conjugation invariants that exist for these values of N . These correspond to orbifolds of the circle with radius $R^2 = 2p_1/p_2$. For each automorphism invariant we find two consistent Klein bottles, and for each Klein bottle we find a complete (and probably unique) set of boundary states. The two Klein bottles are in each case related to each other by simple currents, but surprisingly for the automorphism of the charge conjugation invariant neither of the Klein bottle choices is the canonical (symmetric) one.

1 Introduction

The method of open descendants [1] enables one to construct an open string theory out of a closed one. Simply put, in this method one starts from the torus partition function of a modular invariant and left-right symmetric closed string theory. Then, since such theory has in general discrete symmetries, one can, by means of a Klein bottle projection, mod out a discrete symmetry group that contains at least the left-right symmetry (sometimes called world-sheet parity) as a subgroup. The resulting theory is however normally inconsistent due to massless tadpoles of unphysical fields. Consistency can then be restored by adequately adding open string sectors to the theory.

At first, and for some time, how to construct open descendants was known only for the case of the charge conjugation modular invariant, the so-called Cardy case. Gradually, attempts at extending the method to more general modular invariants began to spawn [2]. Recently, more progress has been achieved. In [3], universal formulas for open descendants of simple current invariants were given. Another class of solutions was studied in [4], where a procedure for deriving the open descendants of modular invariants of the extension type was presented.

These two developments cover a large number of cases but do not yet exhaust all possibilities. Very little is known about the exceptional invariants of automorphism type (apart from the E_7 -invariant of $SU(2)$, level 16 studied in [5]). In this letter we make a first attempt at solving this problem by explicitly constructing the open descendants of a theory with such exceptional invariants, namely orbifolds of $U(2N)$ conformal field theories. These theories have a large set of exceptional modular invariants, related on the one hand to radius-reducing extensions of the circle, and circles of rational radii $R^2 = 2p_1/p_2$ [6], and on the other hand to exceptional modular invariants of $SO(N)$, level 2 [7] [8]. We will focus on two of these invariants here.

Some work on boundary states for orbifold models was already done in [9]. There the orbifold chiral algebra was studied in the continuum limit and boundary coefficients for arbitrary radius were derived. This was done however in the absence of crosscaps. In this letter we consider the maximally extended chiral algebra at a rational point, along with the extra constraints imposed by crosscaps.

2 Brief review of open descendants

Before moving on to the $U(2N)$ orbifold theory we review some facts about open descendants and present key formulas.

The closed sector of a string theory is described by the 1-loop torus modular invariant partition function (MIPF), which is, dismissing integrations over the modular space and character arguments,

$$T = \sum_{ij} \chi_i Z_{ij} \bar{\chi}_j. \quad (1)$$

i, j label the chiral primary fields and $\chi_i, \bar{\chi}_j$ are their respective characters, the bar standing for anti-holomorphic. The matrix Z_{ij} characterizes the various MIPF's of the conformal field theory (CFT). It is symmetric, has positive integer entries and commutes with the modular matrices T and S of the CFT.

The Klein bottle amplitude acts on this closed string spectrum as a projection. The Klein bottle can be seen in two different ways: direct and transverse channels. In the direct channel, it is viewed as a closed string propagating in a loop, coming back to itself with opposite orientation; this is interpreted as a partition function. In the transverse channel, the Klein bottle is viewed as a closed string propagating between two crosscap states; this is interpreted as a transition amplitude (and is denoted by a tilde). The two channels are related via modular transformations acting on the characters. This transformation is S for the Klein bottle, which takes the form

$$K = \sum_i K^i \chi_i \xrightarrow{S} \tilde{K} = \sum_m \Gamma_m^2 \chi_m, \quad (2)$$

with the requirement that $|K^i| = Z_{ii}$, which insures positivity and integrality of the un-oriented closed sector partition function. Only fields that couple to their charge conjugate on the torus can propagate in the transverse channel [10]. These are called transverse fields and are labeled m . The crosscap coefficient Γ_m is their 1-point function on the crosscap. Non-transverse fields always have $\Gamma = 0$.

The open sector is described by the annulus and Moebius strip diagrams. In the direct channel, the annulus and Moebius strip have the interpretation of open strings propagating in a loop and coming back to itself, without or with an orientation flip respectively. They are the partition functions of the open sector. In the transverse channel, the interpretation is that of transition amplitudes for closed strings propagating between two boundary states and a boundary and crosscap state respectively. The channel transformations are S for the annulus, and $P = \sqrt{T}ST^2S\sqrt{T}$ for the Moebius strip. The expressions for the open sector are (transverse channels quantities marked with a tilde)

$$A_{ab} = \sum_i A_{ab}^i \chi_i \xrightarrow{S} \tilde{A}_{ab} = \sum_m B_{am} B_{bm} \chi_m, \quad (3)$$

$$M_a = \pm \sum_i M_a^i \hat{\chi}_i \xrightarrow{P} \tilde{M}_a = \pm \sum_m \Gamma_m B_{bm} \hat{\chi}_m, \quad (4)$$

where a, b label the possible boundary conditions (types of D-branes). The annulus coefficients A_{ab}^i are positive integers, as well as M_a^i , which are furthermore constrained by $M_a^i \leq A_{aa}^i$, $|M_a^i| = A_{aa}^i \pmod{2}$, which insures positivity and integrality of the open sector partition function. The hatted characters in the Moebius denote the usual shifted argument. B_{am} are the boundary coefficients. They are the 1-point functions of the transverse fields m on the disk with boundary condition a and must satisfy the sewing constraints of [11]. The open and closed sectors are further related via Chan-Paton factors and tadpole cancellation, but this is not relevant for our discussion and will not be exploited here.

Examining (2-4) we see that the relevant quantities on the direct and transverse channels are related by

$$K^i = \sum_m S_m^i \Gamma_m^2, \quad A_{ab}^i = \sum_m S_m^i B_{am} B_{bm}, \quad M_a^i = \sum_m P_m^i \Gamma_m B_{am}. \quad (5)$$

The problem boils down to solving these equations, under the restrictions of positivity and integrality, crosscap and sewing constraints. While positivity and integrality are straightforward to check, the crosscap and sewing constraints require knowledge of the fusing and

brading matrices, which are not yet available for the orbifold theory. Nevertheless, the constraints imposed by positivity and integrality are in general very strong, and, as we shall see, are already enough to determine the crosscap and boundary coefficients in our cases.

For future use, we now define U-crosscap and reflection coefficients as

$$U_m = \Gamma_m \sqrt{S_{0m}}, \quad R_{am} = B_{am} \sqrt{S_{0m}}. \quad (6)$$

To find the solutions we start with an allowed Klein bottle and compute the corresponding crosscap coefficients from it. Then we look for non-negative integer annuli A^i_{aa} , such that the corresponding boundary coefficients vanish for fields that are not allowed in the transverse channel, and such that a Moebius amplitude exists that is consistent with it. Finally we try to assemble these boundary coefficients into a complete set, satisfying the condition $R_{am}R^*_{bm} = \delta_{ab}$ [12], and such that the off-diagonal annuli A^i_{ab} are non-negative integers. This can only be done in practice for small N , but it was possible to extrapolate the solution to arbitrary N and prove that it satisfies all relevant constraints in general. Furthermore for small N we found only one solution in each of the four cases (two modular invariants, each with two Klein bottles). The search for diagonal annuli is however necessarily limited to small integers, and therefore we cannot make rigorous claims regarding the uniqueness of our solutions.

3 The orbifold theory

$U(2N)$ conformal field theories describe a free boson compactified on a circle of radius $R^2 = 2N$, $N \in \mathbf{Z}$. Using the procedure of [13] we can build a new theory, the so-called orbifold theory, by modding the \mathbf{Z}_2 symmetry out of the $U(2N)$ CFT [14]. The procedure can be inverted: by extending the orbifold theory with an appropriate simple current one gets back the original $U(2N)$ CFT. The orbifold theory has exceptional MIPF's of the automorphism type when N is both odd and a product of at least two different prime factors [6].

The fields of the orbifold theory are as follows:

Field	Weight	Description
O	0	vacuum
J	1	simple current
ϕ^l	$N/4$	complex fields
ϕ_k	$k^2/4N$	fixed points of J
σ_l	$1/16$	twisted sector
τ_l	$9/16$	twisted sector

Label l takes values 1,2 and label k runs from 1 to $N - 1$. The modular matrices S and P for this theory are presented in appendix B.

The fusion rules of the orbifold theory are invariant under the simultaneous action of an automorphism ω on all labels of the fusion coefficients: $N_{ij}^k = N_{\omega(i)\omega(j)}^{\omega(k)}$. This automorphism acts non-trivially only on the ϕ_k . Its action is described explicitly on

appendix A. For our purpose, the important thing about it is that it can be used to write down an exceptional torus MIPF:

$$T = \sum_{ij} \chi_i \delta_{i\omega(j)} \bar{\chi}_j. \quad (7)$$

We will focus on the case $N = p_1 \times p_2$, $p_1 < p_2$ and both prime. For this case it turns out that the fields¹ $\phi_k, k=p_1, p_2$ self-couple on the torus, whereas the remaining ϕ_k couple amongst themselves crosswise. Since the ϕ_k are real, the transverse fields are then O, J and $\phi_k, k=p_1, p_2$, in total $p_1 + p_2$ of them.

We can construct another exceptional MIPF replacing $\delta_{i\omega(j)}$ with the charge conjugation matrix $C_{i\omega(j)}$ in (7). We will call the two MIPF's "diagonal + automorphism" (D+A) and "Cardy + automorphism" (C+A) respectively. In (C+A) ϕ^l, σ_l, τ_l become transverse fields, raising the total number of these to $p_1 + p_2 + 6$.

Having the torus MIPF, the next step is finding the Klein bottle projection. The Klein bottle has to be such that non-transverse fields do not propagate in the transverse channel; in other words, that their Γ vanishes. Since Γ can be immediately calculated by inverting (2), we conclude that the signs $\varepsilon_j = K^j/Z_{jj}$ of the direct channel Klein bottle have to be such that

$$\Gamma_i^2 = \sum_j S_{ij} \varepsilon_j Z_{jj} = 0, \quad \forall_i : Z_{ii^c} = 0. \quad (8)$$

In the (D+A) case this condition is satisfied for the "trivial" Klein bottle projection, $K^i = 1$ for all the fields coupling diagonally on the torus. In addition there is a second Klein bottle with $K^i = -1$ on the four twist fields σ_j and τ_j and $K^i = 1$ for the other diagonal fields. There are several other Klein bottle choices that satisfy the sum rule (8), but if in addition we impose the Klein bottle constraint of [15] [16] these are the only two that are allowed. These two Klein bottle choices are related by the action of the simple current J , by the mechanism explained in [16]. In the (C+A) case we observe first of all that surprisingly the trivial choice $K^i = 1$ for all the fields coupling diagonally on the torus violates the sum rule (8). There are however several Klein bottle choices satisfying the sum rule, and two of them also satisfy the Klein bottle constraint. One has $K^{\phi_k} = -1$ when k is an odd multiple of p_1 and $K^i = 1$ for the remaining fields coupling diagonally on the torus, and the other has $K^{\phi_k} = -1$ when k is an odd multiple of p_2 and $K^i = 1$ for the remaining fields coupling diagonally on the torus. These two Klein bottle choices are again related by the action of a simple current, but this time by ϕ^l .

For the two aforementioned Klein bottles of (D+A) and (C+A) it was possible to construct a complete set of boundary states, which are now presented.

4 D+A invariant

¹We denote by \dot{p} multiples of p ; zero included, when possible.

4.1 D+A U-crosscap and reflection coefficients

For the (D+A) case and trivial Klein bottle, the U-crosscap coefficients are

$$\begin{array}{c|cccccccc}
 & O & J & \phi^j & & \phi_k & & \sigma_j & \tau_j \\
 \hline
 U_i & \frac{1}{2} \left(\frac{1}{\sqrt{p_1}} + \frac{1}{\sqrt{p_2}} \right) & \frac{1}{2} \left(\frac{1}{\sqrt{p_1}} - \frac{1}{\sqrt{p_2}} \right) & 0 & \dots, & \underbrace{\frac{1}{\sqrt{p_2}}}_{k=2p_1}, \dots, & \underbrace{\frac{1}{\sqrt{p_1}}}_{k=2p_2}, \dots & 0 & 0
 \end{array} \quad (9)$$

The dots stand for zero entries. We defined the coefficients in such a way that for this Klein bottle they are all positive². The Klein bottle with -1 for the twist fields has U_O and U_J interchanged and also a minus sign for $\phi_{k,k=2p_1}$. Note that for both cases all non-transverse fields have vanishing U-crosscap coefficient, as expected.

In order to determine the reflection coefficients, we first have to classify the possible boundary conditions. It turns out that three types of boundary conditions are possible:

- Type b . This type contains two boundary conditions, b_1 and b_2 , regardless of N .
- Type a_1 . These split further into two subsets, a_{1f} and a'_{1f} (with f an odd integer ranging from 1 to $p_1 - 2$). Each of these subsets contains thus $(p_1 - 1)/2$ boundary conditions, amounting to $p_1 - 1$ boundary conditions coming from this type of boundary.
- Type a_2 . Similar to type a_1 . It splits into subsets a_{2f} and a'_{2f} (with odd f ranging from 1 to $p_2 - 2$ this time). It contains $p_2 - 1$ boundary conditions.

There are $p_1 + p_2$ boundary conditions in total, as many as the transverse fields.

²We can always do this, since the problem is invariant under $U_m \rightarrow \varepsilon_m U_m$, $R_{am} \rightarrow \varepsilon_m R_{am}$.

For both Klein bottle projections the reflection coefficients are

	O	J	ϕ^l	ϕ_k	σ_l	τ_l
$R_{b_1,i}$	$\frac{1}{\sqrt{2p_1}}$	$\frac{-1}{\sqrt{2p_1}}$	0	$\dots, \underbrace{\frac{2(-1)^n}{\sqrt{2p_1}}}_{k=2np_2}, \dots$	0	0
$R_{b_2,i}$	$\frac{1}{\sqrt{2p_2}}$	$\frac{1}{\sqrt{2p_2}}$	0	$\dots, \underbrace{\frac{2(-1)^n}{\sqrt{2p_2}}}_{k=2np_1}, \dots$	0	0
$R_{a_{1f},i}$	$\frac{1}{\sqrt{2p_1}}$	$\frac{-1}{\sqrt{2p_1}}$	0	$\dots, \underbrace{\frac{2 \cos(\frac{\pi n f}{2p_1})}{\sqrt{2p_1}}}_{k=np_2}, \dots$	0	0
$R_{a'_{1f},i}$	$\frac{1}{\sqrt{2p_1}}$	$\frac{-1}{\sqrt{2p_1}}$	0	$\dots, \underbrace{\frac{2(-1)^n \cos(\frac{\pi n f}{2p_1})}{\sqrt{2p_1}}}_{k=np_2}, \dots$	0	0
$R_{a_{2f},i}$	$\frac{1}{\sqrt{2p_2}}$	$\frac{1}{\sqrt{2p_2}}$	0	$\dots, \underbrace{\frac{2 \cos(\frac{\pi n f}{2p_2})}{\sqrt{2p_2}}}_{k=np_1}, \dots$	0	0
$R_{a'_{2f},i}$	$\frac{1}{\sqrt{2p_2}}$	$\frac{1}{\sqrt{2p_2}}$	0	$\dots, \underbrace{\frac{2(-1)^n \cos(\frac{\pi n f}{2p_2})}{\sqrt{2p_2}}}_{k=np_1}, \dots$	0	0

(10)

As expected, the non-transverse fields have vanishing reflection coefficients. Suppressing the zero columns in (10), we come to an orthogonal square matrix. Orthogonality of R is important, since it ensures that the completeness conditions of [10] are satisfied. Note that for the limiting case $f = p_l$ the boundaries a_{lf} and a'_{lf} are both equal to b_l . It is however convenient to treat the b -boundaries separately.

4.2 D+A annulus and Moebius strip

With the knowledge of the U-crosscap and reflection coefficients, we can now calculate the direct channel quantities, the annulus and Moebius strip coefficients. For brevity we present only the diagonal annulus A^i_{aa} , which contains the most important information. Using (3) we get

	O	J	ϕ^l	ϕ_k	σ_l	τ_l
$A^i_{b_1,b_1}$	1	1	1	$\dots, \underbrace{2}_{k=p_1}, \dots$	0	0
$A^i_{b_2,b_2}$	1	1	1	$\dots, \underbrace{2}_{k=p_2}, \dots$	0	0
$A^i_{a_{1f},a_{1f}} = A^i_{a'_{1f},a'_{1f}}$	1	1	0	$\dots, \underbrace{1}_{k \pm f = \text{even } p_1}, \dots, \underbrace{2}_{k = \text{even } p_1}, \dots$	0	0
$A^i_{a_{2f},a_{2f}} = A^i_{a'_{2f},a'_{2f}}$	1	1	0	$\dots, \underbrace{1}_{k \pm f = \text{even } p_2}, \dots, \underbrace{2}_{k = \text{even } p_2}, \dots$	0	0

(11)

Since the reflection coefficients are the same for the two Klein bottles, this is the annulus for both cases. Evaluating (4) for the trivial Klein bottle projection gives us the Moebius strip

$$\begin{array}{c|cccccc}
& O & J & \phi^l & \phi_k & \sigma_l & \tau_l \\
\hline
M_{b_1}^i & 1 & 1 & 1 & \dots, \underbrace{2}_{k=\text{odd } p_1}, \dots & 0 & 0 \\
M_{b_2}^i & 1 & 1 & 1 & \dots, \underbrace{2}_{k=\text{odd } p_2}, \dots & 0 & 0 \\
M_{a_{1f}}^i = M_{a'_{1f}}^i & 1 & 1 & 0 & \dots, \underbrace{1}_{k\pm f=\text{even } p_1}, \dots & 0 & 0 \\
M_{a_{2f}}^i = M_{a'_{2f}}^i & 1 & 1 & 0 & \dots, \underbrace{1}_{k\pm f=\text{even } p_2}, \dots & 0 & 0
\end{array} \tag{12}$$

For the Klein bottle with -1 for the twist fields the Moebius is

$$\begin{array}{c|cccccc}
& O & J & \phi^l & \phi_k & \sigma_l & \tau_l \\
\hline
M_{b_1}^i & -1 & -1 & 1 & \dots, \underbrace{2}_{k=\text{odd } p_1}, \dots & 0 & 0 \\
M_{b_2}^i & 1 & 1 & -1 & \dots, \underbrace{-2}_{k=\text{odd } p_2}, \dots & 0 & 0 \\
M_{a_{1f}}^i = M_{a'_{1f}}^i & -1 & -1 & 0 & \dots, \underbrace{1}_{k\pm f=\text{even } p_1}, \dots & 0 & 0 \\
M_{a_{2f}}^i = M_{a'_{2f}}^i & 1 & 1 & 0 & \dots, \underbrace{-1}_{k\pm f=\text{even } p_2}, \dots & 0 & 0
\end{array} \tag{13}$$

Inspection shows that (11-13) respect the positivity and integrality conditions for the open sector. One can also show that (10) leads to a positive integer off-diagonal annulus A_{ab}^i .

5 C+A invariant

5.1 C+A U-crosscap and reflection coefficients

We present the results for the Klein bottle with $K^{\phi_k} = -1$ when k is an odd multiple of p_1 and $K^i = 1$ for remaining the fields that couple diagonally on the torus. Since in the (C+A) case no quantity is proportional to the difference $p_1 - p_2$, the results for the second Klein bottle are obtained by simply interchanging $p_1 \leftrightarrow p_2$ on the formulas below.

The U-crosscap coefficients are

$$\begin{array}{c|cccccc}
& O & J & \phi^l & \phi_k & \sigma_l & \tau_l \\
\hline
U_i & \frac{1}{2\sqrt{p_2}} & \frac{1}{2\sqrt{p_2}} & \frac{1}{2\sqrt{p_1}} & \dots, \underbrace{\frac{1}{\sqrt{p_1}}}_{k=\text{odd } p_2}, \dots, \underbrace{\frac{1}{\sqrt{p_2}}}_{k=\text{even } p_1}, \dots & 0 & 0
\end{array} \tag{14}$$

Next we classify the possible boundaries conditions. As in the (D+A) case we get a certain number fixed boundaries along with an expanding part which depends on the odd parameter f .

- Type a, b, c, d . There are two a -type boundaries, a and a' . These are real (in the sense that their reflection coefficients turn out to be real). The boundaries b, c and d are unique but complex, thus admitting conjugate boundaries b', c' and d' respectively. In total these four types generate eight boundary conditions.
- Type E_f . This type of boundary is real and splits into subtypes E_f and E'_f , with odd f ranging from 1 to $p_1 - 2$, and contributes with a total of $p_1 - 1$ boundary conditions.
- Type F_f . This type is complex and thus splits into F_f and its conjugate F'_f , with off f ranging from 1 to $p_2 - 2$, and contributes with $p_2 - 1$ boundary conditions.

The total number of boundaries is thus $p_1 + p_2 + 6$, as many as the transverse fields, as expected.

The reflection coefficients are

	O	J	ϕ^j	ϕ_k	σ_j	τ_j
$R_{a,i}$	$\frac{1}{\sqrt{8p_1}}$	$\frac{1}{\sqrt{8p_1}}$	$\frac{1}{\sqrt{8p_1}}$	$\dots, \underbrace{\frac{1}{\sqrt{2p_1}}}_{k=np_2}, \dots$	$\frac{1}{\sqrt{8}}$	$\frac{1}{\sqrt{8}}$
$R_{a',i}$	$\frac{1}{\sqrt{8p_1}}$	$\frac{1}{\sqrt{8p_1}}$	$\frac{1}{\sqrt{8p_1}}$	$\dots, \underbrace{\frac{1}{\sqrt{2p_1}}}_{k=np_2}, \dots$	$\frac{-1}{\sqrt{8}}$	$\frac{-1}{\sqrt{8}}$
$R_{b,i}$	$\frac{1}{\sqrt{8p_1}}$	$\frac{1}{\sqrt{8p_1}}$	$\frac{-1}{\sqrt{8p_1}}$	$\dots, \underbrace{\frac{(-1)^n}{\sqrt{2p_1}}}_{k=np_2}, \dots$	$\frac{i\sigma_{1j}}{\sqrt{8}}$	$\frac{i\sigma_{1j}}{\sqrt{8}}$
$R_{c,i}$	$\frac{1}{\sqrt{8p_2}}$	$\frac{-1}{\sqrt{8p_2}}$	$\frac{i\epsilon\sigma_{1j}}{\sqrt{8p_2}}$	$\dots, \underbrace{\frac{(-1)^n}{\sqrt{2p_2}}}_{k=2np_1}, \dots, \underbrace{\frac{i}{\sqrt{2p_2}}}_{k=(2n-1)p_1}, \dots$	$\frac{e^{i\pi\sigma_{1j}/4}}{\sqrt{8}}$	$\frac{-e^{i\pi\sigma_{1j}/4}}{\sqrt{8}}$
$R_{d,i}$	$\frac{1}{\sqrt{8p_2}}$	$\frac{-1}{\sqrt{8p_2}}$	$\frac{i\epsilon\sigma_{1j}}{\sqrt{8p_2}}$	$\dots, \underbrace{\frac{(-1)^n}{\sqrt{2p_2}}}_{k=2np_1}, \dots, \underbrace{\frac{i}{\sqrt{2p_2}}}_{k=(2n-1)p_1}, \dots$	$\frac{-e^{i\pi\sigma_{1j}/4}}{\sqrt{8}}$	$\frac{e^{i\pi\sigma_{1j}/4}}{\sqrt{8}}$
$R_{E_f,i}$	$\frac{1}{\sqrt{2p_1}}$	$\frac{1}{\sqrt{2p_1}}$	$\frac{1}{\sqrt{2p_1}}$	$\dots, \underbrace{\frac{2(-1)^n \cos(\frac{\pi 2nf}{2p_1})}{\sqrt{2p_1}}}_{k=2np_2}, \dots, \underbrace{\frac{2\delta_f (-1)^n \sin(\frac{\pi(2n-1)f}{2p_1})}{\sqrt{2p_1}}}_{k=(2n-1)p_2}, \dots$	0	0
$R_{E'_f,i}$	$\frac{1}{\sqrt{2p_1}}$	$\frac{1}{\sqrt{2p_1}}$	$\frac{-1}{\sqrt{2p_1}}$	$\dots, \underbrace{\frac{2(-1)^n \cos(\frac{\pi 2nf}{2p_1})}{\sqrt{2p_1}}}_{k=2np_2}, \dots, \underbrace{\frac{-2\delta_f (-1)^n \sin(\frac{\pi(2n-1)f}{2p_1})}{\sqrt{2p_1}}}_{k=(2n-1)p_2}, \dots$	0	0
$R_{F_f,i}$	$\frac{1}{\sqrt{2p_2}}$	$\frac{-1}{\sqrt{2p_2}}$	$\frac{i\sigma_{1j}}{\sqrt{2p_2}}$	$\dots, \underbrace{\frac{2 \cos(\frac{\pi 2nf}{2p_2})}{\sqrt{2p_2}}}_{k=2np_1}, \dots, \underbrace{\frac{-2i\delta_f (-1)^n \sin(\frac{\pi(2n-1)f}{2p_2})}{\sqrt{2p_2}}}_{k=(2n-1)p_1}, \dots$	0	0

$$\epsilon = (-1)^{(N+1)/2}, \quad \delta_f = (-1)^{(p_1+f)/2}, \quad \sigma_{1j} = 2\delta_{1j} - 1. \quad (15)$$

Again, both the U-crosscap and reflection coefficients vanish on the non-transverse fields. The reflection coefficients for boundary conditions b', c', d' and F'_f are simply the complex conjugates of their unprimed counterparts.

5.2 C+A annulus and Moebius strip

6 C+A annulus and Moebius strip

The diagonal annulus and Moebius strip that go with the U-crosscap and reflection coefficients of (14-15) are

	O	J	ϕ^1	ϕ^2	ϕ_k	σ	σ_2	τ_1	τ_2
$A^i_{a,a} = A^i_{a',a'}$	1	0	0	0	$\dots, \underbrace{1}_{k=\text{even } p_1}, \dots$	0	0	0	0
$A^i_{b,b} = A^i_{b',b'}$	0	1	0	0	$\dots, \underbrace{1}_{k=\text{even } p_1}, \dots$	0	0	0	0
$A^i_{c,c} = A^i_{d,d}$	0	0	1	0	$\dots, \underbrace{1}_{k=\text{odd } p_2}, \dots$	0	0	0	0
$A^i_{c',c'} = A^i_{d',d'}$	0	0	0	1	$\dots, \underbrace{1}_{k=\text{odd } p_2}, \dots$	0	0	0	0
$A^i_{E_f, E_f} = A^i_{E'_f, E'_f}$	1	1	0	0	$\dots, \underbrace{1}_{k \pm f = \text{odd } p_1}, \dots, \underbrace{2}_{k = \text{even } p_1}, \dots$	0	0	0	0
$A^i_{F_f, F_f} = A^i_{F'_f, F'_f}$	0	0	1	1	$\dots, \underbrace{1}_{k \pm f = \text{even } p_2}, \dots, \underbrace{2}_{k = \text{odd } p_2}, \dots$	0	0	0	0

(16)

	O	J	ϕ^1	ϕ^2	ϕ_k	σ	σ_2	τ_1	τ_2
$M^i_a = M^i_{a'}$	1	0	0	0	$\dots, \underbrace{(-1)^{k/2}}_{k=\text{even } p_1}, \dots$	0	0	0	0
$M^i_b = M^i_{b'}$	0	1	0	0	$\dots, \underbrace{-(-1)^{k/2}}_{k=\text{even } p_1}, \dots$	0	0	0	0
$M^i_c = M^i_d$	0	0	1	0	$\dots, \underbrace{1}_{k=\text{odd } p_2}, \dots$	0	0	0	0
$M^i_{c'} = M^i_{d'}$	0	0	0	1	$\dots, \underbrace{1}_{k=\text{odd } p_2}, \dots$	0	0	0	0
$M^i_{E_f}$	1	1	0	0	$\dots, \underbrace{(-1)^{k/2}}_{k \pm f = \text{odd } p_1}, \dots$	0	0	0	0
$M^i_{E'_f}$	1	1	0	0	$\dots, \underbrace{-(-1)^{k/2}}_{k \pm f = \text{odd } p_1}, \dots$	0	0	0	0
$M^i_{F_f}$	0	0	ϵ	$-\epsilon$	$\dots, \underbrace{1}_{k \pm f = \text{even } p_2}, \dots$	0	0	0	0
$M^i_{F'_f}$	0	0	$-\epsilon$	ϵ	$\dots, \underbrace{1}_{k \pm f = \text{even } p_2}, \dots$	0	0	0	0

(17)

Positivity and integrality of the diagonal annulus and Moebius strip is explicit. Again it can be shown that (15) leads to positive integer off-diagonal annulus.

7 Outlook

Formulas (9-15) together with (16-17) summarize the information on the open descendants of the $N = p_1 \times p_2$ class of $U(2N)$ orbifold theories and they are the main result of this letter. The open descendants thus obtained satisfy the strong constraining requirements of positivity and integrality of the open and closed sector partitions functions, as well as the completeness condition and the Klein bottle constraint. This provides strong evidence for the correctness of our proposal. It is also interesting to note that in the (C+A) case the natural Klein bottle choice $K^i = 1$ for all the fields coupling diagonally on the torus is inconsistent.

A natural next step would be the extension of our results to more general automorphisms. It might also be worthwhile try and understand these results from the point of view of the $U(2N)$ theory. In the $U(2N)$ theory the exceptional invariants are mapped into simple current invariants, whose properties are well under control. Relating the boundary and crosscap coefficients of the orbifold theory exceptional MIPF with those of the $U(2N)$ theory simple current MIPF would undoubtedly help understanding the construction of open descendants of exceptional MIPF's in more general cases. This is currently under study.

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A Automorphisms of the orbifold theory

In this appendix we describe the fusion rule automorphisms present in the orbifold theory.

We denote the automorphism by ω . It acts trivially on all fields other than ϕ_k : $\omega(O; J; \phi^l; \sigma_l; \tau_l) = O; J; \phi^l; \sigma_l; \tau_l$. On the ϕ_k the action is as follows. Write ϕ_k as

$$\phi_k = \begin{cases} N - 2g & \text{for } k \text{ odd} \\ 2g & \text{for } k \text{ even} \end{cases} \quad (18)$$

with g taking the values $1, \dots, (N - 1)/2$. Now look for the smallest integer $m > 1$ such that $m^2 = 1 \pmod N$. For any x define the (unique) number $[x]_N$, $0 \leq [x]_N \leq N/2$, such that $x \equiv \pm[x]_N \pmod N$ for some choice of sign and define the permutation

$$\pi_m : \pi_m(g) = [mg]_N. \quad (19)$$

The automorphism then acts on ϕ_k as

$$\begin{aligned} \omega(\phi_{2g}) &= \phi_{2[mg]_N} \\ \omega(\phi_{N-2g}) &= \phi_{N-2[mg]_N}. \end{aligned} \quad (20)$$

The (D+A) torus is then

$$T = \sum_g \chi_{\phi_{2g}} \bar{\chi}_{\phi_{2[mg]_N}} + \chi_{\phi_{N-2g}} \bar{\chi}_{\phi_{N-2[mg]_N}} + \text{diagonal in the other fields.} \quad (21)$$

The (C+A) torus is obtained applying charge conjugation to (21).

B The modular matrices S and P of the orbifold theory for N odd

$$S = \frac{1}{\sqrt{8N}} \times$$

	O	J	ϕ^l	$\phi_{k'}$	$\sigma_{l'}$	$\tau_{l'}$
O	1	1	1	2	\sqrt{N}	\sqrt{N}
J	1	1	1	2	$-\sqrt{N}$	$-\sqrt{N}$
ϕ^l	1	1	-1	$2(-1)^{k'}$	$i\sigma_{l'}\sqrt{N}$	$i\sigma_{l'}\sqrt{N}$
ϕ_k	2	2	$2(-1)^k$	$4 \cos(2\pi \frac{kk'}{2N})$	0	0
σ_l	\sqrt{N}	$-\sqrt{N}$	$i\sigma_{l'}\sqrt{N}$	0	$\sqrt{N}e^{\frac{i\pi}{4}\sigma_{l'}(-1)^{\frac{N+1}{2}}}$	$-\sqrt{N}e^{\frac{i\pi}{4}\sigma_{l'}(-1)^{\frac{N+1}{2}}}$
τ_l	\sqrt{N}	$-\sqrt{N}$	$i\sigma_{l'}\sqrt{N}$	0	$-\sqrt{N}e^{\frac{i\pi}{4}\sigma_{l'}(-1)^{\frac{N+1}{2}}}$	$\sqrt{N}e^{\frac{i\pi}{4}\sigma_{l'}(-1)^{\frac{N+1}{2}}}$

We define $\sigma_{l'} = 2\delta_{l'} - 1$. This matrix is obtained from [14], with a correction for the fixed point entries.

$$P = \frac{1}{2} \times$$

	O	J	ϕ^l	$\phi_{k'}$	$\sigma_{l'}$	$\tau_{l'}$
O	1	1	$\frac{1}{\sqrt{N}}$	$\frac{1-(-1)^{k'}}{\sqrt{N}}$	0	0
J	1	1	$-\frac{1}{\sqrt{N}}$	$-\frac{1-(-1)^{k'}}{\sqrt{N}}$	0	0
ϕ^l	$\frac{1}{\sqrt{N}}$	$-\frac{1}{\sqrt{N}}$	$-\sigma_{l'} e^{i\pi \frac{N}{2}}$	$\frac{(1+(-1)^k)(-1)^{k'/2}}{\sqrt{N}}$	0	0
ϕ_k	$\frac{1-(-1)^k}{\sqrt{N}}$	$-\frac{1-(-1)^k}{\sqrt{N}}$	$\frac{(1+(-1)^k)(-1)^{k/2}}{\sqrt{N}}$	$\frac{2(1-(-1)^{k+k'}) \cos(2\pi \frac{kk'}{4N})}{\sqrt{N}}$	0	0
σ_l	0	0	0	0	$e^{\frac{-i\pi}{8}\sigma_{l'}}$	$e^{\frac{3i\pi}{8}\sigma_{l'}}$
τ_l	0	0	0	0	$e^{\frac{3i\pi}{8}\sigma_{l'}}$	$-e^{\frac{-i\pi}{8}\sigma_{l'}}$

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