AXIOMS FOR UNARY SEMIGROUPS VIA DIVISION OPERATIONS

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In memory of Takayuki Tamura (1919-2009)

Abstract. When a semigroup has a unary operation, it is possible to define two binary operations, namely, left and right division. In addition it is well known that groups can be defined in terms of those two divisions. The aim of this paper is to extend those results to other classes of unary semigroups. In the first part of the paper we provide characterizations for several classes of unary semigroups, including (a special class of) E-inversive, regular, completely regular, inverse, Clifford, etc., in terms of left and right division. In the second part we solve a problem that was posed elsewhere. The paper closes with a list of open problems.

1. Introduction

A unary semigroup $(S, \cdot, ')$ is a semigroup $(S, \cdot)$ together with an additional unary operation $' : S \to S$. Naturally linked to any unary semigroup are two binary operations, called *left* and *right divisions*, respectively, and defined by:

$$x \backslash y = x'y \qquad \text{and} \qquad x/y = xy' . \tag{1.1}$$

This defines a bimagma (a set with two binary operations) $(S, \backslash, /)$. We will refer to $(S, \backslash, /)$, with $\backslash$ and $/$ defined by (1.1), as the *division bimagma* of the unary semigroup $(S, \cdot, ')$. Thus (1.1) defines a functor $(S, \cdot, ') \rightsquigarrow (S, \backslash, /)$ from unary semigroups to their division bimagmas.

One of the goals of defining a class of algebras in terms of a class of algebras of a different type is that many properties/problems become obvious in the new setting, while difficult to spot in the first. This type of research, very popular some years ago, is now attracting renewed interest as assistance from computational tools allows attacks on problems that not long ago seemed difficult if not impossible. Therefore the main theme of this paper is to characterize various classes of unary semigroups in terms of their division bimagmas. There are many such characterizations for groups, *e.g.*, [2]. Tamura [10] seems to have been the first to find characterizations for more general classes of unary semigroups, specifically, regular involuted and inverse semigroups in terms of their division bimagmas. Tamura's work was followed up in [1].

This paper has two main parts. In §2, we offer new characterizations of several classes of unary semigroups. In §3, we address one of the problems raised in [1].

The division bimagmas of all unary semigroups we consider in this paper will satisfy

$$(x\backslash y)/z = x\backslash(y/z) \tag{B1}$$

$$x/y' = x'\backslash y\,. \tag{B2}$$

In fact, (B1) clearly holds in any unary semigroup; it is just a consequence of associativity and (1.1). Properties (B1) and (B2) allow us to reconstruct the unary operation $'$ and the semigroup multiplication $\cdot$ by

$$x' = (x\backslash x)/x = x\backslash(x/x) \qquad \text{and} \qquad x \cdot y = x/y' = x'\backslash y\,. \tag{1.2}$$

Here we think of (B1) as implying that x' is well-defined in (1.2). In (B2), we view x' as a shorthand for $(x\backslash x)/x = x\backslash(x/x)$. Thus we do not consider $'$ to be part of the signature of a bimagma $(S, \backslash, /)$.

It is easy to see that starting with a bimagma $(S, \backslash, /)$ satisfying (B1), (B2), and defining $'$ and $\cdot$ by (1.2), we obtain a unary semigroup $(S, \cdot, ')$. Indeed, $(xy)z = (x'\backslash y)/z' \overset{(B1)}{=} x'\backslash(y/z') = x(yz)$. Thus we have a functor $(S, \backslash, /) \rightsquigarrow (S, \cdot, ')$ from bimagmas satisfying (B1) and (B2) to unary semigroups.

We now put some of the main results of the first part of this paper together into the following summary. More precise statements will be given in the next section, where we will also recall the definitions of the various classes of unary semigroups.

Theorem. *In the following table, the class of unary semigroups $(S, \cdot, ')$ on the left is characterized in terms of their division bimagmas $(S, \backslash, /)$ by the identities on the right.*

$(S, \cdot, ')$	$(S, \backslash, /)$
E-inversive, (2.1)	*(B1),* $x'\backslash y' = x/y$, $x'/y' = x\backslash y$
regular, $x'' = x$	*(B1), (B2),* $x'\backslash(x\backslash x) = x$
regular involuted	*(B1), (B2),* $x/(x\backslash x) = x$, $(x/y)' = y/x$
regular involuted, $(x(xx)'x)' = x(xx)'x$	*(B1), (B2),* $x/(x\backslash x) = x$, $(x/y)' = y/x$, $(x/(y\backslash x))/(y\backslash y) = ((x/(y\backslash x))/y)\backslash y$
inverse	*(B1), (B2),* $x/(x\backslash x) = x$, $(x\backslash x)/(y/y) = (y/y)/(x\backslash x)$
completely regular	*(B1), (B2),* $x/(x\backslash x) = x$, $x/x = x\backslash x$
Clifford	*(B1), (B2),* $x/(x\backslash x) = x$, $x/x = x\backslash x$ $(x\backslash x)/(y/y) = (y/y)/(x\backslash x)\,.$

In each case, the functors $(S, \cdot, ') \rightsquigarrow (S, \backslash, /)$ given by (1.1) *and $(S, \backslash, /) \rightsquigarrow (S, \cdot, ')$ given by* (1.2) *are mutually inverse.*

Further, in each case, the identities on the right side of the table are independent.

The second part of this paper follows more directly the work of Tamura. Tamura was motivated by work of Kimura and Sen [2], who showed that (in our terminology) groups are precisely those unary semigroups whose division bimagmas satisfy the following two identities:

$$(x/y)\backslash z = y/(z\backslash x) \qquad \text{and} \qquad (x/y)\backslash x = y\,. \tag{1.3}$$

Tamura [10] characterized regular involuted unary semigroups and inverse semigroups in a similar way. The identities he considered are the following:

$$(x/y)\backslash z = y/(z\backslash x) \quad \text{(T1)} \qquad (x/x)\backslash x = x \quad \text{(T2)}$$

$$x/y' = x'\backslash y \quad \text{(T3)} \qquad (x/y)' = y/x \quad \text{(T4)}$$

$$(x/x)/(y/y) = (y/y)/(x/x)\,. \quad \text{(T5)}$$

(Note that (T3) is just (B2).) Tamura's characterizations are as follows ([10], Theorem 3.1).

Proposition 1.1. 1) *Let $(S, \cdot, ')$ be a regular involuted unary semigroup. Then the division bimagma $(S, \backslash, /)$ satisfies (T1)–(T4). Conversely, let $(S, \backslash, /)$ be a bimagma* *satisfying (T1)–(T4). Then* $(S,\cdot,')$*, with* $'$ *and* $\cdot$ *defined by* (1.2)*, is a regular involuted unary semigroup.*

2) *Let* $(S,\cdot,')$ *be an inverse semigroup. Then the division bimagma* $(S,\backslash,/)$ *satisfies the identities (T1)–(T5). Conversely, let* $(S,\backslash,/)$ *be a bimagma satisfying (T1)–(T5). Then* $(S,\cdot,')$*, with* $'$ *and* $\cdot$ *defined by* (1.2)*, is an inverse semigroup.*

Tamura raised the question of the independence of the axioms (T1)–(T5). This question was answered by the first-named author and McCune ([1], Theorem 5.2).

Proposition 1.2. *A bimagma* $(S,\backslash,/)$ *satisfying (T1)–(T3) satisfies (T4).*

The proof in [1] is the output of a proof found by the automated deduction tool Prover9 [8]. In fact, the statement of the result in [1] is that (T1)–(T3) and (T5) imply (T4), but an examination of the proof output shows that axiom (T5) was never used.

Thus axioms (T1)–(T3) characterize division bimagmas of regular involuted unary semigroups and axioms (T1)–(T3), (T5) characterize division bimagmas of inverse semigroups. In addition, axioms (T1)–(T3), (T5) are all independent [1].

A problem posed in [1] is to find a humanly readable proof of Proposition 1.2. In §3, we give such a proof.

The paper closes with a section of open problems.

We conclude this introduction with a remark on terminology. Both [10] and [1] adapted the classical term "groupoid" (a set with a binary operation) to the present setting, referring to the structures $(S,\backslash,/)$ as "bigroupoids". However, the categorical usage of "groupoid" (a small category with inverses) seems to have generally eclipsed the older usage. Thus we have moved to the Bourbaki term *magma* for a set with a binary operation, with the obvious adaptation *bimagma* for our particular situation.

2. New Characterizations

We recall the definitions of the classes of unary semigroups we will consider. They are defined in terms of various subsets of the following properties:

$$x'xx' = x' \qquad \text{(I1)} \qquad\qquad (x(xx)'x)' = x(xx)'x \qquad \text{(I5)}$$

$$xx'x = x \qquad \text{(I2)} \qquad\qquad (xy)' = y'x' \qquad \text{(I6)}$$

$$x'' = x \qquad \text{(I3)} \qquad\qquad xx' = x'x \qquad \text{(I7)}$$

$$(xx')' = xx', \quad (x'x)' = x'x \qquad \text{(I4)} \qquad\qquad xx'y'y = y'yxx' \qquad \text{(I8)}$$

A unary semigroup $(S,\cdot,')$ is *E-inversive* if it satisfies (I1). In this case, $(S,\cdot)$ is E-inversive in the usual sense and conversely, each E-inversive semigroup has a choice of weak inverse $' : S \to S$ satisfying (I1). Note that comparing (1.1) with our reconstruction of $'$ in (1.2) already indicates why E-inversive semigroups are the most basic class we will consider.

$(S,\cdot,')$ is *regular* if it satisfies (I1) and (I2). In this case, $(S,\cdot)$ is regular in the usual sense, and conversely, each regular semigroup has a choice of inverse $' : S \to S$ satisfying (I1), (I2).

Every regular semigroup $(S,\cdot)$ has a choice of inverse $'$ that fixes idempotents, that is, $e' = e$ for every idempotent $e \in S$; simply redefine $'$ on the idempotents if necessary. It is easy to see that in a regular semigroup this property is equivalent to (I5) being an identity. In fact in a unary regular semigroup (thus the unary operation satisfies (I1) and (I2)) $x(xx)'x$ is idempotent and hence, if $'$ fixes idempotents, we have (I5); conversely, for $e^2 = e \in S$ we claim that $e' = e$. In fact, (I5) implies that $e(ee)'e = (e(ee)'e)'$ and hence $ee'e = (ee'e)'$ thus proving the claim.

The equations that make up (I4) can be considered to be weak versions of (I5); they are not of much independent interest, but we will use them to facilitate proofs in other cases.

A unary semigroup $(S, \cdot, ')$ is *involuted* if it satisfies (I3) and (I6). (See [4].)

Among the most distinguished classes of semigroups are *inverse* semigroup and *completely regular* semigroups. Inverse semigroups, to which various books have been dedicated [3, 5, 6], are involuted regular unary semigroups satisfying (I8). Completely regular semigroups, which have also been the subject of a book [7], are regular unary semigroups satisfying (I7). Both inverse and completely regular semigroups have idempotent-fixing inverses. Finally, a *Clifford* semigroup is a completely regular, inverse semigroup.

In order to keep our investigations within reasonable bounds, we will insist that the functors $(S, \cdot, ') \rightsquigarrow (S, \backslash, /)$ defined by (1.1) and $(S, \backslash, /) \rightsquigarrow (S, \cdot, ')$ defined by (1.2) are inverses of each other. Then it follows that $x' = x\backslash(x/x) = x'xx'$, that is, (I1) holds. Thus from now on, we will work in classes of unary semigroups which are, at the very least, E-inversive. In addition, we have $xy = x'\backslash y = x''y$ and $xy = x/y' = xy''$, that is, we assume

$$x''y = xy = xy''\,. \tag{2.1}$$

(This condition will be subsumed by (I3) in all of our regular classes of unary semigroups.)

Finally, recall that $'$ does not appear in the signature of a bimagma; as said after (1.2), x' should be understood as a shorthand for $(x\backslash x)/x$ (and also to $x\backslash(x/x)$, by (B1)), whenever x' appears in an identity involving $\backslash$ and $/$. As the reader will realize soon without this shorthand the proofs below would be much more difficult to follow.

Theorem 2.1. *Let* $(S, \cdot, ')$ *be an* E*-inversive unary semigroup satisfying* (2.1)*. Then the division bimagma* $(S, \backslash, /)$ *satisfies the independent identities* (B1) *and*

$$x'\backslash y' = x/y \tag{2.2}$$
$$x'/y' = x\backslash y\,. \tag{2.3}$$

Conversely, let $(S, \backslash, /)$ *be a bimagma satisfying* (B1), (2.2) *and* (2.3)*. Then* $(S, \cdot, ')$*, with* $'$ *and* $\cdot$ *defined by* (1.2)*, is an* E*-inversive unary semigroup satisfying* (2.1)*.*

The functors $(S, \cdot, ') \rightsquigarrow (S, \backslash, /)$ *defined by* (1.1) *and* $(S, \backslash, /) \rightsquigarrow (S, \cdot, ')$ *defined by* (1.2) *are inverses*

Proof. Suppose $(S, \backslash, /)$ is a bimagma satisfying *(B1)*, (2.2) and (2.3). We claim that the unary semigroup $(S, \cdot, ')$ whose operations are defined by (1.2) is an E-inversive unary semigroup satisfying (2.1). We must first show that the two possible definitions of $\cdot$ in (1.2) coincide. Thus we have to prove that (B2) holds. Note that by (2.2) and (2.3)

$$x''\backslash y'' = x'/y' = x\backslash y \qquad \text{and} \qquad x''/y'' = x'\backslash y' = x/y\,. \tag{2.4}$$

Next we show

$$x''' = x' \tag{2.5}$$

as follows:

$$\begin{aligned}
x''' &\overset{(1.2)}{=} (x''\backslash x'')/x'' && \overset{(2.2)}{=} (x'/x')/x'' && \overset{(1.2)}{=} [(x\backslash(x/x))/x']/x'' \\
&\overset{(B1)}{=} [x\backslash((x/x)/x')]/x'' && \overset{(2.3)}{=} [x\backslash((x'\backslash x')/x')]/x'' && \overset{(B1)}{=} [x\backslash(x'\backslash(x'/x'))]/x'' \\
&\overset{(1.2)}{=} [x\backslash x'']/x'' && \overset{(B1)}{=} x\backslash[x''/x''] && \overset{(2.4)}{=} x\backslash[x/x] \\
&\overset{(1.2)}{=} x' \,.
\end{aligned}$$

Thus $x/y' \overset{(2.4)}{=} x''/y''' \overset{(2.5)}{=} x''''/y''' \overset{(2.4)}{=} x''/y' \overset{(2.3)}{=} x'\backslash y$, so that (B2) holds as claimed.

Since (B1) and (B2) hold, $(S,\cdot,')$ is a semigroup. For (2.1), we compute $xy'' = x'\backslash y'' \overset{(2.2)}{=} x/y' = xy$, and a similar calculation gives the other identity. We have $x\backslash y = x'/y' = x'y$ by (2.3) and similarly, $x/y = xy'$. Then E-inversivity follows easily: $x'xx' = x'(xx') = x\backslash(x/x) = x'$.

Conversely, assume that $(S,\cdot,')$ is an E-inversive semigroup satisfying (2.1) and let $(S,\backslash,/)$ be the associated division bimagma. As noted before, (B1) holds for any unary semigroup. Also, $x'\backslash y' = x''y' = xy' = x/y$, which is (2.2), and (2.3) is similarly proved.

The assertion regarding inverse functors is now clear.

Finally, we check independence of (B1), (2.2) and (2.3). On a two-element set $S = \{a,b\}$, define $x/y = b$, $x\backslash a = a$ and $x\backslash b = b$ for all $x,y \in S$. This is easily seen to satisfy (B1) and (2.2), but not (2.3). Reversing the roles of \ and / gives a model showing the independence of (2.2). Next, on a three-element set $S = \{a,b,c\}$, define $x\backslash x = x$, $a\backslash b = b\backslash a = c$, $a\backslash c = c\backslash a = b\backslash c = c\backslash b = a$ and $x/y = x\backslash y$ for all $x \in S$. This model can be seen to satisfy (2.2) and (2.3) but not (B1). □

We will use the assertion about inverse functors implicitly in what follows. Once we have established that some class of bimagmas gives, at the very least, an E-inversive semigroup satisfying (2.1), then we will usually rewrite the division operations in terms of the semigroup multiplication and unary map.

Next we consider the case of a regular unary semigroup $(S,\cdot,')$. Our choice of inverse is somewhat restricted by (2.1).

Lemma 2.2. *Let $(S,\cdot,')$ be an E-inversive unary semigroup satisfying* (2.1). *Then* (I3) *holds if and only if $(S,\cdot,')$ is a regular unary semigroup.*

Proof. The "only if" direction is clear. For the converse, $x = xx'x \overset{(I3)}{=} x''x'x'' \overset{(2.1)}{=} x''$. □

Theorem 2.3. *Let $(S,\cdot,')$ be a regular unary semigroup satisfying* (I3). *Then the division bimagma $(S,\backslash,/)$ satisfies the independent identities* (B1), (B2) *and*

$$x'\backslash(x\backslash x) = x\,. \tag{2.6}$$

Conversely, let $(S,\backslash,/)$ be a bimagma satisfying (B1), (B2) and (2.6). *Then $(S,\cdot,')$, with $'$ and $\cdot$ defined by* (1.2), *is a regular unary semigroup satisfying (I3).*

Note that (2.6) can be replaced by its "mirror image" $(x/x)/x' = x$.

Proof. Suppose $(S,\backslash,/)$ is a bimagma satisfying (B1), (B2) and (2.6). By a sequence of calculations, we will show that (I3) holds. Firstly, we compute

$$x'\backslash(x''\backslash x) \overset{(B2)}{=} x'\backslash(x'/x') \overset{(1.2)}{=} x''\,. \tag{2.7}$$

Next, we have

$$x'\backslash x' \overset{(1.2)}{=} x'\backslash((x\backslash x)/x) \overset{(B1)}{=} (x'\backslash(x\backslash x))/x \overset{(2.6)}{=} x/x\,. \tag{2.8}$$

Third, we compute

$$\begin{aligned} x''\backslash y \overset{(B2)}{=} x'/y' &\overset{(1.2)}{=} (x\backslash(x/x))/y' \\ \overset{(B1)}{=} x\backslash((x/x)/y') &\overset{(2.8)}{=} x\backslash((x'\backslash x')/y') \\ \overset{(B1)}{=} x\backslash(x'\backslash(x'/y')) &\overset{(B2)}{=} x\backslash(x'\backslash(x''\backslash y))\,. \end{aligned} \tag{2.9}$$

Now in (2.9), set $y = x$ to get

$$x''\backslash x = x\backslash(x'\backslash(x''\backslash x)) \overset{(2.7)}{=} x\backslash x''\,. \tag{2.10}$$

Next, we compute

$$x''\backslash x'' \overset{(2.8)}{=} x'/x' \overset{(B2)}{=} x''\backslash x \overset{(2.10)}{=} x\backslash x''\,. \tag{2.11}$$

Next, we have

$$\begin{aligned} x'\backslash(x\backslash(x'\backslash y)) &\overset{(B2)}{=} x'\backslash(x\backslash(x/y')) \overset{(B1)}{=} x'\backslash((x\backslash x)/y') \\ &\overset{(B1)}{=} (x'\backslash(x\backslash x))/y') \overset{(2.6)}{=} x/y' \\ &\overset{(B2)}{=} x'\backslash y\,. \end{aligned} \tag{2.12}$$

Now in (2.12), take $y = x''\backslash x$ and use (2.7) as follows:

$$x'\backslash(x\backslash x'') \overset{(2.7)}{=} x'\backslash(x\backslash(x'\backslash(x''\backslash x))) \overset{(2.12)}{=} x'\backslash(x''\backslash x) \overset{(2.7)}{=} x''\,. \tag{2.13}$$

Next we show

$$\begin{aligned} x'''\backslash x''' \overset{(2.8)}{=} x''/x'' &\overset{(1.2)}{=} (x'\backslash(x'/x'))/x'' \\ \overset{(B1)}{=} x'\backslash((x'/x')/x'') &\overset{(B2)}{=} x'\backslash((x''\backslash x)/x'') \\ \overset{(B1)}{=} x'\backslash(x''\backslash(x/x'')) &\overset{(B2)}{=} x'\backslash(x''\backslash(x'\backslash x')) \\ \overset{(2.6)}{=} x'\backslash x'\,. \end{aligned} \tag{2.14}$$

Now we have

$$x''' \overset{(2.13)}{=} x''\backslash(x'\backslash x''') \overset{(2.11)}{=} x''\backslash(x'''\backslash x''') \overset{(2.14)}{=} x''\backslash(x'\backslash x') \overset{(2.6)}{=} x'\,. \tag{2.15}$$

Next, we have

$$\begin{aligned} x\backslash(x'\backslash y) &\overset{(2.15)}{=} x\backslash(x'''\backslash y) \overset{(2.9)}{=} x\backslash(x'\backslash(x''\backslash(x'''\backslash y))) \\ &\overset{(2.9)}{=} x''\backslash(x'''\backslash y) \overset{(2.15)}{=} x''\backslash(x'\backslash y)\,. \end{aligned} \tag{2.16}$$

Take $y = x\backslash x$ in (2.16) to get

$$x\backslash x \overset{(2.6)}{=} x\backslash(x'\backslash(x\backslash x)) \overset{(2.16)}{=} x''\backslash(x'\backslash(x\backslash x)) \overset{(2.6)}{=} x''\backslash x\,, \tag{2.17}$$

and so

$$x\backslash x'' \overset{(2.10)}{=} x''\backslash x \overset{(2.17)}{=} x\backslash x\,. \tag{2.18}$$

Finally,

$$x'' \overset{(2.13)}{=} x'\backslash(x\backslash x'') \overset{(2.18)}{=} x'\backslash(x\backslash x) \overset{(2.6)}{=} x\,.$$

Thus (I3) holds.

Now we may compute

$$x'\backslash y' \overset{(B2)}{=} x/y'' \overset{(I3)}{=} x/y \quad\text{and}\quad x'/y' \overset{(B2)}{=} x''\backslash y \overset{(I3)}{=} x\backslash y\,.$$

Thus (2.2) and (2.3) hold. By Theorem 2.1, $(S,\cdot,')$ is an E-inversive unary semigroup. Since (I3) holds, $(S,\cdot,')$ is a regular unary semigroup by Lemma 2.2.

Suppose now $(S,\cdot,')$ is a regular unary semigroup and that (I3) holds. Then $(S,\cdot,')$ is E-inversive and satisfies (2.1), so that (B1) and (B2) hold by Theorem 2.1. For (2.6), we compute $x'\backslash(x\backslash x) = (x'(xx'))'(x'x) \overset{(I1)}{=} x''x'x \overset{(I3)}{=} xx'x \overset{(I2)}{=} x$.

Finally, we check independence of the axioms. On a two-element set $S=\{a,b\}$, define $x/x = x$, $a/b = a$, $b/a = b$ and $x\backslash y = y/x$ for all $x,y\in S$. Then $(S,\backslash,/)$ is a model satisfying (B1), (2.6) but not (B2). Next, on $S=\{a,b\}$, define $x\backslash y = x/y = b$ for all $x,y\in S$. This model satisfies (B1), (B2) but not (2.6). Finally, on $S=\{a,b\}$, define $x/x = b$, $a/b = b/a = a$, $x\backslash a = a$ and $x\backslash b = b$ for all $x\in S$. This model satisfies (2.6), (B2) but not (B1). $\square$

The next class of regular unary semigroups we consider is probably not of much independent interest. In particular, we do not see that (I4) reveals any structural features about the underlying regular semigroup as, for instance, (I5) does, because in general, not every idempotent in a regular unary semigroup satisfying (I4) will have the form xx' or $x'x$. A minimal example (unique up to isomorphism) is

*	0	1	2	3
0	0	3	0	3
1	2	1	2	1
2	2	1	2	1
3	0	3	0	3

with $0' = 1$ and $x' = 0$ elsewhere; in this band $00' = 01 = 3 \neq 0$ and $0'0 = 10 = 2 \neq 0$.

In any case, we include the characterization of this class here because it facilitates the proof in the case of a regular unary semigroup with an idempotent fixing inverse. First, for convenience, we introduce a key identity:

$$x/(x\backslash x) = x\,. \tag{B3}$$

Theorem 2.4. *Let* $(S,\cdot,')$ *be a regular unary semigroup satisfying* (I3) *and* (I4). *Then the division bimagma* $(S,\backslash,/)$ *satisfies the independent identities* (B1), (B2), (B3) *and*

$$(x/x)\backslash x = x\,. \tag{2.19}$$

Conversely, let $(S,\backslash,/)$ *be a bimagma satisfying* (B1), (B2), (B3) *and* (2.19). *Then* $(S,\cdot,')$, *with* $'$ *and* $\cdot$ *defined by* (1.2), *is a regular unary semigroup satisfying (I3) and (I4).*

Note that (2.19) is just (T2) from the Introduction.

Proof. Suppose $(S,\backslash,/)$ satisfies (B1), (B2), (B3) and (2.19). We will verify (2.6). Firstly,

$$\begin{aligned}(x\backslash x)' &\overset{(1.2)}{=} (x\backslash x)\backslash[(x\backslash x)/(x\backslash x)] &&\overset{(B1)}{=} (x\backslash x)\backslash[x\backslash(x/(x\backslash x))]\\ &\overset{(B3)}{=} (x\backslash x)\backslash[x\backslash x] &&\overset{(B3)}{=} [x\backslash(x/(x\backslash x))]\backslash[x\backslash x]\\ &\overset{(B1)}{=} [(x\backslash x)/(x\backslash x)]\backslash[x\backslash x] &&\overset{(2.19)}{=} x\backslash x\,.\end{aligned} \tag{2.20}$$

A dual calculation gives

$$(x/x)' = x/x\,. \tag{2.21}$$

Thus $x'\backslash(x\backslash x) \overset{(B2)}{=} x/(x\backslash x)' \overset{(2.20)}{=} x/(x\backslash x) \overset{(B3)}{=} x$. This is (2.6). By Theorem 2.3, $(S,\cdot,')$ is a regular unary semigroup satisfying (I3). By (2.20), we have $(x'x)' = (x\backslash x)' = x\backslash x = x'x$, and (2.21) similarly gives $(xx')' = xx'$. Thus (I4) holds.

Conversely, suppose $(S,\cdot,')$ is a regular unary semigroup satisfying (I3) and (I4). By Theorem 2.3, (B1) and (B2) hold. For (B3), $x/(x\backslash x) = x(x'x)' \overset{(I4)}{=} xx'x \overset{(I2)}{=} x$, and (2.19) is proved similarly.

On $S = \{a,b\}$, define $a/x = b/b = a$, $b/a = b$ and $x\backslash y = y/x$ for all $x,y \in S$. This gives a model satisfying (B1), (B3) and (2.19), but not (B2). Again on set $S = \{a,b\}$, define $a/x = a$, $b/x = b$, $a\backslash x = b$, $b\backslash x = a$ for all $x \in S$. This is a model satisfying (B1), (B2) and (B3), but not (2.19). Exchanging the roles of $\backslash$ and $/$ gives a model satisfying (B1), (2.19) and (B2), but not (B3). Finally, on $S = \{a,b,c\}$, define $a/a = b/a = a$, $a/b = a/c = b$, $b/b = c/c = c$, $b/c = c/a = b$ and $x\backslash y = y/x$ for all $x,y \in S$. This is a model satisfying (B2), (B3) and (2.19), but not (B1). ☐

Next we characterize regular unary semigroups with an idempotent fixing inverse.

Theorem 2.5. *Let $(S,\cdot,')$ be a regular unary semigroup satisfying* (I3) *and* (I5)*. Then the division bimagma $(S,\backslash,/)$ satisfies* (B1)*,* (B2)*,* (B3) *and*

$$(x/(y\backslash x))/(y\backslash y) = ((x/(y\backslash x))/y)\backslash y\,. \tag{2.22}$$

Conversely, let $(S,\backslash,/)$ be a bimagma satisfying (B1)*,* (B2)*,* (B3) *and* (2.22)*. Then $(S,\cdot,')$, with $'$ and $\cdot$ defined by* (1.2) *is a regular unary semigroup satisfying* (I3) *and* (I5)*.*

We will defer discussing the independence of the identities (B1), (B2), (B3) and (2.22) until Proposition 2.7.

Proof. Suppose $(S,\backslash,/)$ satisfies (B1), (B2), (B3) and (2.22). Setting $y = x$ in (2.22) and using (B3) twice on the left side and once on the right side, we get $(x/x)\backslash x = x$, that is, (2.19) holds. Thus all conditions of Theorem 2.4 hold, and so it follows that $(S,\cdot,')$ is a regular unary semigroup satisfying (I3) and (I4). What remains is to show that (I5) holds. We are going to show that $'$ fixes idempotents. Thus let $e \in S$ be an idempotent. We translate (2.22) into the semigroup language as $x(y'x)'(y'y)' = (x(y'x)'y')'y$. Apply (I4) to the left side and then replace y with y' and use (I3) to get

$$x(yx)'yy' = (x(yx)'y)'y'\,. \tag{2.23}$$

Setting $x = y = e$ in (2.23) and using $e^2 = e$, we get $ee'ee' = (ee'e)'e'$. Apply (I2) to both sides to get

$$ee' = e'e'\,. \tag{2.24}$$

Now take $x = y = e'$ in (2.23). On the left side, we get $e'(e'e')'e'e'' \overset{(2.24)}{=} e'(ee')'e'e'' \overset{(I4)}{=} e'ee'e'e'' \overset{(2.24)}{=} e'eee'e'' = e'ee'e'' \overset{(I3)}{=} e'ee'e \overset{(I1)}{=} e'e$. On the right side, we have $(e'(e'e')'e')'e'' \overset{(2.24)}{=} (e'(ee')'e')'e'' \overset{(I4)}{=} (e'ee'e')'e'' \overset{(I1)}{=} (e'e')'e'' \overset{(2.24)}{=} (ee')'e'' \overset{(I4)}{=} ee'e'' \overset{(I3)}{=} ee'e \overset{(I2)}{=} e$. Thus $e'e = e$, and so $e' = (e'e)' \overset{(I4)}{=} e'e = e$, as claimed.

Now suppose $(S,\cdot,')$ is a regular unary semigroup satisfying (I3) and (I5). Since (I4) necessarily holds, Theorem 2.4 implies that $(S,\backslash,/)$ satisfies (B1), (B2) and (B3). For (2.22), we compute

$$(x/(y\backslash x))/(y\backslash y) = x(y'x)'(y'y)' = x(y'x)'y'y = (x(y'x)'y')'y = ((x/(y\backslash x))/y)\backslash y\,,$$

where the third equality follows since $x(y'x)'y'$ is an idempotent. □

Next we consider regular involuted semigroups.

Theorem 2.6. *Let $(S,\cdot,')$ be a regular involuted unary semigroup. Then the division bimagma $(S,\backslash,/)$ satisfies* (B1), (B2), (B3) *and*

$$(x/y)' = y/x\,. \tag{2.25}$$

Conversely, let $(S,\backslash,/)$ be a bimagma satisfying (B1), (B2), (B3) *and* (2.25). *Then $(S,\cdot,')$, with $'$ and $\cdot$ defined by* (1.2), *is a regular involuted unary semigroup.*

We will defer discussing the independence of the identities (B1), (B2), (B3) and (2.25) until Proposition 2.7.

Proof. Suppose $(S,\backslash,/)$ satisfies (B1), (B2), (B3) and (2.25). Then $x \overset{(B3)}{=} x/(x\backslash x) \overset{(2.25)}{=} [(x\backslash x)/x]' \overset{(1.2)}{=} x''$. Thus (I3) holds. By Theorem 2.3, $(S,\cdot,')$ is a regular unary semigroup. Finally $(xy)' \overset{(I3)}{=} (xy'')' \overset{(1.1)}{=} (x/y')' \overset{(2.25)}{=} y'/x \overset{(1.1)}{=} y'x'$, so that (I6) holds.

Conversely, if $(S,\cdot,')$ is regular and involuted, then (B1), (B2) and (B3) follow from Theorem 2.4, while (2.25) is just $(x/y)' \overset{(1.1)}{=} (xy')' \overset{(I6)}{=} y''x' \overset{(I3)}{=} yx' \overset{(1.1)}{=} y/x$. □

Putting together the identities of the last two results, we have the following.

Proposition 2.7. *The identities* (B1), (B2), (B3), (2.22) *and* (2.25) *are independent.*

Proof. On $S=\{a,b\}$, set $a/x=a$, $b/x=b$ and $x\backslash y = x/y$ for all $x,y\in S$. This is a model satisfying (B1), (B2), (B3) and (2.22), but not (2.25).

On $S=\{a,b\}$, set $x/y=b$, $a\backslash a=a$, $a\backslash b=b$ and $b\backslash x=b$ for all $x,y\in S$. This is a model satisfying (B1), (B2), (2.22), (2.25) but not (B3).

The pair of tables on the left below give a model satisfying, respectively, (B2), (B3), (2.22), (2.25) but not (B1), The tables on the right give a model satisfying (B1), (B3), (2.22), (2.25) but not (B2).

/	0	1	2
0	0	2	0
1	2	1	2
2	0	2	2

\	0	1	2
0	0	2	0
1	2	1	2
2	0	2	2

/	0	1	2
0	0	2	2
1	2	1	2
2	2	2	2

\	0	1	2
0	0	1	2
1	0	1	2
2	2	2	2

Finally, the following tables give a model satisfying (B1), (B2), (B3) and (2.25), but not (2.22).

□

Corollary 2.8. *Let $(S,\cdot,')$ be a regular involuted unary semigroup satisfying (I5). Then the division bimagma $(S,\backslash,/)$ satisfies the independent identities* (B1), (B2), (B3), (2.22) *and* (2.25).

/	0	1	2	3
0	0	2	0	2
1	3	1	3	1
2	0	2	0	2
3	3	1	3	1

\	0	1	2	3
0	0	2	2	0
1	3	1	1	3
2	3	1	1	3
3	0	2	2	0

Conversely, let $(S, \backslash, /)$ *be a bimagma satisfying* (B1), (B2), (B3), (2.22) *and* (2.25). *Then* $(S, \cdot, ')$, *with* $'$ *and* $\cdot$ *defined by* (1.2), *is a regular involuted unary semigroup satisfying (I5).*

Next we consider inverse semigroups providing a basis different from Tamura's.

Theorem 2.9. *Let* $(S, \cdot, ')$ *be an inverse semigroup. Then the division bimagma* $(S, \backslash, /)$ *satisfies the identities* (B1), (B2), (B3) *and*

$$(x\backslash x)/(y/y) = (y/y)/(x\backslash x)\,. \tag{2.26}$$

Conversely, let $(S, \backslash, /)$ *be a bimagma satisfying* (B1), (B2), (B3) *and* (2.26). *Then* $(S, \cdot, ')$, *with* $'$ *and* $\cdot$ *defined by* (1.2), *is an inverse semigroup.*

We defer discussing the independence of (B1), (B2), (B3) and (2.26) until Proposition 2.11.

Proof. Now suppose $(S, \backslash, /)$ satisfies (B1), (B2), (B3) and (2.26). Firstly, we compute

$$(x\backslash x)/(x\backslash x) \overset{(B1)}{=} x\backslash[x/(x\backslash x)] \overset{(B3)}{=} x\backslash x\,. \tag{2.27}$$

Next, we show

$$(x/x)' = x/x \tag{2.28}$$

and

$$(x\backslash x)' = x\backslash x\,. \tag{2.29}$$

Indeed, we have

$$(x/x)' \overset{(1.2)}{=} [(x/x)\backslash(x/x)]/(x/x) \overset{(2.26)}{=} (x/x)/[(x/x)\backslash(x/x)] \overset{(B3)}{=} x/x\,,$$

and

$$(x\backslash x)' \overset{(B3)}{=} [x\backslash(x/(x\backslash x))]' \overset{(B1)}{=} [(x\backslash x)/(x\backslash x)] \overset{(2.27)}{=} x\backslash x\,.$$

Next, we have

$$x''(1.2)(x'\backslash x')/x' \overset{(B2)}{=} (x'\backslash x')'\backslash x \overset{(2.29)}{=} (x'\backslash x')\backslash x \overset{(B2)}{=} (x/x'')\backslash x\,,$$

that is,

$$x'' = (x/x'')\backslash x\,. \tag{2.30}$$

This gives us

$$x'' \overset{(2.30)}{=} (x/x'')\backslash x \overset{(B3)}{=} (x/x'')\backslash(x/(x\backslash x)) \overset{(B1)}{=} [(x/x'')\backslash x]/(x\backslash x) \overset{(2.30)}{=} x''/(x\backslash x)\,,$$

that is,

$$x'' = x''/(x\backslash x)\,. \tag{2.31}$$

For the next step, we show

$$x''/x'' = x'\backslash x'\,. \tag{2.32}$$

Indeed, we have

$$\begin{aligned}
x''/x'' &\overset{(B3)}{=} x''/(x'/(x'\backslash x'))' &&\overset{(B2)}{=} x''/(x'/(x/x''))' \\
&\overset{(B2)}{=} x'''\backslash(x'/(x/x'')) &&\overset{(B1)}{=} (x'''\backslash x')/(x/x'') \\
&\overset{(B2)}{=} (x''/x'')/(x/x'') &&\overset{(2.30)}{=} ([(x/x'')\backslash x]/x'')/(x/x'') \\
&\overset{(B1)}{=} [(x/x'')\backslash(x/x'')]/(x/x'') &&\overset{(1.2)}{=} (x/x'')' \\
&\overset{(B2)}{=} (x'\backslash x')' &&\overset{(2.29)}{=} x'\backslash x' \,.
\end{aligned}$$

Now

$$x'''(1.2)x''\backslash(x''/x'')(2.32)x''\backslash(x'\backslash x') \overset{(B2)}{=} x'/(x'\backslash x')' \overset{(2.28)}{=} x'/(x'\backslash x') \overset{(B3)}{=} x' \,,$$

that is,

$$x''' = x'' \,. \tag{2.33}$$

And now we can verify (I3) as follows:

$$\begin{aligned}
x'' &\overset{(2.31)}{=} x''/(x\backslash x) \overset{(2.29)}{=} x''/(x\backslash x)' \overset{(B2)}{=} x'''\backslash(x\backslash x) \overset{(2.33)}{=} x'\backslash(x\backslash x) \\
&\overset{(B2)}{=} x/(x\backslash x)' \overset{(2.29)}{=} x/(x\backslash x) \overset{(B3)}{=} x \,.
\end{aligned}$$

It follows from Theorem 2.3 that $(S,\cdot,')$ is a regular unary semigroup satisfying (I3). Thus we have $(x'x)' = (x\backslash x)' \overset{(2.29)}{=} x\backslash x = x'x$ and $(xx')' = (x/x)' \overset{(2.28)}{=} x/x = xx'$, so that (I4) holds.

Finally, we compute

$$\begin{aligned}
(xx')(y'y) &\overset{(I4)}{=} (xx')(y'y)' \overset{(1.2)}{=} (x/x)/(y\backslash y) \overset{(2.26)}{=} (y\backslash y)/(x/x) \\
&\overset{(1.2)}{=} (y'y)'(xx') \overset{(I4)}{=} (y'y)(xx') \,,
\end{aligned}$$

and so (I8) holds. Finally, it is well-known that (I2), (I3) and (I8) are sufficient to imply that a unary semigroup is an inverse semigroup in which the unary operation is the natural inverse; the identity (I6) is, in fact, dependent (see *e.g.*, [1]).

Conversely, if $(S,\cdot,')$ is an inverse semigroup, then (B1), (B2) and (B3) follow from Theorem 2.6. For (2.26), we have $(x\backslash x)/(y/y) \overset{(1.1)}{=} (x'x)(yy')' \overset{(I4)}{=} (x'x)(yy') \overset{(I8)}{=} (yy')(x'x) \overset{(I4)}{=} (yy')(x'x)' \overset{(1.1)}{=} (y/y)/(x\backslash x)$. □

Finally, we turn to completely regular semigroups.

Theorem 2.10. *Let $(S,\cdot,')$ be a completely regular semigroup. Then the division bimagma $(S,\backslash,/)$ satisfies the identities* (B1), (B2), (B3) *and*

$$x/x = x\backslash x \,. \tag{2.34}$$

Conversely, let $(S,\backslash,/)$ be a bimagma satisfying (B1), (B2), (B3) *and* (2.34). *Then $(S,\cdot,')$, with $'$ and $\cdot$ defined by* (1.2), *is a completely regular semigroup.*

We defer discussing the independence of (B1), (B2), (B3) and (2.34) until Proposition 2.11.

Proof. Suppose $(S,\backslash,/)$ satisfies (B1), (B2), (B3) and (2.34). Firstly, we have

$$\begin{aligned}(x\backslash x)' &\overset{(1.2)}{=} (x\backslash x)\backslash((x\backslash x)/(x\backslash x)) &&\overset{(B1)}{=} (x\backslash x)\backslash(x\backslash(x/(x\backslash x)))\\ &\overset{(B3)}{=} (x\backslash x)\backslash(x\backslash x) &&\overset{(2.34)}{=} (x\backslash x)/(x\backslash x)\\ &\overset{(B1)}{=} x\backslash(x/(x\backslash x)) &&\overset{(B3)}{=} x\backslash x\,. \end{aligned}\tag{2.35}$$

Thus $x'\backslash(x\backslash x) \overset{(B2)}{=} x/(x\backslash x)' \overset{(2.35)}{=} x/(x\backslash x) \overset{(B3)}{=} x$ and hence (2.6) holds. Therefore the conditions of Theorem 2.3 are satisfied, and so $(S,\cdot,')$ is a regular semigroup satisfying (I3). In addition, $xx' = x/x \overset{(2.34)}{=} x\backslash x = x'x$ and so (I7) holds. Therefore $(S,\cdot,')$ is completely regular.

Conversely, suppose $(S,\cdot,')$ is a completely regular semigroup. Then (B1), (B2) and (B3) hold by Theorem 2.4, while (2.34) is just (I7) rewritten. □

Proposition 2.11. *The identities* (B1)*,* (B2)*,* (B3)*,* (2.26) *and* (2.34) *are independent.*

Proof. On $S = \{a,b\}$, define $a/x = a$, $b/x = b$ and $x\backslash y = x/y$ for all $x,y\in S$. This is a model satisfying (B1), (B2), (B3) and (2.34), but not (2.26).

On $S = \{a,b\}$, define $x/y = x\backslash y = b$ for all $x,y\in S$. This gives a model satisfying (B1), (B2), (2.26) and (2.34), but not (B3).

On $S = \{a,b\}$, define $a/a = b$, $a/b = a$, $b/x = b$, $x\backslash y = b$ for all $x,y\in S$. This gives a model satisfying (B1), (B3), (2.26) and (2.34), but not (B2).

On $S = \{a,b\}$, define $a/a = b$, $a/b = a$, $b/x = b$ and $x\backslash y = x/y$ for all $x,y\in S$. This gives a model satisfying (B2), (B3), (2.26) and (2.34), but not (B1).

Finally, the table below gives a model satisfying (B1), (B2), (B3) and (2.26), but not (2.34).

/	0	1	2	3	4
0	2	4	4	0	4
1	4	3	1	4	4
2	4	0	2	4	4
3	1	4	4	3	4
4	4	4	4	4	4

\	0	1	2	3	4
0	3	4	1	4	4
1	4	2	4	0	4
2	0	4	2	4	4
3	4	1	4	3	4
4	4	4	4	4	4

□

Corollary 2.12. *Let $(S,\cdot,')$ be a Clifford semigroup. Then the division bimagma $(S,\backslash,/)$ satisfies the independent identities* (B1)*,* (B2)*,* (B3)*,* (2.26) *and* (2.34)*.*

Conversely, let $(S,\backslash,/)$ be a bimagma satisfying (B1)*,* (B2)*,* (B3)*,* (2.26) *and* (2.34)*. Then $(S,\cdot,')$, with $'$ and $\cdot$ defined by* (1.2)*, is a Clifford semigroup.*

3. Tamura's Problem

Now we turn the problem that arose from Tamura's work [10]. We repeat here the relevant identities for the reader's convenience:

$$(x/y)\backslash z = y/(z\backslash x) \tag{T1}$$
$$(x/x)\backslash x = x \tag{T2}$$
$$x/y' = x'\backslash y \tag{T3}$$
$$(x/y)' = y/x \tag{T4}$$

There is a tacit assumption here that $'$ is well-defined, and we address this now along with other useful facts.

Lemma 3.1. *Let* $(S, \backslash, /)$ *be a bimagma satisfying* (T1) *and* (T2)*. Then the identities*

$$x/(x\backslash x) = x \tag{3.1}$$

$$x\backslash(x/x) = (x\backslash x)/x \tag{3.2}$$

hold. Define $' : S \to S$ *by* (1.2)*. Then the following identities also hold:*

$$(x\backslash x)' = x\backslash x \tag{3.3}$$

$$(x/x)' = x/x\,. \tag{3.4}$$

(Note that (3.1) is just (B3).)

Proof. For (3.1), we have $x/(x\backslash x) \overset{(T1)}{=} (x/x)\backslash x \overset{(T2)}{=} x$. Thus we obtain (3.2) as follows:

$$x\backslash(x/x) \overset{(3.1)}{=} (x/(x\backslash x))\backslash(x/x) \overset{(T1)}{=} (x\backslash x)/(\underbrace{(x/x)\backslash x}) \overset{(T2)}{=} (x\backslash x)/x\,.$$

For (3.3), we compute

$$\begin{aligned}(x\backslash x)' &\overset{(1.2)}{=} (x\backslash x)\backslash(\underbrace{(x\backslash x)/(x\backslash x)}) &&\overset{(T1)}{=} (x\backslash x)\backslash((\underbrace{x/(x\backslash x)})\backslash x)\\ &\overset{(3.1)}{=} (x\backslash x)\backslash(x\backslash x) &&\overset{(3.1)}{=} (\underbrace{(x/(x\backslash x))\backslash x})\backslash(x\backslash x)\\ &\overset{(T1)}{=} ((x\backslash x)/(x\backslash x))\backslash(x\backslash x) &&\overset{(T2)}{=} (x\backslash x)\,.\end{aligned}$$

The proof of (3.4) is dual to this. □

With this lemma in place, we may now use the shorthand $x' = x\backslash(x/x) = (x\backslash x)/x$ from (1.2).

Our first goal in this section is to give a humanly readable proof of Proposition 1.2, which we restate here.

Proposition 3.2. *Let* $(S, \backslash, /)$ *be a bimagma satisfying* (T1), (T2) *and* (T3)*. Then* (T4) *also holds.*

We begin with two auxiliary lemmas.

Lemma 3.3. *Under the assumptions of Proposition 3.2, (I3) holds.*

Proof. We compute

$$\begin{aligned}x'' &\overset{(1.2)}{=} x'\backslash(x'/x') &&\overset{(1.2)}{=} x'\backslash(\underbrace{x'/(x\backslash(x/x))}) &&\overset{(T1)}{=} x'\backslash((\underbrace{(x/x)/x'})\backslash x)\\ &\overset{(T3)}{=} x'\backslash((\underbrace{(x/x)'}\backslash x)\backslash x) &&\overset{(3.4)}{=} x'\backslash((\underbrace{(x/x)\backslash x})\backslash x) &&\overset{(T2)}{=} x'\backslash(x\backslash x)\\ &\overset{(T3)}{=} x/(x\backslash x)' &&\overset{(3.3)}{=} x/(x\backslash x) &&\overset{(3.1)}{=} x\,.\end{aligned}$$

□

The next lemma provides a number of handy ways of expressing the products of two or three elements, and how the inversion relates with the two binary operations.

Lemma 3.4. *Under the assumptions of Proposition 3.2 the following identities hold:*

$$x/y = (x/y)/(y/y) \tag{3.5}$$

$$x/y = (x/x)\backslash(x/y) \tag{3.6}$$

$$x\backslash y = ((y/y)\backslash x)\backslash y \tag{3.7}$$

$$x\backslash y = x'/y' \qquad \textit{and} \qquad x'\backslash y' = x/y \tag{3.8}$$

$$x\backslash y' = x'/y \tag{3.9}$$

$$x'\backslash(y/z) = x/(z/y) \tag{3.10}$$

$$x\backslash(y/z)' = x\backslash(z/y) \tag{3.11}$$

$$(x/y)\backslash z = (y/x)'/z \tag{3.12}$$

Proof. We start by proving (3.9).

$$x'/y \overset{(I3)}{=} x'/y'' \overset{(T3)}{=} x''\backslash y' \overset{(I3)}{=} x\backslash y',$$

and so, in particular, (3.8) follows:

$$x'/y' \overset{(3.9)}{=} x\backslash y'' \overset{(I3)}{=} x\backslash y \qquad \text{and} \qquad x'\backslash y' \overset{(3.9)}{=} x''/y \overset{(I3)}{=} x/y\,.$$

Regarding (3.6),

$$\begin{aligned}(x/x)\backslash(x/y) &\overset{(T1)}{=} x/(\underbrace{(x/y)\backslash x}) \overset{(T1)}{=} x/(y/\underbrace{(x\backslash x)}) \\ &\overset{(3.3)}{=} x/(\underbrace{y/(x\backslash x)'}) \overset{(T3)}{=} x/(y'\backslash(x\backslash x)) \\ &\overset{(T1)}{=} (\underbrace{(x\backslash x)/x})\backslash y' \overset{(1.2)}{=} x'\backslash y' \\ &\overset{(3.8)}{=} x/y\,.\end{aligned}$$

For (3.7) we have

$$\begin{aligned}x\backslash y \overset{(3.8)}{=} x'/y' &\overset{(1.2)}{=} x'/(y\backslash(y/y)) \overset{(T1)}{=} ((y/y)/x')\backslash y \\ &\overset{(T3)}{=} ((y/y)'\backslash x)\backslash y \overset{(3.4)}{=} ((y/y)\backslash x)\backslash y\,.\end{aligned}$$

Regarding (3.10),

$$\begin{aligned}x/(z/y) &\overset{(3.8)}{=} x/(z'\backslash y') && \overset{(T1)}{=} (\underbrace{y'/x})\backslash z' && \overset{(3.9)}{=} (y\backslash x')\backslash z' \\ &\overset{(3.7)}{=} (\underbrace{((x'/x')\backslash y)\backslash x'})\backslash z' && \overset{(3.9)}{=} (((x'/x')\backslash y)'/x)\backslash z' && \overset{(T1)}{=} x/(\underbrace{z'\backslash((x'/x')\backslash y)'}) \\ &\overset{(3.8)}{=} x/(z/((\underbrace{x'/x'})\backslash y)) && \overset{(3.8)}{=} x/(\underbrace{z/((x\backslash x)\backslash y)}) && \overset{(T1)}{=} x/((y/z)\backslash(x\backslash x)) \\ &\overset{(T1)}{=} ((x\backslash x)/x)\backslash(y/z) && \overset{(1.2)}{=} x'\backslash(y/z)\,.\end{aligned}$$

From this, we get (3.11):

$$x\backslash(y/z)' \overset{(3.9)}{=} x'/(y/z) \overset{(3.10)}{=} x''\backslash(z/y) \overset{(I3)}{=} x\backslash(z/y)\,.$$

For (3.12),

$$(x/y)'/z \overset{(3.7)}{=} (\underbrace{(z/z)\backslash(x/y)'})\backslash z \overset{(3.11)}{=} ((z/z)\backslash(y/x))\backslash z \overset{(3.7)}{=} (y/x)\backslash z\,.$$

Finally we prove (3.5):

$$\begin{aligned}(x/y)/(y/y) &\overset{(3.10)}{=} (x/y)'\backslash(y\backslash y) \overset{(3.12)}{=} (y/x)\backslash(y/y)\\ &\overset{(T1)}{=} x/((y/y)\backslash y) \quad \overset{(T2)}{=} x/y\,.\end{aligned}$$

□

We have everything we need to prove Proposition 3.2.

Proof. (Proposition 3.2)
We claim that $(x/y)' = y/x$. In fact,

$$\begin{aligned}(x/y)' &\overset{(1.2)}{=} (\underbrace{x/y})\backslash((x/y)/(x/y)) &&\overset{(3.5)}{=} ((x/y)/(y/y))\backslash((x/y)/(x/y))\\ &\overset{(T1)}{=} (y/y)/(\underbrace{((x/y)/(x/y))\backslash(x/y)}) &&\overset{(T2)}{=} (y/y)/(x/y)\\ &\overset{(3.8)}{=} (y/y)'\backslash(x/y)' &&\overset{(3.11)}{=} (y/y)'\backslash(y/x)\\ &\overset{(3.4)}{=} (y/y)\backslash(y/x) &&\overset{(3.6)}{=} y/x\,.\end{aligned}$$

Thus we have established (T4), completing the proof. □

4. Open Problems

We begin by restating a couple of problems from [1]. A set of identities in two binary operations is said to be *semi-separated* if at most two of the identities involve both operations.

Problem 4.1. *Is there a semi-separated set of identities characterizing inverse semigroups in terms of their division bimagmas?*

Of course, one can also ask this sort of question about the other varieties of unary semigroups that we considered in this paper.

A 3-basis characterizing inverse semigroups in terms of their division bimagmas was presented in [1].

Proposition 4.2. *Inverse semigroups are characterized in terms of their division bimagmas by the independent identities* (T1), (T2) *and*

$$(x/x)\backslash(y\backslash y) = y\backslash(y/(x/x))\,. \tag{T6}$$

The proof, found by Prover9, was left to the companion website of [1]. That proof shows that the axiom set {(T1), (T2), (T6)} is equivalent to {(T1),...,(T5)}. We tried to find a shorter automated proof that would be easier to translate into humanly readable form, but were unable to find anything reasonable. As current research in automated deduction aims to find tools that provide mathematical insight into theorems and their proofs, we feel that this problem should be posed as a test question for those researchers.

Problem 4.3. *Find a humanly understandable proof of Proposition 4.2. Alternatively, find another* 3*-basis for the division bimagmas of inverse semigroups with a humanly understandable proof.*

As noted above, one of the goals of defining a class of algebras in terms of a class of algebras of a different type is that many properties/problems become obvious in the new setting, while difficult to spot in the first. For instance, Tamura proved that for the division bimagmas of regular involuted semigroups the following are equivalent:

(1) $(S, \backslash)$ is associative;
(2) $(S, /)$ is associative;
(3) $(S, /) = (S, \backslash)$;

In addition the class of regular involuted semigroups in which $(S, /) = (S, \backslash)$ is the class of commutative semigroups satisfying $x^3 = x$. Therefore the following problems are natural.

Problem 4.4. *For each class of semigroups defined in this paper, characterize when:*

(1) $(S, \backslash)$ *[*$(S, /)$*] is associative [commutative, idempotent, with identity, with zero, nilpotent* $(x\backslash(x\backslash \ldots (x\backslash x)) \ldots)) = 0$ *or its mirror image, E-unitary];*
(2) $(S, \backslash)$ *and* $(S, /)$ *are equal [dual* $(x/y = y\backslash x)$*, isomorphic, anti-isomorphic];*
(3) $(S, \backslash)$ *distributes over* $(S, /)$ *(that is, for example,* $(x/y)\backslash z = (x\backslash z)/(y\backslash z)$*).*

Problem 4.5. *For the classes of semigroups discussed in this paper, characterize the natural partial order on a semigroup in the language of bimagmas. Similarly characterize Green's relations.*

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