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Research paper

Analysis of a class of boundary value problems depending on left and right Caputo fractional derivatives☆

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ABSTRACT

In this work we study boundary value problems associated to a nonlinear fractional ordinary differential equation involving left and right Caputo derivatives. We discuss the regularity of the solutions of such problems and, in particular, give precise necessary conditions so that the solutions are $C^1([0,\,1])$. Taking into account our analytical results, we address the numerical solution of those problems by the *augmented*-RBF method. Several examples illustrate the good performance of the numerical method.

1. Introduction

The second order differential equation $y'' = f(t, y)$ is widely studied in the literature, whether one looks for analytical or numerical/computational results.

The fractional calculus (cf. Section 2 for basic definitions and results) is nowadays a topic of intensive research (see [8,16] and the references therein). In the fractional calculus theory the intuitive way of generalizing the previous differential equation is substituting the classical operator y'' by a fractional one, say, the Caputo fractional derivative of order $1 < \alpha \leq 2$, ${}_0^CD^\alpha y$, i.e. to consider the following fractional differential equation ${}_0^CD^\alpha y = f(t, y)$.

A not so obvious, yet possible, way to generalize the second order differential equation is to consider the following fractional ordinary differential equation (FODE):

$$ {}^CD_1^\beta {}_0^CD^\alpha y(t) = f(t, y(t)), \quad t \in [0, 1], \quad 0 < \alpha, \beta \leq 1. \tag{1}$$

(usually this equation will be subjected to some boundary conditions). Perhaps the main reason to use the left and right fractional differential operators as in (1) is the resemblance of equation in (1) with the Euler–Lagrange equation that arises

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from fractional calculus of variations problems. Indeed, if one consider the following variational problem,

$$J(y) = \int_0^1 L(t, y(t), {}_0^C D^\alpha y(t)) dt \to \min,$$

subject to the boundary conditions $y(0) = y_0$ and $y(1) = y_1$, then its Euler–Lagrange equation is given by the formula (cf. [18]):

$$\partial_2 L(t, y(t), {}_0^C D^\alpha y(t)) + {}^C D_1^\alpha \partial_3 L(t, y(t), {}_0^C D^\alpha y(t)) = 0, \quad t \in [0, 1],$$

where ∂_i is the partial derivative of a function with respect to its ith argument.

After searching within the literature on the subject, we found some works for the linear case of Eq. (1), specifically for Sturm–Liouville eigenvalue problems [17,23] and also some works dealing exclusively with fractional calculus of variations problems [2,4] (in some of them a Riemann–Liouville fractional derivative is used instead of the Caputo derivative). Existence results and/or numerical methods were presented in the aforementioned works. One particular issue came up to our minds, that was, the question about the smoothness of the solutions to (1) specially when one want to develop numerical methods to solve it. We are aware that when only left fractional derivatives are involved in a certain differential equation one should not expect too much regularity of their solutions at the initial point (cf. e.g. [1,8]). With that in mind we suspected that *adding* a right fractional derivative to an equation should also raise smoothness problems at the final point of the interval [0, 1]. This is not to say that fractional differential equations do not have smooth solutions, rather to say that one should not expect to have too much differentiability at the boundary points. This issue is analyzed to some extent in Section 3. The importance of knowing how a solution and their derivatives behave will be more than evident to the reader in Sections 4 and 5. We believe that the analysis we made in order to be able to construct our numerical methods (cf. Sections 3 and 4) will be very useful for the researchers. In particular we note that some numerical methods appeared previously in the literature but they are scarce using nonlinear functions f (see [3,13,23]) and, moreover, assuming $C^1[0, 1]$ solutions or even higher smoothness (see [2] and the references therein).

The plan of the paper is as follows: in Section 2 we present the fractional calculus concepts and basic results that are being used in this manuscript. In Section 3 we present our analysis with respect to the smoothness of the solutions of (1) and refer the reader to some problems we couldn't solve but that undoubtedly deserve the attention of the research community. In Section 4 we describe our numerical approach and, in particular, we deduce (novel) formulas for calculating the fractional derivative ${}^C D_1^\beta {}_0^C D^\alpha$ of monomials of the type t^p and $(1-t)^p$, $p > 0$, that are essential to implement our numerical methods. Finally, in Section 5, we present some examples of our work.

2. Fractional calculus

The left and right *Riemann–Liouville fractional integrals* of order $\alpha > 0$ are defined, respectively, by

$$ {}_0 I^\alpha y(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \tag{2}$$

and

$$ I_1^\alpha y(t) := \frac{1}{\Gamma(\alpha)} \int_t^1 (s-t)^{\alpha-1} y(s) ds, \tag{3}$$

provided the integrals exist.

In this work we will use what is known in the literature as *Caputo fractional operators*. We start by introducing the concepts of left and right fractional derivatives.

The left Caputo fractional derivative of order $0 < \alpha < 1$ is defined by

$$ {}_0^C D^\alpha y(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} (y(s) - y(a)) ds, \tag{4}$$

and the right Caputo fractional derivative of order $0 < \alpha < 1$ by

$$ {}^C D_1^\alpha y(t) := -\frac{1}{\Gamma(1-\alpha)} \int_t^1 (s-t)^{-\alpha} (y(s) - y(b)) ds, \tag{5}$$

provided the integrals exist.

It is well known that if $f \in C[0, 1]$, then:

$$ {}_0^C D^\alpha {}_0 I^\alpha f(t) = f(t), \quad {}^C D_1^\alpha I_1^\alpha f(t) = f(t), \quad 0 < \alpha < 1. \tag{6}$$

Moreover, if ${}_0^C D^\alpha f \in C[0, 1]$ and ${}^C D_1^\alpha f \in C[0, 1]$, then:

$$ {}_0 I^\alpha {}_0^C D^\alpha f(t) = C + f(t), \quad I_1^\alpha {}^C D_1^\alpha f(t) = C + f(t), \quad 0 < \alpha < 1, \quad C \in \mathbb{R}. \tag{7}$$

Moreover, we will use the following formulas repeatedly:

$$ {}_0^C D^\alpha t^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}, \quad p > 0,$$

and

$$^{C}D_{1}^{\alpha}(1-t)^{p}=\frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}(1-t)^{p-\alpha},\quad p>0.$$

3. Some results and open problems about the smoothness of the solutions

In this section we present a necessary condition for a solution y of (1) to be differentiable in the closed interval [0, 1]. We then discuss through a series of remarks some problems that we would like to be solved, that would surely be of much importance for a more general study of the properties of the solutions of Eq. (1).

We start with a simple but useful observation.

Lemma 3.1. *Let $0<\alpha,\ \beta\le 1$. For $t\in[0,1)$, the integral*

$$\int_0^t r^{\alpha-1}(1-t+r)^{\beta-1}dr \tag{8}$$

is finite. Moreover for $t=1$, (8) converges if $\alpha+\beta>1$ and diverges if $\alpha+\beta\le 1$.

Proof. Let $t\in(0,1)$. Then,

$$\int_0^t r^{\alpha-1}(1-t+r)^{\beta-1}dr\le(1-t)^{\beta-1}\int_0^t r^{\alpha-1}dr=(1-t)^{\beta-1}\frac{t^{\alpha}}{\alpha},$$

which shows the first claim. Now we fix $t=1$. Then,

$$\int_0^1 r^{\alpha-1}(1-1+r)^{\beta-1}dr=\int_0^1 r^{\alpha+\beta-2}dr,$$

and the conclusion is immediate. □

It follows the main result of this section.

Theorem 1. *Let $0<\alpha\le 1$ and $0<\beta<1$. Suppose that $y\in C^1[0,1]$ is a solution of Eq. (1) with f being a continuously differentiable function of its arguments. Then,*

$$f(1,y(1))=0\vee\alpha+\beta>1. \tag{9}$$

Proof. It is well known (see Section 2) that y will satisfy the integral equation

$$y(t)=a+b\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t(t-s)^{\alpha-1}\int_s^1(\tau-s)^{\beta-1}f(\tau,y(\tau))d\tau ds, \tag{10}$$

where a, b are some real constants. With some change of variables we can transform (10) into

$$y(t)=a+b\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t r^{\alpha-1}\int_0^{1-t+r}x^{\beta-1}f(x+t-r,y(x+t-r))dxdr.$$

Now, since $y\in C^1[0,1]$, we get

$$\begin{aligned}
y'(t)&=b\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\Bigg[t^{\alpha-1}\int_0^1 x^{\beta-1}f(x,y(x))dx\\
&\quad+\int_0^t r^{\alpha-1}\frac{d}{dt}\int_0^{1-t+r}x^{\beta-1}f(x+t-r,y(x+t-r))dxdr\Bigg]\\
&=b\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\Bigg[t^{\alpha-1}\int_0^1 x^{\beta-1}f(x,y(x))dx\\
&\quad+\int_0^t r^{\alpha-1}\left\{-(1-t+r)^{\beta-1}f(1,y(1))+\int_0^{1-t+r}x^{\beta-1}\frac{d}{dt}f(x+t-r,y(x+t-r))dx\right\}dr\Bigg]\\
&=b\frac{t^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\Bigg[t^{\alpha-1}\int_0^1 x^{\beta-1}f(x,y(x))dx\\
&\quad-f(1,y(1))\int_0^t r^{\alpha-1}(1-t+r)^{\beta-1}dr+\int_0^t r^{\alpha-1}\int_0^{1-t+r}x^{\beta-1}\frac{d}{dt}f(x+t-r,y(x+t-r))dxdr\Bigg].
\end{aligned}$$

By the hypothesis on y and f we have that

$$\int_0^t r^{\alpha-1}\int_0^{1-t+r} x^{\beta-1}\frac{d}{dt}f(x+t-r, y(x+t-r))dxdr,$$

is continuous on [0, 1]. Finally, using Lemma 3.1, we conclude on one hand that

$$\frac{b}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 x^{\beta-1}f(x, y(x))dx = 0,$$

and on the other hand that

$$f(1, y(1)) = 0 \vee \alpha+\beta > 1.$$

The result now follows since by [16, Theorem 2.2] we have that ${}^{C}_{0}D^{\alpha}y(0) = 0$, i.e.

$$b = -\frac{1}{\Gamma(\beta)}\int_0^1 x^{\beta-1}f(x, y(x))dx.$$

□

Remark 3.2. The condition in (9) is not sufficient for $y \in C^1[0, 1]$. To see this, just consider $f = 0$ and impose the boundary conditions $y(0) = 0$ and $y(1) = 1$. Then, the unique solution of ${}^{C}D^{\beta}_{1}\,{}^{C}_{0}D^{\alpha}y(t) = 0$ is $y(t) = t^{\alpha}$, which is not differentiable at $t = 0$.

Remark 3.3. From the proof of Theorem 1 it is clear that to obtain more information about the differentiability of a solution to (1) on $I \subseteq [0, 1]$ depends on obtaining more information about the integral

$$\int_0^t r^{\alpha-1}\int_0^{1-t+r} x^{\beta-1}\frac{d}{dt}f(x+t-r, y(x+t-r))dxdr.$$

For example, we conjecture that every solution y of (1) is $C^1(0, 1)$. We actually assume implicitly in the next section that $y \in C^{\infty}(0, 1)$, however, we couldn't find a proof for first order differentiability let alone infinite differentiability.

Remark 3.4. Note that Theorem 1 may be used to show that certain variational problems do not have a $C^1[0, 1]$ solution.[1] However it does not necessarily tell us the points of the interval [0, 1] where the differentiability ceases to exist. Of course that, if $f = c \in \mathbb{R}$, then we can characterize the differentiability of y completely: the only points where y may be not differentiable are at $t = 0$ and $t = 1$ (and it is clearly $C^{\infty}(0, 1)$). We definitely would like to see a result with the complete characterization with respect to the (at least) C^1-differentiability of the solutions of (1).

4. Numerical approach

In this section we will consider the application of the *augmented-RBF* method to solve boundary value problems involving the fractional ordinary differential Eq. (1). This numerical method was introduced in [1] in the context of Sturm–Liouville problems involving fractional derivatives. We assume that the solution can be decomposed as a sum of smooth (denoted by "reg") and singular (denoted by "sing") parts,

$$y(t) = y^{reg}(t) + y^{sing}(t). \tag{11}$$

The regular part y^{reg} is approximated by a standard Radial Basis Functions (RBF) method (eg. [5,10,12,14,15,21,22]) which involves a *global approximation* in the sense that each basis function is nonzero over the whole interval. This property allows to reproduce the global character of the fractional derivative operator and it is possible to obtain highly accurate approximations for fractional derivatives of smooth functions, even with very small matrices (see Section 5 and [1,20]).

On the other hand, the solutions of problems depending on left fractional derivatives may be non-smooth at the origin (eg. [1,6,19,23,24]). Besides the singular behavior at the origin, the solutions of (1) may also be non-smooth for $t = 1$. Therefore, the functional space generated by the (smooth) *RBF* basis functions was augmented with some *fractional* polynomials aiming to approximate the singular part y^{sing}. Roughly speaking, the idea is to approximate *globally* the quantities whose nature is global and *locally*, the features that are local in essence. A Caputo fractional derivative is a global operator [8, Remark 3.2]. Thus, a quite natural numerical approach is to consider the approximation made by global basis functions, as in the RBF method. On the other hand, as was mentioned, the solutions of fractional order differential equations may be singular at some points. Thus, we augment the functional space for the approximations, with some basis functions that can reproduce locally the behavior of the solution in a neighborhood of the points of singularity.

[1] With this respect it would be nice to check (which is out of the scope of this work) if it is possible to obtain the same Euler–Lagrange equation as in [18] but under weaker conditions on the space of functions (which is $C^1[0, 1]$ in mentioned work).

Denote by $\mathcal{C} = \{c_i,\ i = 1, \dots, N\} \subseteq [0, 1]$ a set of centers. The RBF approximation is a linear combination

$$y^{RBF}(t) = \sum_{j=1}^{N} \alpha_j \varphi_j(t), \tag{12}$$

where $\varphi_j(t) = \eta_\epsilon(|t - c_j|)$, for some function $\eta_\epsilon : \mathbb{R}_0^+ \to \mathbb{R}$. In this work, we will use the Gaussian ($\eta_\epsilon(r) = e^{-(\epsilon r)^2}$), but many other RBF's were used in the literature. It is well known in the context of RBF approximations that for smooth solutions, typically the best approximations are obtained for small parameter ϵ, that is, for almost flat RBF's. However, in that case the condition number increases exponentially and this prevents to obtain accurate results. This ill conditioning can be circumvented performing a change of basis. This can be done automatically by using the RBF-QR method (cf. [10]).

The procedure is a consequence of the expansion ($y, y_k \in [-1, 1]$),

$$e^{-\epsilon^2(y-y_k)^2} = \sum_{j=0}^{\infty} \epsilon^{2j} c_j(y_k) e^{-\epsilon^2 y^2} T_j(y),$$

where $c_j(y_k) = t_j e^{-\epsilon^2 y_k^2} y_k^j\ {}_0F_1([], j+1, \epsilon^4 y_k^2)$, for $t_0 = \frac{1}{2}$ and $t_j = 1,\ j > 0$ and T_j are Chebyshev polynomials.

This expansion is truncated for some $M \in \mathbb{N}$, in such a way that the higher terms that are excluded are smaller than machine precision, and we can write

$$\begin{pmatrix} \phi_1(y) \\ \phi_2(y) \\ \vdots \\ \phi_N(y) \end{pmatrix} \approx \underbrace{\begin{pmatrix} c_1(y_1) & c_2(y_1) & \dots & c_M(y_1) \\ c_1(y_2) & c_2(y_2) & \dots & c_M(y_2) \\ \vdots & \vdots & \ddots & \vdots \\ c_1(y_N) & c_2(y_N) & \dots & c_M(y_N) \end{pmatrix}}_{C} \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_M \end{pmatrix} \underbrace{\begin{pmatrix} \psi_1(y) \\ \psi_2(y) \\ \vdots \\ \psi_M(y) \end{pmatrix}}_{\Psi(x)},$$

where $d_j = \dfrac{2\epsilon^{2j}}{j!}$ and $\psi_j(y) = e^{-\epsilon^2 y^2} T_j(y)$. Thus, instead of the RBF basis functions given by the Gaussian, we consider the new basis defined in the interval [0,1]:

$$\Phi(t) = \tilde{C}\ \Psi(2t - 1), \tag{13}$$

where $\tilde{C}$ is a matrix obtained from the QR factorization of the matrix C (see [10] for details) and will denote by $\Phi_j(\bullet)$, the jth component of the vectorial function $\Phi(\bullet)$. These new functions span the same space as the RBF basis functions, but are much more well conditioned. Thus, instead of (12) we consider the linear combination

$$y^{QR}(t) = \sum_{j=1}^{N} \alpha_j^{QR} \Phi_j(t). \tag{14}$$

The *augmented-RBF* approximation is given by

$$y(t) \approx \tilde{y}(t) = \sum_{j=1}^{N} \alpha_j \Phi_j(t) + \sum_{j=1}^{S_0} \beta_j t^{(p_0)_j} + \sum_{j=1}^{S_1} \gamma_j (1-t)^{(p_1)_j}, \tag{15}$$

for some $S_0, S_1 \in \mathbb{N}$, which are respectively the number of singular terms at $t = 0$ and $t = 1$. The exponents are defined by

$$\mathcal{P}_0 = \{p = i + j\alpha + k\beta : i, j, k \in \mathbb{N}_0, p \notin \mathbb{N}_0\}$$

and

$$\mathcal{P}_1 = \{p = i + j\alpha + k\beta : i, j, k \in \mathbb{N}_0, p \notin \mathbb{N}_0, p > \alpha, p - \alpha \notin \mathbb{N}\}$$

and the sequences $(p_0)_i$ and $(p_1)_i$ are obtained by sorting (respectively) the sets $\mathcal{P}_0$ and $\mathcal{P}_1$ in ascending order. We excluded the case $i + j\alpha + k\beta \in \mathbb{N}_0$, since the corresponding monomials are smooth and, therefore, already taken into account by the span of RBF basis. The remaining conditions $p > 0$ and $p - \alpha \notin \mathbb{N}$, for the exponents in $\mathcal{P}_1$, arise from technical reasons (cf. Proposition 4.3 below).

Remark 4.1. We shall give a brief explanation of why we chose the monomials t^p and $(1-t)^p$ for some number p. Recall from equality (10) that the singularities of the derivative of the solutions of (1) *may*[2] arise from the derivatives of the functions

$$t^\alpha, \text{ at } t = 0,$$

and

$$\int_0^t (t-s)^{\alpha-1} \int_s^1 (\tau - s)^{\beta-1} d\tau ds = \frac{1}{\beta} \int_0^t (t-s)^{\alpha-1} (1-s)^\beta ds, \text{ at } t = 1. \tag{16}$$

[2] We use the word "may" in virtue of the discussion done in Remark 3.3.

The reader might be wondering why we chose $(1-t)^p$ to treat the case $t=1$. Well, it turns out that there is no computational advantage in choosing (16) instead of an asymptotic expansion, involving terms of type $(1-t)^p$. These terms can also reproduce the singular behavior of the solution in the neighborhood of the point $t=1$, provided the exponents are correctly chosen. Moreover, since (16) depends on two parameters it would not be obvious in which order the corresponding singular terms would appear in the *augmented*-RBF method.

By linearity of the fractional derivative operators we have

$$ {}^C D_1^{\beta}{}_0^C D^{\alpha} \tilde{y}(t) = \sum_{j=1}^{N} \alpha_j {}^C D_1^{\beta}{}_0^C D^{\alpha} \Phi_j(t) + \sum_{j=1}^{S_0} \beta_j {}^C D_1^{\beta}{}_0^C D^{\alpha} t^{(p_0)_j} + \sum_{j=1}^{S_1} \gamma_j {}^C D_1^{\beta}{}_0^C D^{\alpha} (1-t)^{(p_1)_j} \tag{17} $$

and

$$ {}^C D_1^{\beta}{}_0^C D^{\alpha} \Phi(t) = \tilde{C}\, {}^C D_1^{\beta}{}_0^C D^{\alpha} (\Psi(2t-1)). $$

To the best of our knowledge, it is not known a closed form for the fractional derivatives ${}^C D_1^{\beta}{}_0^C D^{\alpha}(\Psi(2t-1))$. Thus, we expanded the functions $\Psi(2t-1)$ in a Mac-Laurin series and then, applied the fractional derivative operators term by term to each monomial. The expansion was truncated, once the contributions of the higher terms were smaller than machine precision.

Next we derive the formula for the fractional derivatives of each monomial.

We will use the notation

$$ (r,k) = \frac{r(r-1)\dots(r-k+1)}{k!}, \quad k \in \mathbb{N}_0, \quad r \in \mathbb{R}. $$

Note that if r is not a negative integer, (r, k) is given by a quotient of Gamma functions, namely (cf. [16, (1.5.25)]):

$$ (r,k) = \frac{\Gamma(r+1)}{k!\,\Gamma(r-k+1)}. $$

We recall the following expansion provided by the Binomial Theorem: for $|x| > |y|$ and r a complex number

$$ (x+y)^r = \sum_{k=0}^{\infty} (r,k) x^{r-k} y^k. $$

We will make use of the following identity

$$ -\Gamma(n+c)\Gamma(-n+1-c) = \Gamma(1+c)\Gamma(-c)(-1)^n, \quad n \in \mathbb{N}_0, c \notin \mathbb{Z} \tag{18} $$

which can be proved by induction.

Proposition 4.2. *Let* $p \in \mathbb{R}_0^+$, $0 < \alpha \leq 1$, $0 < \beta \leq 1$ *and* $0 < t < 1$. *Then,*

$$ {}^C D_1^{\beta}{}_0^C D^{\alpha} t^p = \begin{cases} 0, & p = 0 \vee p = \alpha \\ -\frac{\Gamma(p+1)(1-t)^{1-\beta}}{\Gamma(p-\alpha)\Gamma(2-\beta)}\; {}_2F_1(1, 1+\alpha-p; 2-\beta; 1-t), & p \in \mathbb{R}^+ \setminus \{\alpha\}. \end{cases} \tag{19} $$

Proof. If $p=0$, we conclude immediately that

$$ {}^C D_1^{\beta}{}_0^C D^{\alpha} t^p = {}^C D_1^{\beta}{}_0^C D^{\alpha} 1 = {}^C D_1^{\beta} 0 = 0. $$

Otherwise ($p > 0$), we have

$$ {}_0^C D^{\alpha} t^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}. $$

Now using the Binomial Theorem,

$$ (1-(1-s))^{p-\alpha} = \sum_{k=0}^{\infty} (-1)^k (p-\alpha, k)(1-s)^k. $$

Therefore,

$$ \begin{aligned} {}^C D_1^{\beta}{}_0^C D^{\alpha} t^p &= {}^C D_1^{\beta} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} \\ &= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} {}^C D_1^{\beta} t^{p-\alpha} := T. \end{aligned} $$

If $p = \alpha$, then we have

$$ T = \Gamma(\alpha+1) {}^C D_1^{\beta} t^0 = \Gamma(\alpha+1) {}^C D_1^{\beta} 1 = 0. $$

Otherwise,

$$T = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} {}^C D_1^\beta \sum_{k=0}^{\infty} (-1)^k (p-\alpha,k)(1-t)^k$$
$$= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} \sum_{k=1}^{\infty} (-1)^k (p-\alpha,k) \frac{k!}{\Gamma(1+k-\beta)} (1-t)^{k-\beta}.$$

Now, if $p-\alpha := n \in \mathbb{N}$,

$$T = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \sum_{k=1}^{\infty} (-1)^k \frac{n!}{(n-k)!\Gamma(1+k-\beta)} (1-t)^{k-\beta}$$
$$= \frac{\Gamma(n+\alpha+1) n (1-t)^{1-\beta}}{\Gamma(n+1)} \sum_{k=1}^{\infty} (-1)^k \frac{(n-1)!}{(n-k)!\Gamma(1+k-\beta)} (1-t)^{k-1}$$
$$= \frac{\Gamma(n+\alpha+1)(1-t)^{1-\beta}}{(n-1)!\Gamma(2-\beta)} \sum_{k=0}^{\infty} (-1)^{k+1} \frac{(n-1)!\Gamma(2-\beta)}{(n-k-1)!\Gamma(2+k-\beta)} (1-t)^k$$
$$= -\frac{\Gamma(n+\alpha+1)(1-t)^{1-\beta}}{(n-1)!\Gamma(2-\beta)} \, {}_2F_1(1, 1-n; 2-\beta; 1-t).$$

Finally, to perform the calculations in the case $p-\alpha \notin \mathbb{N}$, we use the identity (18), for $c = \alpha - p$; $n = k$, and we get

$$\frac{(-1)^k}{\Gamma(-k+1-\alpha+p)} = -\frac{\Gamma(k+\alpha-p)}{\Gamma(1+\alpha-p)\Gamma(p-\alpha)}. \tag{20}$$

Thus,

$$T = \Gamma(p+1) \sum_{k=1}^{\infty} \frac{(-1)^k}{\Gamma(1+k-\beta)\ \Gamma(p-\alpha-k+1)} (1-t)^{k-\beta} \underbrace{=}_{eq.\ (20)}$$
$$= -\frac{\Gamma(p+1)}{\Gamma(p-\alpha)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\alpha-p)}{\Gamma(1+\alpha-p)\Gamma(1+k-\beta)} (1-t)^{k-\beta}$$
$$= -\frac{\Gamma(p+1)}{\Gamma(p-\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(k+1+\alpha-p)}{\Gamma(1+\alpha-p)\Gamma(2+k-\beta)} (1-t)^{k+1-\beta}$$
$$= -\frac{\Gamma(p+1)(1-t)^{1-\beta}}{\Gamma(2-\beta)\Gamma(p-\alpha)} \sum_{k=0}^{\infty} \frac{\Gamma(2-\beta)\Gamma(k+1+\alpha-p)}{\Gamma(1+\alpha-p)\Gamma(2+k-\beta)} (1-t)^k$$
$$= -\frac{\Gamma(p+1)(1-t)^{1-\beta}}{\Gamma(2-\beta)\Gamma(p-\alpha)} \, {}_2F_1(1, 1+\alpha-p; 2-\beta; 1-t).$$

□

The previous Proposition provides the formula for the derivative ${}^C D_1^\beta {}^C_0 D^\alpha t^p$, for $p \in \mathbb{R}_0^+$. In particular, it allows to calculate some of the terms in (17), namely ${}^C D_1^\beta {}^C_0 D^\alpha t^{(p_0)_j}$ and ${}^C D_1^\beta {}^C_0 D^\alpha \Phi_j(t)$, by using the Mac-Laurin expansion and applying the operators term by term to each monomial. The next result provides the formula that allows to calculate the remaining terms, ${}^C D_1^\beta {}^C_0 D^\alpha (1-t)^{(p_1)_j}$.

Proposition 4.3. *Let* $0 < \alpha < 1$, $p > \alpha$, $p - \alpha \notin \mathbb{N}$ *and* $0 < \beta \leq 1$. *Then, for* $0 < t < 1$ *we have that:*

$$\begin{aligned} {}^C D_1^\beta {}^C_0 D^\alpha (1-t)^p = \\ &- \frac{p(1-t)^{1-\beta}}{(1+\alpha-p)\Gamma(2-\beta)\Gamma(-\alpha)} \, {}_3F_2(1, 1+\alpha, 1+\alpha-p; 2-\beta, 2+\alpha-p; 1-t) \\ &+ p(1-t)^{-\alpha+p-\beta} \frac{\Gamma(-\alpha+p+1)}{\Gamma(-\alpha+p+1-\beta)} \left(\frac{1}{(-\alpha+p)\Gamma(-\alpha+1)} \right. \\ &\left. + \frac{{}_3F_2(1, 1+\alpha, 1+\alpha-p; 2, 2+\alpha-p; 1)}{(1+\alpha-p)\Gamma(-\alpha)} \right). \end{aligned} \tag{21}$$

Proof. We have that:

$$\begin{aligned}
{}_0^C D^{\alpha}(1-t)^p &= \frac{-p}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}(1-s)^{p-1}ds\\
&= \frac{-p}{\Gamma(1-\alpha)}\int_0^t (1-s+(-(1-t)))^{-\alpha}(1-s)^{p-1}ds\\
&= \frac{-p}{\Gamma(1-\alpha)}\int_0^t \sum_{k=0}^{\infty}(-\alpha,k)(1-s)^{-\alpha-k}(-1)^k(1-t)^k(1-s)^{p-1}ds\\
&= \frac{-p}{\Gamma(1-\alpha)}\sum_{k=0}^{\infty}(-1)^k(-\alpha,k)(1-t)^k\int_0^t (1-s)^{-\alpha-k+p-1}ds\\
&\underbrace{=}_{p-\alpha\notin\mathbb{N}} \frac{p}{\Gamma(1-\alpha)}\sum_{k=0}^{\infty}(-1)^k(-\alpha,k)(1-t)^k\frac{(1-t)^{-\alpha-k+p}-1}{-\alpha-k+p}\\
&= \frac{p}{\Gamma(1-\alpha)}\sum_{k=0}^{\infty}(-1)^k(-\alpha,k)\frac{(1-t)^{-\alpha+p}-(1-t)^k}{-\alpha-k+p}.
\end{aligned}$$

Now,

$$\begin{aligned}
{}^C D_1^{\beta}\,{}_0^C D^{\alpha}(1-t)^p &= \frac{p}{\Gamma(1-\alpha)}\sum_{k=0}^{\infty}(-1)^k(-\alpha,k)\,{}^C D_1^{\beta}\frac{(1-t)^{-\alpha+p}-(1-t)^k}{-\alpha-k+p}\\
&= \frac{p}{\Gamma(1-\alpha)}\left(\sum_{k=0}^{\infty}(-1)^k(-\alpha,k)\,{}^C D_1^{\beta}\frac{(1-t)^{-\alpha+p}}{-\alpha-k+p}-\sum_{k=1}^{\infty}(-1)^k(-\alpha,k)\,{}^C D_1^{\beta}\frac{(1-t)^k}{-\alpha-k+p}\right)\\
&= \frac{p}{\Gamma(1-\alpha)}\left(\sum_{k=0}^{\infty}\frac{(-1)^k(-\alpha,k)}{-\alpha-k+p}\frac{\Gamma(-\alpha+p+1)}{\Gamma(-\alpha+p+1-\beta)}(1-t)^{-\alpha+p-\beta}\right.\\
&\quad\left.-\sum_{k=1}^{\infty}\frac{(-1)^k(-\alpha,k)}{-\alpha-k+p}\frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}(1-t)^{k-\beta}\right) = T_1 - T_2,
\end{aligned}$$

where

$$T_1 := \frac{p}{\Gamma(1-\alpha)}\sum_{k=0}^{\infty}\frac{(-1)^k(-\alpha,k)}{-\alpha-k+p}\frac{\Gamma(-\alpha+p+1)}{\Gamma(-\alpha+p+1-\beta)}(1-t)^{-\alpha+p-\beta},$$

and

$$T_2 := \frac{p}{\Gamma(1-\alpha)}\sum_{k=1}^{\infty}\frac{(-1)^k(-\alpha,k)}{-\alpha-k+p}\frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}(1-t)^{k-\beta}.$$

Before we proceed, we need to prove the following identity:

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{(-1)^k}{(-\alpha-k+p)\Gamma(k+1)\Gamma(-\alpha-k+1)}\\
&= \frac{1}{(-\alpha+p)\Gamma(-\alpha+1)}+\frac{{}_3F_2(1,1+\alpha,1+\alpha-p;2,2+\alpha-p;1)}{(1+\alpha-p)\Gamma(-\alpha)}.
\end{aligned}$$

Indeed, using (18) with $n=k$ and $c=\alpha$, we get

$$-\Gamma(1+\alpha)\Gamma(-\alpha)(-1)^k = \Gamma(k+\alpha)\Gamma(-k+1-\alpha). \tag{22}$$

which implies that

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{(-1)^k}{(-\alpha-k+p)\Gamma(k+1)\Gamma(-\alpha-k+1)} = \sum_{k=1}^{\infty}\frac{-\Gamma(\alpha+k)}{(-\alpha-k+p)\Gamma(k+1)\Gamma(-\alpha)\Gamma(\alpha+1)}\\
&= \frac{1}{\Gamma(-\alpha)\Gamma(\alpha+1)}\sum_{k=1}^{\infty}\frac{\Gamma(\alpha+k)}{(\alpha+k-p)\Gamma(k+1)} = \frac{1}{\Gamma(-\alpha)\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k+1)}{(\alpha+k+1-p)\Gamma(k+2)}\\
&= \frac{1}{\Gamma(-\alpha)(1+\alpha-p)}\frac{\Gamma(2)\Gamma(2+\alpha-p)}{\Gamma(1)\Gamma(\alpha+1)\Gamma(1+\alpha-p)}\sum_{k=0}^{\infty}\frac{\Gamma(1+k)\Gamma(\alpha+k+1)\Gamma(1+\alpha-p+k)}{\Gamma(2+k)\Gamma(2+\alpha-p+k)k!}\\
&= \frac{{}_3F_2(1,1+\alpha,1+\alpha-p;2,2+\alpha-p;1)}{(1+\alpha-p)\Gamma(-\alpha)},
\end{aligned}$$

which proves our claim.

Now,

$$
\begin{aligned}
T_1 &= \frac{p}{\Gamma(1-\alpha)}\sum_{k=0}^{\infty}\frac{(-1)^k(-\alpha,k)}{-\alpha-k+p}\frac{\Gamma(-\alpha+p+1)}{\Gamma(-\alpha+p+1-\beta)}(1-t)^{-\alpha+p-\beta}\\
&= \frac{p(1-t)^{-\alpha+p-\beta}}{\Gamma(1-\alpha)}\frac{\Gamma(-\alpha+p+1)}{\Gamma(-\alpha+p+1-\beta)}\sum_{k=0}^{\infty}\frac{(-1)^k(-\alpha,k)}{-\alpha-k+p}\\
&= \frac{p(1-t)^{-\alpha+p-\beta}}{\Gamma(1-\alpha)}\frac{\Gamma(-\alpha+p+1)}{\Gamma(-\alpha+p+1-\beta)}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(-\alpha+1)}{(-\alpha-k+p)\Gamma(k+1)\Gamma(-\alpha-k+1)}\\
&= \frac{p(1-t)^{-\alpha+p-\beta}}{\Gamma(1-\alpha)}\frac{\Gamma(-\alpha+p+1)}{\Gamma(-\alpha+p+1-\beta)}\Gamma(-\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{(-\alpha-k+p)\Gamma(k+1)\Gamma(-\alpha-k+1)}\\
&= p(1-t)^{-\alpha+p-\beta}\frac{\Gamma(-\alpha+p+1)}{\Gamma(-\alpha+p+1-\beta)}\left(\frac{1}{(-\alpha+p)\Gamma(-\alpha+1)}\right.\\
&\quad\left.+\frac{{}_3F_2(1,1+\alpha,1+\alpha-p;2,2+\alpha-p;1)}{(1+\alpha-p)\Gamma(-\alpha)}\right).
\end{aligned}
$$

For T_2 we have,

$$
\begin{aligned}
T_2 &= \frac{(1-t)^{1-\beta}(1+\alpha-p)\Gamma(2-\beta)}{(1-t)^{1-\beta}(1+\alpha-p)\Gamma(2-\beta)}\frac{p}{(-\alpha)\Gamma(-\alpha)}\sum_{k=1}^{\infty}\frac{(-1)^k(-\alpha,k)}{-\alpha-k+p}\frac{\Gamma(k+1)}{\Gamma(k+1-\beta)}(1-t)^{k-\beta}\\
&= \frac{p(1-t)^{1-\beta}}{(1+\alpha-p)\Gamma(2-\beta)\Gamma(-\alpha)}\sum_{k=1}^{\infty}\frac{(-1)^k(-\alpha,k)\Gamma(2-\beta)}{(-\alpha)(-\alpha-k+p)}\frac{\Gamma(k+1)(1+\alpha-p)}{\Gamma(k+1-\beta)}(1-t)^{k-1}\\
&= \frac{p(1-t)^{1-\beta}}{(1+\alpha-p)\Gamma(2-\beta)\Gamma(-\alpha)}\sum_{k=1}^{\infty}\frac{(-1)^k\Gamma(-\alpha+1)\Gamma(2-\beta)}{\Gamma(k+1)\Gamma(-\alpha-k+1)(-\alpha)(-\alpha-k+p)}\\
&\quad\cdot\frac{\Gamma(k+1)(1+\alpha-p)}{\Gamma(k+1-\beta)}(1-t)^{k-1}\\
&= \frac{p(1-t)^{1-\beta}}{(1+\alpha-p)\Gamma(2-\beta)\Gamma(-\alpha)}\sum_{k=1}^{\infty}\frac{-(-1)^k\Gamma(-\alpha)\Gamma(2-\beta)}{\Gamma(-\alpha-k+1)(\alpha+k-p)}\frac{(1+\alpha-p)}{\Gamma(k+1-\beta)}(1-t)^{k-1}\\
&= \frac{p(1-t)^{1-\beta}}{(1+\alpha-p)\Gamma(2-\beta)\Gamma(-\alpha)}\sum_{k=1}^{\infty}\frac{-(-1)^k\Gamma(1+\alpha)\Gamma(-\alpha)\Gamma(2-\beta)}{\Gamma(1+\alpha)\Gamma(-\alpha-k+1)(\alpha+k-p)}\frac{(1+\alpha-p)}{\Gamma(k+1-\beta)}(1-t)^{k-1}
\end{aligned}
$$

Thus, using (22) we get

$$
\begin{aligned}
T_2 &= \frac{p(1-t)^{1-\beta}}{(1+\alpha-p)\Gamma(2-\beta)\Gamma(-\alpha)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\alpha)\Gamma(-k+1-\alpha)\Gamma(2-\beta)}{\Gamma(1+\alpha)\Gamma(-\alpha-k+1)(\alpha+k-p)}\frac{(1+\alpha-p)}{\Gamma(k+1-\beta)}(1-t)^{k-1}\\
&= \frac{p(1-t)^{1-\beta}}{(1+\alpha-p)\Gamma(2-\beta)\Gamma(-\alpha)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\alpha)\Gamma(2-\beta)}{\Gamma(1+\alpha)(\alpha+k-p)}\frac{(1+\alpha-p)}{\Gamma(k+1-\beta)}(1-t)^{k-1}\\
&= \frac{p(1-t)^{1-\beta}}{(1+\alpha-p)\Gamma(2-\beta)\Gamma(-\alpha)}\sum_{k=0}^{\infty}\frac{\Gamma(k+1+\alpha)\Gamma(2-\beta)}{\Gamma(1+\alpha)(\alpha+k+1-p)}\frac{(1+\alpha-p)}{\Gamma(k+2-\beta)}(1-t)^{k}\\
&= \frac{p(1-t)^{1-\beta}}{(1+\alpha-p)\Gamma(2-\beta)\Gamma(-\alpha)}\sum_{k=0}^{\infty}\frac{\frac{\Gamma(k+1+\alpha)}{\Gamma(1+\alpha)}\frac{\Gamma(k+1+\alpha-p)}{\Gamma(1+\alpha-p)}}{\frac{\Gamma(k+2-\beta)}{\Gamma(2-\beta)}\frac{\Gamma(k+2+\alpha-p)}{\Gamma(2+\alpha-p)}}(1-t)^{k}\\
&= \frac{p(1-t)^{1-\beta}}{(1+\alpha-p)\Gamma(2-\beta)\Gamma(-\alpha)}\sum_{k=0}^{\infty}\frac{(1)_k(1+\alpha)_k(1+\alpha-p)_k}{(2-\beta)_k(2+\alpha-p)_k}\frac{(1-t)^k}{k!}\\
&= \frac{p(1-t)^{1-\beta}}{(1+\alpha-p)\Gamma(2-\beta)\Gamma(-\alpha)}\,{}_3F_2(1,1+\alpha,1+\alpha-p;2-\beta,2+\alpha-p;1-t),
\end{aligned}
$$

which finishes the proof. □

Remark 4.4. We would like to point out here one key point of our work: the proofs of Propositions 4.2 and 4.3 are similar, however, there is a fundamental difference that enabled us to design and obtain accurate numerical methods. Indeed, by expanding the kernel $(t-s)^{-\alpha}$ as a series instead of the monomial $(1-t)^{p-1}$ in the beginning of the proof of Proposition 4.3, we were then able to obtain a *closed* formula for calculating ${}^{C}D_{1}^{\beta}{}^{C}_{0}D^{\alpha}(1-t)^{p}$ (cf. (21)). Expanding $(1-t)^{p-1}$ as a series would produce a double sum for the final result of the calculation of ${}^{C}D_{1}^{\beta}{}^{C}_{0}D^{\alpha}(1-t)^{p}$, that we were not able to identify with any known function and that from the numerical point of view was almost useless in view of its slow convergence.

We will consider the following class of boundary conditions

$$\begin{cases}\delta_1 y(0)+\delta_2\int_0^1 g(t)y(t)dt=\delta_3,\\ \delta_4 y(1)+\delta_5\int_0^1 h(t)y(t)dt=\delta_6,\end{cases} \tag{23}$$

for given functions $g, h \in C([0, 1])$ and $\delta_i \in \mathbb{R}$, $i = 1, 2, \ldots, 6$. For completeness we prove an existence and uniqueness theorem for the differential Eq. (1) subject to (23) under certain conditions (cf. the Appendix).

Given a vector of coefficients of the linear combination (15),

$$\mathcal{V}=[\alpha_1,\ldots,\alpha_N,\beta_1,\ldots,\beta_{S_0},\gamma_1,\ldots,\gamma_{S_1}]^T$$

and M points equally spaced in the interval (0, 1),

$$x_i=\frac{i}{M+1},\quad i=1,2,\ldots,M,$$

we calculate the vector

$$\mathcal{U}=[u_1,\ldots,u_M]^T,\quad u_i={}^{C}D_{1}^{\beta}{}^{C}_{0}D^{\alpha}\tilde{y}(x_i)-f(x_i,\tilde{y}(x_i)),\quad i=1,2,\ldots,M$$

and define

$$\mathcal{F}(\mathcal{V})=\sum_{i=1}^{M}u_i^2.$$

The vector of the coefficients of the *augmented*-RBF, $\mathcal{V}$, will be determined as the solution of the minimization problem of the function $\mathcal{F}(\mathcal{V})$. Note that $\mathcal{F}(\mathcal{V})=0$ if and only if $\tilde{y}$ satisfies the fractional differential Eq. (1) at all the points x_i. We will also impose the boundary conditions,

$$\begin{cases}\delta_1 \tilde{y}(0)+\delta_2\int_0^1 g(t)\tilde{y}(t)dt=\delta_3,\\ \delta_4 \tilde{y}(1)+\delta_5\int_0^1 h(t)\tilde{y}(t)dt=\delta_6,\end{cases} \tag{24}$$

which can be written as

$$\begin{bmatrix}\mathrm{M}_1 \mid \mathrm{M}_2 \mid \mathrm{M}_3\end{bmatrix}\mathcal{V}=\begin{bmatrix}\delta_3\\ \delta_6\end{bmatrix}, \tag{25}$$

where

$$\mathrm{M}_1=\begin{bmatrix}\delta_1\Phi_1(0)+\delta_2\int_0^1 g(t)\Phi_1(t)dt & \ldots & \delta_1\Phi_N(0)+\delta_2\int_0^1 g(t)\Phi_N(t)dt\\ \delta_4\Phi_1(1)+\delta_5\int_0^1 h(t)\Phi_1(t)dt & \ldots & \delta_4\Phi_N(1)+\delta_5\int_0^1 h(t)\Phi_N(t)dt\end{bmatrix}_{2\times N},$$

$$\mathrm{M}_2=\begin{bmatrix}\delta_2\int_0^1 g(t)t^{(p_0)_1}dt & \ldots & \delta_2\int_0^1 g(t)t^{(p_0)_{S_0}}dt\\ \delta_4+\delta_5\int_0^1 h(t)t^{(p_0)_1}dt & \ldots & \delta_4+\delta_5\int_0^1 h(t)t^{(p_0)_{S_0}}dt\end{bmatrix}_{2\times S_0}$$

and

$$\mathrm{M}_3=\begin{bmatrix}\delta_1+\delta_2\int_0^1 g(t)(1-t)^{(p_1)_1}dt & \ldots & \delta_1+\delta_2\int_0^1 g(t)(1-t)^{(p_1)_{S_1}}dt\\ \delta_5\int_0^1 h(t)(1-t)^{(p_1)_1}dt & \ldots & \delta_5\int_0^1 h(t)(1-t)^{(p_1)_{S_1}}dt\end{bmatrix}_{2\times S_1}.$$

Thus, the coefficients for the *augmented*-RBF approximation are determined by solving a nonlinear optimization problem, with a system of linear equality constraints,

$$Min_{\mathcal{V}}\mathcal{F}(\mathcal{V}):(25)\text{ is satisfied.} \tag{26}$$

Next, we summarize the main steps of the numerical approach for solving FODE (1) under the boundary conditions (23).

Numerical algorithm

1. Define the set of RBF centers $\mathcal{C}$.
2. Calculate the RBF-QR basis functions, defined in (13) and the corresponding derivatives.
3. Choose S_0 and S_1, the number of singular terms at $t=0$ and $t=1$, respectively and define the sets of exponents $\mathcal{P}_0$ and $\mathcal{P}_1$.
4. Calculate ${}^{C}D_{1}^{\beta}{}^{C}_{0}D^{\alpha}t^{(p_0)_i}$, $i=1,\ldots,S_0$ and ${}^{C}D_{1}^{\beta}{}^{C}_{0}D^{\alpha}(1-t)^{(p_1)_i}$, $i=1,\ldots,S_1$, by using Propositions 4.2 and 4.3.
5. Define the linear constraints (25) corresponding to the boundary conditions of the problem.
6. Solve the optimization problem (26).

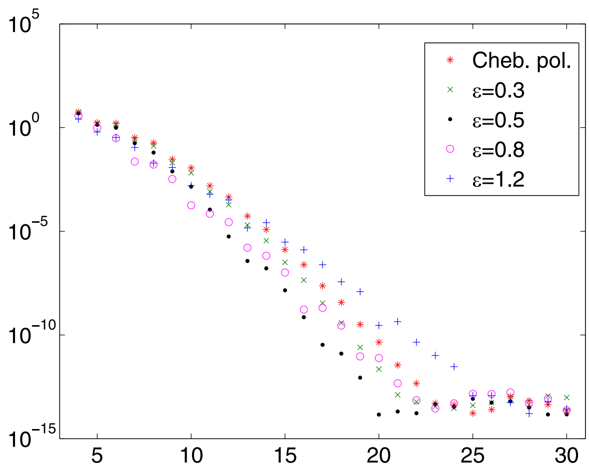

Fig. 1. Plots of $\|e(N,\epsilon)\|_{L^1([0,1])}$, as a function of N, for $\epsilon = 0.3, 0.5, 0.8, 1.2$. We included also the results obtained with the pseudo-spectral method.

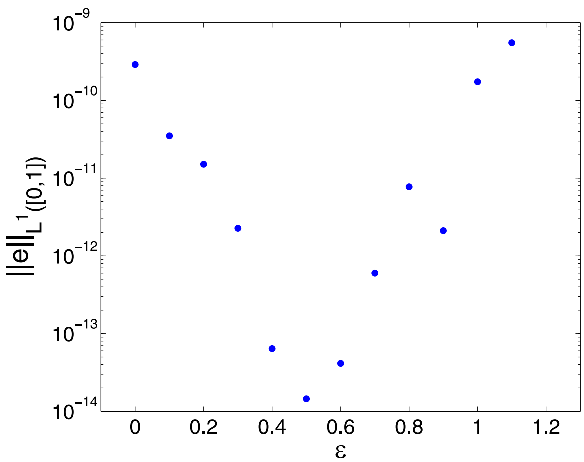

Fig. 2. Plot of $\|e(N,\epsilon)\|_{L^1([0,1])}$, obtained for $N = 20$ and different values of ϵ. The case $\epsilon = 0$ corresponds to the case of the pseudo-spectral method.

5. Numerical results

In this section we present some numerical results that illustrate the performance of the numerical method. In all the simulations we considered $M = 2000$, and the RBF centers have a Chebyshev distribution. Next, we test our numerical algorithm for the calculation of the fractional derivatives of the RBF basis functions. We considered the function $s(t) = \sin\left(\frac{5\pi}{2}t\right)$, for $\alpha = \frac{1}{2}$ and $\beta = \frac{\pi}{7}$. A closed formula for ${}^C D^{\beta}_{1}{}^C_0 D^{\alpha} s(t)$ is not available and again we calculated a Mac-Laurin expansion of s, truncated such that the remaining terms are smaller than machine precision. Then, we applied the fractional operators term by term to each monomial, by using Proposition 4.2. We will analyze the error,

$$e(N,\epsilon) = {}^C D^{\beta}_{1}{}^C_0 D^{\alpha} s(t) - {}^C D^{\beta}_{1}{}^C_0 D^{\alpha} y^{QR}(t)$$

where y^{QR} is the RBF-QR approximation obtained with N base functions and a certain shape parameter ϵ. Fig. 1 shows $\|e(N,\epsilon)\|_{L^1([0,1])}$, as a function of N, for $\epsilon = 0.3, 0.5, 0.8, 1.2$. We included also the case of the pseudo-spectral method, where the basis functions are Chebyshev polynomials, which can be seen as an asymptotic case, when $\epsilon \to 0$. We have spectral convergence and obtain results close to machine precision with a small number of basis functions N.

In Fig. 2 we plot $\|e(N,\epsilon)\|_{L^1([0,1])}$, obtained for $N = 20$ and different values of ϵ. The case $\epsilon = 0$ corresponds to the case of pseudo-spectral method. These results illustrate that in general, the RBF method is more accurate than the pseudo-spectral method, provided a suitable shape parameter is chosen, which was already observed in previous studies (cf. [1,9,20]). In this case we have an optimal shape parameter $\epsilon^{\star} \approx 0.5$.

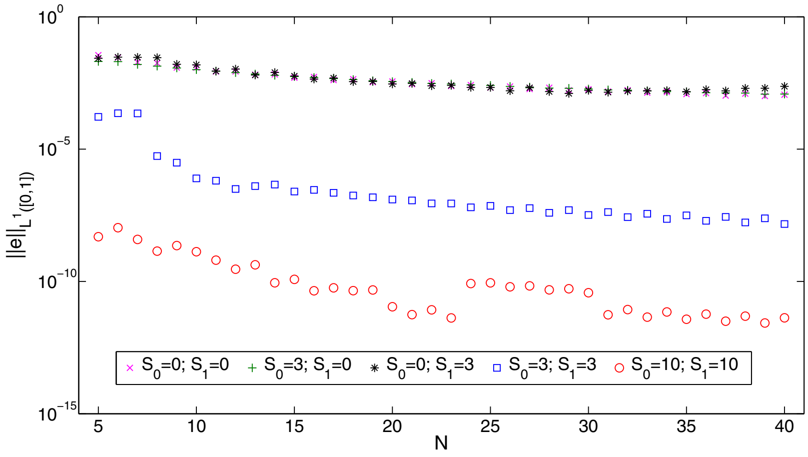

Fig. 3. Plot of $\|e(N,\epsilon)\|_{L^1([0,1])}$, obtained for $N = 4, 5, \dots, 40$, for several choices of S_0 and S_1.

Next we consider a boundary value problem for which we can derive the exact solution,

$$\begin{cases} {}^C D_1^{\beta}\, {}^C_0 D^{\alpha} y(t) = 1, \quad t \in (0,1), \\ y(0) = 0, \\ y(1) = 1. \end{cases} \tag{27}$$

By (10), we have

$$\begin{aligned} y(t) &= a + b\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t (t-s)^{\alpha-1}\int_s^1 (\tau - s)^{\beta-1} d\tau ds \\ &= a + b\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t \frac{(t-s)^{\alpha-1}(1-s)^\beta}{\beta} ds \\ &= a + b\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)\Gamma(\beta+1)}\int_0^1 (t-tw)^{\alpha-1}(1-tw)^\beta t dw \\ &= a + b\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^\alpha}{\Gamma(\alpha)\Gamma(\beta+1)}\int_0^1 (1-w)^{\alpha-1}(1-tw)^\beta dw \\ &= a + \frac{bt^\alpha}{\Gamma(1+\alpha)} + \frac{t^\alpha\ {}_2F_1(1,-\beta,1+\alpha,t)}{\Gamma(1+\beta)\Gamma(1+\alpha)} \end{aligned}$$

(cf. [11, 3.197–3]) and the boundary conditions imply that

$$a = 0;\ \ b = \Gamma(1+\alpha) - \frac{{}_2F_1(1,-\beta,1+\alpha,1)}{\Gamma(1+\beta)}.$$

Next we analyze the convergence of the numerical method applied to the boundary value problem (27), depending of the number of singular terms, for $\alpha = \frac{1}{2}$ and $\beta = \frac{\pi}{7}$ (note that $\alpha + \beta < 1$). Fig. 3 shows results for $\|e(N,\epsilon)\|_{L^1([0,1])}$, as a function of the number of RBF's with shape parameter $\epsilon = 0.5$, for several choices for the number of singular terms, S_0 and S_1. The first three examples correspond to the cases $S_0 = 0; S_1 = 0$, $S_0 = 3; S_1 = 0$ and $S_0 = 0; S_1 = 3$, which means approximations based on RBF method (without singular terms) and on the *augmented*-RBF method with 3 singular terms (respectively) at $t = 0$ and $t = 1$. We can observe that the errors obtained with these three choices are very similar and we could not get errors smaller than a value of order 10^{-2}, even for larger values of N. The RBF basis functions are analytic, while the solution is singular at $t = 0$ and $t = 1$. Thus, it is natural to have a slow convergence with these approaches. Moreover, the *augmented*-RBF with singular terms to deal with the singularity in just one of the points of singularity do not differ much, from the case of the RBF method (without singular terms), because there is always a point of singularity without a suitable numerical treatment. The remaining choices are $S_0 = S_1 = 3$ and $S_0 = S_1 = 10$, for which we have much better results and the method provides approximations whose L^1 norm of the error is of order 10^{-7} and 10^{-12}, respectively.

These numerical results illustrate that the *augmented-RBF* method is a very good approach for solving fractional differential equations, at low computational cost. On one hand, the RBF-QR provides a very accurate discretization of fractional differential operators and allows to calculate fractional derivatives of smooth functions, close to machine precision, even with very small matrices (see Fig. 1 and [1,20]). On the other hand, the singular behavior of the solutions of fractional differential equations can be reproduced by including some singular fractional monomials. This procedure implies to add just a few columns to the matrices associated to the RBF-QR method and does not increase much the computational cost of the method. However, this allows an improvement of several orders of magnitude at the error (see Fig. 3).

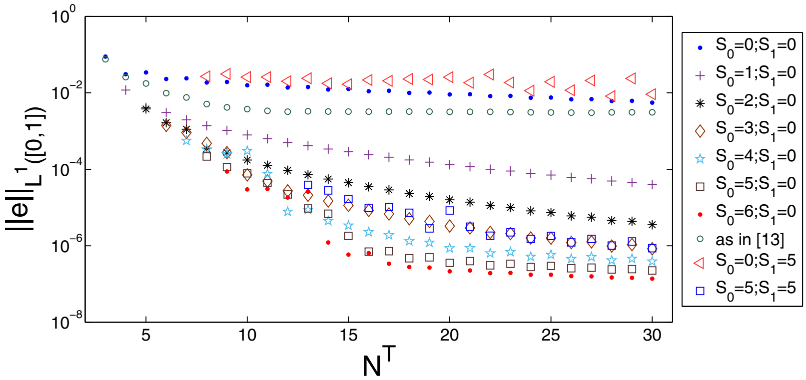

Fig. 4. Plot of the L^1 norm of the error for some choices of S_0 and S_1, together with results obtained with the numerical approach described in [13].

Next, we will test our algorithm with an example already considered in [7,13]. In those references, the authors considered the fractional variational problem

$$\text{minimize } J(y) = \frac{1}{2}\int_0^1 \left({}_0^C D^\alpha y(t) - f(t)\right)^2 dt,$$

with the boundary conditions

$$y(0) = 0, \quad y(1) = 1$$

and $f(t)$ given by

$$f(t) = \frac{\Gamma(\gamma+1)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}.$$

In this case, the exact solution is $y(t) = t^\gamma$ (cf. [7]) and the Euler–Lagrange equation is given by

$${}^C D_1^\alpha {}_0^C D^\alpha y(t) - {}^C D_1^\alpha f(t) = 0, \quad t \in (0,1).$$

We implemented the numerical method described in [13] and will compare with the numerical results that we obtained with our numerical approach. For a fair comparison, we define N^T, which is the total number of basis functions for the numerical approximation. In [13], the numerical approximation is a polynomial of degree n and thus, we have $n+1$ monomials as basis functions, which means that $N^T = n+1$. In our approach, we define $N^T = N + S_0 + S_1$. In Fig. 4 we plot the L^1 norm of the error for some choices of S_0 and S_1, for fixed values of $\gamma = \frac{1}{2}$ and $\alpha = \frac{2}{5}$. We can observe that for $S_0 = S_1 = 0$ and also for the approach described in [13], the convergence is very slow, essentially because in both cases the numerical approximation is an analytic function, while the solution is singular at the origin. Once we include a few singular terms to deal with the singularity at $t = 0$, the error decreases several orders of magnitude. Note that the solution is smooth at $t = 1$, thus, in this particular case, it would not be necessary to include singular terms at $t = 1$. However, assuming that in general we may not know the regularity of the solution at $t = 0$ and $t = 1$, it may be better to include singular terms at both points. For example, we can observe in Fig. 4 that the choice $S_0 = S_1 = 5$ provides good approximations for the solution. Of course, in this example it is crucial to include singular terms at $t = 0$. For example if we consider singular expansion just at $t = 1$ ($S_0 = 0$ and $S_1 = 5$), we obtain results that are even worse than in the case $S_0 = S_1 = 0$.

The last example is a boundary value problem with a nonlinear fractional ordinary differential equation, for which the solution is unknown,

$$\begin{cases} {}^C D_1^\beta {}_0^C D^\alpha y(t) = L\frac{|y|}{1+|y|} + 1, & t \in (0,1), \\ y(0) = \int_0^1 \sin(3t^2) y(t) dt, \\ y(1) = 0, \end{cases} \tag{28}$$

with $L = 0.16\Gamma(\alpha+1)\Gamma(\beta+1)$. It is not difficult to check that $f(t,y) = L\frac{|y|}{1+|y|} + 1$ is Lipschitz with constant L. Moreover, the conditions of Theorem 2 are satisfied and, therefore, (28) has a unique continuous solution.

We applied the *augmented*-RBF method with $N = 30$ RBF basis functions (with $\epsilon = 0.5$) and $S_0 = S_1 = 10$. In Fig. 5 we plot the numerical solutions obtained for some pairs of orders for the fractional derivatives, $(\alpha, \beta) = (2/5, 2/5)$, $(4/5, 4/5)$, $(\sqrt{2}/3, 1/2)$. In this case we do not have any theoretical bound for the error. However, since the numerical approximation satisfies the boundary conditions (23) close to machine precision, we can get an heuristic control of the error simply by analyzing the magnitude of

$$\mathcal{R}(t) := \left| {}^C D_1^\beta {}_0^C D^\alpha y(t) - f(t, y(t)) \right|, \quad \text{for } t \in (0,1).$$

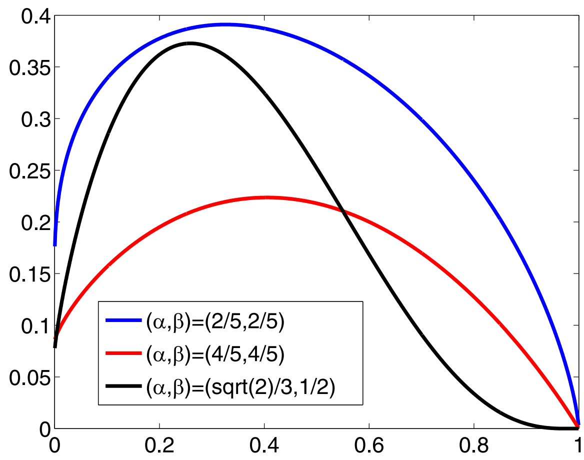

Fig. 5. Plots of the numerical solutions obtained for $(\alpha,\beta)=(2/5,2/5),\ (4/5,4/5),\ (\sqrt{2}/3,1/2)$.

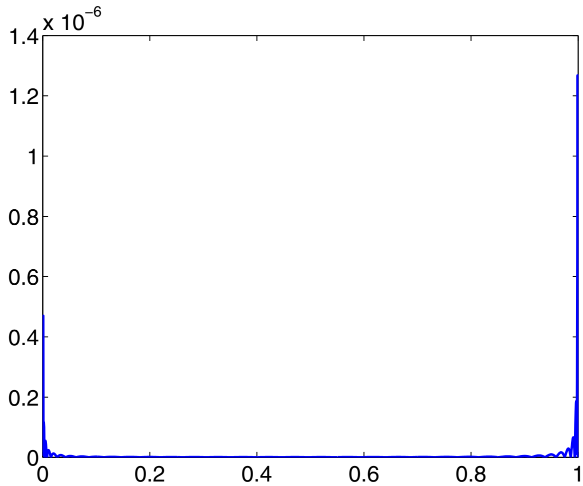

Fig. 6. Plot of the $\mathcal{R}(t)=\left|{}^{C}D_{1}^{\beta}{}_{0}^{C}D^{\alpha}y(t)-f(t,y(t))\right|$, $t\in(0,1)$, for $(\alpha,\beta)=(2/5,2/5)$.

In Fig. 6 we plot $\mathcal{R}(t)$ in the interval (0, 1), for the solution plotted in Fig. 5 corresponding to $\alpha=2/5,\ \beta=2/5$. We can observe that the function $\mathcal{R}(t)$ is almost zero in the whole interval and the maximum values ($\approx 10^{-6}$) are attained close to the points of singularity $t=0$ and $t=1$. These results suggest that we should increase the number of singular terms, S_0 and S_1, in order to improve the accuracy close to the points of singularity.

6. Conclusions

In this work we presented some results and some open problems on the smoothness of the solutions of the following (nonlinear) FODE:

$$ {}^{C}D_{1}^{\beta}{}_{0}^{C}D^{\alpha}y(t)=f(t,y(t)),\quad t\in[0,1],\quad 0<\alpha,\beta\le 1. $$

Based on what we were able to achieve, we developed a meshless numerical method, where the classical RBF method is augmented with some fractional polynomials in order to deal with the singular behavior of the solution. Some numerical examples illustrate that highly accurate results can be obtained, even with small matrices.

Appendix

Consider the following FODE,

$$ {}^{C}D_{1}^{\beta}{}_{0}^{C}D^{\alpha}y(t)=f(t,y(t)),\quad t\in[0,1],\quad 0<\alpha,\beta\le 1, \tag{29} $$

subject to the following boundary conditions,

$$\begin{cases} \delta_1 y(0) + \delta_2 \int_0^1 g(t)y(t)dt = \delta_3, \\ \delta_4 y(1) + \delta_5 \int_0^1 h(t)y(t)dt = \delta_6, \end{cases} \tag{30}$$

where the functions $g, h \in C([0,1])$ and $\delta_i \in \mathbb{R},\ i = 1, 2, \ldots, 6$.

Lemma A.1. *Suppose that $\delta_1, \delta_4 \neq 0$ in (30). A function $y \in C[0,1]$ is a solution of the BVP (29)–(30) if and only if it satisfies the following integral equation:*

$$\begin{aligned} y(t) = {} & \frac{\delta_3 - \delta_2 \int_0^1 g(t)y(t)dt}{\delta_1}(1 - t^\alpha) \\ & + \frac{t^\alpha}{\delta_4}\left[\delta_6 - \delta_5 \int_0^1 h(t)y(t)dt - \frac{\delta_4}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-s)^{\alpha-1} \int_s^1 (\tau - s)^{\beta-1} f(\tau, y(\tau)) d\tau ds\right] \\ & + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} \int_s^1 (\tau - s)^{\beta-1} f(\tau, y(\tau)) d\tau ds \end{aligned} \tag{31}$$

Proof. Let $y \in C[0,1]$. Then, by (6) and (7), we know that y satisfies the BVP (29)–(30) if and only if y satisfies the integral equation

$$y(t) = a + b\frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} \int_s^1 (\tau - s)^{\beta-1} f(\tau, y(\tau)) d\tau ds,$$

where a, b are real numbers. To complete the proof we need to uniquely determine a and b so that (31) holds. Using the given boundary conditions, we have that

$$\delta_1 a + \delta_2 \int_0^1 g(t)y(t)dt = \delta_3,$$

and

$$\begin{aligned} & \delta_4\left(a + \frac{b}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-s)^{\alpha-1} \int_s^1 (\tau - s)^{\beta-1} f(\tau, y(\tau)) d\tau ds\right) \\ & + \delta_5 \int_0^1 h(t)y(t)dt = \delta_6 \end{aligned}$$

After some somewhat tedious calculations we determine a and b uniquely and, after rearranging some terms, (31) holds. □

Theorem 2. *Let $0 < \alpha, \beta \leq 1$ and $\delta_1, \delta_4 \neq 0$. Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying a Lipscitz condition,i.e. there is a constant L such that*

$$|f(t, y) - f(t, x)| \leq L|y - x|,\ \ t \in [0,1],\ \ x, y \in \mathbb{R}.$$

If

$$\left|\frac{\delta_2}{\delta_1}\right| \int_0^1 |g(t)|dt + \left|\frac{\delta_5}{\delta_4}\right| \int_0^1 |h(t)|dt + \frac{2L}{\Gamma(\alpha+1)\Gamma(\beta+1)} < 1, \tag{32}$$

then the BVP (29)–(30) has a unique solution $y \in C[0,1]$.

Proof. Consider the Banach space $X = C[0,1]$ with norm $||x|| = \sup_{t\in[0,1]} |x(t)|$. Define the operator

$$\begin{aligned} Fy(t) = {} & \frac{\delta_3 - \delta_2 \int_0^1 g(t)y(t)dt}{\delta_1}(1 - t^\alpha) \\ & + \frac{t^\alpha}{\delta_4}\left[\delta_6 - \delta_5 \int_0^1 h(t)y(t)dt - \frac{\delta_4}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 (1-s)^{\alpha-1} \int_s^1 (\tau - s)^{\beta-1} f(\tau, y(\tau)) d\tau ds\right] \\ & + \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-s)^{\alpha-1} \int_s^1 (\tau - s)^{\beta-1} f(\tau, y(\tau)) d\tau ds. \end{aligned} \tag{33}$$

It is clear that F: $C[0,1] \to C[0,1]$. Now,

$$|Fy(t) - Fx(t)| = \left|\frac{\delta_2}{\delta_1} \int_0^1 g(t)[x(t) - y(t)]dt(1 - t^\alpha)\right.$$

$$
\begin{aligned}
&+\frac{t^\alpha}{\delta_4}\left[\delta_5\int_0^1 h(t)[x(t)-y(t)]dt+\frac{\delta_4}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1(1-s)^{\alpha-1}\int_s^1(\tau-s)^{\beta-1}\right.\\
&\cdot[f(\tau,y(\tau))-f(\tau,x(\tau))]d\tau ds]\\
&\left.+\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t(t-s)^{\alpha-1}\int_s^1(\tau-s)^{\beta-1}[f(\tau,y(\tau))-f(\tau,x(\tau))]d\tau ds\right|\\
\le\ &||y-x||\left(\left|\frac{\delta_2}{\delta_1}\right|\int_0^1|g(t)|dt+\left|\frac{\delta_5}{\delta_4}\right|\int_0^1|h(t)|dt\right.\\
&+\frac{L}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1(1-s)^{\alpha-1}\int_s^1(\tau-s)^{\beta-1}d\tau ds\\
&\left.+\frac{L}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t(t-s)^{\alpha-1}\int_s^1(\tau-s)^{\beta-1}d\tau ds\right)\\
\le\ &||y-x||\left(\left|\frac{\delta_2}{\delta_1}\right|\int_0^1|g(t)|dt+\left|\frac{\delta_5}{\delta_4}\right|\int_0^1|h(t)|dt+\frac{2L}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right),
\end{aligned}
$$

and the result follows from the Banach fixed point theorem. □

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