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A Note on Expansiveness and Hyperbolicity for Generic Geodesic Flows

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Abstract In this short note we contribute to the generic dynamics of geodesic flows associated to metrics on compact Riemannian manifolds of dimension ≥ 2. We prove that there exists a C^2-residual subset $\mathscr{R}$ of metrics on a given compact Riemannian manifold such that if $g \in \mathscr{R}$, then its associated geodesic flow φ_g^t is expansive if and only if the closure of the set of periodic orbits of φ_g^t is a uniformly hyperbolic set. For surfaces, we obtain a stronger statement: there exists a C^2-residual $\mathscr{R}$ such that if $g \in \mathscr{R}$, then its associated geodesic flow φ_g^t is expansive if and only if φ_g^t is an Anosov flow.

Keywords Expansiveness · Residual sets · Anosov · Geodesic flows

Mathematics Subject Classification (2010) 37C20 · 37D40 · 53D25

1 Introduction and Statement of the Results

One of the main goals in the theory of dynamical systems is to understand the orbits structure for 'most' maps and flows. Since it is an unattainable task to perceive every dynamical systems people often try to understand large classes of systems like open and dense sets, generic sets (i.e. Baire's second category or residual sets) or even dense sets. The pursue of results for a generic class of systems is very important

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because residual subsets are not only ‘thicker’ than dense ones, but countable intersections of residual subsets is still residual a property no longer true for dense subsets. As a matter of fact there is a well-established and active theory of generic dynamical systems nowadays.

A central branch of dynamical systems interconnected with mathematical physics intend to understand the geodesic flows associated to a certain metric tensor on differentiable Riemannian manifolds. The geodesic flow derived from a given metric describes the flow trajectory of a free particle not subject to external forces - i.e. a mechanical system driven by the kinetic energy solely. When studying the geodesic flow associated to negative curvature manifolds, Anosov (see [2]) came across with a surprising and quite rigid dynamic property - uniform hyperbolicity. Its main characteristic is the uniform rate of contraction and expansion, in the whole manifold, of a full rank fiber decomposition invariant under the tangent map of the flow (see (1) for the formal definition). Despite the geodesic flow on negative curvature manifolds was the first example of these type of dynamics, there are other dynamical systems displaying a global uniform hyperbolicity. This family is the so-called *Anosov dynamical systems*. Yet, geodesic flows trigger a huge challenge since perturbation tools for geodesic flows are constrained, mainly because one can only perturb the metric, meaning that the perturbation is never simply a local issue.

The *expansiveness property* means, in rough terms, that if two points stay near for forward and backward iterates, then they must be equal. Expansive systems can be considered chaotic in the sense that they exhibit sensitivity to the initial conditions. Expansiveness is a property displayed by hyperbolic sets and has its first studies in the seventies [8, 13]. It is well known that the Anosov class (global hyperbolicity) and the Axiom A class (local hyperbolicity) satisfy the expansiveness property.

In the early nineties Ruggiero (see [17]) proved that robustly expansive geodesic flows of compact, boundaryless Riemannian manifolds have the property that the closure of the set of periodic orbits is a hyperbolic set. When the manifold is 2-dimensional Ruggiero was able to obtain a sharpen result; the geodesic flow is robustly expansive if and only if it is an Anosov flow. The proof that robust expansiveness on surfaces is sufficient to get an Anosov flow can be also obtained by combining [14, 18].

Let us consider for a while that M is a surface and take the residual subset of metrics $\mathscr{R}$ such that (i) all closed orbits are hyperbolic or elliptic and (ii) all elliptic closed orbits are irrationally elliptic (see [1] or Section 1.1 for precise definitions). Elementary KAM-type arguments (see e.g. [7, §4]) allows us to obtain that, if the geodesic flow φ_g^t satisfy the expansiveness property, then there are no irrationally elliptic closed orbits. Thus, we obtain that under the expansiveness hypothesis on $g \in \mathscr{R}$ only hyperbolic closed orbits are allowed. Therefore:

Generically, if φ_g^t is expansive, then all closed orbits are hyperbolic.

A much more demanding question seeks to know if the closure of the hyperbolic closed orbits above is a uniformly hyperbolic set. In other words, can we upgrade pointwise hyperbolicity by uniform hyperbolicity on the set of closed orbits for generic expansive geodesic flows? Furthermore, can we hope to obtain global hyperbolicity (Anosov) for generic expansive geodesic flows? These problems are precisely answered in the next two results which are the core of this paper. The first one shows that seemingly different concepts like expansiveness and global hyperbolicity are yet tantamount from a generic point of view, at least in surfaces.

Theorem 1 *Let M be a surface. There exists a C^2-residual $\mathscr{R} \subset \mathcal{R}^2(M)$ such that if $g \in \mathscr{R}$, then φ_g^t is expansive if and only if φ_g^t is Anosov.*

The second one focuses on manifolds of dimension ≥ 3.

Theorem 2 *Let M be a d-dimensional manifold ($d \geq 3$). There exists a C^2-residual $\mathscr{R} \subset \mathcal{R}^2(M)$ such that if $g \in \mathscr{R}$, then φ_g^t is expansive if and only if the closure of the periodic orbits of φ_g^t is a uniformly hyperbolic set.*

The weak statement of Theorem 2 when compared to the literature (see [6] and references therein) and also to Theorem 1, is mainly due to the absence of the C^2-closing lemma for geodesic flows[1]. Fortunately, when M is a surface we can obtain the Anosov property by using the expansiveness hypothesis (cf. [17, §3]).

We also point out that the strategy described in [6] for the higher dimensional case cannot be directly applied to the present setting. Actually, the arguments in [6] rely mightily on getting a different index of two hyperbolic closed orbits and also that topologically transitivity holds generically. In the symplectic setting the index of hyperbolic closed orbits is always constant and equal to the dimension of the manifold so no change of index happens whatsoever. Moreover, no results on genericity of topological transitivity are available for geodesic flows[2]. Instead, we will use a different approach based on a symplectic reasoning. Needless to say that our arguments can be applied to Hamiltonian flows and also symplectomorphisms and thus obtaining the stronger conclusion (global hyperbolicity) by using Pugh and Robinson closing lemma [15].

We end the introduction by calling attention to a problem posed by Mañé in the early nineties - *Is there any expansive geodesic flow in a compact Riemannian manifold with conjugate*[3] *points?* In surfaces, and by a result of Paternain (see [14]), we know that the answer is negative. Later, in [19], Ruggiero was able to prove that if a metric g exhibiting an expansive geodesic φ_g^t, is close enough to $\hat{g}$ with no conjugate points such that $\varphi_{\hat{g}}^t$ is expansive too, then g do not have conjugate points as well. If we were able, like in Theorem 1, to obtain a stronger conclusion on Theorem 2, namely that φ_g^t is Anosov, then we would conclude that metrics displaying expansive geodesic flows and having conjugate points is a meager set. In overall, examples of

[1]The closing lemma for geodesic flows is only available in the C^1 case (see [5, 16]) and this version is useless to us.

[2]It is unlikely that it can be obtained a statement like the one in [4] for geodesic flows without new breakthrough techniques. Indeed, like we told above the C^2-closing lemma is still an open problem and seems to be more treatable when compared with a connecting lemma for pseudo-orbits (the key step to get transitivity).

[3]Recall that a manifold has no conjugate points if the exponential map at some point, and thus at every point, is non-singular.

metrics displaying expansive geodesic flows and having conjugate points would be a quite rare event from topological viewpoint.

1.1 Basic Definitions for Geodesic Flows

Let (M, g) be a boundaryless, connected and C^∞ Riemannian manifold of dimension $d \geq 2$, with $g \in \mathcal{R}^r(M)$ where $\mathcal{R}^r(M)$ stands for the set of C^r metrics on M with $2 \leq r \leq \infty$. Given a tangent vector $v \in T_x M$ at a point $x \in M$, denote by $\gamma_{x,v} : \mathbb{R} \to M$ the geodesic such that $\gamma_{x,v}(0) = x$ and $\dot{\gamma}_{x,v}(0) = v$. The *geodesic flow* of g is the one-parameter family of diffeomorphisms on the tangent bundle $\varphi_g^t : TM \to TM$ defined by $\varphi_g^t(x, v) = \big(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)\big)$. Since geodesics travel with constant speed, the unit tangent bundle $S_g M$ defined by the points $(x, v) \in TM$ such that $g_x(v, v) = 1$ is preserved by φ_g^t. By writing the canonical projection $\pi : S_g M \to M$, we have that geodesics $\gamma \subset M$ lift to orbits of the geodesic flow $\pi^{-1}\gamma \subset S_g M$. Since M is compact so is $S_g M$. A *transversal* Σ to the geodesic flow at a regular point (x, v) in $S_g M$ is an $(2d - 2)$-dimensional smooth submanifold satisfying

$$T_{(x,v)} S_g M = T_{(x,v)} \Sigma \oplus \mathbb{R} X_g(x, v)$$

where $\mathbb{R} X_g(x, v)$ stands for the one-dimensional subspace spanned by the vector $\dot{\gamma}_{x,v}(t)$. Note that Σ is a symplectic submanifold.

Consider a C^1-family of transversals $\Sigma_t := \Sigma_t(x, v)$ to the flow at $\varphi_g^t(x, v)$, $t \geq 0$, and of small enough neighborhoods $U_t \subset S_g M$ of (x, v). The *transversal Poincaré flow* of g at (x, v) is defined to be the family of C^1-symplectomorphisms $P_g^t : \Sigma_0 \cap U_t \to \Sigma_t$ given by $P_g^t(y, u) = \varphi_g^{\Theta(y,u,t)}(y, u)$ with

$$\Theta(y, u, t) = \min\{s \geq 0 : \varphi_g^s(y, u) \in \Sigma_t\}.$$

We assume that U_t is sufficiently small such that, by the implicit function theorem, Θ is C^1 and $\Theta(U_t, t)$ is bounded for a fixed $t > 0$. The *transversal linear Poincaré flow* of g at (x, v) is the derivative of P_g^t at (x, v),

$$DP_g^t(x, v) : T_{(x,v)} \Sigma_0 \to T_{\varphi_g^t(x,v)} \Sigma_t.$$

Given a regular point (x, v), we say that (x, v) is a *periodic point* of the geodesic flow φ_g^t if $\varphi_g^t(x, v) = (x, v)$ for some t. The smallest $t_0 > 0$ satisfying the condition above is called *period* of (x, v); in this case, we say that the orbit of (x, v) is a *closed orbit* of period t_0. Nontrivial closed geodesics on M for g are in one-to-one correspondence with the closed orbits of φ_g^t. When (x, v) is periodic of period $\ell > 0$ we call $P_g := P_g^\ell(x, v)$ the *Poincaré map* and Σ the *Poincaré section*.

Given a C^2-metric g and a φ_g^t-invariant, compact and regular set $\Lambda \subset S_g M$, we say that Λ is *uniformly hyperbolic* if there exist $\theta \in (0, 1)$, $C > 0$ and a DP_g^t-invariant splitting $E_\Lambda^s \oplus E_\Lambda^u$ of $T_\Lambda \Sigma$ such that for any $(x, v) \in \Lambda$ we have

$$\|DP_g^t(x, v)|_{E^s_{(x,v)}}\| \leq C\theta^t \quad \text{and} \quad \|DP_g^{-t}(\varphi_g^t(x, v))|_{E^u_{\varphi_g^t(x,v)}}\| \leq C\theta^t. \tag{1}$$

A periodic point (x, v) is called *hyperbolic* if its whole orbit is a uniform hyperbolic set. Equivalently, a closed geodesic is hyperbolic if its transversal linear Poincaré flow on the period has no eigenvalue of modulus 1. If the eigenvalues are non-real and with modulus 1 the closed orbit is said to be *elliptic*, and if they are irrational we say that the orbit is *irrationally elliptic*. The *parabolic* closed orbits have real eigenvalues equal to 1 or -1.

We say that a geodesic flow φ_g^t is *expansive* if for any $\epsilon > 0$, there is $\delta > 0$ such that if, for any $(x_1, v_1), (x_2, v_2) \in S_g M$ we have

$$d\left(\varphi_g^t(x_1, v_1), \varphi_g^{\alpha(t)}(x_2, v_2)\right) \leq \delta,$$

for all $t \in \mathbb{R}$ and for some increasing homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$, then $(x_2, v_2) = \varphi_g^s(x_1, v_1)$, where $|s| \leq \epsilon$. This definition asserts that any two points whose orbits remain indistinguishable, up to any continuous time displacement, must be in the same orbit. Observe that the reparametrization α is not assumed to be close to identity.

2 Proof of Theorem 1

In order to obtain the non-trivial part of the proof of Theorem 1 (the *'only if'* part) on surfaces we begin by proving the following simple preliminary result.

Lemma 2.1 *Let $g \in \mathcal{R}^2(M)$. If φ_g^t has a non-hyperbolic periodic point (x, v) of period ℓ, then for all $\epsilon, \nu > 0$, there exists $\hat{g} \in \mathcal{R}^2(M)$, such that:*

(1) *$\hat{g}$ is ϵ-C^2-close to g;*
(2) *$\varphi_{\hat{g}}^t$ has two hyperbolic periodic points (x_1, v_1) and (x_2, v_2) (not belonging to the same orbit) with equal period for $P_g^\ell(x, v)$ and*
(3) $\sup_{t\in\mathbb{R}} d\left(\varphi_g^t(x_1, v_1), \varphi_g^t(x_2, v_2)\right) < \nu.$

Proof Let be given $\epsilon, \nu > 0$. The metric g can be slightly perturbed in order to obtain an elliptic periodic point $(\hat{x},\hat{v})$ of period ℓ arbitrarily near (x, v). Indeed, we can use the Kupka-Smale theorem for geodesic flows [9, §2] or else [1, 11] together. Generic assumptions on the higher order derivatives at the elliptic periodic point can be assumed due to Klingenberg and Takens theorem [11]. These generic assumptions allow the existence of an irrationally elliptic closed orbit, which implies the existence of invariant curves[4] Now, the proof follows directly from the Lyapunov stability of generic elliptic periodic orbits and the existence of full shifts in every neighborhood of the elliptic point (see [12, 20]). Indeed, it is well-known that near generic elliptic points there are a profusion of homoclinic phenomena with invariant subsets (horseshoes) which are topologically equivalent to the full shift automorphism of symbolic dynamics. Lastly, we just have to take an adequate metric $\hat{g}$ ϵ-C^2-close to g and pick a full shift $\hat{\nu}$-close to the elliptic point (where $\hat{\nu}$ depends on ν and ℓ). The existence of

[4] The existence of invariant curves imply Lyapunov stability, i.e. orbits near the irrationally elliptic closed orbit stay near it for all future iterates. around the corresponding fixed point of the Poincaré map (see [7, §4.1]).

a full shift guarantees the existence of two hyperbolic periodic points of equal period fulfilling (2). Then, by Lyapunov stability, both hyperbolic periodic points are in a 'trapping region' without any chance of diffusion and so accomplishing (3). □

Consider now the so-called *star systems* defined by the C^2-interior of the metrics such that all closed orbits are hyperbolic:

$$\mathscr{H}(M) := \{g \in \mathcal{R}^\infty(M) \colon \text{all closed orbits } \gamma \text{ are hyperbolic}\} \text{ and } \mathscr{F}^2(M) := \text{int}_{C^2}\mathscr{H}(M).$$

Clearly, $g \in \mathscr{F}^2(M)$ means that $g \in \mathscr{H}(M)$ and for any $\hat{g} \in \mathcal{R}^\infty(M)$, C^2 close enough to g, we also have that $\hat{g} \in \mathscr{H}(M)$. We will make use of the following crucial and quite useful result by Contreras.

Theorem 2.2 *(*[9, Theorem E]*) There exists a C^2-open and C^∞-dense set $\mathcal{G} \subset \mathcal{R}^\infty(M)$, such that if $g \in \mathscr{F}^2(M) \cap \mathcal{G}$, then the closure of the periodic orbits of φ_g^t is a uniformly hyperbolic set.*

Since, at this point, we are considering M a surface the next result only needs a 2-dimensional version of Theorem 2.2 proved in [10].

Lemma 2.3 *There is a residual set $\mathscr{R} \subset \mathcal{R}^2(M)$, where* $\dim M = 2$*, such that for any $g \in \mathscr{R} \setminus \mathscr{F}^2(M)$, φ_g^t has two sequences of periodic points $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ such that for each n, p_n and q_n have distinct orbits and*

$$\lim_{n\to+\infty} \sup_{t\in\mathbb{R}} d\left(\varphi_g^t(p_n), \varphi_g^t(q_n)\right) = 0.$$

Proof For each $n \in \mathbb{N}$, we denote by $\mathscr{N}_n$ the subset of $\mathcal{R}^2(M)$ such that any $g \in \mathscr{N}_n$ has a C^2-neighborhood $\mathcal{U}$ in $\mathcal{R}^2(M)$ with the following C^2-open property: for every $\hat{g} \in \mathcal{U}$, there are hyperbolic periodic points p_n, q_n of $\hat{g}$ such that

$$\sup_{t\in\mathbb{R}} d\left(\varphi_{\hat{g}}^t(p_n), \varphi_{\hat{g}}^t(q_n)\right) < \frac{1}{n}.$$

Let $\mathscr{O}_n$ be the C^2-complementary of the C^2-closure of $\mathscr{N}_n$. Clearly, $\mathscr{N}_n \cup \mathscr{O}_n$ is C^2-open and C^2-dense in $\mathcal{R}^2(M)$. We define the C^2-residual subset in the statement of the lemma by:

$$\mathscr{R} = \bigcap_{n\in\mathbb{N}} \left(\mathscr{O}_n \cup \mathscr{N}_n\right).$$

If $g \in \mathscr{R} \setminus \mathscr{F}^2(M)$, then there is a sequence of metrics g_k converging to g in the C^2 sense and a sequence of non-hyperbolic periodic orbits $\tilde{p}_k$ of $\varphi_{g_k}^t$. Then, for any $n \in \mathbb{N}$, by Lemma 2.1, we have $g \notin \mathscr{O}_n$, and so $g \in \mathscr{N}_n$. Then, for each n, g has a C^2-neighborhood $\mathcal{U}$ in $\mathcal{R}^2(M)$ with the following property: for all $\hat{g} \in \mathcal{U}$ and in particular for g, there are hyperbolic periodic points p_n, q_n for $\hat{g}$ such that

$$\sup_{t\in\mathbb{R}} d\left(\varphi_{\hat{g}}^t(p_n), \varphi_{\hat{g}}^t(q_n)\right) < \frac{1}{n}.$$

In conclusion, we can define two sequences of periodic orbits for φ_g^t, $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ such that

$$\sup_{t\in\mathbb{R}} d\left(\varphi_g^t(p_n), \varphi_g^t(q_n)\right) < \frac{1}{n},$$

and so

$$\lim_{n\to+\infty}\sup_{t\in\mathbb{R}} d\left(\varphi_g^t(p_n), \varphi_g^t(q_n)\right) = 0.$$

And the proof of the lemma is complete. □

From Lemma 2.3, we know that if g belong to the residual $\mathscr{R}\cap\mathcal{G}$ and g has the expansiveness property, g must be in $\mathscr{F}^2(M)$. From Theorem 2.2, we know that the closure of the periodic orbits of φ_g^t is a uniformly hyperbolic set. Finally, from [17, Corollary 3.1] if the geodesic flow of a compact surface is expansive, then the closed orbits are dense and so φ_g^t is Anosov. This ends the proof of Theorem 1 when $\dim M = d = 2$.

3 Proof of Theorem 2

Lemma 3.1 below was proved in [9], as part of the proof of Theorem A and Theorem C. See the end of [9, Section 3]. As it is mentioned in [9] the techniques to obtain this result were developed by Arnaud and Herman in [3]. We recall that a *q-elliptic* closed orbit ($q = 1, \ldots, d$) is defined by the property of having its transversal linear Poincaré flow on the period with precisely q pairs of non-real eigenvalues of modulus 1 and the remaining ones outside the unit circle. These orbits can be viewed as partial hyperbolic ones with central manifold of dimension q. It is well-known that, fixing any $T > 0$, for an open and dense subset of metrics all closed orbits with period less that T are either q-elliptic or hyperbolic. This is basically due to the bumpy metric theorem in [1] (see also [10, §2] after [11, Theorem 2]).

The set $\mathcal{G}_0$ in next lemma is a set described by quite intricate generic conditions (see [9, Theorem C, Theorem 4.1] for full details).

Lemma 3.1 *Let $g \in \mathcal{G}_0 \cap \mathcal{R}^\infty(M)$ and γ be a q-elliptic closed orbit for φ_g^t with $q > 1$. Then, there exists a 1-elliptic periodic point $\tilde{\gamma}$ near γ such that the Poincaré map (on the period) restricted to the 2-dimensional central manifold is a twist map.*

Next result is Lemma 2.1 revisited now considering that M has dimension ≥ 3.

Lemma 3.2 *Let be given $g \in \mathcal{R}^2(M)$. If φ_g^t has a non-hyperbolic periodic point (x, v) of period ℓ, then for all $\epsilon, \nu > 0$, there exists $\hat{g} \in \mathcal{R}^2(M)$, such that:*

(1) *$\hat{g}$ is ϵ-C^2-close to g;*
(2) *$\varphi_{\hat{g}}^t$ has two hyperbolic periodic points (x_1, v_1) and (x_2, v_2) (not belonging to the same orbit) with equal period for $P_g^\ell(x, v)$ and*
(3) $\sup_{t\in\mathbb{R}} d\left(\varphi_g^t(x_1, v_1), \varphi_g^t(x_2, v_2)\right) < \nu.$

Proof The proof is made in two steps: first we use Lemma 3.1 to produce a 1-elliptic closed orbit for an arbitrarily near metric. Indeed, we can easily perturb a metric in order to transform a non-hyperbolic periodic point (x, v) into a q-elliptic one. If after an initial perturbation we obtain a 1-elliptic point we are done. Otherwise, we use Lemma 3.1 to obtain it. Second, we apply directly the 2-dimensional arguments in Lemma 2.1 inside the 2-dimensional invariant central manifold of this 1-elliptic closed orbit. Notice that this 2-dimensional submanifold is normally hyperbolic, thus persistent. □

Now, Lemma 3.2 allow us to obtain Lemma 2.3 when M has dimension ≥ 3 and Theorem 2 follows.

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