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Existence and Uniqueness of Density Conserving Solutions to the Coagulation–Fragmentation Equations with Strong Fragmentation

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1. INTRODUCTION

The time evolution of the size distribution of a system of clusters can be modeled by an infinite system of ordinary differential equations, each equation describing the rate of change of the concentration $c_j(t)$ of a cluster of size j (j-cluster), for $j = 1, 2, \ldots$.

Assuming only binary reactions between clusters (i.e., a given cluster can fragment only to give two smaller ones, and, reciprocally, can only be formed by coagulation of two smaller clusters) the *coagulation-fragmentation equations* describing the evolution of cluster sizes are

$$\dot{c}_j(t) = \frac{1}{2}\sum_{k=1}^{j-1} W_{j-k,k}(c(t)) - \sum_{k=1}^{\infty} W_{j,k}(c(t)), \qquad j = 1, 2, 3, \ldots, \tag{1}$$

where $W_{j,k}(c(t)) = a_{j,k}c_j(t)c_k(t) - b_{j,k}c_{j+k}(t)$. The coagulation and the fragmentation coefficients $a_{j,k}$ and $b_{j,k}$, respectively, are time independent parameters satisfying the basic assumptions of being non-negative and symmetric with respect to permutation of the subscripts.

Since $c_j(t)$ is physically interpreted as the concentration of j-clusters it is of special relevance to consider solutions to the initial value problem

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for (1) which are non-negative, $c_j(t) \geq 0$, and have finite *density*, $\rho_c(t) = \sum_{j=1}^{\infty} jc_j(t) < \infty$. This motivates the study of the Cauchy problem for (1) in the Banach space of finite density sequences,

$$X_1 = \left\{ c = (c_j) : \|c\|_1 \stackrel{\text{def}}{=} \sum_{j=1}^{\infty} j \mid c_j \mid < \infty \right\}. \tag{2}$$

Due to the physical assumptions underlying the model one expects density to be a constant of motion. When this is, indeed, what happens, the solution is called *density conserving*.

Although an easy formal computation of $\dot{\rho}_c(t)$ gives $\dot{\rho}_c(t) = 0$, and hence $\rho_c(t) = \text{const}$, for all $t \geq 0$, it is well known that, for some rate coefficients, density can vary along solutions [1, 6, 7, 11]. When it decreases, the solution is called a *geling solution* and the breakdown of density conservation at a time $\tau_g \geq 0$ is physically interpreted as corresponding to the occurrence of a phase transition in the system (gelation): the formation of a "macroscopic" gel phase with mass, at time $t > \tau_g$, proportional to $\rho_c(0) - \rho_c(t)$. In terms of the size distribution of clusters, this finite time phase transition implies a very rapid formation of large size clusters and the transport of part of the mass of the system to a cluster of infinite size (i.e., $j \to \infty$) in finite time.

In this paper we are only concerned with existence and uniqueness of density conserving solutions.

Proofs of existence of solutions to (1) have been obtained for a variety of rate coefficients. The method of proof consists in taking the finite n-dimensional truncations of (1), for which solutions c^n are density conserving. Standard methods in the theory of ordinary differential equations give existence and uniqueness of non-negative globally defined solutions to the truncated system. To prove that (1) has a solution, one proves the existence of a subsequence (c^{n_k}) of solutions to the truncated systems, converging to some function c, and then that c satisfies (1) in some convenient sense (usually the integral version of the system). Following Carr [3], we say that a solution to (1) is an *admissible solution* if it is obtained as the limit as $n \to \infty$ of solutions to the n-dimensional density conserving truncations of (1). This method of proof was first used by McLeod [9, 10] in the study of the special case of (1) with $b_{j,k} \equiv 0$ and has been used ever since in almost all studies of existence for this type of equations (for an exception see [12]).

The first study of (1) for which all coefficients $a_{j,k}$ and $b_{j,k}$ could be non-zero seems to have been [13]. In it, Spouge assumed coagulation coefficients satisfying $a_{j,k} \leq r_j r_k$, where (r_j) is a non-negative sequence such that $r_j \sim o(j)$ as $j \to \infty$ and with fragmentation coefficients satisfying a condition that, for binary fragmentation, is equivalent to boundedness.

With these assumptions, existence of non-negative global solutions to (1) was established. Questions of uniqueness and density conservation were not discussed in [13].

In a more recent paper, Ball and Carr [1] dropped the assumptions on the $b_{j,k}$'s (except the basic ones) and considered a slightly stronger condition on the coagulation, namely $a_{j,k} \leq K_a(j + k)$, where $K_a \geq 0$ is a constant. Under these conditions they proved the existence of non-negative, globally defined, density conserving admissible solutions. To get density conservation of all solutions, or to prove uniqueness, further assumptions were needed, in particular, some restrictions on the growth rate of the fragmentation coefficients.

Comparing the assumptions on the rate coefficients in these two studies we conclude that [1] gives a definitive improvement on [13] when $r_j \sim \mathcal{O}(j^\alpha)$, as $j \to \infty$, for $\alpha \in [0, \frac{1}{2}]$. However, for $\alpha > \frac{1}{2}$ the proofs in [1] are no longer valid and the only existence result for this case seems to have been that of [13], where the (binary) fragmentation coefficients $b_{j,k}$ need to be bounded.

In this paper we prove the existence and uniqueness of admissible solutions for coagulation coefficients satisfying

$$\text{(H1)} \quad a_{j,k} \leq K_a(jk)^\alpha \text{ with } \alpha \in (\tfrac{1}{2}, 1],$$

and for fragmentation coefficients satisfying the following *strong fragmentation condition*, first introduced by Carr in [3]:

There exists a constant $\gamma > 0$ such that, for all $\mu \geq 1$, there exists $K_f(\mu) > 0$ such that

$$\text{(H2)} \quad \sum_{j=1}^{h(r)} j^\mu b_{j,r-j} \geq K_f(\mu) r^{\gamma+\mu}$$

for all $r \geq 3$, where $H(r)$ is the integer part of $(r - 1)/2$. Furthermore, we prove that admissible solutions are density conserving. This result is somewhat interesting since for the case $a_{j,k}$ satisfying (H1) and $b_{j,k} \equiv 0$ solutions do not conserve density, in general [6, 7, 11]. Heuristically, the effect of fragmentation is to reduce the rate of formation of large clusters and, taking into account the interpretation of the geling behaviour given above, we expect that increasing fragmentation strength can prevent gelation. This is proved to occur for admissible solutions if (H2) holds and $\gamma > \alpha$.

Fragmentation coefficients satisfying (H2) include, for example, $b_{j,k} = (j + k)^\beta$ and $b_{j,k} = (jk)^\beta$ for $\beta > -1$, which have been used in studies of degradation of polymers [8]. Other examples are given in [3].

The paper is organized as follows. In Section 2 basic definitions and some preliminary results needed afterwards are stated.

In Section 3 we prove a regularity result analogous to the one presented in [3] for the case $a_{j,k} \le K_a(j + k)$, namely that (H1)–(H2) and $\gamma > 2\alpha - 1$ imply the μ-moment of a solution c^n to the n-dimensional truncated system, defined by $\|c^n(t)\|_\mu = \sum_{j=1}^n j^\mu |c_j^n(t)|$, is finite and bounded above by a function independent of n, for all $t > 0$ and $\mu \ge 1$. Since the bound is independent of n this property is valid for all functions $c^\infty : \mathbb{R}^+ \to X_1$ obtained as pointwise limits of subsequences of (c^n); i.e., $c_j^{n_k} \to c_j^\infty$ as $n_k \to \infty$, for some sequence (n_k) and all j.

This gives an a priori bound on $\|c^\infty(t)\|_\mu$ for $t > 0$ and $\mu \ge 1$ that allow us to prove, in Section 4, that $\|c^\infty\|_\mu$ is integrable in $[0, t)$ for all $t \ge 0$ and $\mu < 1 + \gamma$. In particular, this holds for admissible solutions. Still in Section 4 this integrability property is shown to hold for all solutions to (1).

In Section 5 we prove the existence of admissible solutions. The starting point of the proof is as in [1]: considering the solutions (c^n) to the n-dimensional density conserving truncation of (1) we prove, by Helly's theorem, the existence of a subsequence (c^{n_k}) and a function c such that $c_j^{n_k}(t) \to c_j(t)$ as $n_k \to \infty$, for all j and t. To prove that c is an admissible solution of (1) we use the integrability property of Section 4, the Fatou–Lebesgue theorem, and the dominated convergence theorem to pass to the limit $n_k \to \infty$ in the truncated system and to conclude that c satisfies the integral version of (1). Density conservation of admissible solutions is an easy consequence of the integrability of $\|c\|_\mu$.

Finally, in Section 6, we use the integrability of $\|c\|_\mu$, for all solutions c of (1), to get the uniqueness of density conserving solutions.

2. Preliminaries

2.1. *Definitions*

We study the existence of solutions to the Cauchy problem for (1) in the Banach space X_1 defined by (2). In fact, we are only interested in nonnegative solutions, i.e., solutions $c(t)$ that, for all $t \ge 0$, are in

$$X^+ = \{c \in X_1 : c_j \ge 0 \text{ for all } j\}.$$

As pointed out in the Introduction, we need to consider μ-moments of solutions, with $\mu \ge 1$, and so we define the following subspaces of X_1:

$$X_\mu = \left\{ c = (c_j) : \|c\|_\mu \overset{\text{def}}{=} \sum_{j=1}^\infty j^\mu |c_j| < \infty \right\}, \qquad \mu \ge 1. \tag{3}$$

The subspaces X_μ are themselves Banach spaces with the norm $\|\cdot\|_\mu$. In X_1 we need to use the following notion of convergence

DEFINITION 1 (Weak* convergence in X_1). Let (c^n) be a sequence in X_1. We say that c^n converges weak* to c in X_1 as $n \to \infty$, and we write $c^n \stackrel{*}{\rightharpoonup} c$, if

1. $\sup_n \|c^n\|_1 < \infty$, and
2. $c_j^n \to c_j$ as $n \to \infty$ for all $j \in \mathbb{N}$.

Following Ball and Carr [1], we use the definition of the solution.

DEFINITION 2 (Solution to (1)). Let $T \in (0, \infty]$ and $c_0 = (c_{0j}) \in X^+$. A solution $c = (c_j)$ of Eq. (1) on $[0, T)$, with initial data $c(0) = c_0$, is a function $c: [0, T) \to X^+$ such that

1. For all $j \in \mathbb{N}$, $c_j: [0, T) \to \mathbb{R}$ is continuous, and $\sup_{t\in[0,T)} \|c(t)\|_1 < \infty$.
2. For all $j \in \mathbb{N}$, $t \in [0, T)$,

$$\sum_{k=1}^{\infty} a_{j,k}c_k, \sum_{k=1}^{\infty} b_{j,k}c_{j+k} \in L^1(0, t).$$

3. For all $j \in \mathbb{N}$, $t \in [0, T)$,

$$c_j(t) = c_{0j} + \int_0^t \left[\frac{1}{2}\sum_{k=1}^{j-1} W_{j-k,k}(c(s)) - \sum_{k=1}^{\infty} W_{j,k}(c(s))\right] ds.$$

We say that a solution c is density conserving on $[0, \tau]$ if $\|c(t)\|_1 = \|c(0)\|_1$ for all $t \in [0, \tau]$.

Throughout this paper we make use of the following n-dimensional density conserving truncation of (1),

$$\dot{c}_j(t) = \frac{1}{2}\sum_{k=1}^{j-1} W_{j-k,k}(c(t)) - \sum_{k=1}^{n-j} W_{j,k}(c(t)), \qquad 1 \le j \le n, \tag{4}$$

where for $j = 1$ ($j = n$) the first (last) sum is defined to be zero. Defining $c_j^n = 0$ for $j > k$, solutions c^n to the n-dimensional system (4) can be considered to be in X_μ for any μ.

DEFINITION 3 (Admissible solution). A solution $c = (c_j)$ to (1) on $[0, T)$ is an admissible solution if there exists a sequence $n_k \to \infty$ such that solutions c^{n_k} to the n_k-dimensional truncations (4) converge weak* to $c(t)$ in X_1 for all $t \in [0, T)$.

2.2. Basic Results

We now state without proofs some preliminary results needed afterwards. The first two propositions deal with properties of the spaces X_μ.

PROPOSITION 2.1 [4, Lemma 1.2.1]. *For all $\beta > \alpha$, X_β is densely and compactly embedded in X_α and $\|c\|_\alpha \leq \|c\|_\beta$ for all $c \in X_\beta$.*

PROPOSITION 2.2 [4, Lemma 1.2.2]. *For all $\alpha < \beta < \gamma$, and all $c \in X_\gamma$, the following inequality holds:*

$$\|c\|_\beta^{\gamma-\alpha} \leq \|c\|_\alpha^{\gamma-\beta}\|c\|_\gamma^{\beta-\alpha}.$$

For integers m, n such that $1 \leq m \leq n$ define the following subsets of $\mathbb{N} \times \mathbb{N}$:

$$T^1_{m,n} = \{(j,k) : j, k \geq m, j + k \leq n\}$$

$$T^2_{m,n} = \{(j,k) : m \leq j + k \leq n, j, k \leq m - 1\}$$

$$T^3_{m,n} = \{(j,k) : 1 \leq j \leq m - 1, k \geq m, j + k \leq n\}$$

$$T^4_{m,n} = \{(j,k) : m \leq j \leq n, j + k \geq n + 1\}.$$

The next two results are basic tools for the manipulation of solutions.

PROPOSITION 2.3 [1, Lemma 2.1]. *Let (g_j) be a sequence and let m, n be integers satisfying $1 \leq m \leq n$. Let (c_j) be a solution of (4). Then*

$$\sum_{j=m}^{n} g_j \dot{c}_j = \frac{1}{2}\sum_{T^1_{m,n}} (g_{j+k} - g_j - g_k) W_{j,k}(c) + \frac{1}{2}\sum_{T^2_{m,n}} g_{j+k} W_{j,k}(c)$$
$$+ \sum_{T^3_{m,n}} (g_{j+k} - g_k) W_{j,k}(c),$$

where the sums are defined to be zero if the associated region is empty.

Setting $g_j = j$ and $m = 1$, Proposition 2.3 implies that solutions to (4) are density conserving.

PROPOSITION 2.4 [1, Lemma 3.1]. *Let (g_j), m and n be as in Proposition 2.3. Let c be a solution of (1) on $[0, T)$ for some $T > 0$. Then, for all t_1, t_2 such that $0 \leq t_1 \leq t_2 < T$ we have*

$$\sum_{j=m}^{n} g_j(c_j(t_2) - c_j(t_1)) = \int_{t_1}^{t_2} \Bigg[\frac{1}{2} \sum_{T^1_{m,n}} (g_{j+k} - g_j - g_k)$$

$$+ \frac{1}{2} \sum_{T^2_{m,n}} g_{j+k} + \sum_{T^3_{m,n}} (g_{j+k} - g_k) - \sum_{T^4_{m,n}} g_j \Bigg] W_{j,k}(c(s))\, ds,$$

where the sums are defined to be zero if the associated region is empty.

Using this proposition it is easy to see that a solution to (1) is density conserving on $[0, \tau]$ if and only if

$$\lim_{n\to\infty} \int_0^t \sum_{j=1}^{n} \sum_{k=n-j+1}^{\infty} jW_{j,k}(c(s))\, ds = 0, \tag{5}$$

for all $t \in [0, \tau]$.

Two further technical results are presented here.

PROPOSITION 2.5 [3, Lemmas 2.3 and 3.3]. *Let $\sigma \geq 1$ and $0 \leq \lambda \leq 1$ be arbitrary. Then*

1. *for $r \geq 3$ and $1 \leq j \leq H(r)$, $r^\sigma - j^\sigma - (r-j)^\sigma \geq (2^\sigma - 2)j^\sigma$.*
2. *there exists a $C_\sigma > 0$ such that, $(i+j)[(i+j)^\sigma - i^\sigma - j^\sigma] \leq C_\sigma(i^\sigma j + ij^\sigma)$, for all $i, j \geq 1$.*
3. *for all $i, j \geq 1$, $0 \leq i^\lambda + j^\lambda - (i+j)^\lambda \leq (ij)^\lambda(i+j)^{-\lambda}$.*

PROPOSITION 2.6 [1, Lemma 2.3]. *Assume* (H1) *and let c^n be a solution of Eq.* (4) *on* $[0, \infty)$. *Define*

$$\theta^n_m(t) = e^{-t} \Bigg[\sum_{j=m}^{n} jc_j(t) + 2mK_a \|c^n(0)\|_1^2 \Bigg].$$

Then $\dot\theta^n_m(t) \leq 0$, for all $m \leq n$, $t \geq 0$.

3. A Regularity Property

The main result of this section is that, under (H1), (H2), and an additional relation between γ and α, a finite moment property is valid; namely, for all μ, $n \geq 1$, $t > 0$, $\|c^n(t)\|_\mu < \infty$ can be bounded above by a function independent of n.

LEMMA 3.1. *Assume* (H1), (H2). *Let $c_0 = (c_{0j}) \in X^+$, $\|c_0\|_1 = \rho_0 > 0$, and let c^n be the solution of* (4) *with initial data $c^n(0) = (c_{01}, \ldots, c_{0n})$.*

Then, for every $\mu \geq 1 + \alpha$, there exists constants $\mathscr{C}_0$, $\mathscr{C}_1$, $\mathscr{C}_2$, depending only on α, γ, μ, and ρ_0, such that

$$\frac{d}{dt}\|c^n\|_\mu \leq \mathscr{C}_0 + \mathscr{C}_1\|c^n\|_\mu^{\alpha_1} - \mathscr{C}_2\|c^n\|_\mu^{\alpha_2}, \tag{6}$$

where $\alpha_1 = 1 + (2\alpha - 1)/(\mu - 1)$ and $\alpha_2 = 1 + \gamma/(\mu - 1)$.

Proof. The proof is similar to that of Lemma 2.4 in [3]. By Proposition 2.3 we have

$$\frac{d}{dt}\|c^n\|_\mu = \frac{1}{2}\sum_{j+k\leq n}[(j+k)^\mu - j^\mu - k^\mu]W_{j,k}(c^n). \tag{7}$$

We are going to analyse separately the contributions of the coagulation and fragmentation terms to the right-hand side of Eq. (7):

(a) Coagulation terms. Using (H1), Proposition 2.5.2, and the symmetry of the summation region, we obtain

$$\begin{aligned}\frac{1}{2}\sum_{j+k\leq n}&[(j+k)^\mu - j^\mu - k^\mu]a_{j,k}c_j^n c_k^n\\ &\leq \frac{1}{2}C_\mu K_a \sum_{j+k\leq n}\frac{j^{\mu+\alpha}k^{\alpha+1} + j^{\alpha+1}k^{\mu+\alpha}}{j+k}c_j^n c_k^n\\ &\leq C_\mu K_a \sum_{j+k\leq n} j^{\mu+\alpha-1}k^{\alpha+1}c_j^n c_k^n\\ &\leq C_\mu K_a\|c^n\|_{\alpha+1}\|c^n\|_{\mu+\alpha-1},\end{aligned}$$

and applying Proposition 2.2 we get, for $\mu \geq 1 + \alpha$,

$$C_\mu K_a\|c^n\|_{\alpha+1}\|c^n\|_{\mu+\alpha-1} \leq \mathscr{C}_1\|c^n\|_\mu^{\alpha_1},$$

where $\mathscr{C}_1 = \mathscr{C}_1(\alpha, \mu, \rho_0)$ and $\alpha_1 = 1 + (2\alpha - 1)/(\mu - 1)$ are constants.

(b) Fragmentation terms.

$$\begin{aligned}\frac{1}{2}\sum_{j+k\leq n}&[(j+k)^\mu - j^\mu - k^\mu]b_{j,k}c_{j+k}^n\\ &= \frac{1}{2}\sum_{r=2}^{n}\sum_{j=1}^{r-1}[r^\mu - j^\mu - (r-j)^\mu]b_{j,r-j}c_r^n\\ &\geq (2^{\mu-1} - 1)b_{1,1}c_2^n + 2(2^{\mu-1} - 1)\sum_{r=3}^{n}\sum_{j=1}^{H(r)} j^\mu b_{j,r-j}c_r^n\end{aligned}$$

$$\begin{aligned}
&\geq (2^{\mu-1}-1)b_{1,1}c_2^n + 2(2^{\mu-1}-1)K_f(\mu)\sum_{r=3}^{n} r^{\mu+\gamma}c_r^n \\
&\geq \kappa(\mu,\gamma)\|c^n\|_{\mu+\gamma} - \kappa(\mu,\gamma)c_1^n \\
&\geq \kappa(\mu,\gamma)\|c^n\|_{\mu+\gamma} - \mathscr{C}_0(\mu,\gamma,\rho_0),
\end{aligned}$$

where $\kappa(\mu, \gamma) \stackrel{\text{def}}{=} \min\{2(2^{\mu-1} - 1)K_f(\mu), 2^{-\mu-\gamma}(2^{\mu-1} - 1)b_{1,1}\}$ and $\mathscr{C}_0 = \mathscr{C}_0(\mu, \gamma, \rho_0) \stackrel{\text{def}}{=} \kappa(\mu, \gamma)\rho_0$. Using Proposition 2.2 and defining $\mathscr{C}_2 = \mathscr{C}_2(\mu, \gamma, \rho_0) \stackrel{\text{def}}{=} \kappa(\mu, \gamma)\rho_0^{-\gamma/(\mu-1)}$, we conclude that the contribution of the fragmentation terms can be bounded below by $\mathscr{C}_2\|c^n\|_\mu^{\alpha_2} - \mathscr{C}_0$ with $\alpha_2 = 1 + \gamma/(\mu - 1)$. This concludes the proof. ∎

LEMMA 3.2. *Suppose that the assumptions in Lemma* 3.1 *still hold and, furthermore,* $\gamma > 2\alpha - 1$. *Then* $\|c^n(t)\|_\mu < \infty$ *for every* $t > 0$, $\mu \geq 1$, *and* $n = 1, 2, \ldots$. *If* (c^n) *has a subsequence* (c^{n_k}) *weak* convergent to some* $c \in X^+$ *on* $[0, T)$, *then also* $\|c(t)\|_\mu < \infty$ *for* $0 < t < T$.

Proof. Assume $\gamma \geq 1 + \alpha$. To simplify notation let $x(t) = \|c^n(t)\|_\mu$ for $t \geq 0$, and $x_0 = \|c^n(0)\|_\mu$. Consider the differential inequality (6) in Lemma 3.1. Since $\gamma > 2\alpha - 1$ we have $\alpha_2 > \alpha_1 > 1$. Let $\beta > 0$ be the unique positive solution of $\mathscr{C}_0 + \mathscr{C}_1x^{\alpha_1} - \mathscr{C}_2x^{\alpha_2} = 0$. If $x_0 \leq \beta$ then $x(t) \leq \beta$ for all $t > 0$. Suppose now that $x_0 > \beta$. Then, for all $\nu \in (1, \alpha_2)$, we can choose $A > 0$ and a maximal $\delta > 0$ such that, for all $x > x_0 - \delta$,

$$\mathscr{C}_0 + \mathscr{C}_1x^{\alpha_1} - \mathscr{C}_2x^{\alpha_2} < -Ax^\nu. \tag{8}$$

By a maximal δ we mean the unique δ such that $x_\delta = x_0 - \delta$ satisfies

$$\mathscr{C}_0 + \mathscr{C}_1x_\delta^{\alpha_1} - \mathscr{C}_2x_\delta^{\alpha_2} = -Ax_\delta^\nu. \tag{9}$$

Let $z(t) = \zeta(t, x_0)$ be the solution of $\dot{z} = -Az^\nu$ with $z(0) = x_0$. A solution $x(t)$ of the differential inequality (6) with $x(0) = x_0$ satisfies $x(t) \leq z(t)$, provided $t > 0$ is such that (8) holds. Thus,

$$x(t) \leq \zeta(t, x_0) = \frac{1}{[x_0^{1-\nu} + (\nu - 1)At]^{1/(\nu-1)}} \leq \frac{1}{[(\nu - 1)At]^{1/(\nu-1)}},$$

and so $x(t) < \infty$ for all initial data $x_0 > \beta$ and for $t > 0$ such that (8) holds. Suppose there exists a $\tau > 0$ such that $x(\tau) = x_\delta$. Then, by (9), we conclude that $x(t) < x_\delta$ for all $t > \tau$. Hence, for all $\mu \geq 1 + \alpha$, all positive integers n, and all $t > 0$ we have $\|c^n(t)\|_\mu < \infty$ and, furthermore, for $t > 0$ sufficiently small,

$$\|c^n(t)\|_\mu \le \frac{1}{[(\nu - 1)At]^{1/(\nu-1)}} \tag{10}$$

independently of the initial data, provided A and ν are appropriately chosen. Since the bound in the right-hand side of (10) is independent of n, if $c^n \xrightarrow{*} c^\infty$ for some $c^\infty \in X$, then $\|c^\infty\|_\mu < \infty$ for all $t > 0$ and $\mu \ge 1 + \alpha$, and $\|c^\infty\|_\mu$ is bounded above by the right-hand side of (10) for sufficiently small $t > 0$. The result for $\mu \in [1, 1 + \alpha)$ follows by Proposition 2.1. ▮

4. Integrability of Higher Moments

Let $I_\mu(t) \stackrel{\text{def}}{=} \int_0^t \|c(s)\|_\mu \, ds$. In this section we prove two integrability results that are of crucial importance for the rest of the paper. The first one establishes, for $\mu < 1 + \gamma$, the integrability of the μ-moments of functions that are weak* limits of subsequences of c^n. This result is used in the proof of the existence of admissible solutions (Theorem 5.1). In the second we prove that the same integrability property is valid for all solutions of (1), which is needed for the uniqueness result.

Theorem 4.1. *Assume* (H1), (H2), *and* $\gamma > \alpha$. *Let* $c \in X^+$ *be the weak* limit of a subsequence* (c^{n_k}) *of solutions to the* n_k*-truncated system* (4) *on* $[0, T)$, $T > 0$, *with initial data* $c_0 \in X^+$. *Then, for every* $t \in [0, T)$ *and every* $\varepsilon > 0$, $I_{1+\gamma-\varepsilon}(t) < \infty$.

Proof. Generally $c_0 \in X^+$ is not necessarily an element of X_μ for $\mu > 1$, in which case $\|c(0)\|_\mu = \infty$ and $\|c(t)\|_\mu$ can be non-integrable in a neighbourhood of the origin. Assume that $\|c(0)\|_\mu = \infty$. The proof of Lemma 3.1, and in particular (10), can be used to estimate the behaviour of $\|c(t)\|_\mu$ for t near zero. From (10) we get, for $t > 0$ small enough,

$$\|c(t)\|_\mu \le \frac{1}{[(\nu - 1)At]^{1/(\nu-1)}}; \tag{11}$$

thus, for integrability of $\|c\|_\mu$ near the origin it is sufficient to have $\nu > 2$ and, since we can take any $\nu \in (1, 1 + \gamma/(\mu - 1))$, this implies that we must have $1 + \gamma/(\mu - 1) > 2$, i.e., $\mu < 1 + \gamma$, in order to choose $\nu > 2$. Thus, for any $\mu < 1 + \gamma$ we have integrability of $\|c(t)\|_\mu$ for $t \in [0, T)$, which proves the result. ▮

Remark 4.1. In particular, Theorem 4.1 is valid if c is an admissible solution of (1). Moreover, Theorem 4.1 still holds with $I_\mu(t)$ changed to

$I^n_\mu(t) = \int_0^t \|c^n(s)\|_\mu \, ds$, and we conclude that, under the above assumptions, there exists a function $g_\mu(\cdot) \in L^1_{\mathrm{loc}}([0, T))$ such that $I^n_\mu(t) \leq \int_0^t g_\mu(s)\,ds < \infty$ for all $0 \leq t < T$ and $n \in \mathbb{N}$.

THEOREM 4.2. *Assume* (H1), (H2), *and* $\gamma > \alpha$. *Let* c *be any solution of Eq.* (1) *on* $[0, T)$, $T > 0$. *Then, for every* $t \in [0, T)$ *and every* $\varepsilon > 0$, $I_{1+\gamma-\varepsilon}(t) < \infty$.

This theorem was proved in [3] for coagulation rates of the type $a_{j,k} \leq K_a(j^\alpha + k^\alpha)$ and the present proof follows the one in that paper closely. The proof consists in a repeated application of the following lemma

LEMMA 4.1. *Assume* (H1), (H2), *and* $\gamma > \alpha$. *Let* c *be any solution of Eq.* (1) *on* $[0, T)$, $T > 0$, *and assume* $I_\sigma(t) < \infty$ *for all* $t \in [0, T)$ *and some* $\sigma \geq 1$. *Then, for all* $t \in [0, T)$, $I_{\gamma+\lambda}(t) < \infty$, *where* $\lambda = \sigma - \alpha$ *if* $\sigma - \alpha < 1$, *and* $\lambda = 1 - \varepsilon$ *for any* $\varepsilon > 0$ *otherwise.*

Proof. Let $\lambda < 1$ be as defined above. Due to the type of bound provided by the strong fragmentation hypothesis (H2) it is natural to consider the equation for the partial λ-moment in order to obtain information on the nature of $I_{\gamma+\lambda}(t)$. Using Proposition 2.4 we have

$$\sum_{j=1}^n j^\lambda (c_j(t) - c_j(0))$$
$$+ \int_0^t \left[\frac{1}{2} \sum_{T^1_{1,n}} (j^\lambda + k^\lambda - (j+k)^\lambda) + \sum_{T^4_{1,n}} j^\lambda \right] a_{j,k} c_j(s) c_k(s)\, ds$$
$$= \int_0^t \left[\frac{1}{2} \sum_{T^1_{1,n}} (j^\lambda + k^\lambda - (j+k)^\lambda) + \sum_{T^4_{1,n}} j^\lambda \right] b_{j,k} c_{j+k}(s)\, ds.$$

The strategy of the proof is to establish the convergence of the left-hand side as $n \to \infty$ and, with this information and the strong fragmentation hypothesis, to get the convergence of $I_{\gamma+\lambda}(t)$.

Since $\lambda < 1$ and $c \in X^+$ the first term in the left-hand side is bounded independently of n and is convergent as $n \to \infty$. For the coagulation contribution to the left-hand side we have

$$\int_0^t \sum_{j,k=1}^n j^\lambda a_{j,k} c_j(s) c_k(s)\, ds \leq K_a \int_0^t \|c(s)\|_\sigma \|c(s)\|_1 \, ds$$
$$\leq K_a I_\sigma(t) \max_{s \in [0,t]} \|c(s)\|_1 < \infty$$

and, using Proposition 2.5,

$$\begin{aligned}\int_0^t \sum_{j,k=1}^{n} (j^\lambda + k^\lambda - (j+k)^\lambda) a_{j,k} c_j(s) c_k(s)\, ds &\le K_a \int_0^t \sum_{j,k=1}^{n} \left(\frac{jk}{j+k}\right)^\lambda (jk)^\alpha c_j(s) c_k(s)\, ds \\ &\le \frac{1}{2} K_a \int_0^t \sum_{j,k=1}^{n} (jk)^{\lambda+\alpha} \left(\frac{1}{j^\lambda} + \frac{1}{k^\lambda}\right) c_j(s) c_k(s)\, ds \\ &\le \frac{1}{2} K_a \int_0^t \sum_{j,k=1}^{n} (j^\alpha k^\sigma + j^\sigma k^\alpha) c_j(s) c_k(s)\, ds \\ &= K_a \int_0^t \sum_{j=1}^{n} j^\sigma c_j(s) \sum_{k=1}^{n} k^\alpha c_k(s)\, ds \\ &\le K_a \int_0^t \|c(s)\|_\sigma \|c(s)\|_1\, ds \\ &\le K_a I_\sigma(t) \max_{s\in[0,t]} \|c(s)\|_1 < \infty.\end{aligned}$$

Hence, the left-hand side converges as $n \to \infty$ and so does the right-hand side. Since all the terms in it are positive this implies that

$$\int_0^t \sum_{j,k=1}^{\infty} (j^\lambda + k^\lambda - (j+k)^\lambda) b_{j,k} c_{j+k}(s)\, ds < \infty. \tag{12}$$

Now the proof follows as in [3]:

$$\sum_{j,k=1}^{\infty} (j^\lambda + k^\lambda - (j+k)^\lambda) b_{j,k} c_{j+k} \ge \sum_{r=3}^{\infty} B_r c_r, \tag{13}$$

where

$$\begin{aligned}B_r &= \sum_{j=1}^{r-1} (j^\lambda + (r-j)^\lambda - r^\lambda) b_{r-j,j} \\ &\ge 2 \sum_{j=1}^{H(r)} (j^\lambda + (r-j)^\lambda - r^\lambda) b_{r-j,j} \\ &\ge 2C'_\lambda \sum_{j=1}^{H(r)} j^\lambda b_{r-j,j} \\ &\ge 2C'_\lambda K_f(\mu) r^{\gamma+\lambda}\end{aligned}$$

with C'_λ is a positive constant such that $j^\lambda + (r-j)^\lambda - r^\lambda \ge C'_\lambda j^\lambda$ (whose existence can be asserted by a proof similar to that of Proposition 2.5).

Thus, substituting this estimate for B_r into (13) and (12), we obtain $I_{\gamma+\lambda}(t) < \infty$. ▌

Proof of Theorem 4.2. Let s be the smallest positive integer satisfying $s(\gamma - \alpha) \geq \alpha$.

(a) Suppose $s > 1$. Then $s - 1$ is a positive integer and

$$0 < (s-1)(\gamma - \alpha) < \alpha.$$

Defining $\sigma_j \overset{\text{def}}{=} 1 + (j-1)(\gamma - \alpha)$, we have

$$1 = \sigma_1 < \sigma_2 < \cdots < \sigma_{s-1} < \sigma_s < 1 + \alpha.$$

Applying Lemma 4.1 s times, starting with $\sigma = \sigma_1 = 1$, gives $I_{\sigma_{s+1}} < \infty$, and one further application of Lemma 4.1 with $\sigma = \sigma_{s+1} = 1 + s(\gamma - \alpha) > 1$ gives the result.

(b) Suppose $s = 1$. Then two applications of Lemma 4.1, starting with $\sigma = 1$, gives the result. ▌

5. Existence and Density Conservation of Admissible Solutions

We start by proving existence of admissible solutions.

THEOREM 5.1. *Assume* (H1), (H2), *and* $\gamma > \alpha$. *Let* $c_0 = (c_{0j}) \in X^+$. *Then there exists a solution* c *of Eq.* (1) *on* $[0, \infty)$ *with* $c(0) = c_0$.

Proof. Let $c^n(0) = (c_{01}, c_{02}, \ldots, c_{0n})$, and c^n be the unique solution of Eq. (4) on $[0, \infty)$ with initial data $c^n(0)$. c^n is nonnegative and conserves density, $\sum_{j=1}^n jc_j^n(t) = \sum_{j=1}^n jc_j^n(0)$ for every $t \geq 0$ [1]. Regard $c^n(t)$ as an element of X^+ by setting $c_j^n(t) \equiv 0$ for $j > n$. We have $\|c^n(t)\|_1 \leq \|c_0\|_1$ and $0 \leq c_j^n(t) \leq j^{-1}\|c_0\|_1$. Using Proposition 2.6 and the density conservation of c^n, the functions θ_m^n, $n \geq m$, are of uniform bounded variation on $[0, \infty)$ for each fixed m. Applying Helly's theorem [5, p. 370–374], there exists a subsequence, still denoted θ_m^n, and a function of bounded variation θ_m such that $\theta_m^n(t) \to \theta_m(t)$ as $n \to \infty$, for each $t \geq 0$. Writing

$$c_j^n(t) = j^{-1}e^t(\theta_j^n(t) - \theta_{j+1}^n(t)) + 2j^{-1}K_a\|c^n(0)\|_1^2,$$

it follows that there exist a subsequence, still denoted c^n, and functions of bounded variation on compact subsets of $[0, \infty)$, $c_j\colon \mathbb{R}_0^+ \to \mathbb{R}$, such that, $c_j^n(t) \to c_j(t)$ as $n \to \infty$ for all $t \geq 0$ and positive integers j, and $c_j(t) \geq 0$, $\|c(t)\|_1 \leq \|c_0\|_1$. The proof is complete once we establish that $c = (c_j)$ is a solution of the coagulation–fragmentation equations. I.e., starting with the integral version of (4), namely

$$
\begin{aligned}
c_j^n(t) - c_j^n(0) = &\int_0^t \left[\frac{1}{2}\sum_{k=1}^{j-1} a_{j-k,k}c_{j-k}^n(s)c_k^n(s) - \sum_{k=1}^{n-j} a_{j,k}c_j^n(s)c_k^n(s)\right] ds \\
&- \int_0^t \left[\frac{1}{2}\sum_{k=1}^{j-1} b_{j-k,k}c_j^n(s) - \sum_{k=1}^{n-j} b_{j,k}c_{j+k}^n(s)\right] ds,
\end{aligned} \tag{14}
$$

we should be able to pass to the limit $n \to \infty$ and obtain the integral version of (1) as the equation satisfied by the limit function c.

(a) Convergence of the left-hand side of Eq. (14). It is clear from the definition of $c_j^n(0)$ and the results above that the left-hand side of (14) converges to $c_j(t) - c_{0j}$ as $n \to \infty$.

(b) Convergence of the right-hand side of Eq. (14); coagulation terms. The limit of the first sum is easily obtained; since the sum has a fixed and finite number of terms and $c_k^n(t) \to c_k(t)$ as $n \to \infty$ for all t and k, we get

$$
\sum_{k=1}^{j-1} a_{j-k,k}c_{j-k}^n(s)c_k^n(s) \to \sum_{k=1}^{j-1} a_{j-k,k}c_{j-k}(s)c_k(s).
$$

As the function in the left-hand side is uniformly bounded in n and s,

$$
\sum_{k=1}^{j-1} a_{j-k,k}c_{j-k}^n(s)c_k^n(s) \le K_a \sum_{k=1}^{j-1} (j-k)^\alpha c_{j-k}^n(s)k^\alpha c_k^n(s) \le K_a\|c_0\|_1^2,
$$

and the result follows by the dominated convergence theorem.

For the second sum we have

$$
\begin{aligned}
&\left|\sum_{k=1}^{n-j} a_{j,k}c_j^n c_k^n - \sum_{k=1}^{\infty} a_{j,k}c_j c_k\right| \\
&\quad \le |c_j^n - c_j| \sum_{k=1}^{n-j} a_{j,k}c_k^n + c_j \left|\sum_{k=1}^{n-j} a_{j,k}c_k^n - \sum_{k=1}^{\infty} a_{j,k}c_k\right| \\
&\quad \le K_a j^\alpha \|c_0\|_1 |c_j^n - c_j| + c_j \left|\sum_{k=1}^{n-j} a_{j,k}c_k^n - \sum_{k=1}^{\infty} a_{j,k}c_k\right|.
\end{aligned}
$$

Consequently, it suffices to prove that

$$
\int_0^t \left|\sum_{k=1}^{n-j} a_{j,k}c_k^n(s) - \sum_{k=1}^{\infty} a_{j,k}c_k(s)\right| ds \to 0 \qquad \text{as } n \to \infty. \tag{15}
$$

Let m be a fixed positive integer and let n be taken sufficiently large so that $n - j > m$. Then

$$\begin{aligned}\int_0^t \left| \sum_{k=1}^{n-j} a_{j,k} c_k^n(s) - \sum_{k=1}^{\infty} a_{j,k} c_k(s) \right| ds \\ \leq K_a j^\alpha \int_0^t \sum_{k=n-j+1}^{\infty} k^\alpha c_k(s)\, ds + K_a j^\alpha \int_0^t \sum_{k=1}^{m-1} k^\alpha |c_k^n(s) - c_k(s)|\, ds \\ + K_a j^\alpha \int_0^t \sum_{k=m}^{n-j} k^\alpha |c_k^n(s) - c_k(s)|\, ds.\end{aligned} \tag{16}$$

For the first integral in the right-hand side of (16) observe that $\sum_{k=n-j+1}^{\infty} k^\alpha c_k(s) \leq \|c(s)\|_1 \leq \|c_0\|_1$ and the integrand converges pointwise to zero as $n \to \infty$, for all s and j. Hence, the integral converges to zero as $n \to \infty$.

For the second integral, again note that the integrand is uniformly bounded above by $2\|c_0\|_1$, for all s and m. Since the sum has only a finite number of terms, independently of n, each converging to zero as $n \to \infty$, we again conclude the convergence to zero of the integral as $n \to \infty$.

Finally, for the last integral the integrand is bounded above by $2\|c_0\|_1$, and, by the Fatou–Lebesgue theorem, we have

$$\begin{aligned}\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \int_0^t \sum_{k=m}^{n-j} k^\alpha |c_k^n(s) - c_k(s)|\, ds \\ \leq \int_0^t \overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \sum_{k=m}^{n-j} k^\alpha (|c_k^n(s)| + |c_k(s)|)\, ds = 0.\end{aligned}$$

Hence, for all $\varepsilon > 0$ we can choose $p(\varepsilon)$ such that for all $m > p(\varepsilon)$

$$\int_0^t \sum_{k=m}^{n-j} k^\alpha |c_k^n(s) - c_k(s)|\, ds < \varepsilon$$

for all n sufficiently large. This concludes the proof of (15).

(c) Convergence of the right-hand side of Eq. (14); fragmentation terms. The convergence of the first sum in the fragmentation terms is easy to obtain since $c_j^n(s) \to c_j(s)$ as $n \to \infty$ for all j and s, and the sum has a fixed and finite number of terms and is uniformly bounded independently of n, s:

$$\frac{1}{2} \sum_{k=1}^{j-1} b_{j-k,k} c_j^n(s) = j B_j c_j^n(s) \leq B_j \|c^n(s)\|_1 \leq B_j \|c_0\|_1$$

where $B_j \stackrel{\text{def}}{=} \frac{1}{2j} \sum_{k=1}^{j-1} b_{j-k,k}$. Thus, the dominated convergence theorem gives

$$\int_0^t \frac{1}{2} \sum_{k=1}^{j-1} b_{j-k,k} c_j^n(s)\, ds \to \int_0^t \frac{1}{2} \sum_{k=1}^{j-1} b_{j-k,k} c_j(s)\, ds$$

The last part of the proof consists in establishing that, for each fixed j, the following limit holds:

$$\int_0^t \sum_{k=1}^{n-j} b_{j,k} c_{j+k}^n(s)\, ds \to \int_0^t \sum_{k=1}^{\infty} b_{j,k} c_{j+k}(s)\, ds, \qquad \text{as } n \to \infty. \tag{17}$$

It is clear from (14) and from results in (a) and (b) that there exists a function $f_j(t)$ such that, for each j and t,

$$f_j(t) \stackrel{\text{def}}{=} \lim_{n\to\infty} \int_0^t \sum_{k=1}^{n-j} b_{j,k} c_{j+k}^n(s)\, ds.$$

We need to prove that

$$f_j(t) = \int_0^t \sum_{k=1}^{\infty} b_{j,k} c_{j+k}(s)\, ds.$$

The basic strategy is the following: if we could prove that, given $\varepsilon > 0$ arbitrary, there exists a positive integer p such that for any integer $m \geq p$ is valid,

$$\int_0^t \sum_{k=m}^{n-j} b_{j,k} c_{j+k}^n(s)\, ds < \varepsilon, \qquad \text{for all } n \text{ sufficiently large}, \tag{18}$$

then we could write

$$\left| \int_0^t \sum_{k=1}^{n-j} b_{j,k} c_{j+k}^n(s)\, ds - \int_0^t \sum_{k=1}^{m-1} b_{j,k} c_{j+k}^n(s)\, ds \right| < \varepsilon$$

and, since the second sum has a finite number of terms, $m - 1$, independently of n, and is bounded above by $\tilde{B}_{m,j}\|c_0\|_1$, where $\tilde{B}_{m,j} \stackrel{\text{def}}{=} \sum_{k=1}^{m-1} b_{j,k}$, we can pass to the limit $n \to \infty$ and obtain, for all fixed m sufficiently large and all j and t,

$$\left| \lim_{n\to\infty} \int_0^t \sum_{k=1}^{n-j} b_{j,k} c_{j+k}^n(s)\, ds - \int_0^t \sum_{k=1}^{m-1} b_{j,k} c_{j+k}(s)\, ds \right| < \varepsilon$$

and, hence, (17) holds.

In order to obtain (18) we use the equation for the first moment of c^n. By Proposition 2.3 with $g_j = j$ and $n > 2m$ we have

$$\sum_{j=m}^{n} j\dot{c}_j^n = \left[\frac{1}{2}\sum_{T_{m,n}^2} (j+k) + \sum_{T_{m,n}^3} j\right] a_{j,k} c_j^n c_k^n - \left[\frac{1}{2}\sum_{T_{m,n}^2} (j+k) + \sum_{T_{m,n}^3} j\right] b_{j,k} c_{j+k}^n$$

and, since

$$\begin{aligned}\left[\frac{1}{2}\sum_{T_{m,n}^2} (j+k) + \sum_{T_{m,n}^3} j\right] b_{j,k} c_{j+k}^n &\geq \sum_{T_{m,n}^3} j b_{j,k} c_{j+k}^n \\ &\geq \sum_{k=m}^{n-j} b_{j,k} c_{j+k}^n, \qquad \text{for all } j \leq m-1,\end{aligned}$$

we can write

$$\sum_{j=m}^{n} j c_j^n(t) + \int_0^t \sum_{k=m}^{n-j} b_{j,k} c_{j+k}^n(s)\, ds \tag{19}$$

$$\leq \sum_{j=m}^{n} j c_j(0) + \int_0^t \left[\frac{1}{2}\sum_{T_{m,n}^2} (j+k) + \sum_{T_{m,n}^3} j\right] a_{j,k} c_j^n(s) c_k^n(s)\, ds. \tag{20}$$

For the first term in (20) we clearly have that, for all $\varepsilon > 0$, there exists a p_1 such that the sum is less the $\varepsilon/3$ for all $m > p_1$ and all $n > 2m$.

For the term with the sum over $T_{m,n}^3$ observe that, by (10), Theorem 4.1, and Remark 4.1, there exists a function $g_{1+\alpha}(\cdot) \in L_{\text{loc}}^1([0, T))$ independent of m and n, such that

$$\sum_{j=1}^{m} \sum_{k=m}^{n} j^{1+\alpha} c_j^n(s) k^\alpha c_k^n(s) \leq \|c_0\|_1 g_{1+\alpha}(s). \tag{21}$$

Thus, by the Fatou–Lebesgue theorem,

$$\varlimsup_{n\to\infty}\int_0^t \sum_{j=1}^{m}\sum_{k=m}^{n} j^{1+\alpha}c_j^n(s)k^\alpha c_k^n(s)\,ds$$
$$\le \int_0^t \varlimsup_{n\to\infty}\sum_{j=1}^{m}\sum_{k=m}^{n} j^{1+\alpha}c_j^n(s)k^\alpha c_k^n(s)\,ds$$
$$= \int_0^t \left(\sum_{j=1}^{m} j^{1+\alpha}c_j(s)\right)\sum_{k=m}^{n}\varlimsup_{n\to\infty} k^\alpha c_k^n(s)\,ds.$$

By (21) we can apply the Fatou–Lebesgue theorem once more to obtain

$$\varlimsup_{m\to\infty}\varlimsup_{n\to\infty}\int_0^t \sum_{j=1}^{m}\sum_{k=m}^{n} j^{1+\alpha}c_j^n(s)k^\alpha c_k^n(s)\,ds$$
$$\le \int_0^t \|c(s)\|_{1+\alpha}\varlimsup_{m\to\infty}\varlimsup_{n\to\infty}\sum_{k=m}^{n} k^\alpha c_k^n(s)\,ds = 0,$$

since $\sum_{k=m}^{n} k^\alpha c_k^n(s) \le \|c_0\|_1$ for all m, n and s.

Thus, for all $\varepsilon > 0$ there is a $p_2 \in \mathbb{N}$ such that, for all $m > p_2$,

$$\int_0^t \sum_{T_{m,n}^3} ja_{j,k}c_j^n(s)c_k^n(s)\,ds < \frac{\varepsilon}{3}$$

for all $n > 2m$ sufficiently large.

Finally, for the term with the sum over $T_{m,n}^2$ in (20) we have

$$\sum_{j=1}^{m-1}\sum_{k=m-j}^{m-1} j^{1+\alpha}c_j^n(s)k^\alpha c_k^n(s) \le \|c_0\|_1 g_{1+\alpha}(s),$$

and, since the sum has a finite number of terms, for all n, we conclude that

$$\lim_{n\to\infty}\int_0^t \sum_{j=1}^{m-1}\sum_{k=m-j}^{m-1} j^{1+\alpha}c_j^n(s)k^\alpha c_k^n(s)\,ds = \int_0^t \sum_{j=1}^{m-1}\sum_{k=m-j}^{m-1} j^{1+\alpha}c_j(s)k^\alpha c_k(s)\,ds.$$

Moreover, by the Fatou–Lebesgue theorem and the non-negativity of solutions,

$$\varlimsup_{m\to\infty}\int_0^t \sum_{j=1}^{m-1}\sum_{k=m-j}^{m-1} j^{1+\alpha}c_j(s)k^\alpha c_k(s)\,ds = 0,$$

and so, also in this case we can, given $\varepsilon > 0$, choose p_3 such that the integral is less then $\varepsilon/3$ for all $m > p_3$ and all $n > 2m$ sufficiently large.

Hence, if $m > p = \max\{p_1, p_2, p_3\}$ we conclude that (19) is less then ε, proving (18).

Hence, we conclude that c satisfies the integral version of (1) and so each c_j is an absolutely continuous function on $[0, t]$ for all $t < T$, and so c is a solution of (1). ∎

THEOREM 5.2. *With the assumptions of Theorem* 5.1, *admissible solutions conserve density.*

Proof. From the proof of Theorem 5.1 we already know that $\|c(t)\|_1 \leq \|c_0\|_1$ for all $t \geq 0$. Since, by Theorem 4.1, we have

$$\lim_{n\to\infty} \int_0^t \sum_{j=1}^{n} \sum_{k=n-j+1}^{\infty} jW_{j,k}(c(s))\, ds = -\lim_{n\to\infty} \int_0^t \sum_{j=1}^{n} \sum_{k=n-j+1}^{\infty} jb_{j,k}c_{j+k}(s)\, ds \leq 0,$$

we conclude that

$$\|c(t)\|_1 = \|c_0\|_1 - \lim_{n\to\infty} \int_0^t \sum_{j=1}^{n} \sum_{k=n-j+1}^{\infty} jW_{j,k}(c(s))\, ds \geq \|c_0\|_1,$$

proving the result. ∎

6. UNIQUENESS OF DENSITY CONSERVING SOLUTIONS

THEOREM 6.1. *Assume* (H1), (H2), *and* $\gamma > \alpha$. *Then, for any initial data* $c_0 \in X^+$, *there exists only one density conserving solution of Eq.* (1).

Remark 6.1. By Theorem 5.2 we already know that there is at least one density conserving solution, the admissible solution, so Theorem 6.1 implies the uniqueness of admissible solutions and so the sequence (c^n) of approximating solutions, and not only a subsequence, converge to c, the admissible solution of Eq. (1) (see [2]).

Proof of Theorem 6.1. The proof is similar to the correspondent uniqueness proofs in [1, 3]. Let $c = (c_j)$ and $d = (d_j)$ be two density conserving solutions with initial data $c_0 \neq 0$. Let $x = (x_j) = c - d$ and $\psi(t) = \sum_{j=1}^{\infty} j|x_j(t)|$. Then, by Proposition 2.4,

$$\sum_{j=1}^{n} j|x_j(t)| = \int_0^t [U_n(s) + V_n(s)]\, ds, \tag{22}$$

where

$$U_n(s) = \frac{1}{2} \sum_{T^1_{1,n}} (g_{j+k} - g_j - g_k)\Delta_{j,k}(s)$$

$$V_n(s) = -\sum_{T^4_{1,n}} g_j \Delta_{j,k}(s)$$

with

$$\begin{aligned} g_j &= j \operatorname{sgn}(x_j) \\ \Delta_{j,k}(s) &= W_{j,k}(c(s)) - W_{j,k}(d(s)) \\ &= a_{j,k}(c_j(s)x_k(s) + d_k(s)x_j(s)) - b_{j,k}(s)x_{j+k}(s) \end{aligned}$$

and $T^1_{1,n}$, $T^4_{1,n}$ as in Proposition 2.4.

We are going to estimate the right-hand side of (22) as $n \to \infty$. Using the inequalities

$$-(g_{j+k} - g_j - g_k)x_{j+k} \le 0, \qquad (g_{j+k} - g_j - g_k)x_k \le 2j|x_k|,$$

we have

$$\begin{aligned} &\int_0^t U_n(s)\, ds \\ &\quad \le K_a \int_0^t \sum_{T^1_{1,n}} j^{1+\alpha}k^\alpha c_j(s)|x_k(s)|\, ds + K_a \int_0^t \sum_{T^1_{1,n}} j^{1+\alpha}k^\alpha d_j(s)|x_k(s)|\, ds \end{aligned}$$

and

$$\begin{aligned} &K_a \int_0^t \sum_{T^1_{1,n}} j^{1+\alpha}k^\alpha c_j(s)|x_k(s)|\, ds \\ &\quad \le K_a \int_0^t \sum_{j,k=1}^\infty j^{1+\alpha}k^\alpha c_j(s)|x_k(s)|\, ds = K_a \int_0^t \|c(s)\|_{1+\alpha} \sum_{k=1}^\infty k^\alpha |x_k(s)|\, ds \\ &\quad \le K_a \int_0^t \|c(s)\|_{1+\alpha}\psi(s)\, ds = \int_0^t \varphi_c(s)\psi(s)\, ds, \end{aligned}$$

where $\varphi_c(s) \stackrel{\text{def}}{=} K_a\|c(s)\|_{1+\alpha}$. Similarly, defining $\varphi_d(s) \stackrel{\text{def}}{=} K_a\|d(s)\|_{1+\alpha}$, we get

$$K_a \int_0^t \sum_{T^1_{1,n}} j^{1+\alpha} k^\alpha d_j(s) |x_k(s)| \, ds \le \int_0^t \varphi_d(s) \psi(s) \, ds.$$

Thus

$$\int_0^t U_n(s) \, ds \le \int_0^t \varphi(s) \psi(s) \, ds,$$

where $\varphi(s) = \varphi_c(s) + \varphi_d(s)$ is integrable by Theorem 4.2.

For the V_n term in (22) we have

$$\begin{aligned} \left| \int_0^t V_n(s) \, ds \right| &= \left| \int_0^t \sum_{T^4_{1,n}} g_j \Delta_{j,k}(s) \, ds \right| \\ &\le \int_0^t \sum_{T^4_{1,n}} |g_j a_{j,k} (c_j(s) c_k(s) - d_j(s) \, d_k(s))| \, ds \\ &\quad + \int_0^t \sum_{T^4_{1,n}} |g_j b_{j,k} x_{j+k}(s)| \, ds. \end{aligned}$$

Now the first term in the right-hand side is bounded above by

$$K_a \int_0^t \sum_{T^4_{1,n}} j^{1+\alpha} c_j(s) k^\alpha c_k(s) \, ds + K_a \int_0^t \sum_{T^4_{1,n}} j^{1+\alpha} \, d_j(s) k^\alpha \, d_k(s) \, ds$$

which converges to zero as $n \to \infty$ since the $(1 + \alpha)$-moments of all solutions are integrable by Theorem 4.2. Since c and d are density conserving solutions we have, by (5),

$$\lim_{n \to \infty} \int_0^t \sum_{T^4_{1,n}} j W_{j,k}(c(s)) \, ds = 0 = \lim_{n \to \infty} \int_0^t \sum_{T^4_{1,n}} j W_{j,k}(d(s)) \, ds.$$

This implies

$$\lim_{n \to \infty} \int_0^t \sum_{T^4_{1,n}} j a_{j,k} c_j(s) \, ds = \lim_{n \to \infty} \int_0^t \sum_{T^4_{1,n}} j b_{j,k} c_{j+k}(s) \, ds$$

and it follows from what was previously done that the limit in the left-hand side is zero and analogously for the corresponding limit for the solution d. Thus, observing that

$$\begin{aligned}
&\left| \int_0^t \sum_{T^4_{1,n}} g_j b_{j,k} x_{j+k}(s)\, ds \right| \\
&\quad \le \int_0^t \sum_{T^4_{1,n}} j b_{j,k} |x_{j+k}(s)|\, ds \\
&\quad \le \int_0^t \sum_{T^4_{1,n}} j b_{j,k} c_{j+k}(s)\, ds + \int_0^t \sum_{T^4_{1,n}} j b_{j,k}\, d_{j+k}(s)\, ds,
\end{aligned}$$

we obtain

$$\int_0^t V_n(s)\, ds \to 0 \qquad \text{as } n \to \infty.$$

Passing to the limit in (22) we get

$$\psi(t) \le \int_0^t \varphi(s)\psi(s)\, ds$$

and then, by Gronwall's inequality, $\psi(t) \equiv 0$, which implies uniqueness. ∎

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