Results About the Free Kawasaki Dynamics of Continuous Particle Systems in Infinite Volume: Long-time Asymptotics and Hydrodynamic Limit

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In memory of our colleague and friend São

(Maria Conceição Carvalho)

Abstract An infinite particle system of independent jumping particles in infinite volume is considered. Their construction is recalled, further properties are derived, the relation with hierarchical equations, Poissonian analysis, and second quantization are discussed. The hydrodynamic limit for a general initial distribution satisfying a mixing condition is derived. The long-time asymptotics is computed under an

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extra assumption. The relation with constructions based on infinite volume limits is discussed.

Keywords Infinite particle systems · Kawasaki dynamics · Hydrodynamic limit · Long-time asymptotics

1 Introduction

Here we study systems with a large number of particles where the dynamics consists of particles jumping at random times. To gain insights to these systems, one studies them at different scales which are relevant to the system, for us: the scale on which the individual movement is happening, the typical distance between particles, the scale on which we observe the system and finally the size of the system. The key feature of this paper is that we consider the regime where the system size is much larger than all the other scales and try to evaluate what kind of challenges that may bring. The most commonly studied case is where the scale on which the system is observed and the system size are of the same order, which simplifies the situation substantially. Most considerations in this paper become almost trivial in this case, see Sect. 5.3 for a detailed discussion. We are presenting hitherto unpublished results regarding infinite systems, to do so with their proofs would exceed the present framework. The latter can be found in (arXiv:0912.1312).

In this paper, we will walk through mathematical and some conceptual challenges which appear when the system size is much larger than the other scales. For simplicity we model this through an infinite system, as it means that all results derived are automatically independent of the system size. For example, relaxation rates, which depend on the system size, may easily give rise to so slow relaxation of large systems failing to capture the physically relevant relaxation mechanism. Infinite systems are very challenging, see [5, 49] for Hamiltonian dynamics and [13, 50] for gradient diffusions. Despite serious efforts in both cases, our understanding is far from complete and rigorous results are sparse. In order to get a better understanding of the challenges one has to face treating infinite systems, we will consider a non-interacting system where each particle follows a jump process. This allows us to have a very explicit representation of the dynamics and to treat the following three properties in great detail: construction of dynamics (Sects. 3 and 4), long-time asymptotics (Sects. 5.1 and 6.1), and hydrodynamic limits (Sects. 5.2 and 6.2).

The study of the dynamics of an infinite particle system without interaction can be reduced to the analysis of a type of one-particle dynamics. However, this one-particle dynamics being an effective description of the infinite particle system dynamics has to be considered with initial "distributions" which are not integrable, but merely bounded, as they actually describe the particle density of the infinite particle system. To stress this different interpretation, we will call it a pseudo-one-particle system in the following. Each constant initial "distribution" gives rise to a distinct invariant measure of the infinite particle system, in our case, Poisson point random fields with constant intensity. These are all mutually singular probability measures, and hence the system cannot be ergodic. If we want to understand the dynamics of the infinite particle system in full, we are forced to work with initial distributions which are singular towards each other. In this paper, we consider increasingly more general initial distributions corresponding to different physical realms. In Sect. 4 we consider system in equilibrium that is point random fields with density with respect to a Poisson point random field with constant intensity, in Sect. 5 locally equilibrium, that is Poisson point random fields with non-constant intensity and in Sect. 6 non-equilibrium, that is general point random fields with some moment conditions.

The reader may wonder whether, in the free case, not everything is totally obvious. We try to tackle all difficulties using the most straightforward explicit techniques to make the challenges transparent. All these difficulties are innate for large system size, see Sect. 5.3. Though most of the techniques will not extend to the interacting case, the challenges should be maintained if not increased.

Let us give a more technical introduction into the paper. In Sect. 2 we describe the generator of the infinite particle dynamics, the state space and associated notions. The dynamic considered here is a version in the continuum of the well established Kawasaki dynamics for lattice gas systems. These are random evolutions of particle systems in which individual particles jump in the space with rates leading to a Gibbs state in the continuum as an invariant measure [2, 17, 31]. Having the particle number as a conserved quantity these dynamics give rise to a continuous family of invariant measures which are important to study the so-called hydrodynamic limits, see e.g. [6, 22]. Continuous versions of Glauber dynamics were introduced in [3, 14, 16, 24, 27, 31, 43]. In Sect. 2.2 we introduce and recall properties of the associated (pseudo)-one-particle operator.

In Sect. 3 we discuss different ways to construct the infinite particle dynamics. In Sect. 3.1 we recall a probabilistic approach worked out in [32]. The free infinite particle dynamics is just the product of countably many copies of the one-particle dynamics. However, to consider this product as a stochastic movement in the configuration space as further step, one has to show that a.s. in any finite time interval only finitely many jump events are visible in a bounded observation window, i.e., the number of particles in the window stays finite and only finitely many particles pass the window. This does not hold for any initial configuration and the processes can only be started in a restricted set of initial configurations or in initial distributions supported therein. We add to the results of [32] the description of the path-space measure corresponding to the process.

This construction method cannot be extended to interacting systems. A probabilistic construction writing the system as a limit of finite particle systems will require a quite detailed control, see e.g. [13]. In Sect. 3.2 we present an analytic approach to construct the semi-group which may also work for interacting systems. The approach is based on writing the dynamics in terms of the system of correlation functions, see [2] for such a construction in the interacting case. If one only starts in probability measure with a density with respect to an invariant measure, that is in the equilibrium case, techniques from second quantisation become available, cf. Sect. 4.

In Sect. 4.1 we give a description of the dynamics in terms of Fock spaces which also allow a comparison with non-relativistic quantum fields type descriptions. In the case of symmetric dynamics, the powerful methods of Dirichlet forms may be applied, in particular, existence can be shown even for general interacting systems, cf. [31] and references therein. In the case of the free Kawasaki process, one can apply in addition second quantization techniques, which give a full description of the L^2-theory, also for non-symmetric jump rates, cf. Sect. 4.2.

In Sect. 5 we consider the case in which one starts the process in a Poisson point random field with non-constant intensity, that is the system is in local equilibrium. The path space measure is then a Poisson point random field on path space. L^p techniques break down already in this case and hence Dirichlet form techniques cannot be applied, see Remark 6. In Sect. 5.1 we consider the long-time asymptotics of the one-dimensional distribution in time and show that it converges to an invariant measure, that is a Poisson random field with constant intensity. The constant is the arithmetic mean of the initial intensity, see Definition 1. This is easily shown in Proposition 3 under the assumption that the Fourier transform of the intensity z is a signed measure. For such z the arithmetic mean exists, see Corollary 1, but not any bounded non-negative function z for which the arithmetic mean exists and the time asymptotics converges has a signed-measure as Fourier transform, see Remark 9. There also exist bounded non-negative functions for which the arithmetic mean does not exist, see Remark 8. In order to establish a general result, one needs better analytic control of the behaviour of z at infinity.

In Sect. 5.2 we scale space and time simultaneously to obtain a macroscopic description in terms of partial differential equation. When the Fourier transform of the activity is a signed measure, the proper choice of a scale and the computation of the limit is direct. For general bounded activities, one has to control higher derivatives, see Proposition 6.

In Sect. 5.3, we compare the results for systems where the system size is of the same order as the scale on which one observes the system and we derive a representation of the continuous Fourier transform in order to see the technical difference with the previous results. The finite system size induces a spectral gap of size L^{-2} which eliminates all regularity requirements on the jump rate and the activity z. We also show that for a mildly larger system size this effect essentially maintains. In [8] the authors showed the convergence to the hydrodynamic limit, with much more sophisticated techniques, uniformly in time for an infinite particle system but of independent Brownian particles. We also discuss why in the infinite volume one cannot give a comparison result between the free Kawasaki dynamics and the system of independent Brownian particles.

The results described above about long-time asymptotics and hydrodynamic limits are extended to general initial distributions in Sect. 6. We require some kind of mixing property for the initial distributions, the so-called decay of correlation, which we formulate in terms of the cumulants (or Ursell functions). For the hydrodynamic limit, we have to require in addition that the first correlation function of the initial measure converges reasonably under the scaling.

We prove in Sect. 6.3 that the conditions on the initial distributions required above are fulfilled, in particular, by Gibbs measures in the high-temperature low-activity regime.

This article is a testimony to the enthusiasm, deep scientific knowledge and commitment of our collaborator, co-author and friend Yuri Kondratiev. He passed away on the 5th of September of 2023.

2 Kawasaki Dynamics

2.1 The Generator

The configuration space $\Gamma := \Gamma_{\mathbb{R}^d}$ over $\mathbb{R}^d$, $d \in \mathbb{N}$, is defined as the set of all locally finite subsets of $\mathbb{R}^d$,

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d : |\gamma_\Lambda| < \infty \text{ for every compact } \Lambda \subset \mathbb{R}^d \right\},$$

where $|\cdot|$ denotes the cardinality of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. As usual we identify each $\gamma \in \Gamma$ with the non-negative Radon measure $\sum_{x\in\gamma} \delta_x \in \mathscr{M}(\mathbb{R}^d)$, where δ_x is the Dirac measure with mass at x, $\sum_{x\in\emptyset} \delta_x$ is, by definition, the zero measure, and $\mathscr{M}(\mathbb{R}^d)$ denotes the space of all non-negative Radon measures on the Borel σ-algebra $\mathscr{B}(\mathbb{R}^d)$. This identification allows to endow Γ with the topology induced by the vague topology on $\mathscr{M}(\mathbb{R}^d)$, i.e., the weakest topology on Γ with respect to which all mappings

$$\Gamma \ni \gamma \longmapsto \langle f, \gamma \rangle := \int_{\mathbb{R}^d} \gamma(dx)\, f(x) = \sum_{x\in\gamma} f(x), \quad f \in C_c(\mathbb{R}^d),$$

are continuous. Here $C_c(\mathbb{R}^d)$ denotes the set of all continuous functions on $\mathbb{R}^d$ with compact support. By $\mathscr{B}(\Gamma)$ we will denote the corresponding Borel σ-algebra on Γ.

Given a non-negative function $a \in L^1(\mathbb{R}^d, dx)$, the generator $L := L_a$ of the free Kawasaki dynamics for an infinite particle system is given by the informal expression

$$(LF)(\gamma) := \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy\, a(x-y)\, (F(\gamma \setminus x \cup y) - F(\gamma)). \tag{1}$$

We proceed to give a rigorous meaning to the right-hand side of (1). Let $\mathscr{O}_c(\mathbb{R}^d)$ denote the set of all open sets in $\mathbb{R}^d$ with compact closure. A $\mathscr{B}(\Gamma)$-measurable function F is called cylinder and exponentially bounded whenever there is a $\Lambda \in \mathscr{O}_c(\mathbb{R}^d)$ such that $F(\gamma) = F(\gamma_\Lambda)$ for all $\gamma \in \Gamma$, and $|F(\gamma)| \le Ce^{c|\gamma_\Lambda|}$, $\gamma \in \Gamma$, for some $C, c > 0$. For such a function F one has

$$\sum_{x\in\gamma}\int_{\mathbb{R}^d} dy\, a(x-y)\,|F(\gamma\setminus x\cup y)-F(\gamma)|$$
$$\leq \left[|\gamma_\Lambda|\int_{\mathbb{R}^d} dy\, a(y)+\int_\Lambda dy\sum_{x\in\gamma}a(x-y)\right]2Ce^{c(|\gamma_\Lambda|+1)}, \tag{2}$$

which is finite provided the configuration γ is an element in $\Gamma_a\subset\Gamma$,

$$\Gamma_a:=\left\{\gamma\in\Gamma : y\mapsto\sum_{x\in\gamma}a(x-y) \text{ is } L^1_{\text{loc}}(\mathbb{R}^d,dy)\right\}.$$

In this case, the sum and the integral in (1) are finite, and thus the operator L is well-defined on the space $\mathscr{F}L^0_{eb}(\Gamma_a)$ of all cylinder functions exponentially bounded on Γ restricted to Γ_a.

Concerning the set Γ_a, we note that $\mu(\Gamma_a)=1$ for any probability measure μ on Γ with first correlation function $k^{(1)}_\mu$ bounded. This follows from the fact that for each closed ball $B(n)\subset\mathbb{R}^d$ centered at 0 and radius $n\in\mathbb{N}$ we have

$$\int_\Gamma \mu(d\gamma)\int_{B(n)}dy\sum_{x\in\gamma}a(x-y)=\int_{\mathbb{R}^d}dx\,k^{(1)}_\mu(x)\int_{B(n)}dy\,a(x-y)$$
$$\leq C\|a\|_{L^1(\mathbb{R}^d,dx)}\text{vol}(B(n))$$

for any constant $C\geq|k^{(1)}_\mu|$. Here $\text{vol}(B(n))$ denotes the volume of $B(n)$ with respect to the Lebesgue measure on $\mathbb{R}^d$. Clearly, the latter implies that $\mu(\Gamma\setminus\Gamma_a)=0$.

In view of these considerations, throughout this work we shall restrict our setting to Γ_a.

Among the elements in $\mathscr{F}L^0_{eb}(\Gamma_a)$ we distinguish the functions $e_B(f)$, called Bogoliubov exponentials,

$$e_B(f,\gamma):=\prod_{x\in\gamma}(1+f(x)),\quad \gamma\in\Gamma, \tag{3}$$

for f being any bounded $\mathscr{B}(\mathbb{R}^d)$-measurable function with compact support ($f\in B_c(\mathbb{R}^d)$). An important reason to single out these functions is due to the especially simple form for the action of L on them, namely, for all $f\in B_c(\mathbb{R}^d)$ and all $\gamma\in\Gamma_a$,

$$(Le_B(f))(\gamma)=\sum_{x\in\gamma}(Af)(x)e_B(f,\gamma\setminus x), \tag{4}$$

where

$$Af(x):=\int_{\mathbb{R}^d}dy\,a(x-y)\,(f(y)-f(x)). \tag{5}$$

In the sequel we call the linear operator A the pseudo-one-particle operator, though we will explain in which way it differs from the genuine one-particle operator. Due to its special role throughout this work, its properties will be studied in more detail in the next subsection.

Given a locally integrable function $z \geq 0$, we remind that the Poisson measure π_z with intensity z is the unique probability measure on Γ for which the Laplace transform is given by

$$\int_\Gamma \pi_z(d\gamma)\, e^{\langle\varphi,\gamma\rangle} = \exp\left(\int_{\mathbb{R}^d} dx\, \left(e^{\varphi(x)} - 1\right) z(x)\right), \tag{6}$$

for all φ in the Schwartz space $\mathscr{D}(\mathbb{R}^d) := C_c^\infty(\mathbb{R}^d)$ of all infinitely differentiable functions with compact support. We recall that a measure μ on Γ_a is called infinitesimally reversible with respect to an operator L, whenever L is symmetric in $L^2(\Gamma_a, \mu)$.

Lemma 1 *Assume that a is an even function. Then for any real number $z > 0$ the Poisson measure π_z with constant intensity z is an infinitesimally reversible measure with respect to L.*

Remark 1 It is clear that the linear space spanned by the class of functions $e_B(f)$, $f \in B_c(\mathbb{R}^d)$, is in $\mathscr{F}L^0_{eb}(\Gamma_a)$. The space $\mathscr{F}L^0_{eb}(\Gamma_a)$ also contains the class of coherent states $e_{\pi_z}(f)$ corresponding to $f \in B_c(\mathbb{R}^d)$,

$$e_{\pi_z}(f) := \exp\left(-\int_{\mathbb{R}^d} dx\, z(x) f(x)\right) e_B(f), \tag{7}$$

see Sect. 4.1 for a use of this relation.

2.2 The One-particle Operator

In the sequel the pseudo-one-particle operator A introduced in (5) will play an essential role. Because of this, in this subsection we shall collect its main properties used below. We observe that in stochastic analysis the operator A is known as a relatively simple example of a bounded generator of a Markov jump process on $\mathbb{R}^d$ (see e.g. [11, Sect. 4.2]).

In terms of the usual convolution $*$ of functions, we also note that

$$Af = a * f - a^{(0)} f, \quad a^{(0)} := \int_{\mathbb{R}^d} a(x)\, dx. \tag{8}$$

Therefore, the properties of convolution of functions (namely, Young's inequality) lead straightforwardly to the L^p results stated in the next proposition. There, we also consider the real Banach spaces $B(\mathbb{R}^d)$ and $C_\infty(\mathbb{R}^d)$, respectively, of all bounded measurable functions and of all continuous functions vanishing at infinity, both with the supremum norm $\|f\|_u := \sup_{x\in\mathbb{R}^d} |f(x)|$. We recall that a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ is called sub-Markovian whenever for $0 \leq f \leq 1$ it follows $0 \leq T_t f \leq 1$ for every $t \geq 0$. If, in addition, $T_t f_n \nearrow 1$ for some sequence $f_n \nearrow 1$, then $(T_t)_{t\geq 0}$ is called Markovian. Here, for the spaces $B(\mathbb{R}^d)$ and $C_\infty(\mathbb{R}^d)$ the convergence is pointwise, and for an L^p-space the convergence is almost everywhere. A strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ is called positivity preserving, if $f \geq 0$ implies $T_t f \geq 0$ for every $t \geq 0$.

Proposition 1 *The linear operator A is a bounded operator on $L^p(\mathbb{R}^d, z dx)$, for $z > 0$ constant and $1 \leq p \leq \infty$ (on $B(\mathbb{R}^d)$ and on $C_\infty(\mathbb{R}^d)$). As a consequence, A is the generator of a uniformly continuous semigroup $(e^{tA})_{t\geq 0}$ on $L^p(\mathbb{R}^d, z dx)$, $1 \leq p \leq \infty$ (on $B(\mathbb{R}^d)$ and on $C_\infty(\mathbb{R}^d)$),*

$$e^{tA} := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}, \tag{9}$$

the sum converging in norm for every $t \geq 0$. Moreover, $(e^{tA})_{t\geq 0}$ is a positivity preserving (contraction) semigroup on each $L^p(\mathbb{R}^d, z dx)$ space, $1 \leq p \leq \infty$ (on $B(\mathbb{R}^d)$ and on $C_\infty(\mathbb{R}^d)$), Markovian on $L^p(\mathbb{R}^d, z dx)$ for $1 \leq p < \infty$.

For the proof see e.g. [11, Sect. 4.2, 21].

Due to (8), the operator A, as well as the semigroup $(e^{tA})_{t\geq 0}$, both either on a $L^p(\mathbb{R}^d, z dx)$ space, $1 \leq p < \infty$, or on $C_\infty(\mathbb{R}^d)$, may be expressed in terms of the Fourier transform,

$$\hat{f}(k) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dx\, e^{-i\langle x,k\rangle} f(x),$$

see e.g. [21].

Proposition 2 *For every function φ in the Schwartz space $\mathscr{S}(\mathbb{R}^d)$ of tempered test functions one has*

$$\widehat{A\varphi}(k) = (2\pi)^{d/2} \big(\hat{a}(k) - \hat{a}(0)\big)\, \hat{\varphi}(k), \quad k \in \mathbb{R}^d,$$

and

$$\left(e^{tA}\varphi\right)(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dk\, e^{i\langle k,x\rangle} e^{t(2\pi)^{d/2}(\hat{a}(k)-\hat{a}(0))} \hat{\varphi}(k), \quad x \in \mathbb{R}^d. \tag{10}$$

Remark 2 Since a is non-negative, for all $k \in \mathbb{R}^d$ one has

$$\mathrm{Re}(\hat{a}(k) - \hat{a}(0)) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dx\, (\cos(\langle k, x\rangle) - 1) a(x) \leq 0. \tag{11}$$

The equality in (11) holds only for $k = 0$. This follows from the fact that the set $\{x : \langle k, x\rangle = 2n\pi, n \in \mathbb{Z}\}$ has zero Lebesgue measure if and only if $k \neq 0$.

Remark 3 According to Proposition 2, the semigroup $(e^{tA})_{t\geq 0}$ is defined by a kernel μ_t, $t \geq 0$, which Fourier transform is given by

$$\hat{\mu}_t(k) = \frac{1}{(2\pi)^{d/2}} e^{t(2\pi)^{d/2}(\hat{a}(k)-\hat{a}(0))}.$$

Because a is non-negative, $\hat{a}$ and thus $\hat{\mu}_t$, $t \geq 0$, are positive definite functions. Bochner's theorem yields that each μ_t is a non-negative finite measure on $\mathbb{R}^d$. We note that the Markovian property of $(e^{tA})_{t\geq 0}$ means that each μ_t is actually a probability measure. For more details see e.g. [21].

Remark 4 It is easy to check that on the space $L^2(\mathbb{R}^d, zdx)$ the adjoint operator A^* of A is defined for all $f \in L^2(\mathbb{R}^d, zdx)$ by

$$(A^* f)(x) = \int_{\mathbb{R}^d} dy\, a(y-x)(f(y) - f(x)) + \int_{\mathbb{R}} dy\big(a(y) - a(-y)\big)\, f(x).$$

As a consequence, if a is an even function, then A is a bounded self-adjoint operator on $L^2(\mathbb{R}^d, zdx)$. In this case, it follows from (9) that the semigroup $(e^{tA})_{t\geq 0}$ is a self-adjoint contraction on $L^2(\mathbb{R}^d, zdx)$.

Remark 5 For a non-constant activity parameter z, the operator A is still bounded but, in general, the semigroup $(e^{tA})_{t\geq 0}$ is not any longer a contraction. For instance, on $\mathbb{R}$ consider $a(x) = e^{-x^2}$ and $z(x) = 1 + e^{-x^2+4x}$. A simple calculation shows that for $\varphi(x) = \pi^{-1/2}e^{-x^2}$ one has $\int_{\mathbb{R}} dx\, \varphi(x)(A\varphi)(x)z(x) > 0$, proving that the semigroup $(e^{tA})_{t\geq 0}$ cannot be a contraction in $L^2(\mathbb{R}^d, zdx)$.

3 Independent Infinite Particle Processes

3.1 A Probabilistic Approach

3.1.1 One-particle Process

As we mentioned at the beginning of Sect. 2.2, the operator A is the generator of a Markov jump process on $\mathbb{R}^d$. Following e.g. [11], we can explicitly construct this process as follows. Consider the Markov chain $(Y_k)_{k\in\mathbb{N}_0}$ on $\mathbb{R}^d$, with transition density function $\mu(x, y) = \frac{a(x-y)}{a^{(0)}}$. Let $(Z_t)_{t\geq 0}$ be a Poisson process with parameter $a^{(0)}$ independent of the Markov chain. We then define the Markov process $(X_t)_{t\geq 0}$ by $X_t := Y_{Z_t}$, $t \geq 0$. This process has A as generator and, by construction, it has *cadlag* paths in $\mathbb{R}^d$. We denote by $D([0,\infty), \mathbb{R}^d)$ the set of all *cadlag* paths from $[0,\infty)$ to $\mathbb{R}^d$ and by P^x the path-space measure corresponding to the process X starting at $x \in \mathbb{R}^d$. By E_x we denote the expectation w.r.t. this measure. Hence, for $\varphi \in \mathscr{D}(\mathbb{R}^d)$ we have that

$$E_x[\varphi(X_t)] = e^{tA}\varphi(x). \tag{12}$$

3.1.2 Construction

The process on Γ corresponding to L is the following random evolution: each particle evolves according to the above jump process, independently of the other particles, cf. Lemma 2 below. This independent infinite particle process was rigorously constructed in [32] (see also [25]). In the following we present the main results therein.

We need further assumptions on the jump kernel a to make this construction rigorous. It is sufficient for the construction done in [32] to assume that there exist an $\alpha > 2d$ and a $C > 0$ such that

$$0 \le a(y) \le \frac{C}{(1+|y|)^\alpha}, \qquad \text{for all } y \in \mathbb{R}^d. \tag{13}$$

Then [32] shows that there exists a Markov process $\big(D([0,\infty),\Theta), (\mathbf{X}_t)_{t\ge 0}, (\mathbf{P}_\gamma)_{\gamma\in\Theta}\big)$ on the set

$$\Theta := \left\{\gamma \in \Gamma \;:\; \exists m \in \mathbb{N} \text{ such that } |\gamma_{B(n)}| \le m \operatorname{vol}(B(n)), \quad \forall n \in \mathbb{N}\right\}.$$

Here $D([0,\infty),\Theta)$ denotes the set of all *cadlag* paths from $[0,\infty)$ to Θ, the process $\mathbf{X}_t : D([0,\infty),\Theta) \to \Theta$ is the canonical one, that is, $\mathbf{X}_t(\omega) := \omega(t)$, $\omega \in D([0,\infty),\Theta)$, and each $\mathbf{P}_\gamma$ is the path-space measure of the process starting at a $\gamma \in \Theta$. By $\mathbf{E}_\gamma$ we denote the expectation w.r.t. $\mathbf{P}_\gamma$.

Note, that it is impossible to let the process start from an arbitrary initial configuration $\gamma \in \Gamma$, as follows implicitly from the discussion before Theorem 2.2 in [32]. One is obliged to restrict the set of possible initial configurations, in our case to Θ. Hence we can start the process from configurations in Θ and initial distributions μ supported on Θ. The restriction that $\mu(\Theta) = 1$ is not very severe. For example, it is sufficient that for all n the random variable $|\gamma_{B(n)}|$ has all exponential moments finite with respect to μ and $|\gamma_{B(n)}|/\operatorname{vol}(B(n))$ has a μ integral uniformly bounded in n. In particular, for any probability measure μ on Γ which correlation functions $k_\mu^{(n)}$, $n \in \mathbb{N}$, fulfill the so-called Ruelle bound, i.e., there is a $C > 0$ such that $k_\mu^{(n)} \le C^n$ for every $n \in \mathbb{N}$. This holds, for instance, for Gibbs measures w.r.t. superstable, lower and upper regular potentials, cf. [48]. Trivially, it holds for Poisson measures which have as intensity a non-negative bounded measurable function. For a constant intensity this result was shown using ergodicity in [41]. However, finiteness of exponential moments or even finiteness of all moments are not necessary, one may assume instead decay of correlation type properties.

3.1.3 Starting from a Configuration and Transition Propabilities

Choosing, for a fixed initial configuration $\gamma \in \Theta$, an enumeration $\{x_n\}_{n\in\mathbb{N}}$, the infinite particle process can be described more explicitly. For each n let us consider an independent copy of the one-particle jump process $((X_t^{(n)})_{t\ge 0}, P^{x_n})$ introduced at the beginning of the section. In [32] it is shown that if one starts in a $\gamma = \{x_n\}_{n\in\mathbb{N}} \in \Theta$, then $\bigotimes_{n=1}^{\infty} P^{x_n}$-a.s. for all times the sequence $(X_t^{(n)})_{n\in\mathbb{N}}$ has neither accumulation points nor two entries of the sequence coincide and hence it can be identified with the configuration $\{X_t^{(n)}\}_{n\in\mathbb{N}} \in \Gamma$. Furthermore, it has been shown that this configuration lies in Θ. Hence $(\{X_t^{(n)}\}_{n\in\mathbb{N}})_{t\geq 0}$ is the corresponding process on the configuration space Θ and the path-space measure is the "symmetrization" of $\bigotimes_{n=1}^{\infty} P^{x_n}$. Moreover, the transition probability $(\mathbf{P}_t)_{t\geq 0}$ of the process $(\mathbf{X}_t)_{t\geq 0}$ is just the product of the one-particle transition probabilities $e^{tA}(x-y)dy$, i.e., $\prod_{n=1}^{\infty} e^{tA}(x_n - y_n)dy_n$.

As a consequence, we can give explicit formulas for a lot of expressions. The Fourier-Laplace transform of the distribution of the processes at a fixed time, is for all non-positive $\varphi \in \mathscr{D}(\mathbb{R}^d)$ given by

$$\mathbf{E}_\gamma\left[e^{\langle\varphi,\mathbf{X}_t\rangle}\right] = \int_\Theta \mathbf{P}_t(\gamma, d\xi)\, e^{\langle\varphi,\xi\rangle} = \prod_{x\in\gamma} E_x\left[e^{\varphi(X_t)}\right] = \prod_{x\in\gamma} e^{tA}\varphi(x), \tag{14}$$

see also [32] (and also Lemma 3 below for an extension to a wider class of φ), alternatively we can use the Bogoliubov exponentials introduced in (3) for $\varphi \in \mathscr{D}(\mathbb{R}^d)$ with $-1 < \varphi \leq 0$,

$$\mathbf{E}_\gamma\left[e_B(\varphi, \mathbf{X}_t)\right] = \prod_{x\in\gamma} E_x\left[\varphi(X_t) + 1\right] = e_B(e^{tA}\varphi, \gamma). \tag{15}$$

More generally, it follows that the Fourier-Laplace transform of the joint distribution of the processes at different times can be expressed as

$$\mathbf{E}_\gamma\left[e^{\langle\varphi_1,\mathbf{X}_{t_1}\rangle}\cdots e^{\langle\varphi_n,\mathbf{X}_{t_n}\rangle}\right] = \prod_{x\in\gamma} E_x\left[e^{\varphi_1(X_{t_1})}\cdots e^{\varphi_n(X_{t_n})}\right],$$

for all $0 \leq t_1 < \ldots < t_n$ and all non-positive $\varphi_1, \ldots, \varphi_n \in \mathscr{D}(\mathbb{R}^d)$, $n \geq 2$. By a monotone approximation procedure using Riemannian sums, this relation allows us to calculate the Laplace transform of the path-space itself measure $\mathbf{P}_\gamma$, i.e.,

$$\mathbf{E}_\gamma\left[e^{-\int dt\langle\varphi(t,\cdot),\mathbf{X}_t\rangle}\right] = \prod_{x\in\gamma} E_x\left[e^{-\int dt\varphi(t,X_t)}\right], \tag{16}$$

for all non-negative continuous functions φ from $[0,\infty)\times\mathbb{R}^d$ to $\mathbb{R}$ with compact support.

The next result gives a relation between the process $\mathbf{X}$ and the operator L.

Lemma 2 *For all $F \in \mathscr{F}L_{eb}^0(\Theta)$ and all $\gamma \in \Theta$ there holds*

$$\mathbf{E}_\gamma\left[F(\mathbf{X}_t) - F(\mathbf{X}_0) - \int_0^t ds\, LF(\mathbf{X}_s)\right] = 0. \tag{17}$$

Lemma 3 *Let $\chi(y) = \frac{C'}{(1+|y|)^{\alpha/2}}$, for some $C' > 0$ and $\alpha > 0$ as in (13). Then there exists a $c > 0$ such that $A\chi \le c\chi$ and $e^{tA}\chi \le e^{ct}\chi$ for all $t \ge 0$. Moreover, for all $\gamma \in \Theta$ and all measurable functions φ such that $\frac{|\varphi|}{\chi}$ is bounded, one has that the product $\prod_{x\in\gamma}(1+\varphi(x))$ is absolutely convergent and*

$$\mathbf{E}_\gamma\left[e^{\langle\varphi,\mathbf{X}_t\rangle}\right] < \infty.$$

3.1.4 Starting from a Distribution

In Sects. 5 and 6 one considers the one-dimensional in time distributions of processes starting with initial distributions μ which are probability measures on Θ. The path-space measure $\mathbf{P}^\mu$ corresponding to such a process is given by $\int_\Theta \mathbf{P}_\gamma \mu(d\gamma)$. Its one-dimensional distribution is a probability measure $P^{\mathbf{X}}_{\mu,t}$ on Θ defined for all non-negative measurable functions F by

$$\int_\Theta P^{\mathbf{X}}_{\mu,t}(d\gamma)F(\gamma) := \int_\Theta \mu(d\gamma)\mathbf{E}_\gamma\left[F(\mathbf{X}_t)\right]. \tag{18}$$

In particular, for $\mu = \delta_\gamma$, $\gamma \in \Theta$, the one-dimensional distribution coincides with the transition kernel $\mathbf{P}_t(\gamma, \cdot)$ described above.

For functions F being Bogoliubov exponentials, definition (18) leads to the so-called Bogoliubov functionals [4]. By definition, the Bogoliubov functional corresponding to a probability measure μ on Γ is defined by

$$B_\mu(\varphi) := \int_\Gamma e_B(\varphi, \gamma)\mu(d\gamma), \qquad \varphi \in \mathscr{D}(\mathbb{R}^d).$$

Such a functional is an analogue of the Fourier-Laplace transform on configuration spaces, cf. [36]. Due to (15) there is an interesting relation between the Bogoliubov functional corresponding to the initial distribution and the Bogoliubov functional corresponding to the one-dimensional distribution of the process at a time $t > 0$, namely,

$$\int_\Theta P^{\mathbf{X}}_{\mu,t}(d\gamma)e_B(\varphi, \gamma) = \int_\Theta \mu(d\gamma)e_B(e^{tA}\varphi, \gamma) = B_\mu(e^{tA}\varphi). \tag{19}$$

In particular, for $\mu = \pi_z$ for some bounded intensity function $z \ge 0$ one finds

$$\int_\Theta e_B(\varphi, \gamma)P^{\mathbf{X}}_{\pi_z,t}(d\gamma) = \int_\Theta e_B(e^{tA}\varphi, \gamma)\pi_z(d\gamma) = \exp\left(\int_{\mathbb{R}^d} e^{tA}\varphi(x)z(x)dx\right), \tag{20}$$

for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$. In Sect. 5 we shall consider this special case in more detail.

One may ask why one does not work with the more standard Fourier-Laplace transform. Reconsidering (14) one sees that at least formally

$$\mathbf{E}_\gamma \left[e^{\langle \varphi, \mathbf{X}_t \rangle} \right] = e^{\langle \ln(e^{tA} e^{\varphi}), \gamma \rangle} \tag{21}$$

and notes that the pseudo-one-particle dynamic obtained in this way $\varphi \mapsto \ln(e^{tA} e^{\varphi})$ is non-linear.

3.1.5 Pseudo-one-Particle Operator

We call in this paper A the pseudo-one-particle operator. This is motivated by (19) which gives us the possibility to study time development of the infinite particle system in terms of the semi-group e^{tA}. Indeed, the Bogoliubov exponentials, like the usual exponentials, form a class of function large enough to describe the dynamics uniquely. The difference with the dynamics of a system of one particle is, beside the exponential in (19), the interpretation of z. In the one-particle system, z would describe the initial distribution of the particle and hence given by an $L^1(\mathbb{R}^d, dx)$ function. In the infinite particle system z describes the intensity of the Poisson measure, that is the density of particles and hence it is natural to consider $z \in L^\infty(\mathbb{R}^d, dx)$. The difference is in which space we consider the semi-group to act, and we will see in Sect. 5 and in Sect. 6 that the qualitative behaviour is quite different. Note that for a Poisson measure on the configuration space the number of particles is a.s. finite if and only if $z \in L^1(\mathbb{R}^d, dx)$.

3.2 An Analytic Approach

Within the framework of infinite dimensional analysis on configuration spaces [26] one may derive alternative representations and constructions of the dynamics. Instead of describing the infinite particle dynamics on Γ through the Kolmogorov equation $\frac{\partial}{\partial t} F_t = L F_t$, the so-called K-transform [37, 38] allows an alternative description for the action of L on $\mathscr{F} L^0_{eb}(\Gamma_a)$.

Given the space $\Gamma_0 := \{\gamma \in \Gamma : |\gamma| < \infty\}$ of finite configurations endowed with the metrizable topology as described in [26], let $B_{exp,ls}(\Gamma_0)$ be the space of all exponentially bounded Borel measurable functions G (i.e., $|G(\eta)| \leq C_1 e^{C_2|\eta|}$ for some $C_1, C_2 > 0$) with local support (i.e., there is a $\Lambda \in \mathscr{O}_c(\mathbb{R}^d)$ such that $G(\eta) = 0$ for all $\eta \in \Gamma_0$ with $|\eta \cap (\mathbb{R}^d \setminus \Lambda)| \neq 0$). The K-transform of a $G \in B_{exp,ls}(\Gamma_0)$ is the mapping $KG : \Gamma \to \mathbb{R}$ defined for all $\gamma \in \Gamma$ by $(KG)(\gamma) := \sum_{\substack{\eta \subset \gamma \\ |\eta| < \infty}} G(\eta)$. Note that $K\left(B_{exp,ls}(\Gamma_0)\right) \subset \mathscr{F} L^0_{eb}(\Gamma_a)$. Given a $G \in B_{exp,ls}(\Gamma_0)$, then in terms of the operator L we obtain

$$\begin{aligned}(L(KG))(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d \setminus (\gamma \setminus x)} dy\, a(x-y) \big[&(KG)(\gamma \setminus x) + (KG(\cdot \cup y))(\gamma \setminus x) \\ &-(KG)(\gamma \setminus x) - (KG(\cdot \cup x))(\gamma \setminus x) \big],\end{aligned}$$

which leads to the so-called symbol acting on quasi-observables $\hat{L}$,

$$(\hat{L}G)(\eta) := \sum_{x\in\eta} \int_{\mathbb{R}^d} dy\, a(x-y)\,(G(\eta\setminus x\cup y) - G(\eta))\,, \quad G \in B_{exp,ls}(\Gamma_0),$$

and the corresponding time evolution equation $\frac{\partial}{\partial t}G_t = \hat{L}G_t$. The transition kernel $\hat{P}_t$ corresponding to $\hat{L}$ is then for $\eta = \{x_1, \dots, x_n\}$ given by

$$\int_{\Gamma_0} \hat{P}_t(\eta', d\eta)\, G(\eta) = \int_{\mathbb{R}^{dn}} dy_1 \dots dy_n \prod_{i=1}^{n} e^{tA}(x_i - y_i) G(\{y_1, \dots, y_n\}).$$

This allows us to extend the explicit formula for transition kernels of $\mathbf{X}$ to the class of all so-called observables of additive type

$$\int_{\Theta} \mathbf{P}_t(\gamma, d\xi)\,(KG)(\xi) = K\left(\int_{\Gamma_0} \hat{P}_t(\cdot, d\eta)\, G(\eta)\right)(\gamma).$$

By duality one may extend the dynamical description to correlation functions. For this purpose, on the space $\Gamma_0 = \cup_{n=0}^{\infty}\{\gamma \in \Gamma : |\gamma| = n\}$ let us consider the so-called Lebesgue-Poisson measure $\lambda_z := \sum_{n=0}^{\infty} \frac{1}{n!}(zm)^{(n)}$, where m denotes the Lebesgue measure and each $(zm)^{(n)}$, $n \in \mathbb{N}$, is the image measure on $\{\gamma \in \Gamma : |\gamma| = n\}$ of the product measure $z(x_1)dx_1 \cdots z(x_n)dx_n$ under the mapping $(\mathbb{R}^d)^n \ni (x_1, ..., x_n) \mapsto \{x_1, ..., x_n\}$. If for some probability measure μ on Γ there is a function k_μ on Γ_0 such that the equality $\int_{\Gamma} \mu(d\gamma)\,(KG)(\gamma) = \int_{\Gamma_0} \lambda(d\eta)\, G(\eta) k_\mu(\eta)$ holds for all $G \in B_{exp,ls}(\Gamma_0)$, then k_μ are the correlation function corresponding to μ. Here we abbreviate $\lambda := \lambda_1$. Denoting by $\hat{L}^*$ the dual operator of $\hat{L}$ in the sense

$$\int_{\Gamma_0} \lambda_z(d\eta)\,(\hat{L}G)(\eta)k(\eta) = \int_{\Gamma_0} \lambda_z(d\eta)\, G(\eta)(\hat{L}^*k)(\eta),$$

one obtains the following expression, cf. [12] for more details,

$$(\hat{L}^*k)(\eta) = \sum_{x\in\eta} \int_{\mathbb{R}^d} dy\, a(y-x) k(\eta\setminus x\cup y) - |\eta| k(\eta) a^{(0)}.$$

The corresponding time evolution equation for correlation functions, $\frac{\partial}{\partial t}k_t = \hat{L}^*k_t$, is the analogue of the BBGKY-hierarchy for the case of the free Kawasaki dynamics. In our case this equation can be explicitly solved, namely,

$$k_t(\{x_1, \dots, x_n\}) = \int_{\mathbb{R}^{dn}} dy_1 \cdots dy_n\, k_\mu(\{y_1, \dots, y_n\}) \prod_{i=1}^{n} e^{tA}(y_i - x_i) \tag{22}$$

for the initial condition k_μ. Let us note that if one assumes a Ruelle bound for the initial correlation function k_μ, then all the above considerations can be made rigorous. Indeed, e^{tA} is a contraction in $L^\infty(\mathbb{R}^d, dx)$ and hence by (22) the correlation functions retain the Ruelle bound with time. In order to show that these correlation functions correspond to a probability measure μ_t on $\Theta \subset \Gamma$ one can either use similar arguments as in [23] or consider it as a moment problem on configuration space and try to verify the criteria in [19] for a version of the infinite dimensional moment problem giving a localization on the configuration space based on techniques established in [20] and guarantee uniqueness using Carleman bounds, see e.g. [18].

According to the previous considerations, the time evolution of the particle system may also be described in terms of Bogoliubov functionals, cf. [29], through the time evolution equation

$$\frac{\partial}{\partial t} B_{\mu,t}(\varphi) = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, a(x-y)(\varphi(y)-\varphi(x)) \frac{\delta B_{\mu,t}(\varphi)}{\delta \varphi(x)}, \quad \varphi \in \mathscr{D}(\mathbb{R}^d).$$

Here $\frac{\delta B_{\mu,t}(\varphi)}{\delta \varphi(x)}$ denotes the first variational derivative of $B_{\mu,t}$ at φ. Actually, $B_{\mu,t}(\varphi)$ is the Bogoliubov functional corresponding to the one-dimensional distribution of the process starting in μ. Hence, the previous equation has an explicit solution given by (19), i.e., $B_{\mu,t}(\varphi) = B_\mu(e^{tA}\varphi)$. If one develops the Bogoliubov functional in powers of φ then one recover the equations for the correlation functions derived above.

4 Equilibrium Dynamics

In this section we are interested in the representation of the generator L and its semigroup in terms of the creation, annihilation and second quantization operators as analytic expressions. These representations are possible because there is a well-known canonical unitary isomorphism between the (symmetric) Fock space and the Poisson space. The operator theoretic side can be only made rigorous for constant activity z. When a is symmetric then π_z for constant activity is a reversible measure, which gave rise to the name of the section. We start by recalling the aforementioned canonical unitary isomorphism. Our approach is based on [1, 33], but see also [28] and references therein.

4.1 The $L^2(\pi_z)$ Space and the Fock Space Representation

Let z be a non-negative locally integrable function, which we denote by $z(\cdot)$ in order to distinguish it from the case of constant intensity. We will explain in this sub-section why in order to use the Fock space representation to establish existence on has to assume that z is constant.

We consider the complex Hilbert space $L^2(\pi_{z(\cdot)}) := L^2(\Gamma, \mathscr{B}(\Gamma), \pi_{z(\cdot)})$ of square integrable complex valued functions on Γ with respect to the Poisson measure $\pi_{z(\cdot)}$. The coherent states introduced in Remark 1 generate the system of so-called Charlier polynomials, namely,

$$e_{\pi_{z(\cdot)}}(\varphi, \gamma) = \exp\left(-\int_{\mathbb{R}^d} dx\, z(x) f(x)\right) e_B(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle C^n_{z(\cdot)}(\gamma), \varphi^{\otimes n}\rangle,$$

where $C^n_{z(\cdot)}(\gamma) \in (\mathscr{D}')^{\hat{\otimes} n}$ and $(\mathscr{D}')^{\hat{\otimes} n}$ is the n-th symmetric tensor product of the Schwartz distributions space $\mathscr{D}' := \mathscr{D}'(\mathbb{R}^d)$. This system is orthogonal and any $F \in L^2(\pi_{z(\cdot)})$ can be expanded in terms of Charlier polynomials

$$F(\gamma) = \sum_{n=0}^{\infty} \langle C^n_{z(\cdot)}(\gamma), f^{(n)}\rangle, \qquad f^{(n)} \in L^2(z(\cdot)dx)^{\hat{\otimes} n}, \tag{23}$$

where $L^2(z(\cdot)dx) := L^2(\mathbb{R}^d, z(\cdot)dx)$. This yields a unitary isomorphism $I_{\pi_{z(\cdot)}}$ between $L^2(\pi_{z(\cdot)})$ and the so-called symmetric Fock space

$$\mathscr{F}(L^2(z(\cdot)dx)) := \bigoplus_{n=0}^{\infty} n! L^2(z(\cdot)dx)^{\hat{\otimes} n}, \qquad L^2(z(\cdot)dx)^{\hat{\otimes} 0} := \mathbb{C}.$$

More precisely, for each $F \in L^2(\pi_{z(\cdot)})$ of the form (23) one has $I_{\pi_{z(\cdot)}}(F) = (f^{(n)})_{n=0}^{\infty}$. Next we recall the definition of annihilation, creation and second quantization operators on the total subset of Fock space vectors of the form $f^{(n)} = f_1 \hat{\otimes} \ldots \hat{\otimes} f_n$. The action of the creation operator, denoted by $a^+(h)$, $h \in L^2(z(\cdot)dx)$, on elements $f^{(n)} \in L^2(z(\cdot)dx)^{\hat{\otimes} n}$ is given by $a^+(h) f^{(n)} = h \hat{\otimes} f^{(n)}$. The annihilation operator $a^-(h)$ is the adjoint operator.

Given a semigroup $(e^{tA})_{t\geq 0}$ on the Hilbert space $L^2(z(\cdot)dx)$ one can construct potentially unbounded operator $\mathrm{Exp}(e^{tA})$ on $\mathscr{F}(L^2(z(\cdot)dx))$ defined by $e^{tA} \otimes ... \otimes e^{tA}$ on each space $L^2(z(\cdot)dx)^{\hat{\otimes} n}$. Note that whenever the operator norm of e^{tA} is greater than one $\mathrm{Exp}(e^{tA})$ becomes an unbounded operator. Hence in order to be in the realm of semi-group theory we have to assume that $(e^{tA})_{t\geq 0}$ is a contraction semi-group. In our case, for non-constant functions $z(\cdot)$ the semi-group e^{tA} is in general not a contraction, see Remark 5 and indeed the previous construction gives not rise to a semi-group, cf. Remark 6. So the constructions below are not mathematically rigorous for non-constant activity as they stand but well-defined on the coherent states, Bogoliubov exponentials respectively.

The generator of $\mathrm{Exp}(e^{tA})$ is the so-called second quantization operator $d\mathrm{Exp}A$ corresponding to A. The image of the Fock coherent state $e(f) := (f^{\otimes n}/n!)_{n=0}^{\infty}$, $f \in L^2(z(\cdot)dx)$, under $\mathrm{Exp}(e^{tA})$ is given by

$$\mathrm{Exp}(e^{tA})(e(f)) = e(e^{tA} f). \tag{24}$$

Through the unitary isomorphism $I_{\pi_z(\cdot)}$ a contraction semigroup $(\mathrm{Exp}_{\pi_{z(\cdot)}}(e^{tA}))_{t\geq 0}$ on $L^2(\pi_{z(\cdot)})$ is obtained. In particular, since $I^{-1}_{\pi_z(\cdot)}e(f) = e_{\pi_{z(\cdot)}}(f)$, it follows from (7) that

$$\mathrm{Exp}_{\pi_{z(\cdot)}}\left(e^{tA}\right) e_B(f) = e_B(e^{tA} f) \exp\left(-\int_{\mathbb{R}} (e^{tA} f(x) - f(x)) z(x) dx\right). \tag{25}$$

The operator L, differentiating (25), can be expressed in terms of creation and annihilation operators, namely,

$$L = d\mathrm{Exp}_{\pi_{z(\cdot)}}(A) + a^-_{\pi_{z(\cdot)}}(A^* z(\cdot)), \tag{26}$$

where $d\mathrm{Exp}_{\pi_{z(\cdot)}}(\mathrm{A})$ and $a^-_{\pi_{z(\cdot)}}$ are, respectively, the image of $d\mathrm{Exp}(A)$ and a^- under $I_{\pi_z(\cdot)}$, and hence

$$a^-_{\pi_{z(\cdot)}}(h)F(\gamma) = \int dx z(x) \left(F(\gamma \cup x) - F(\gamma)\right). \tag{27}$$

For more details see e.g. [28].

Finally, we would like to present the "annihilation and creation operators" in the form more common in physical literature. For each $x \in \mathbb{R}^d$ we define an operator $a^-(x)$ acting on $\boldsymbol{f} = (f^{(n)})_n$ by

$$(a^-(x)\boldsymbol{f})^{(n)}(y_1, \ldots, y_n) = \sqrt{n+1} f^{(n+1)}(x, y_1, \ldots, y_n).$$

The adjoint of the operator $a^-(x)$ is formally given by

$$(a^+(x)\boldsymbol{f})^{(n)}(y_1, \ldots, y_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \delta(x - y_k) f^{(n-1)}(y_1, \ldots, \hat{y}_k, \ldots, y_n),$$

where $\hat{y}_k$ means that the k-th coordinate is excluded. Actually, the expression for $a^+(x)$ is well-defined as a quadratic form. It is easy to check the relations $a^{\pm}(f) = \int_{\mathbb{R}^d} a^{\pm}(x) f(x) dx$ in the sense of quadratic forms. Rewritten in a style more similar to the common usage in physics, the right hand side of (26) takes the form

$$\int dx\, z(x) \int dy \left(a(x-y) - a^{(0)}\delta(x-y)\right) \left(a^+_{\pi}(x) a^-_{\pi}(y) - a^-_{\pi}(y)\right),$$

where $a^{\pm}_{\pi}(x)$ are the image of $a^{\pm}(x)$ under $I_{\pi_z(\cdot)}$ for more details see [28].

4.2 Second Quantization Operator Approach

4.2.1 The Symmetric Case

If a is an even function, then the generator L is symmetric (see Lemma 1) and one can straightforwardly give an alternative construction to the one presented in Sect. 3 for the infinite particle dynamics. As a matter of fact, for a an even function, and for each constant $z > 0$, the operator L defined in (1) gives rise to a Dirichlet form on $L^2(\Gamma, \pi_z)$,

$$\int_\Gamma \pi_z(d\gamma)(LF)(\gamma)F(\gamma) = -\frac{1}{2}\int_\Gamma \pi_z(d\gamma)\sum_{x\in\gamma}\int_{\mathbb{R}^d} dy\, a(x-y)\,|F(\gamma\setminus x\cup y) - F(\gamma)|^2 .$$

This allows the use of Dirichlet forms techniques to derive a Markov process on Γ with *cadlag* paths and having π_z as an invariant measure [31]. Actually, one can show that in this situation L is the second quantization operator corresponding to the non-positive self-adjoint operator A on $L^2(\mathbb{R}^d, zdx)$ (Remark 4). Hence, L is a non-positive essentially self-adjoint operator on $L^2(\Gamma, \pi_z)$, and it is the generator of a contraction semigroup on $L^2(\Gamma, \pi_z)$.

4.2.2 The Asymmetric Case

According to (15) and (7) the action of the transition kernel on coherent states corresponding π_z is given by

$$\mathbf{E}_\gamma\left[e_{\pi_z}(\varphi, \mathbf{X}_t)\right] = e_{\pi_z}(e^{tA}\varphi, \gamma)\exp\left(\int_{\mathbb{R}^d} dx\ \left(e^{tA}\varphi(x) - \varphi(x)\right)z\right). \tag{28}$$

This allows to express the action of the transition probability on coherent states in terms of annihilation and creation operators

$$\mathrm{Exp}_{\pi_z}(e^{tA})e^{a^-_{\pi_z}(e^{tA^*}z - z)}. \tag{29}$$

Observe that the right-hand side of (28) gives the action of a semigroup which preserves coherent states. Lemma 4 shows that L is the generator of a strongly continuous Markov semigroup.

Lemma 4 *Let $\mathcal{D}_{\mathrm{coh}}$ be the vector space spanned by all functions $e_B(\varphi)$ with $\varphi \in L^1(\mathbb{R}^d, zdx) \cap L^2(\mathbb{R}^d, zdx)$. The operator L restricted to $\mathcal{D}_{\mathrm{coh}}$ is closable in $L^2(\Gamma, \pi_z)$ and its closure is an extension of the operator $(L, \mathcal{F}L^0_{\mathrm{eb}}(\Gamma_a))$ defined in Sect. 2. Moreover, it is the generator of a Markov semigroup and when A is symmetric then $e^{tL} = \mathrm{Exp}_{\pi_z}(e^{tA})$.*

The proof is an adaptation for a non-symmetric generator of the proof of Proposition 4.1 in [1] and see also [30] for details. Technically, in our case the proof is simpler, because A is a bounded operator and we consider functions of the type $e_B(\varphi)$.

5 Local Equilibrium Dynamics

The considerations in Sect. 4 were essentially for Poisson measures π_z which have as activity parameter a constant function z. According to Lemma 1 and Sect. 2, such measures are reversible measures for the free Kawasaki process. As a first step towards other initial distributions, in this section we consider the so-called local equilibrium case, that is, Poisson measures with a non-constant activity parameter. The choice of the name “local equilibrium” is motivated by the following situation: Assume the activity z is varying on a scale much larger than the scale of the range of the jump kernel a. One considers the dynamics in a part of the volume with an extension much larger than the range of a but much smaller than the scale of variation of z. Then the Poisson measure effectively has a constant activity and thus is effectively a reversible measure for the dynamics. Thus, in this situation, the system is in local equilibrium. However, in this section, we do not consider an extra length scale of the variation of z, but it will appear in Sect. 5.2.

Remark 6 Although for bounded non-constant functions z the process starting in π_z can be constructed explicitly, cf. Sect. 3.1, one cannot expect that the expression given in (19) can be extended to a semigroup either in the $L^1(\Gamma_a, \pi_z)$ or in the $L^2(\Gamma_a, \pi_z)$ sense. The existence of the semigroup in an L^p-sense can only be expected w.r.t. an invariant measure. Indeed, this can be seen from the following equality

$$\frac{\|e^{tL} e_B(f)\|_{L^p(\pi_z)}}{\|e_B(f)\|_{L^p(\pi_z)}} = \exp\left(\frac{1}{p}\int_{\mathbb{R}^d}\left(|1+e^{tA}f(x)|^p - |1+f(x)|^p\right)z(x)dx\right),$$

and the fact that the operator norm of e^{tL} from $L^p(\pi_z)$ to itself admits the bound

$$\|e^{tL}\|_{\mathrm{Op}} \geq \sup_{f\in\mathscr{D}(\mathbb{R}^d), f\geq 0,} \exp\left(\frac{1}{p}\int_{\mathbb{R}^d}\left((1+e^{tA}f(x))^p - (1+f(x))^p\right)z(x)dx\right),$$

where the right-hand side is infinite if $(e^{tA})_{t\geq 0}$ is not a contraction semigroup (see Remark 5).

Let $z \geq 0$ be a bounded measurable function. According to (20), for the one-dimensional distribution of the free Kawasaki process $(\mathbf{X}_t)_{t\geq 0}$ with initial distribution π_z one finds

$$\int_{\Theta} e_B(\varphi, \gamma) P^{\mathbf{X}}_{\pi_z, t}(d\gamma) = \exp\left(\int_{\mathbb{R}^d} dx\ e^{tA}\varphi(x)z(x)\right), \tag{30}$$

for every $\varphi \in \mathscr{D}(\mathbb{R}^d)$ and every $t \geq 0$.

For each $t \geq 0$ fixed, let us now consider the linear functional defined on $L^1(\mathbb{R}^d, dx)$ by

$$L^1(\mathbb{R}^d, dx) \ni f \mapsto \int_{\mathbb{R}^d} dx\, (e^{tA} f)(x) z(x).$$

Due to the contractivity property of the semigroup e^{tA} in $L^1(\mathbb{R}^d, dx)$, see Proposition 1, and to the boundedness of z, this functional is bounded on $L^1(\mathbb{R}^d, dx)$, and thus it is defined by a kernel $z_t \in L^\infty(\mathbb{R}^d, dx)$, that is,

$$\int_{\mathbb{R}^d} dx\, e^{tA} f(x) z(x) = \int_{\mathbb{R}^d} dx\, f(x) z_t(x), \tag{31}$$

for all $f \in L^1(\mathbb{R}^d, dx)$. Moreover, since e^{tA} is positivity preserving in $L^1(\mathbb{R}^d, dx)$, it follows from (31) that $z_t \geq 0$.

This shows by (30) that the one-dimensional distribution $P^{\mathbf{X}}_{\pi_z, t}$ is just the Poisson measure π_{z_t} (see also [7, 9]).

Furthermore, the path-space measure $\mathbf{P}^{\pi_z}$ corresponding to the process is also Poissonian. It is a Poisson measure on $\Gamma_{D([0,\infty),\mathbb{R}^d)}$ with intensity P^z, where P^z is the path-space measure on $D([0,\infty), \mathbb{R}^d)$ of the one-particle jump process corresponding to A with initial distribution $z(x)dx$. Using the fact that the path-space measure $\mathbf{P}_\gamma$ is supported on $D([0,\infty), \Theta)$ one easily sees that this implies that $\mathbf{P}^{\pi_z}$ is actually supported on a subset of $\Gamma_{D([0,\infty),\mathbb{R}^d)}$ which can be naturally identified with $D([0,\infty), \Theta)$.

Lemma 5 *The path-space measure $\mathbf{P}^{\pi_z}$ of the process $\mathbf{X}$ starting with initial distribution π_z is the Poisson measure π_{P^z} on $\Gamma_{D([0,\infty),\mathbb{R}^d)}$. In particular, for any continuous function φ on $[0,\infty) \times \mathbb{R}^d$ with compact support we have*

$$\int_{\Gamma_{D([0,\infty),\mathbb{R}^d)}} \mathbf{P}^{\pi_z}(d\omega) e^{-\int_0^\infty \langle \varphi(t,\cdot), \omega(t)\rangle dt} = \exp\left(\int_{D([0,\infty),\mathbb{R}^d)} \left[e^{-\int_0^\infty \varphi(t,\omega(t))dt} - 1 \right] P^z(d\omega) \right).$$

5.1 Long-time Asymptotics

In this subsection we want to study the behavior of the one-dimensional distribution of the infinite particle dynamics for large times. As mentioned above (31), the free Kawasaki dynamic leaves the Poissonian structure of the initial distribution unchanged and only the underlying intensity develops with time. Thus, the analysis of the long-time asymptotics behavior reduces to the long-time asymptotics of the intensity, see Lemma 6 for details. In other words, we reduced the problem to a pseudo-one-particle problem where the intensity plays the role of the initial distribution.

Definition 1 One says that a function $z \in L^1_{\mathrm{loc}}(\mathbb{R}^d, dx)$ has arithmetic mean, denoted by mean(z), whenever the following limit exists

$$\lim_{R\to+\infty} \frac{1}{\mathrm{vol}(B(R))} \int_{B(R)} dx\, z(x). \tag{32}$$

The following proposition gives a condition on the activity under which the long-time asymptotics of the one time distribution of the infinite particle dynamics exists, it is still Poissonian with constant intensity; the constant being the arithmetic mean of the initial intensity.

Proposition 3 *Let $z \geq 0$ be a bounded measurable function which Fourier transform is a signed measure. Then z has arithmetic mean and the one-dimensional distribution $P^{\mathbf{X}}_{\pi_z,t}$ converges weakly to $\pi_{\mathrm{mean}(z)}$ when t goes to infinity.*

Under the assumptions of Proposition 3, the activity z has arithmetic mean, cf. Corollary 1. Then, using Fourier transform, one can easily derive the long-time asymptotics for the pseudo-one-particle dynamics, cf. Lemma 7. By Lemma 6 this implies the existence of the long-time asymptotics for the infinite particle dynamics and identifies the limit. Using the same technique one can show that the long-time asymptotics for the infinite particle system only depends on the space asymptotics of the initial intensity z, cf. Corollary 2.

In Proposition 3, the assumption concerning the Fourier transform is actually not very transparent, because we cannot reasonably restate it in terms of z in the position variables. Therefore, we give several examples for illustration, cf. Example 1, and we derive certain properties of the arithmetic mean. Some classes of activities with reasonable asymptotic behavior do not fulfill the aforementioned Fourier transform assumption, e.g. Example 1(v). Note that also not every bounded function has an arithmetic mean, cf. Remark 8.

Let us discuss the properties of the arithmetic mean in some more details before we prove the lemmas mentioned above.

Remark 7 If two functions $z_1, z_2 \in L^1_{\mathrm{loc}}(\mathbb{R}^d, dx)$ have arithmetic mean, then for every $\alpha_1, \alpha_2 \in \mathbb{R}$ the function $\alpha_1 z_1 + \alpha_2 z_2$ also has arithmetic mean and $\mathrm{mean}(\alpha_1 z_1 + \alpha_2 z_2) = \alpha_1 \mathrm{mean}(z_1) + \alpha_2 \mathrm{mean}(z_2)$.

Example 1 To be more concrete we give some examples:

(i) If z is a constant function then $\mathrm{mean}(z) = z$.
(ii) If z decays to zero, i.e., for every $\varepsilon > 0$ there exists an $R > 0$ such that $|z(x)| \leq \varepsilon$ for $x \notin B(R)$, then $\mathrm{mean}(z) = 0$.
(iii) If $z \in L^p(\mathbb{R}^d, dx)$, $p \in [1, \infty)$, then $\mathrm{mean}(z) = 0$.
(iv) Also trigonometric functions have no influence, e.g., for $d = 1$ and $z(x) = 1 + \varepsilon \sin(x)$ we have $\mathrm{mean}(z) = 1$.
(v) A less trivial example is the following one. Given $z_0, z_1 \geq 0$, let z be the function defined at each $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ by

$$z(x)=\begin{cases} z_1 \text{ if } x_1 \geq 0 \\ z_0 \text{ otherwise} \end{cases}.$$

In this case one finds $\text{mean}(z)=\frac{z_0+z_1}{2}$. Note that in this case $\hat{z}$ is not a signed measure.

Remark 8 The arithmetic mean does not exist for all bounded non-negative functions.

1. For $d=1$ and

$$z(x)=\begin{cases} 1 \text{ if } 2^{2k} \leq |x| \leq 2^{2k+1} \text{ for a } k \in \mathbb{N}_0 \\ 0 \text{ otherwise} \end{cases},$$

the integral in (32) oscillates between $1/3$ and $2/3$. Thus z does not have arithmetic mean.

2. Another example is given by the following slowly oscillating function

$$z(x)=\cos(\ln(1+|x|))+z_0, \quad x \in \mathbb{R}^d,$$

where z_0 is a constant greater or equal to 1. Then for large R it holds that

$$\frac{1}{\text{vol}(B(R))}\int_{B(R)} z(x)\,dx \sim \frac{d}{\sqrt{1+d^2}}\,\sin\Big(\ln(R+1)+\arctan(d)\Big)+z_0.$$

In general slowly varying functions will show in general spurious behavior.

First, we prove that one may reduce the proof of Proposition 3 to a pseudo-one-particle system. This follows from the independent movement of the particles discussed in Sect. 3.

Lemma 6 *Let $0 \leq z \in L^1_{\text{loc}}(\mathbb{R}^d, dx)$ be such that*

$$\lim_{t\to+\infty}\int_{\mathbb{R}^d} dx\, e^{tA}\varphi(x)\, z(x) =: \int_{\mathbb{R}^d} z_\infty(dx)\varphi(x)$$

exists for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$. If z_∞ is a (non-negative) Radon measure on $\mathbb{R}^d$, then the one-dimensional distribution $P^{\mathbf{X}}_{\pi_z,t}$ converges weakly to π_{z_∞} when t tends to infinity.

It remains to derive the long-time asymptotics for the pseudo-one-particle system.

Lemma 7 *Let $z \geq 0$ be a bounded measurable function such that its Fourier transform $\hat{z}$ is a signed measure. For all $\varphi \in \mathscr{S}(\mathbb{R}^d)$ one has*

$$\lim_{t\to+\infty}\int_{\mathbb{R}^d} dx\, e^{tA}\varphi(x)\, z(x) = (2\pi)^{-d/2}\,\hat{z}(\{0\})\int_{\mathbb{R}^d} dx\,\varphi(x). \quad (33)$$

The next lemma shows that the existence of the arithmetic mean is stable under L^1-convergence. In particular, it yields that the condition assumed in Proposition 3 on the Fourier transform of the intensity is sufficient to ensure the existence of the arithmetic mean.

Lemma 8 *Let $z \geq 0$ be a bounded measurable function. Assume that there exists a total subset of $L^1(\mathbb{R}^d, dx)$ and a $C > 0$ such that for all φ in that total subset we have*

$$\lim_{R\to\infty} R^{-d} \int_{\mathbb{R}^d} dx\, \varphi(x/R)z(x) = C \int_{\mathbb{R}^d} \varphi(x)dx. \tag{34}$$

Then z has an arithmetic mean. In addition, equality (34) holds for $C = \mathrm{mean}(z)$ and for any $\varphi \in L^1(\mathbb{R}^d, dx)$.

Corollary 1 *Given a bounded measurable function $z \geq 0$ the following two results hold:*

1. *If z has arithmetic mean, then the limit in (34) exists for all $\varphi \in L^1(\mathbb{R}^d, dx)$ and $C = \mathrm{mean}(z)$.*
2. *If the Fourier transform of z is a signed measure, then z has arithmetic mean and*

$$\mathrm{mean}(z) = (2\pi)^{-d/2}\, \hat{z}(\{0\}).$$

Remark 9 Let us underline the difference between $\hat{z}(\{0\})$ and the evaluation of $\hat{z}$ as a function at zero. We demonstrate this difference for the case when z is a bounded L^1-function. The Fourier transform of z is then a continuous function which we denote for the moment by $\tilde{z}$. Evaluation at zero makes sense in this case. However, we want to consider the Fourier transform as a generalized function, i.e., as a linear form on a function space. We assume that this linear form is regular enough to be expressed by a signed measure. If, as in our case, z is an L^1-function, then its Fourier transform is the measure $\hat{z}(dk) = \tilde{z}(k)dk$. Thus $\hat{z}(\{0\}) = 0$, which is totally different from $\tilde{z}(0) = \int_{\mathbb{R}^d} dx z(x)$.

Below we list the Fourier transforms (interpreted as generalized functions, signed measures, respectively) of the examples given in Example 1, provided they exist and with the same enumeration:

(i) $(2\pi)^{d/2} z\delta_0(dk)$.
(ii) a continuous function for $p = 1$ or an $L^{p/(p-1)}$-function for $1 < p \leq 2$ multiplied in both cases by the Lebesgue measure.
(iii) $(2\pi)^{d/2}\, (\delta_0(dk) + i\varepsilon/2\delta_1(dk) - i\varepsilon/2\delta_{-1}(dk))$.
(iv) In the one-dimensional case the Fourier transform is the following generalized function $\hat{z}(k) = \sqrt{2\pi}(z_0 + z_1)/2\delta_0(k) + i(z_0 - z_1)/\sqrt{2\pi}\,\mathscr{P}(1/k)$, where $\mathscr{P}(1/k)$ denotes the Cauchy principal value of $1/k$. Using this explicit formula the conclusion of Proposition 3 can be shown although the assumptions of Proposition 3 are not fulfilled.

Applying the same technique as in Lemma 7 we can prove that the time asymptotics depends only on the behavior of z at infinity.

Corollary 2 *Let $z_1, z_2 \geq 0$ be two bounded measurable functions. If $z_1 - z_2 \in L^p(\mathbb{R}^d, dx)$ for some $1 \leq p \leq 2$, then the free Kawasaki dynamics with initial distribution π_{z_1} and the free Kawasaki dynamics with initial distribution π_{z_2} have the same long-time asymptotic limit.*

Throughout this subsection, the study of the time asymptotic behavior of the free Kawasaki process with an initial distribution π_z was based on the analysis of the Laplace transform

$$\int_\Gamma \pi_z(d\gamma)\, \mathbf{E}_\gamma[e^{\langle \varphi, \mathbf{X}_t \rangle}], \tag{35}$$

cf. Lemma 6. Another way to interpret this is as follows: we have studied the time asymptotic behavior of the so-called empirical field corresponding to a $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$n_t(\varphi, \mathbf{X}) := \langle \varphi, \mathbf{X}_t \rangle = \sum_{x \in \mathbf{X}_t} \varphi(x)$$

and showed that the limit $t \to \infty$ exists and identified the limiting distribution. The special role of the empirical fields is essential in the next subsection.

5.2 Hydrodynamic Limits

In the sequel let $z \geq 0$ be a bounded measurable function. In order to obtain a macroscopic description of our system, we rescale simultaneously the empirical field $n_t(\varphi, \mathbf{X}) = \langle \varphi, \mathbf{X}_t \rangle$, $\varphi \in \mathcal{D}(\mathbb{R}^d)$, in space and in time. The scale transformation in space is given by $\langle \varphi, \gamma \rangle \to \varepsilon^d \langle \varphi(\varepsilon \cdot), \gamma \rangle$, and in time by $t \to \varepsilon^{-\kappa} t$ for some $\kappa > 0$. To obtain non-trivial macroscopic density profiles, one has to scale the initial intensity as well, $z \to z(\varepsilon \cdot)$. This scaling yields a scaling of the Laplace transform of the empirical field, in other words, the Laplace transform of the one-dimensional distribution of the scaled process. For each $t \geq 0$ and each $\varphi \in \mathcal{D}(\mathbb{R}^d)$ we obtain from (15) the following form

$$\begin{aligned} &\int_\Gamma \pi_{z(\varepsilon \cdot)}(d\gamma)\, \mathbf{E}_\gamma \left[e^{\varepsilon^d \langle \varphi(\varepsilon \cdot), \mathbf{X}_{\varepsilon^{-\kappa} t} \rangle} \right] \\ &= \int_\Gamma \pi_{z(\varepsilon \cdot)}(d\gamma)\, e_B \left(e^{\varepsilon^{-\kappa} t A} \left(e^{\varepsilon^d \varphi(\varepsilon \cdot)} - 1 \right), \gamma \right) \\ &= \exp\left(\int_{\mathbb{R}^d} dx\, \left(e^{\varepsilon^{-\kappa} t A} \left(e^{\varepsilon^d \varphi(\varepsilon \cdot)} - 1 \right) \right)(x) z(\varepsilon x) \right). \end{aligned} \tag{36}$$

In the sequel we denote the scaled empirical field by

$$n_t^{(\varepsilon)}(\varphi, \mathbf{X}) := n_{\varepsilon^{-\kappa}t}(\varepsilon^d \varphi(\varepsilon\cdot), \mathbf{X}) = \varepsilon^d \left\langle \varphi(\varepsilon\cdot), \mathbf{X}_{\varepsilon^{-\kappa}t} \right\rangle . \tag{37}$$

According to the independent movement of the particles, we are again able to reduce the study of the infinite particle system to an effective pseudo-one-particle system. Again technical difficulties arise from the fact that the scale of the system size is much larger than the scale of space and time considered in the empirical field. Actually, the system size is infinite. Under the additional assumption that the Fourier transform of the activity z is a signed measure, the hydrodynamic limit can be derived rather directly using Fourier techniques, cf. Propositions 4 and 5. The general case of just bounded activities requires more technical involved considerations (postponed to Proposition 6).

Proposition 4 *Let $z \geq 0$ be a bounded measurable function such that its Fourier transform is a signed measure. For each $t \geq 0$ the following limit exists for all $\varphi \in \mathscr{D}(\mathbb{R}^d)$*

$$\lim_{\varepsilon \to 0^+} \int_{\Theta} \pi_{z(\varepsilon\cdot)}(d\gamma)\, \mathbf{E}_\gamma \left[e^{n_t^{(\varepsilon)}(\varphi, \mathbf{X})} \right] = \int_{\mathscr{D}'(\mathbb{R}^d)} \delta_{\rho_t}(d\omega) e^{\langle \varphi, \omega \rangle} \tag{38}$$

whenever one of the following conditions is fulfilled:

1. If

$$a_i^{(1)} := \int_{\mathbb{R}^d} dx\, x_i a(x) < \infty, \quad \forall\, i = 1, \ldots, d,$$

and $a^{(1)} := (a_1^{(1)}, \ldots, a_d^{(1)}) \neq 0$, then for $\kappa = 1$ the limiting density ρ_t is given by

$$\int_{\mathbb{R}^d} dx\, \rho_t(x)\varphi(x) := \int_{\mathbb{R}^d} dx\, z(x + ta^{(1)})\varphi(x), \quad \varphi \in \mathscr{D}(\mathbb{R}^d);$$

2. If $a^{(1)} = 0$, and

$$a_{ij}^{(2)} := \int_{\mathbb{R}^d} dx\, x_i x_j a(x) < \infty, \quad \forall\, i, j = 1, \ldots, d,$$

then for $\kappa = 2$ the limiting density ρ_t is given, for any $\varphi \in \mathscr{D}(\mathbb{R}^d)$, by

$$\int_{\mathbb{R}^d} dx\, \rho_t(x)\varphi(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dx\, z(x) \int_{\mathbb{R}^d} dk\, e^{i\langle k, x\rangle} e^{-\frac{t}{2}\langle a^{(2)}k, k\rangle} \hat{\varphi}(k),$$

where $a^{(2)}$ denotes the $d \times d$ matrix with coefficients $a_{ij}^{(2)}$.

Remark 10 The limiting density $\rho_t(x) = z(x + ta^{(1)})$ obtained in Proposition 4 is the solution of the linear partial differential equation $\frac{\partial}{\partial t}\rho_t(x) = \langle a^{(1)}, \nabla \rho_t(x) \rangle =$

$\mathrm{div}(a^{(1)}\rho_t(x))$ with the initial condition $\rho_0 = z$. In the same way, the second case stated in Proposition 4 yields a limiting density which is the solution of the heat equation

$$\frac{\partial}{\partial t}\rho_t(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}^{(2)} \frac{\partial^2}{\partial x_i \partial x_j}\rho_t(x)$$

with the same initial condition.

Given the function a, we may decompose a into a sum of an even function p and an odd function q, $a = p + q$. Note that one always has $p^{(1)} = 0$. Submitting also the function a itself to a proper scale transformation (beyond the previous scale transformations in space and in time), one may complement the statement of Proposition 4. In the proof of Proposition 4 one has to control $\lim_{\varepsilon\to 0^+} \frac{\hat{a}(-\varepsilon k)-\hat{a}(0)}{\varepsilon^\kappa}$ which suggests the scaling $a_\varepsilon := p + \varepsilon q$ and $\kappa = 2$. Note that the assumptions on the function a (i.e., $0 \le a \in L^1(\mathbb{R}^d, dx)$), carry over to a_ε, $\varepsilon > 0$. Hence all considerations done in Sect. 2 and following sections for the one-particle operator A and its underlying dynamics still hold for the operator A_ε defined by (5) with a replaced by a_ε. Denote by

$$a^{(1)} := \int_{\mathbb{R}^d} dx\, x a(x) = \int_{\mathbb{R}^d} dx\, x q(x) = \varepsilon^{-1}\int_{\mathbb{R}^d} dx\, x a_\varepsilon(x).$$

Proposition 5 (“weak asymmetry”) *Let $z \ge 0$ be a bounded measurable function such that its Fourier transform is a signed measure. Under the above conditions, if $0 \ne q^{(1)} = a^{(1)} \in \mathbb{R}^d$ and $p_{ij}^{(2)} < \infty$ for every $i, j = 1, \ldots, d$, then for each $t \ge 0$ and each $\varphi \in \mathscr{D}(\mathbb{R}^d)$ the following limit exists, and it is given by*

$$\lim_{\varepsilon\to 0^+} \int_\Gamma \pi_{z(\varepsilon\cdot)}(d\gamma)\, \mathbf{E}_\gamma[e^{n_t^{(\varepsilon)}(\varphi,\mathbf{X})}]$$
$$= \exp\left(\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} dx\, z(x)\int_{\mathbb{R}^d} dk\, e^{i\langle k,x\rangle} e^{-it\langle a^{(1)},k\rangle - \frac{t}{2}\langle a^{(2)}k,k\rangle}\hat{\varphi}(k)\right).$$

Remark 11 Similarly to Remark 10, one may then conclude that for continuous differentiable z Proposition 5 leads to a limiting density ρ_t which is a solution of the partial differential equation

$$\frac{\partial}{\partial t}\rho_t(x) = \mathrm{div}(a^{(1)}\rho_t(x)) + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}^{(2)} \frac{\partial^2}{\partial x_i \partial x_j}\rho_t(x)$$

with the initial condition $\rho_0 = z$.

Proposition 6 *Assume that a has all moments finite. Then the results stated in Propositions 4 and 5 hold for all non-negative bounded measurable functions z.*

5.3 Finite Volume System

The situation is much easier when one starts with a particle system on a compact manifold. Let us explain this in the case, when we consider particle configurations on $[-L/2, L/2]^d$ with periodic boundary conditions, that is we consider the system on the torus. Let us start by giving a description of the system in global coordinates coming from the covering space of the torus. For any sufficiently decaying function φ on $\mathbb{R}^d$, denote by $\varphi_L(x) := \sum_{j\in\mathbb{Z}^d} \varphi(jL)$. Any function on the torus can be written in this way. Consider the particle system $\Gamma_L := \left\{\gamma \subset [-L/2, L/2]^d \,:\, |\gamma| < \infty\right\}$ with the jump rate given by a_L. The probabilistic construction is the finite product and hence automatically a configuration at least if one allows coinciding points. The particle dynamics extend to this case and can also be reduced in the same way to a pseudo-one-particle dynamics. Denote by A_L the generator of the latter. An easy calculation show that $A_L\varphi_L = (A\varphi)_L$ and hence $e^{tA_L}\varphi_L = (e^{tA}\varphi)_L$. Therefore, the semi-group of the pseudo-one-particle on the torus can be described in the following way

$$\int_{[-L/2,L/2]^d} \left(e^{tA_L}\varphi_L\right)(x)\, z(x)dx = \int_{\mathbb{R}^d} (e^{tA}\varphi)(x)\, P_L z(x)dx, \tag{39}$$

where

$$P_L z(x) := \sum_{j\in\mathbb{Z}} 1\!\!1_{[-L/2,L/2]^d}(x - jL)z(x - jL) \tag{40}$$

is the extension of z to a $(L\mathbb{Z})^d$ periodic function. Note that if φ has compact support then $\varphi_L = \varphi$ for L large enough. In order to connect with our previous consideration we take the Fourier transform and obtain the expression

$$(\sqrt{2\pi}L)^{-d} \sum_{j\in\mathbb{Z}^d} e^{t(2\pi)^{d/2}(\hat{a}(-j/L)-\hat{a}(0))}\hat{\varphi}(-j/L)\,\widehat{1\!\!1_{[-L/2,L/2]^d}} * \hat{z}(j/L). \tag{41}$$

5.3.1 Long-time Asymptotics

The behaviour at large time for general z can be easily established in the finite volume. As $\widehat{1\!\!1_{[-L/2,L/2]^d}} * \hat{z}$ is a continuous function one sees directly that for fixed L, the above converges to

$$(\sqrt{2\pi}L)^{-d}\,\widehat{1\!\!1_{[-L/2,L/2]^d}} * \hat{z}(0) = \frac{1}{L^d}\int_{[-L/2,L/2]^d} z(x)dx$$

with an exponential rate. When it converges the latter in turn converges to mean(z) for $L \to \infty$. We only used that $a \in L^1$ and that $z \in L^\infty$ is such that the arithmetic mean exists in contrast to the extra regularity required in Lemma 7, namely $\hat{z}$ is a signed measure.

In other words, we showed that the convergence is equivalent to the existence of the arithmetic mean without any regularity assumption of z except being bounded. Whereas, when one first sends L to infinity then (39) converges to (31) and then considering the time asymptotics afterwards can become much more involved. In the finite volume case one has a spectral gap proportional to L^{-2}, which will disappear in the $L \to \infty$. Let us first consider what happens when one sends t and L simultaneously to infinity but t is sufficiently larger than L. We need first a result to control the spectral behaviour of the semi-group near $k = 0$.

Lemma 9 *Assume that a has finite second moments and the matrix $(a^{(2)}_{i,j})_{i,j=1,\ldots,d}$ is positive definite in the sense that there exists a constant $\sigma' > 0$ such that for all $k \in \mathbb{R}^d$ holds that $\sum_{i,j=1}^d k_i k_j a^{(2)}_{i,j} \geq (\sigma')^{-2}|k|^2$. Then for each $\sigma > \sigma'$ there exists a constant $c > 0$ such that*

$$\left| e^{t(2\pi)^{d/2}(\hat{a}(-k)-\hat{a}(0))} \right| \leq \exp\left(-t \frac{\min\{|k|^2, c\}}{2\sigma^2} \right). \tag{42}$$

Let us use this lemma and the bound

$$\left| (\sqrt{2\pi} L)^{-d} \widehat{\mathbb{1}_{[-L/2,L/2]^d}} * \hat{z}(j/L) \right| \leq \frac{1}{L^d} \int_{[-L/2,L/2]^d} z(x)dx \leq \|z\|_\infty \tag{43}$$

in order to estimate

$$\begin{aligned}
&(\sqrt{2\pi} L)^{-d} \sum_{j \in \mathbb{Z}^d : j \neq 0} e^{t(2\pi)^{d/2}\mathrm{Re}(\hat{a}(-j/L)-\hat{a}(0))} |\hat{\varphi}(-j/L)| \widehat{\mathbb{1}_{[-L/2,L/2]^d}} * \hat{z}(j/L) \\
&\leq \|z\|_\infty \sum_{j \in \mathbb{Z}^d : j \neq 0} e^{t(2\pi)^{d/2}\mathrm{Re}(\hat{a}(-j/L)-\hat{a}(0))} |\hat{\varphi}(-j/L)| \\
&\leq C\|z\|_\infty L^d e^{-t\frac{\min\{L^{-2},c\}}{2\sigma}} \to 0,
\end{aligned}$$

for some constant C. The latter converges to zero whenever $L^2 \ln L/t \to 0$.

One may think that one can use Lemma 9 to show that the time asymptotics of the semi-group $e^{tA}\varphi$ is the same as the one of the heat semi-group also in the infinite particle system, that is $L = \infty$. However, even when one ignores that Lemma 9 gives different exponential rates, one sees that the difference between the kernels can at best be bounded near $k = 0$ by a term of the form $|k|t^M e^{-ct|k|^2}$, where M is the order of the derivative considered. In the limit $t \to \infty$ this can be uniformly bounded by a term $|k|^{-2M+1}$, which becomes worse when one uses higher derivatives.

5.3.2 Hydrodynamic Limit

In finite volume also the study of the hydrodynamic limit is rather easy. In Fourier coordinates the exponent in (36) takes the form

$$(\sqrt{2\pi}L)^{-d}\sum_{j\in\mathbb{Z}^d} e^{t(2\pi)^{d/2}\varepsilon^{-2}(\hat{a}_\varepsilon(-\varepsilon j/L)-\hat{a}(0))}\hat{\varphi}(-j/L)\,\widehat{1\!\!1_{[-L/2,L/2]^d}}*\hat{z}(j/L). \tag{44}$$

The only ε dependence is in the jump kernel. As a is integrable, $\hat{a}_\varepsilon(-\varepsilon x)-\hat{a}(0)$ converges locally uniformly and without loss of generality we can assume that $\hat{\varphi}$ has compact support, the hydrodynamic limit is established immediately without any further assumptions on z. Again because of the spectral gap we only need that a has second moment and no further assumptions on z beside boundedness in contrast to Propositions 4 and 5 for the infinite volume system.

In Proposition 6 we were able to establish the hydrodynamic limit in the infinite volume system for a general z, but only under the assumption that a has all higher moments finite with increasing requirements for larger dimension d of the one-particle system.

However, one can establish the hydrodynamic limit with only little stronger requirements on a, namely that the third moment is finite for example (any power larger than 2 should be sufficient as well), if L does not grow too much compared to ε^-1. Indeed, we have to improve the bound a bit in the following way. As all derivatives of $t(2\pi)^{d/2}\varepsilon^{-2}(\hat{a}_\varepsilon(-\varepsilon k)-\hat{a}(0))$ in k are polynomially bounded in k and uniformly bounded in ε, one can use the mean value theorem to control the difference between the Fourier transform of the transition group and the limiting heat kernel, that is, to bound

$$\left|e^{\varepsilon^{-\kappa}t(2\pi)^{d/2}(\hat{a}_\varepsilon(-\varepsilon k)-\hat{a}(0))}-e^{it\langle a^{(1)},k\rangle-\frac{0^{2-\kappa}}{2}t\langle a^{(2)}k,k\rangle}\right|\le C(1+|k|^2)^{1/2}\varepsilon. \tag{45}$$

Using that the term $\widehat{1\!\!1_{[-L/2,L/2]^d}}*\hat{z}(j/\varepsilon L)$ can be bounded by $\|z\|_\infty L^d$ and assuming that $\varepsilon L^d\to 0$ we can replace the exponent in (44) by $it\langle a^{(1)},k\rangle-\frac{0^{2-\kappa}}{2}t\langle a^{(2)}k,k\rangle$. Taking the inverse Fourier transform we obtain

$$\int_{\mathbb{R}^d} e^{-t\langle a^{(1)},\nabla\rangle-t\langle\nabla,a^{(2)}\nabla\rangle/2}\varphi(x)P_L z(x)dx, \tag{46}$$

which converges for example for bounded z and $\varphi\in L^1$ to the same expression with $P_L z$ replaced by z.

In both cases, long-time asymptotics and hydrodynamic limit, the spectral gap coming from the finite volume is simplifying the situation a lot. The spectral gap being of order L^{-2} is too small to describe the physical relevant decay of the system. However, the infinite volume variant of the two aforementioned cases show that even for $L=\infty$ one has the convergence result. The mechanism it is based on, must be very different, as the finite volume results are based on a spectral gap that disapears for $L = \infty$.

6 Non-equilibrium Dynamics

In this section we widen the class of initial distributions to measures far from equilibrium, that is we consider all probability measures μ on Θ as initial distributions subject only to a mild mixing condition. This means that we consider the processes constructed as in Sect. 3 but not necessarily with a Poissonian initial distribution. Assuming enough mixing of the initial measure μ, namely (48), and incorporating ideas from [8] we are able to generalize Proposition 3, see Proposition 7 in Sect. 6.1 and Proposition 6, see Theorem 1 in Sect. 6.2.

We formulate the mixinig requirement in terms of the second Ursell function (factorial cumulant), which can be expressed in terms of the first and second correlation function (factorial moments) defined in Sect. 3.2, namely

$$u_\mu^{(2)}(x, y) := k_\mu(\{x, y\}) - k_\mu(\{x\})k_\mu(\{y\}).$$

We denote in the following the first correlation function $x \mapsto k_\mu(\{x\})$ by ρ_μ. The condition on the second Ursell function, (48), is a rather weak mixing or decay of correlation condition. In Sect. 6.3 we show that this condition is fulfilled, in particular, by Gibbs measures in the high-temperature low-activity regime. The condition will also hold beyond that regime even in the presence of a phase transition, cf. e.g. [44].

6.1 Long-time Asymptotics

Recalling (19), the Laplace transform of the one-dimensional distribution $P^{\mathbf{X}}_{\mu,t}$ can be expressed in terms of a one-particle system, i.e., for all non-negative $f \in \mathscr{S}(\mathbb{R}^d)$

$$\int_\Gamma e^{-\langle f,\gamma\rangle} P^{\mathbf{X}}_{\mu,t}(d\gamma) = \int_\Theta e_B(e^{tA}(e^{-f} - 1), \gamma)\mu(d\gamma) = \int_\Theta e^{\langle \ln(e^{tA}(e^{-f}-1)+1),\gamma\rangle}\mu(d\gamma). \quad (47)$$

Proposition 7 *Let μ be a measure on Θ which has first and second correlation function. Assume that μ fulfills the following mixing condition*

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} u_\mu^{(2)}(x, y)dy < \infty. \quad (48)$$

In addition, we assume that the Fourier transform of the first correlation function ρ_μ *is a signed measure. Then, the one-dimensional distribution* $P^{\mathbf{X}}_{\mu,t}$ *converges weakly to* $\pi_{\mathrm{mean}(\rho_\mu)}$ *when* t *tends to infinity.*

6.2 Hydrodynamic Limits

As in Sect. 5.2 we want to study the rescaled empirical field $n_t^{(\varepsilon)}(\varphi, \mathbf{X}) = \varepsilon^d \langle \varphi(\varepsilon\cdot), \mathbf{X}_{\varepsilon^{-\kappa}t} \rangle$. Since we do not have any longer a natural parameter associated to the initial measure, one cannot formulate something like slowly varying intensities. However, one sees that a possible framework is to work with a quite arbitrary sequence of initial measures $(\mu_\varepsilon)_{\varepsilon>0}$. The main restriction on this sequence is that one has to assume a particular convergence for the first correlation measure described below in more details and the mixing condtion (48) uniformly in ε. In Corollary 4 we prove that these conditions are fulfilled by Gibbs measures in the high-temperature low-activity regime for slowly varying intensity $z(\varepsilon\cdot)$. Furthermore, the limit is identified.

Theorem 1 *Assume that* a *has all moments finite. Let* $(\mu_\varepsilon)_{\varepsilon>0}$ *be a sequence of measures on* Θ *such that* ρ_{μ_ε} *is uniformly bounded in* ε*, for all* $x \in \mathbb{R}^d$ *the limit* $\lim_{\varepsilon\to 0^+} \rho_{\mu_\varepsilon}(\{x/\varepsilon\}) =: \rho_0(x)$ *exists, and the following mixing condition*

$$\sup_{x\in\mathbb{R}^d, \varepsilon>0} \int_{\mathbb{R}^d} u^{(2)}_{\mu_\varepsilon}(x, y)dy < \infty$$

holds. Then, for each $t \geq 0$*, the following limit exists for all non-negative* $\varphi \in \mathscr{D}(\mathbb{R}^d)$

$$\lim_{\varepsilon\to 0^+} \int_\Gamma \mu_\varepsilon(d\gamma)\, \mathbf{E}_\gamma \left[e^{-n_t^{(\varepsilon)}(\varphi,\mathbf{X})} \right] =: \int_{\mathscr{D}'(\mathbb{R}^d)} \delta_{\rho_t}(d\omega) e^{-\langle\varphi,\omega\rangle}, \tag{49}$$

and the conclusions of Sect. 5.2 continue to hold.

6.3 Application to Gibbs Measures

In this subsection we prove that the hypothesis for the results of the previous subsection are fulfilled for a concrete class of non-equilibrium measures, namely, for Gibbs measures in the high-temperature low-activity regime.

In order to recall the definition of a Gibbs measure, first we have to introduce a pair potential $V : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, that is, a measurable function such that $V(-x) = V(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$. For $\gamma \in \Gamma$ and $x \in \mathbb{R}^d \setminus \gamma$ we define a relative energy of interaction between a particle located at x and the configuration γ by

$$E(x,\gamma) := \begin{cases} \sum_{y\in\gamma} V(x-y), \text{ if } \sum_{y\in\gamma} |V(x-y)| < \infty \\ +\infty, \qquad \text{otherwise} \end{cases}.$$

A probability measure μ on Γ is called a Gibbs measure corresponding to V, an intensity function $z \geq 0$, and an inverse of temperature β whenever it fulfills the Georgii-Nguyen-Zessin equation [42, Theorem 2]

$$\int_\Gamma \mu(d\gamma) \sum_{x\in\gamma} H(x,\gamma) = \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} dx\, z(x) H(x, \gamma \cup \{x\}) e^{-\beta E(x,\gamma)} \tag{50}$$

for all positive measurable functions $H : \mathbb{R}^d \times \Gamma \to \mathbb{R}$. This definition is equivalent to the definition via DLR-equation, see [15, 34, 42, 46]. We observe that for $V \equiv 0$ (50) reduces to the Mecke identity, which yields an equivalent definition of the Poisson measure π_z [40, Theorem 3.1]. We also note that for either $V \equiv 0$ and z not being a constant or $V \not\equiv 0$, a Gibbs measure neither is a reversible nor an invariant initial distribution for the free Kawasaki dynamics under consideration. In order to have thermodynamical behavior we assume that V is stable, i.e., there exists a $B > 0$ such that $\sum_{\{x,y\}\subset\eta} V(x-y) \geq -B|\eta|$ for all configurations $\eta \in \Gamma_0$. Furthermore, we shall assume that the parameters β, z are small (high-temperature low-activity regime), i.e.,

$$\|z\|_u e^{2\beta B+1} C(\beta) < 1,$$

where $C(\beta) := \int_{\mathbb{R}^d} dx\, (1 - e^{-\beta|V(x)|})$. These conditions are, in particular, sufficient to insure the existence of Gibbs measures, cf. [45, 47]. Moreover, the correlation functions corresponding to such measures exist and fulfil a Ruelle bound defined in Sect. 3, and thus, as noted there, they are supported on Θ.

In the high-temperature low-activity regime one has rather detailed information about the Ursell functions (factorial cumulants) $u_\mu : \Gamma_0 \to \mathbb{R}$ corresponding to μ, which are the bounded measurable functions such that for all $f \in \mathscr{D}(\mathbb{R}^d)$ holds

$$\int_\Gamma e^{\langle f,\gamma\rangle} \mu(d\gamma) = \exp\left(\int_{\Gamma_0} \prod_{y\in\eta} (e^{f(y)} - 1) u_\mu(\eta) \lambda(d\eta) \right). \tag{51}$$

The function $x \to u_\mu(\{x\})$ coincides with the first correlation function of μ.

We present the results necessary for the following. For further details, see e.g. [10, 39] and see also [35].

The Ursell functions can be expressed in terms of a sum over all connected graphs weighted by the Meyer-functions

$$k(\xi) := \sum_{G\in\mathscr{G}_c(\xi)} \prod_{\{x,y\}\in G} (e^{-\beta V(x-y)} - 1), \tag{52}$$

where $\mathscr{G}_c(\xi)$ denotes the set of all connected graphs with vertex set ξ:

$$u_\mu(\eta) := \int_{\Gamma_0} \lambda_z(d\xi)k(\eta \cup \xi) \prod_{x \in \eta} z(x). \tag{53}$$

Actually, the following bound is the key result of the cluster expansion of Penrose–Ruelle type

$$|k(\xi)| \le e^{2\beta B|\xi|} \sum_{T \in \mathscr{T}(\xi)} \prod_{\{x,y\} \in G} (e^{-\beta|V(x-y)|} - 1), \tag{54}$$

where $\mathscr{T}(\xi)$ denotes the set of all trees with set of vertices ξ. This leads to the following integrability bound

$$\int_{\mathbb{R}^{dn}} |u_\mu(\{x, y_1, \ldots, y_n\})| z(y_1)dy_1 \ldots z(y_n)dy_n$$
$$\le e^{(2\beta B+1)(n+1)} (\|z\|_u C(\beta))^n \sum_{m=0}^{\infty} \frac{(n+m+1)!}{m!} \left(e^{2\beta B+1} \|z\|_u C(\beta)\right)^m. \tag{55}$$

In particular, the mixing condition (48) of Proposition 7 and Theorem 1 holds.

We show that Gibbs measures in the high-temperature low-activity regime with a translation invariant potential fulfill the assumptions of Proposition 7.

Corollary 3 *Let $z \ge 0$ be a bounded measurable function which Fourier transform is a bounded signed measure. Let μ be a Gibbs measure corresponding to a translation invariant potential V described above, inverse temperature β and activity z which are in the high-temperature low-activity regime. Then the first correlation function ρ_μ has as Fourier transform a measure and the arithmetic mean*

$$\mathrm{mean}(\rho_\mu) = \frac{1}{(2\pi)^{d/2}} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^{dn}} \overline{\hat{k_r}(p_1, \ldots, p_n)} \hat{z}(\{p_1 + \ldots + p_n\}) \hat{z}(dp_1) \cdot \ldots \cdot \hat{z}(dp_n),$$

where

$$k_r(y_1, \ldots, y_n) := \sum_{G \in \tilde{\mathscr{G}}_c} \prod_{\{x_1,x_2\} \in G} (e^{-\beta V(x_1 - x_2)} - 1)$$

and where $\tilde{\mathscr{G}}_c$ denotes the set of all connected graphs with vertex set $(0, y_1, \ldots, y_n)$. As a consequence, all the assumptions of Proposition 7 are fulfilled.

As in Sect. 5.2, we consider initial measures with a slowly varying intensity, i.e., Gibbs measures corresponding to β, V, and $z(\varepsilon\cdot)$, which we denote by μ_ε. As $\|z\|_u$ is unchanged, all scaled measures μ_ε remain in the high-temperature low-activity regime and the bound (55) holds uniformly in $\varepsilon > 0$. For Gibbs measures, which are not Poisson measures, the first correlation function is not any longer just the intensity. The function appearing as the initial value in the limiting partial differential equation is the scaling limit of the first correlation function and not just the unscaled activity. Let us describe what the scaling limit of the first correlation function is. Denote by ρ_c^{equi} the correlation function corresponding to the Gibbs measure with constant activity c, temperature β and potential V. Due to the translation invariance of V this correlation function is a constant function. Given a function $z \geq 0$ denote by $x \mapsto \rho_{z(x)}^{\text{equi}}$ the function that associates to each x the constant value of the first correlation function of the Gibbs measure with constant activity $z(x)$. This function is the scaling limit of the first correlation function of μ_ε. Note that $\rho_\mu(x) = u_\mu(\{x\})$.

Corollary 4 *Given a bounded measurable function $z \geq 0$, a potential V, and an inverse temperature β fulfilling the conditions of the high-temperature and low-activity regime, the corresponding Gibbs measures fulfill all the assumptions of Theorem 1. Moreover,*

$$\rho_0(x) := \lim_{\varepsilon \to 0^+} u_{\mu_\varepsilon}(\{x/\varepsilon\}) = \rho_{z(x)}^{\text{equi}}. \tag{56}$$

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Appendix

We give some estimates necessary to establish the hydrodynamic limit.

Lemma 10 *Let $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and $0 \leq \varepsilon \leq 1$ be given. Then*

$$\begin{aligned} \|e^{tA}(e^{\varepsilon^d \varphi(\varepsilon \cdot)} - 1)\|_u &\leq \varepsilon^d \|\varphi\|_u e^{\|\varphi\|_u} \\ \|e^{tA}(e^{\varepsilon^d \varphi(\varepsilon \cdot)} - 1)\|_{L^1(\mathbb{R}^d, dx)} &\leq \|\varphi\|_{L^1(\mathbb{R}^d, dx)} e^{\|\varphi\|_{L^1(\mathbb{R}^d, dx)}} \\ \|e^{tA}(e^{\varepsilon^d \varphi(\varepsilon \cdot)} - 1 - \varepsilon^d \varphi(\varepsilon \cdot))\|_{L^1(\mathbb{R}^d, dx)} &\leq \varepsilon^d \|\varphi\|^2_{L^1(\mathbb{R}^d, dx)} e^{\|\varphi\|_{L^1(\mathbb{R}^d, dx)}} \end{aligned} \tag{57}$$

Let us introduce the following two equivalent systems of norms for the locally convex topological vector space $\mathscr{S}(\mathbb{R}^d)$ for $A \in \mathbb{N}$ and $M \geq 0$:

$$\|f\|_{A,M,u} := \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \leq A}} \sup_{x \in \mathbb{R}^d} |D^\alpha f(x)| \left(1 + |x|^2\right)^M, \tag{58}$$

$$\|f\|_{A,M,2} := \sum_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \leq A}} \left(\int_{\mathbb{R}^d} |D^\alpha f(x)|^2 \left(1 + |x|^2\right)^M dx \right)^{1/2}. \tag{59}$$

Lemma 11 *Let $f_1, f_2 \in C^\infty(\mathbb{R}^d; \mathbb{C})$ be two C^∞-functions with non-positive real part such that for an $A \in \mathbb{N}$ and a $M \geq 0$ one has $\|f_i\|_{A,-M,u} < \infty$. Then there exists a constant C depending on A and on $\|f_i\|_{A,-M,u}$ such that*

$$\|e^{f_1} - e^{f_2}\|_{A,-(A+1)M,2} \leq C\|f_1 - f_2\|_{A,-M,2}. \tag{60}$$

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