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RESEARCH ARTICLE

Regular centralizers of idempotent transformations

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Abstract Denote by $T(X)$ the semigroup of full transformations on a set X. For $\varepsilon \in T(X)$, the centralizer of ε is a subsemigroup of $T(X)$ defined by $C(\varepsilon) = \{\alpha \in T(X) : \alpha\varepsilon = \varepsilon\alpha\}$. It is well known that $C(\mathrm{id}_X) = T(X)$ is a regular semigroup. By a theorem proved by J.M. Howie in 1966, we know that if X is finite, then the subsemigroup generated by the idempotents of $C(\mathrm{id}_X)$ contains all non-invertible transformations in $C(\mathrm{id}_X)$.

This paper generalizes this result to $C(\varepsilon)$, an arbitrary regular centralizer of an idempotent transformation $\varepsilon \in T(X)$, by describing the subsemigroup generated by the idempotents of $C(\varepsilon)$. As a corollary we obtain that the subsemigroup generated by the idempotents of a regular $C(\varepsilon)$ contains all non-invertible transformations in $C(\varepsilon)$ if and only if ε is the identity or a constant transformation.

Keywords Idempotent transformations · Regular centralizers · Generators

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1 Introduction

Let X be an arbitrary nonempty set. The semigroup $T(X)$ of full transformations on X consists of the functions from X to X with composition as the semigroup operation. It is a subsemigroup of the semigroup $P(X)$ of partial transformations on X (functions whose domain and image are included in X). Both $T(X)$ and $P(X)$ have the symmetric group $\mathrm{Sym}(X)$ of permutations on X as their group of units.

Let S be a semigroup. For $a \in S$, the *centralizer* $C(a)$ of a is the subsemigroup of S consisting of all elements $b \in S$ that commute with a, that is, $C(a) = \{b \in S : ab = ba\}$. If S contains the identity 1, then clearly $1 \in C(a)$ for every $a \in S$. An element $a \in S$ is called *regular* if $a = axa$ for some $x \in S$. If all elements of S are regular, we say that S is a *regular semigroup* [19, p. 50].

In 1966, Howie [18] determined the subsemigroup generated by the idempotents of $T(X)$. This seminal paper has given rise to many investigations of idempotent generated semigroups and, together with its companion paper written by Erdos [11], prompted many generalizations to various structures. (See, for example, [1–3, 5, 8–10, 12–17, 20–28]; more than one hundred references from various branches of mathematics could be provided.)

Suppose that X is finite. In this case, Howie's theorem says that the idempotent generated subsemigroup of $T(X)$ consists of the identity transformation on X together with all singular (non-invertible) transformations. (See [4] for a very short and direct proof of this result.) The same result holds for the semigroup $P(X)$. Indeed, every singular transformation $\alpha \in P(X)$ with domain $A \subseteq X$ can be written as $\alpha = \mathrm{id}_A\, \beta$, where id_A is the identity on A (which is an idempotent in $P(X)$) and β is a full transformation on X such that $x\beta = x\alpha$ (if $x \in A$) and $x\beta = x_0$ (if $x \notin A$), where x_0 is a fixed element in $X\alpha$. Thus the result for $P(X)$ follows from the result for $T(X)$.

Now, both semigroups of full and partial transformations on a set can be viewed as regular centralizers of idempotents in a full transformation semigroup. Indeed, $T(X) = C(\mathrm{id}_X)$, where id_X is the identity on X. Fix an element $r \in X$ and let $X' = X - \{r\}$. Then $P(X')$ is isomorphic to the centralizer of $C(\varepsilon_r)$ in $T(X)$, where ε_r is the constant transformation on X whose image is $\{r\}$. It is well known that both $T(X) = C(\mathrm{id}_X)$ and $P(X') \cong C(\varepsilon_r)$ are regular semigroups [19, Exercise 15, p. 63].

Many authors generalized versions of Howie's Theorem for various subsemigroups of $T(X)$ (see for example [21] and [28], and the references in them). The purpose of this paper is to generalize Howie's theorem for a finite X in the following way. We will consider an arbitrary regular centralizer $C(\varepsilon)$, where ε is an idempotent in $T(X)$, and determine the subsemigroup generated by the idempotents in $C(\varepsilon)$. As noted above, two of those centralizers will be $T(X)$ itself and (up to isomorphism) $P(X')$, where X' is the set X with one element removed. We find that these two are the only regular centralizers $C(\varepsilon)$ in $T(X)$ whose idempotent generated subsemigroup consists of id_X together with the singular transformations in $C(\varepsilon)$.

Finally, it is worth observing that the centralizers of idempotent transformations are very interesting transformation semigroups. They have a structure slightly more complex than $T(X)$, but many ideas, approaches and techniques used to study $T(X)$ also hold for the centralizers of its idempotents. More importantly, semigroup theory is deeply linked to centralizers, since, viewed as maps, every column of the Cayley table of a semigroup commutes with every row; and conversely, every groupoid whose Cayley table satisfies this property is in fact a semigroup. It is therefore not surprising that the study of centralizers in semigroups appears in the literature in many different branches of mathematics and computer science, under various different names (such as graph monoids, free partially commutative monoids, right-angled Artin monoids, trace monoids, etc.).

2 Centralizers of idempotents in $T(X)$

It will be convenient to view the centralizers of idempotents of $T(X)$ as the semigroups of transformations on X that preserve an equivalence relation and a cross-section. In this section, we describe these semigroups and identify the centralizers of idempotents of $T(X)$ that are regular semigroups.

Let $f : A \to B$ be a function and let $A_1 \subseteq A$. We denote by $\operatorname{im}(f)$ the image of f; by $\ker(f)$ the kernel of f, that is, the equivalence relation $\{(a_1, a_2) \in A \times A : a_1 f = a_2 f\}$ on A; by $f|_{A_1}$ the restriction of f to A_1; and by $A_1 f$ the image of A_1 under f. We will write functions on the right and compose from left to right, that is, for $f : A \to B$, $g : B \to C$, and $x \in A$, we will write xf (not $f(x)$) and $x(fg) = (xf)g$ (not $(gf)(x) = g(f(x))$).

Let ρ be an equivalence relation on X and let R be a cross-section of the partition X/ρ induced by ρ. Then

$$T(X, \rho, R) = \{\alpha \in T(X) : R\alpha \subseteq R \text{ and } \forall_{x,y\in X} ((x, y) \in \rho \Rightarrow (x\alpha, y\alpha) \in \rho)\}$$

is a subsemigroup of $T(X)$. It has been proved in [6] that the semigroups $T(X, \rho, R)$ are precisely the centralizers of idempotents of $T(X)$. More precisely, $T(X, \rho, R)$ is the centralizer of the unique idempotent $\varepsilon \in T(X)$ such that $\rho = \ker(\varepsilon)$ and $R = \operatorname{im}(\varepsilon)$.

The regular semigroups $T(X, \rho, R)$ have been characterized in [7].

Definition 2.1 Let ρ be an equivalence relation on X, let m be a positive integer. We say that ρ is *m-bounded* if all ρ-classes have at most m elements. We say that ρ is a *$\top$-relation* if there is at most one ρ-class with 2 or more elements.

The following result has been proved in [7, Theorem 3.7].

Theorem 2.2 *A semigroup $T(X, \rho, R)$ is regular if and only if ρ is 2-bounded or a $\top$-relation.*

We note that the only equivalence relations that are both 2-bounded and $\top$-relations are the equality relation $\{(x, x) : x \in X\}$ (which is the only 1-bounded relation) and the relations that have exactly one equivalence class with 2 elements and all other classes with one element.

Recall the regular centralizers $C(\mathrm{id}_X) = T(X)$ and $C(\varepsilon_r) \cong P(X')$ we considered in Sect. 1. In the language of binary relations with cross-sections, they are

$T(X, \Delta, X)$ and $T(X, \omega, \{r\})$, where $\Delta = \{(x, x) : x \in X\}$ is the equality relation on X and $\omega = X \times X$ is the universal relation on X. Thus, we have

$$\begin{aligned} &T(X, \Delta, X) = C(\mathrm{id}_X) = T(X), \\ &T(X, \omega, \{r\}) = C(\varepsilon_r) \cong P(X'), \quad \text{where } X' = X - \{r\}. \end{aligned} \tag{2.1}$$

For the remainder of the paper, we assume that X is a nonempty finite set.

3 The 2-bounded relations

Throughout this section, we assume that ρ is a 2-bounded relation on a finite set X, that is, each ρ-class has at most 2 elements. The purpose of this section is to prove a series of lemmas about 2-bounded relations that will be needed in the proof of the characterization theorem (Theorem 4.3).

We split the set X into three subsets:

$$\begin{aligned} &R_1 = \{s \in R : s\rho = \{s\}\}, \\ &R_2 = \{t \in R : t\rho = \{t, x\} \text{ with } t \neq x\}, \\ &X_2 = X - R. \end{aligned} \tag{3.1}$$

We agree that $R_{1+1} = R_2$ and $R_{2+1} = R_1$. We also note that for every $x \in X$,

$$x \in X_2 \Leftrightarrow \text{ there is } t \in R_2 \text{ such that } t\rho = \{t, x\}.$$

Let $\alpha, \beta \in T(X, \rho, R)$ such that $\ker(\alpha) = \ker(\beta)$. Then we define the following subset of R:

$$R(\alpha, \beta) = \{r \in R : r\alpha \in R_2 \Leftrightarrow r\beta \in R_2\}. \tag{3.2}$$

The set $R(\alpha, \beta)$ and Lemma 3.3 below will be crucial in proving the characterization theorem in the case when ρ is a 2-bounded relation. The following two lemmas will be used in the proof of Lemma 3.3.

Lemma 3.1 *Let* $\alpha, \beta \in T(X, \rho, R)$ *with* $\ker(\alpha) = \ker(\beta)$. *Suppose that for some* $i \in \{1, 2\}$, *there are* $r \in R$ *and* $u \in R_i$ *such that* $r\alpha \in R_i$, $r\beta \in R_{i+1}$, *and* $u \notin \mathrm{im}(\beta)$. *Then there is an idempotent* $\varepsilon \in T(X, \rho, R)$ *such that* $\ker(\alpha) = \ker(\beta\varepsilon)$ *and* $|R(\alpha, \beta\varepsilon)| > |R(\alpha, \beta)|$.

Proof Define an idempotent $\varepsilon \in T(X, \rho, R)$ by $((r\beta)\rho)\varepsilon = \{u\}$ and $x\varepsilon = x$ for all other $x \in X$.

We first prove that $\ker(\alpha) = \ker(\beta\varepsilon)$. We have $\ker(\alpha) = \ker(\beta) \subseteq \ker(\beta\varepsilon)$. For the reverse inclusion, let $(x, y) \in \ker(\beta\varepsilon)$. If $x\beta = y\beta$, then $(x, y) \in \ker(\beta)$, and so $(x, y) \in \ker(\alpha)$. Suppose $x\beta \neq y\beta$. Since $u \notin \mathrm{im}(\beta)$, we have $x\beta \neq u$ and $y\beta \neq u$. But then, since $(x\beta)\varepsilon = (y\beta)\varepsilon$, we must have $x\beta, y\beta \in (r\beta)\rho$. Since $x\beta \neq y\beta$, $(r\beta)\rho$ must have two elements, say $(r\beta)\rho = \{t, z\}$, where $t \in R_2$. Moreover, exactly one of $x\beta$ and $y\beta$ equals t, say $x\beta = t$ and $y\beta = z$. But then $x\beta = t = r\beta$, and so $(x, r) \in \ker(\beta) =$

$\ker(\alpha)$, implying $x\alpha = r\alpha$. Since $y\beta = z \notin R$, there is $t_1 \in R_2$ such that $t_1\rho = \{t_1, y\}$ and $t_1\beta = t$. Then $t_1\beta = r\beta$, and so $t_1\alpha = r\alpha$. Thus $y\alpha \in (t_1\alpha)\rho = (r\alpha)\rho = \{r\alpha\}$. (The last equality is true since, under the current assumptions, $r\beta \in R_2$, and so $r\alpha \in R_1$.) Hence $y\alpha = r\alpha$, and so $x\alpha = r\alpha = y\alpha$, implying $(x, y) \in \ker(\alpha)$. We proved that $\ker(\beta\varepsilon) = \ker(\alpha)$.

Let $r_1 \in R(\alpha, \beta)$. If $r_1\beta \neq r\beta$, then $r_1(\beta\varepsilon) = r_1\beta$, and so $r_1 \in R(\alpha, \beta\varepsilon)$. Suppose $r_1\beta = r\beta$. Then $r_1\alpha = r\alpha$ since $\ker(\alpha) = \ker(\beta)$. Thus $r_1(\beta\varepsilon) = (r_1\beta)\varepsilon = (r\beta)\varepsilon = u \in R_i$ and $r_1\alpha = r\alpha \in R_i$. Hence $r_1 \in R(\alpha, \beta\varepsilon)$. We proved that $R(\alpha, \beta) \subseteq R(\alpha, \beta\varepsilon)$. The inclusion is proper because $r\alpha \in R_i$, $r\beta \in R_{i+1}$, and $r(\beta\varepsilon) = (r\beta)\varepsilon = u \in R_i$, which implies that $r \in R(\alpha, \beta\varepsilon) - R(\alpha, \beta)$. It follows that $|R(\alpha, \beta\varepsilon)| > |R(\alpha, \beta)|$. □

Lemma 3.2 *Let* $\alpha, \beta \in T(X, \rho, R)$ *with* $\ker(\alpha) = \ker(\beta)$. *Suppose that for some* $i \in \{1, 2\}$, $R_i \subseteq \operatorname{im}(\beta)$ *and there is* $r \in R$ *such that* $r\alpha \in R_i$ *and* $r\beta \in R_{i+1}$. *Then there is* $r_1 \in R$ *such that* $r_1\alpha \in R_{i+1}$ *and* $r_1\beta \in R_i$.

Proof Suppose to the contrary that such an r_1 does not exist. Then for every $r_1 \in R_i\beta^{-1}$, we have $r_1\alpha \in R_i$. Thus, for $A = R_i\beta^{-1}$, the restrictions $\alpha|_A$ and $\beta|_A$ are mappings from A to R_i. Since $R_i \subseteq \operatorname{im}(\beta)$, the mapping $\beta|_A : A \to R_i$ is onto. Since $\ker(\alpha) = \ker(\beta)$, we also have $\ker(\alpha|_A) = \ker(\beta|_A)$, and so, since R_i is finite, the mapping $\alpha|_A : A \to R_i$ is also onto. Hence, since $r\alpha \in R_i$, there is $r_2 \in A$ such that $r_2\alpha = r\alpha$. Thus $r_2\beta = r\beta$, which is a contradiction since $r_2\beta \in R_i$ and $r\beta \in R_{i+1}$. This completes the proof. □

Lemma 3.3 *Let* $\alpha, \beta \in T(X, \rho, R)$ *with* $\ker(\alpha) = \ker(\beta)$. *Suppose there is* $r_0 \in R$ *such that* $r_0 \notin \operatorname{im}(\alpha)$. *Then either* $R(\alpha, \beta) = R$ *or there exists an idempotent* $\varepsilon \in T(X, \rho, R)$ *such that* $\ker(\alpha) = \ker(\beta\varepsilon)$ *and* $|R(\alpha, \beta\varepsilon)| > |R(\alpha, \beta)|$.

Proof Suppose $R(\alpha, \beta) \neq R$. Then for some $i \in \{1, 2\}$, there is $r \in R$ such that $r\alpha \in R_i$ and $r\beta \in R_{i+1}$. If there is $u \in R_i$ such that $u \notin \operatorname{im}(\beta)$, then the results follows by Lemma 3.1.

Now suppose that such a u does not exist. Then $R_i \subseteq \operatorname{im}(\beta)$, and so, by Lemma 3.2, there is $r_1 \in R$ such that $r_1\alpha \in R_{i+1}$ and $r_1\beta \in R_i$. The restrictions $\alpha|_R$ and $\beta|_R$ are mappings from R to R. Since $r_0 \notin \operatorname{im}(\alpha)$, the mapping $\alpha|_R : R \to R$ is not onto. Since $\ker(\alpha) = \ker(\beta)$, we also have $\ker(\alpha|_R) = \ker(\beta|_R)$, and so, since R is finite, the mapping $\beta|_R : R \to R$ is not onto either. Thus there is $v \in R$ such that $v \notin \operatorname{im}(\beta)$. Since $R_i \subseteq \operatorname{im}(\beta)$, we have $v \in R_{i+1}$.

So now we have $r_1 \in R$ and $v \in R_{i+1}$ such that $r_1\alpha \in R_{i+1}$, $r_1\beta \in R_i$, and $v \notin \operatorname{im}(\beta)$. The result follows again by Lemma 3.1. □

Corollary 3.4 *Let* $\alpha \in T(X, \rho, R)$ *be such that* $r_0 \notin \operatorname{im}(\alpha)$ *for some* $r_0 \in R$. *Then there exist idempotents* $\varepsilon, \varepsilon_1, \ldots, \varepsilon_k \in T(X, \rho, R)$ *such that* $R(\alpha, \varepsilon\varepsilon_1 \ldots \varepsilon_k) = R$.

Proof Since $T(X, \rho, R)$ is regular (recall that ρ is 2-bounded in this section), there is an idempotent $\varepsilon \in T(X, \rho, R)$ such that $\ker(\alpha) = \ker(\varepsilon)$. If $R(\alpha, \varepsilon) = R$, then we are done. Otherwise, by Lemma 3.3, there exists an idempotent $\varepsilon_1 \in T(X, \rho, R)$ such that $|R(\alpha, \varepsilon\varepsilon_1)| > |R(\alpha, \varepsilon)|$ and $\ker(\alpha) = \ker(\varepsilon\varepsilon_1)$. Applying the foregoing argument finitely many times, we obtain the desired idempotents. □

It will be convenient to introduce compact notation for certain idempotents.

Notation 3.5 Let z, y, z_1, y_1 be four distinct elements of X. Then:

(1) $(z, y; z_1, y_1\rangle$ will denote the idempotent in $T(X)$ that maps $z \to z_1$, $y \to y_1$, and fixes all other elements of X.
(2) $(z, y; z_1\rangle$ will denote the idempotent in $T(X)$ that maps $z \to z_1$, $y \to z_1$, and fixes all other elements of X.
(3) $(z; z_1\rangle$ will denote the idempotent in $T(X)$ that maps $z \to z_1$ and fixes all other elements of X.

Lemma 3.6 *Let $\beta \in T(X, \rho, R)$. Suppose there is $t \in R_2$ such that $t \notin \mathrm{im}(\beta)$. Let t_1 and t_2 be distinct elements of R_2 with $t_1\rho = \{t_1, x_1\}$ and $t_2\rho = \{t_2, x_2\}$. Consider the permutation $g = (t_1\, t_2)(x_1\, x_2) \in T(X, \rho, R)$. Then there exist idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in T(X, \rho, R)$ such that $\beta g = \beta\varepsilon_1\varepsilon_2\varepsilon_3$.*

Proof Let $t\rho = \{t, x\}$ and note that $\{t, x\} \cap \mathrm{im}(\beta) = \emptyset$. If $t \in \{t_1, t_2\}$, say $t = t_2$ (and so $x = x_2$), then

$$\beta g = \beta(t_1\, t_2)(x_1\, x_2) = \beta(t_1, x_1; t_2, x_2\rangle.$$

If $t \notin \{t_1, t_2\}$, then

$$\beta g = \beta(t_1\, t_2)(x_1\, x_2) = \beta(t_1, x_1; t, x\rangle(t_2, x_2; t_1, x_1\rangle(t, x; t_2, x_2\rangle.$$

This concludes the proof. □

Lemma 3.7 *Let $\beta \in T(X, \rho, R)$. Suppose there is $t \in R_2$ such that $t \notin \mathrm{im}(\beta)$. Let s_1 and s_2 be distinct elements of R_1. Consider the permutation $g = (s_1\, s_2) \in T(X, \rho, R)$. Then there exist idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in T(X, \rho, R)$ such that $\beta g = \beta\varepsilon_1\varepsilon_2\varepsilon_3$.*

Proof Let $t\rho = \{t, x\}$ and note that $\{t, x\} \cap \mathrm{im}(\beta) = \emptyset$. Then $\beta g = \beta(s_1\, s_2) = \beta(s_1; t\rangle(s_2; s_1\rangle(t, x; s_2\rangle$. □

Lemma 3.8 *Let $\beta \in T(X, \rho, R)$. Suppose there are $s \in R_1$ and $x \in X_2$ such that $s, x \notin \mathrm{im}(\beta)$. Let t_1 and t_2 be distinct elements of R_2 with $t_1\rho = \{t_1, x_1\}$ and $t_2\rho = \{t_2, x_2\}$. Consider the permutation $g = (t_1\, t_2)(x_1\, x_2) \in T(X, \rho, R)$. Then there exist idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 \in T(X, \rho, R)$ such that $\beta g = \beta\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5$.*

Proof Let $t \in R_2$ be such that $x \in t\rho$ and note that $|t\rho \cap \mathrm{im}(\beta)| \le 1$ (since $x \notin \mathrm{im}(\beta)$). If $|t\rho \cap \mathrm{im}(\beta)| = 0$, then $t \notin \mathrm{im}(\beta)$ and the result follows by Lemma 3.6. Suppose $|t\rho \cap \mathrm{im}(\beta)| = 1$, that is, we have $t \in \mathrm{im}(\beta)$ and $x \notin \mathrm{im}(\beta)$. If $t \in \{t_1, t_2\}$, say $t = t_2$ (and so $x = x_2$), then

$$\beta g = \beta(t_1\, t_2)(x_1\, x_2) = \beta(t_2, x_2; s\rangle(t_1, x_1; t_2, x_2\rangle(s; t_1\rangle.$$

If $t \notin \{t_1, t_2\}$, then

$$\beta g = \beta(t_1\, t_2)(x_1\, x_2) = \beta(t, x; s\rangle(t_1, x_1; t, x\rangle(t_2, x_2; t_1, x_1\rangle(t, x; t_2, x_2\rangle(s; t\beta\rangle.$$

This concludes the proof. □

Lemma 3.9 *Let $\beta \in T(X, \rho, R)$. Suppose there is $s \in R_1$ such that $s \notin \mathrm{im}(\beta)$. Let s_1 and s_2 be distinct elements of R_1. Consider the permutation $g = (s_1\, s_2) \in T(X, \rho, R)$. Then there exist idempotents $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in T(X, \rho, R)$ such that $\beta g = \beta\varepsilon_1\varepsilon_2\varepsilon_3$.*

Proof If $s \in \{s_1, s_2\}$, say $s = s_2$, then $\beta g = \beta(s_1\, s_2) = \beta(s_1; s_2\rangle$. If $s \notin \{s_1, s_2\}$, then $\beta g = \beta(s_1\, s_2) = \beta(s_1; s\rangle(s_2; s_1\rangle(s; s_2\rangle$. □

Lemma 3.10 *Let $\alpha, \beta \in T(X, \rho, R)$ be such that $R(\alpha, \beta) = R$. Then*:

(1) *If $R_2 \cap \mathrm{im}(\alpha) \neq R_2$, then $R_2 \cap \mathrm{im}(\beta) \neq R_2$.*
(2) *If $R_1 \cap \mathrm{im}(\alpha) \neq R_1$ and $X_2 \cap \mathrm{im}(\alpha) \neq X_2$, then $R_1 \cap \mathrm{im}(\beta) \neq R_1$ and $X_2 \cap \mathrm{im}(\beta) \neq X_2$.*

Proof Suppose $R_2 \cap \mathrm{im}(\alpha) \neq R_2$. Let $A = R_2\alpha^{-1}$. Since $R(\alpha, \beta) = R$, we have $A = R_2\alpha^{-1} = R_2\beta^{-1}$. Then $\alpha|_A$ and $\beta|_A$ are mappings from A to R_2. Since $R_2 \cap \mathrm{im}(\alpha) \neq R_2$, we have that $\alpha|_A : A \to R_2$ is not onto. Then, since $\ker(\alpha|_A) = \ker(\beta|_A)$ and R_2 is finite, we have that $\beta|_A : A \to R_2$ is not onto either. Hence $R_2 \cap \mathrm{im}(\beta) \neq R_2$. We proved (1). A proof of (2) is similar. □

4 The characterization theorem

In this section, we assume that ρ is a binary relation on a finite set X such that $T(X, \rho, R)$ is a regular semigroup. In other words, ρ is either 2-bounded or a $\top$-relation. For any such ρ, we will characterize the elements of $T(X, \rho, R)$ that are products of idempotents in $T(X, \rho, R)$. (Recall that the regular semigroups $T(X, \rho, R)$ are precisely the regular centralizers of idempotents of $T(X)$.)

We split the set X into three subsets:

$$\begin{aligned} R_1 &= \{s \in R : s\rho = \{s\}\}, \\ R_2 &= \{t \in R : |t\rho| \geq 2\}, \\ X_2 &= X - R. \end{aligned} \tag{4.1}$$

Note that if ρ is a 2-bounded relation, then these sets are precisely the sets R_1, R_2, and X_2 defined in (3.1).

We will need the following two lemmas. For a transformation $\alpha \in T(Y)$, where Y is a nonempty set, we denote by $\mathrm{Fix}(\alpha)$ the set of all fixed points of α, that is, $\mathrm{Fix}(\alpha) = \{y \in Y : y\alpha = y\}$.

Lemma 4.1 *Let α be a singular transformation in $T(Y)$, where Y is a finite nonempty set. Then there are idempotents $\varepsilon_1, \ldots, \varepsilon_k \in T(Y)$ such that $\alpha = \varepsilon_1 \ldots \varepsilon_k$ and $\mathrm{Fix}(\alpha) \subseteq \mathrm{Fix}(\varepsilon_i)$ for all i.*

Proof For every $y \in \mathrm{im}(\alpha)$, select $x_y \in y\alpha^{-1}$ with the restriction that if $y\alpha = y$, then $x_y = y$. Define an injective mapping $g : \mathrm{im}(\alpha) \to Y$ by $yg = x_y$, and extend g to a permutation on Y (in any way). Note that g fixes every element of $\mathrm{Fix}(\alpha)$. Let $x \in Y$ and let $y = x\alpha$. We have $x(\alpha g)^2 = (yg)(\alpha g) = x_y(\alpha g) = yg = x(\alpha g)$, and so $\varepsilon = \alpha g$ is an idempotent such that $\mathrm{Fix}(\alpha) \subseteq \mathrm{Fix}(\varepsilon)$. Then $\alpha = \varepsilon h$, where $h = g^{-1}$. Express h as a product of transpositions: $h = (x_1\, y_1)(x_2\, y_2)\dots(x_m\, y_m)$. Since $\mathrm{Fix}(\alpha) \subseteq \mathrm{Fix}(h)$, we may assume that none of x_i or y_i is a fixed point of α. We now have

$$\alpha = \varepsilon(x_1\, y_1)(x_2\, y_2)\dots(x_m\, y_m).$$

Since α is singular, ε is also singular, that is, there is $z \in Y - \mathrm{im}(\varepsilon)$. If $z \in \{x_1, y_1\}$, say $z = y_1$, then $\varepsilon(x_1\, y_1) = \varepsilon(x_1; y_1\rangle$. If $z \notin \{x_1, y_1\}$, then $\varepsilon(x_1\, y_1) = \varepsilon(x_1; z\rangle(y_1; x_1\rangle(z; y_1\rangle$. Note that, in either case, every fixed point of α is also fixed by the constructed idempotents. We proved that $\varepsilon(x_1\, y_1)$ is a product of idempotents that fix every fixed point of α. Since $\varepsilon(x_1\, y_1)$ is singular, we have by the foregoing argument that $\varepsilon(x_1\, y_1)(x_2\, y_2)$ is a product of idempotents that fix every fixed point of α. Continuing this way, we obtain that $\alpha = \varepsilon(x_1\, y_1)(x_2\, y_2)\dots(x_m\, y_m)$ is a product of idempotents that fix every fixed point of α. □

Lemma 4.2 *Let $\alpha \in T(Y)$, where Y is a finite set. Suppose that $\alpha|_Z \in \mathrm{Sym}(Z) - \{\mathrm{id}_Z\}$ for some $Z \subseteq Y$ and that there are idempotents $\varepsilon_1, \dots, \varepsilon_k \in T(Y)$ such that $\alpha = \varepsilon_1 \dots \varepsilon_k$. Then there are $i, j \in \{1, \dots, k\}$ with $i < j$, $z \in Z$, and $y \in Y - Z$ such that $z\varepsilon_i \in Y - Z$ and $y\varepsilon_j \in Z$.*

Proof Suppose to the contrary that $z\varepsilon_i \in Z$ for all $i \in \{1, \dots, k\}$ and all $z \in Z$. Then each $\varepsilon_i|_Z$ is an idempotent in $T(Z)$ and $\alpha|_Z = \varepsilon_1|_Z \dots \varepsilon_k|_Z$. Since $\alpha|_Z \in \mathrm{Sym}(Z)$, it follows that each $\varepsilon_i|_Z$ is also in $\mathrm{Sym}(Z)$. But the only idempotent in $\mathrm{Sym}(Z)$ is id_Z, and so $\alpha|_Z = \mathrm{id}_Z$. This is a contradiction. Hence there is $i \in \{1, \dots, k\}$ such that $z\varepsilon_i \in Y - Z$ for some $z \in Z$.

Select the smallest $i \in \{1, \dots, k\}$ such that $z\varepsilon_i \in Y - Z$ for some $z \in Z$. Let $y_0 = z\varepsilon_i$. Then the mapping $\varepsilon_1 \dots \varepsilon_{i-1}|_Z$ is in $T(Z)$ and it is injective (otherwise $\alpha|_Z$ would not be injective). Since Z is finite, it follows that $\varepsilon_1 \dots \varepsilon_{i-1}|_Z \in \mathrm{Sym}(Z)$, and so $\varepsilon_1 \dots \varepsilon_{i-1}|_Z = \mathrm{id}_Z$. Suppose to the contrary that $y\varepsilon_j \in Y - Z$ for all $j > i$ and all $y \in Y - Z$. Then

$$\begin{aligned} z\alpha &= z(\varepsilon_1 \dots \varepsilon_k) = (z(\varepsilon_1 \dots \varepsilon_{i-1}))(\varepsilon_i \dots \varepsilon_k) = (z\,\mathrm{id}_Z)(\varepsilon_i \dots \varepsilon_k) \\ &= (z\varepsilon_i)(\varepsilon_{i+1} \dots \varepsilon_k) = y_0(\varepsilon_{i+1} \dots \varepsilon_k) \in Y - Z, \end{aligned}$$

which is a contradiction since $\alpha|_Z$ maps Z to Z. Hence there is $j > i$ such that $y\varepsilon_j \in Z$ for some $y \in Y - Z$. □

We now describe the subsemigroup generated by the idempotents of $T(X, \rho, R)$.

Theorem 4.3 *Let ρ be a binary relation on a finite set X such that $T(X, \rho, R)$ is a regular semigroup, let R_1, R_2, X_2 be the subsets of X defined by (4.1), and let $\alpha \in T(X, \rho, R)$. Then α is a product of idempotents in $T(X, \rho, R)$ if and only if α satisfies the following conditions*:

(1) *$\alpha|_R$ is either a singular transformation on R or the identity on R;*
(2) *If $|R_2| = 1$ or there is $s \in R_1$ such that $s \notin \mathrm{im}(\alpha)$, then either $\alpha|_{X_2} = \mathrm{id}_{X_2}$ or there is $x \in X_2$ such that $x \notin \mathrm{im}(\alpha)$.*

Proof ($\Rightarrow$) We will prove the contrapositive. Suppose α does not satisfy (1). Then $\alpha|_R \in \mathrm{Sym}(R) - \{\mathrm{id}_R\}$, and so α is not a product of idempotents in $T(X, \rho, R)$ by Lemma 4.2 (since for every idempotent $\varepsilon \in T(X, \rho, R)$, $r\varepsilon \in R$ for every $r \in R$).

Suppose α does not satisfy (2). Then $\alpha|_{X_2} \neq \mathrm{id}_{X_2}$ and $x \in \mathrm{im}(\alpha)$ for every $x \in X_2$. The latter can only happen when $\alpha|_{X_2}$ is a permutation on X_2 (since no element of R can be mapped to X_2). Thus $\alpha|_{X_2} \in \mathrm{Sym}(X_2) - \{\mathrm{id}_{X_2}\}$, and so α is not a product of idempotents in $T(X, \rho, R)$ by Lemma 4.2 (since for every idempotent $\varepsilon \in T(X, \rho, R)$ and every $r \in R = X - X_2$, we have $r\varepsilon \notin X_2$).

($\Leftarrow$) Suppose α satisfies (1) and (2). If $\alpha|_R$ is the identity on R and ρ is 2-bounded, then α is an idempotent (since then for every $t \in R_2$ with $t\rho = \{t, x\}$, either $x\alpha = t$ or $x\alpha = x$). Let $\alpha|_R = \mathrm{id}_R$ and let ρ be a $\top$-relation. Then $|R_2| = 1$, and so, by (2), either $\alpha|_{X_2} = \mathrm{id}_{X_2}$ or there is $x \in X_2$ such that $x \notin \mathrm{im}(\alpha)$. In the former case, $\alpha = \mathrm{id}_X$, which is an idempotent; and in the latter case, α is a product of idempotents in $T(X, \rho, R)$ by Lemma 4.1.

Suppose $\alpha|_R$ is a singular transformation on R. We first consider the case when α is a 2-bounded relation. Then, by (2), α satisfies at least one of the following conditions:

(A) There is $t \in R_2$ such that $t \notin \mathrm{im}(\alpha)$;
(B) There are $s \in R_1$ and $x \in X_2$ such that $s, x \notin \mathrm{im}(\alpha)$;
(C) $\alpha|_{X_2} = \mathrm{id}_{X_2}$ and there is $s \in R_1$ such that $s \notin \mathrm{im}(\alpha)$.

Suppose α satisfies (A) or (B). Then, we have by Corollary 3.4 that there are idempotents $\varepsilon, \varepsilon_1, \dots, \varepsilon_k$ ($k \geq 0$) in $T(X, \rho, R)$ such that $R(\alpha, \varepsilon\varepsilon_1 \dots \varepsilon_k) = R$. Let $\beta = \varepsilon\varepsilon_1 \dots \varepsilon_k$. Then $R(\alpha, \beta) = R$ (and so $\ker(\alpha) = \ker(\beta)$). We will define permutations $g_0, g_1 \in \mathrm{Sym}(X)$ such that $\alpha = \beta g_0 g_1$, $g_0|_{R_2 \cup X_2} = \mathrm{id}_{R_2 \cup X_2}$, and $g_1|_{R_1} = \mathrm{id}_{R_1}$.

We first define g_0 on $R_1 \cap \mathrm{im}(\beta)$. Let $s \in R_1 \cap \mathrm{im}(\beta)$. Then $s = s_1\beta$ for some $s_1 \in R_1$ or $s = t\beta$ for some $t \in R_2$. In the former case, set $sg_0 = s_1\alpha$; and in the latter set $sg_0 = t\alpha$. Since $R(\alpha, \beta) = R$, we have that $sg_0 \in R_1$. Moreover, g_0 is well-defined and injective on $R_1 \cap \mathrm{im}(\beta)$ since $\ker(\alpha) = \ker(\beta)$. We extend g_0 to a permutation on R_1 (in any way), and then to a permutation on X by setting $xg_0 = x$ for every $x \in R_2 \cup X_2$. Note that $g_0|_{R_2 \cup X_2} = \mathrm{id}_{R_2 \cup X_2}$ and for every $y \in X$, $y\alpha = y(\beta g_0)$ if $y\beta \in R_1$.

We begin the definition of g_1 by defining it on $R_2 \cap \mathrm{im}(\beta)$. Let $t \in R_2 \cap \mathrm{im}(\beta)$. Then $t = t_1\beta$ for some $t_1 \in R_2$ or $t = s\beta$ for some $s \in R_1$. In the former case, set $tg_1 = t_1\alpha$; and in the latter set $tg_1 = s\alpha$. Since $R(\alpha, \beta) = R$, we have that $tg_1 \in R_2$. Moreover, g_1 is well-defined and injective on $R_2 \cap \mathrm{im}(\beta)$ since $\ker(\alpha) = \ker(\beta)$. We extend g_1 to a permutation on R_2 (in any way). Next, we define g_1 on X_2. Recall that for every $x \in X_2$, there is a unique $t_x \in R_2$ such that $x \in t_x\rho$. For all $x, y \in X_2$, we let

$$xg_1 = y \Leftrightarrow t_x g_1 = t_y.$$

At this point, g_1 is a permutation on $R_2 \cup X_2$. Finally, we extend g_1 to a permutation on X by setting $sg_1 = s$ for every $s \in R_1$.

Let $y \in X$ be such that $y\beta \in R_2 \cup X_2$. We want to prove that $y\alpha = y(\beta g_1)$. Suppose $y\beta = t \in R_2$. Then $t = t_1\beta$ for some $t_1 \in R_2$ or $t = s\beta$ for some $s \in R_1$. In the former case, $y(\beta g_1) = t_1(\beta g_1) = tg_1 = t_1\alpha = y\alpha$, where the last equality is true because $\ker(\alpha) = \ker(\beta)$. Similarly, $y(\beta g_1) = y\alpha$ if $t = s\beta$ where $s \in R_1$. Suppose $y\beta = x \in X_2$. Let $t = t_x$, let $t_2 = tg_1$, and let $x_2 \in t_2\rho \cap X_2$. Since $\beta \in T(X, \rho, R)$, there are $t_1 \in R_2$ and $x_1 \in t_1\rho$ such that $t_1\beta = t$ and $x_1\beta = x$. Then $t_1\alpha = tg_1 = t_2$, and so, $x_1\alpha = x_2$ or $x_1\alpha = t_2$. But the latter is impossible since $t_1\alpha = t_2 = x_1\alpha$ would imply $t = t_1\beta = x_1\beta = x$, which is a contradiction. Thus $x_1\alpha = x_2$, and so

$$y(\beta g_1) = (y\beta)g_1 = xg_1 = x_2 = x_1\alpha = y\alpha,$$

where the last equality is true because $y\beta = x = x_1\beta$ and $\ker(\alpha) = \ker(\beta)$.

Since $g_0|_{R_2 \cup X_2} = \mathrm{id}_{R_2 \cup X_2}$ and $g_1|_{R_1} = \mathrm{id}_{R_1}$, it follows from the above arguments that $\alpha = \beta g_0 g_1$. Since $g_0|_{R_1}$ is a permutation on R_1 and $g_0|_{R_2 \cup X_2}$ is the identity on $R_2 \cup X_2$, we can express g_0 as a product of transpositions: $g_0 = (s_1\, u_1) \dots (s_m\, u_m)$, where $s_i, u_i \in R_1$ for all i. Since $g_1|_{R_2}$ is a permutation on R_2, we can express $g_1|_{R_2}$ as $g_1|_{R_2} = (t_1\, v_1) \dots (t_p\, v_p)$, where $t_i, v_i \in R_2$ for all i. Let x_i, y_i $(1 \le i \le k)$ be the elements of X_2 such that $x_i \in t_i\rho$ and $y_i \in v_i\rho$. Then, by the definition of g_1, we have $x_i g_1 = y_i$ and $y_i g_1 = x_i$ for every $i \in \{1, \dots, k\}$. Hence, since $g_1|_{R_1} = \mathrm{id}_{R_1}$, we have

$$g_1 = (t_1\, v_1) \dots (t_p\, v_p)(x_1\, y_1) \dots (x_p\, y_p) = (t_1\, v_1)(x_1\, y_1) \dots (t_p\, v_p)(x_p\, y_p),$$

where the last equality follows from the fact that R_2 and X_2 are disjoint, and so every $(t_i\, v_i)$ commutes with every $(x_j\, y_j)$. Hence

$$\alpha = \beta g_0 g_1 = \beta(s_1\, u_1) \dots (s_m\, u_m)(t_1\, v_1)(x_1\, y_1) \dots (t_p\, v_p)(x_p\, y_p). \tag{4.2}$$

Since $R(\alpha, \beta) = R$ and α satisfies (A) or (B), β also satisfies (A) or (B) by Lemma 3.10. Since β is a product of idempotents, it follows by Lemmas 3.7 and 3.9 that $\beta_1 = \beta(s_1\, u_1)$ is also a product of idempotents. Moreover, it is clear that β_1 also satisfies (A) or (B). Applying the foregoing argument m times, we obtain that $\beta_m = \beta(s_1\, u_1) \dots (s_m\, u_m)$ is a product of idempotents and it satisfies (A) or (B). Now, by Lemmas 3.6 and 3.8, $\beta_{m+1} = \beta_m(t_1\, v_1)(x_1\, y_1)$ is a product of idempotents. Moreover, it is clear that β_{m+1} also satisfies (A) or (B). Applying the foregoing argument p times, we obtain that $\beta_{m+p} = \beta_m(t_1\, v_1)(x_1\, y_1) \dots (t_p\, v_p)(x_p\, y_p)$ is a product of idempotents. But, by (4.2), $\beta_{m+p} = \alpha$, and so α is a product of idempotents.

Suppose α satisfies (C). Then $\alpha|_R$ is a nonsingular transformation on R that fixes every element of R_2 (since α fixes every element of X_2), and so it follows by Lemma 4.1 that there are idempotents $\varepsilon_1, \dots, \varepsilon_k$ in $T(R)$ such that $\alpha|_R = \varepsilon_1 \dots \varepsilon_k$ and each ε_i fixes every element of R_2. We extend each ε_i to an idempotent in $T(X, \rho, R)$ by setting $x\varepsilon_i = x$ for every $x \in X_2$. Since $\alpha|_{X_2} = \mathrm{id}_{X_2}$, we have $\alpha = \varepsilon_1 \dots \varepsilon_1$. We finished the proof of ($\Leftarrow$) in the case when ρ is a 2-bounded relation.

We now suppose that ρ is a $\top$-relation. Then R_2 has exactly one element, say $R_2 = \{t_1\}$, and $t_1\rho = \{t_1\} \cup X_2$. Recall that we are still under the assumption that $\alpha|_R$ is a singular transformation on R. We consider two possible cases.

Case 1. $t_1\alpha \neq t_1$.

Then $t_1\alpha \in R_1$ and $x\alpha = t_1\alpha$ for every $x \in X_2$. By Lemma 4.1, $\alpha|_R = \varepsilon_1 \dots \varepsilon_k$ for some idempotents $\varepsilon_1, \dots, \varepsilon_k \in T(R)$. We extend each ε_i to an idempotent in $T(X, \rho, R)$ by setting $x\varepsilon_i = t_1\varepsilon_i$ for every $x \in X$. (The extended ε_i is indeed an idempotent since for every $x \in X_2$, $x\varepsilon_i^2 = (x\varepsilon_i)\varepsilon_i = (t_1\varepsilon_i)\varepsilon_i = t_1\varepsilon_i^2 = t_1\varepsilon_i = x\varepsilon_i$.) Then $\alpha = \varepsilon_1 \dots \varepsilon_k$ since for every $x \in X_2$,

$$x\alpha = t_1\alpha = t_1(\varepsilon_1\varepsilon_2 \dots \varepsilon_k) = (t_1\varepsilon_1)(\varepsilon_2 \dots \varepsilon_k) = (x\varepsilon_1)(\varepsilon_2 \dots \varepsilon_k) = x(\varepsilon_1\varepsilon_2 \dots \varepsilon_k).$$

Case 2. $t_1\alpha = t_1$.

Then, by Lemma 4.1, there are idempotents $\varepsilon_1, \dots, \varepsilon_k \in T(R)$ such that $\alpha|_R = \varepsilon_1 \dots \varepsilon_k$ and $t_1\varepsilon_i = t_1$ for every $i \in \{1, \dots, k\}$. By (2), $\alpha|_{t_1\rho}$ is either a singular transformation on $t_1\rho$ or the identity on $t_1\rho$. Thus, again by Lemma 4.1, there are idempotents $\eta_1, \dots, \eta_m \in T(t_1\rho)$ such that $\alpha|_{t_1\rho} = \eta_1 \dots \eta_m$ and $t_1\eta_j = t_1$ for every $j \in \{1, \dots, m\}$. Extend each ε_i and each η_j to an idempotent in $T(X)$ by setting $x\varepsilon_i = x$ for every $x \in X_2$, and $s\eta_j = s$ for every $s \in R_1$. Then it is clear that $\alpha = \varepsilon_1 \dots \varepsilon_k\eta_1 \dots \eta_m$. This concludes the proof of ($\Leftarrow$). □

Let S be a subsemigroup of $T(X)$ (where X is finite) such that $\mathrm{id}_X \in S$. Denote by $E(S)$ the set of idempotents of S, by $\langle E(S)\rangle$ the subsemigroup of S generated by $E(S)$, and by $\mathrm{Sng}(S)$ the semigroup of singular transformations in S. Since the only nonsingular idempotent of $T(X)$ is id_X, we always have $\langle E(S)\rangle \subseteq \mathrm{Sng}(S) \cup \{\mathrm{id}_X\}$.

We already observed in Sect. 1 that if $S = T(X)$ or $S = P(X')$, where X' is X with one element removed, then $\langle E(S)\rangle = \mathrm{Sng}(S) \cup \{\mathrm{id}_X\}$ (see (2.1)). As we will see in the next result, Theorem 4.3 implies that these two semigroups are the only regular centralizers of idempotents in $T(X)$ for which that happens.

Corollary 4.4 *Let ρ be a binary relation on a finite set X such that $T(X, \rho, R)$ is a regular semigroup. Then $\langle E(T(X, \rho, R))\rangle = \mathrm{Sng}(T(X, \rho, R)) \cup \{\mathrm{id}_X\}$ if and only if $\rho = \Delta$ or $\rho = \omega$.*

Proof Let R_1, R_2, and X_2 be as in (4.1).

($\Rightarrow$) We will prove the contrapositive. Suppose $\rho \neq \Delta$ and $\rho \neq \omega$. Then $|R| \geq 2$ and $|X_2| \geq 1$. Hence there are $r_1, r_2 \in R$ with $r_1 \neq r_2$ and $|r_1\rho| \geq 2$. Define $\alpha \in T(X, \rho, R)$ by: $(r_1\rho)\alpha = \{r_2\}$, $(r_2\rho)\alpha = \{r_1\}$, and $y\alpha = y$ for all other $y \in X$. Then $\alpha|_R$ is neither the identity on R nor a singular transformation on R, and so α is not a product of idempotents in $T(X, \rho, R)$ by Theorem 4.3. However, α is singular since it is not injective (it maps all elements of $r_1\rho$ to r_2), and so it is not surjective (since X is finite). This concludes the proof of contrapositive.

($\Leftarrow$) We already know that this implication is true (see Sect. 1 and (2.1)). However, we will show how it follows from Theorem 4.3.

Suppose $\rho = \Delta$. Then $R = R_1 = X$ and $R_2 = X_2 = \emptyset$. Let $\alpha \in T(X, \rho, R)$ be singular. Then $\alpha|_R = \alpha$ is a singular transformation on $R = X$ and $\alpha|_{X_2} = \mathrm{id}_{X_2} = \emptyset$. Thus α is a product of idempotents in $T(X, \rho, R)$ by Theorem 4.3.

Suppose $\rho = \omega$. If $|X| = 1$, then $\omega = \Delta$, and we are done by the foregoing argument. Suppose $|X| \geq 2$. Then there is $t_1 \in X$ such that $R = R_2 = \{t_1\}$, $X_2 = X - \{t_1\}$, and $R_1 = \emptyset$. Let $\alpha \in T(X, \rho, R)$ be singular. Then $t_1\alpha = t_1$, and so $\alpha|_R = \mathrm{id}_R$. Since α is singular, $t_1\alpha = \alpha$, and $X_2 = X - \{t_1\}$, it follows that there is $x \in X_2$ such that $x \notin \mathrm{im}(\alpha)$. Thus α is a product of idempotents in $T(X, \rho, R)$ by Theorem 4.3. □

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