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On the relations between Poissonian white noise analysis and harmonic analysis on configuration spaces

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Abstract

We unify techniques of Poissonian white noise analysis and harmonic analysis on configuration spaces establishing relations between the main structures of both ones. This leads to new results inside of infinite-dimensional analysis as well as in its applications to problems of mathematical physics, e.g., statistical mechanics of continuous systems.

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1. Introduction

This paper complements the work initialized and developed in [KK02,KK04] concerning a particular direction on configuration space analysis. This special

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approach within configuration space analysis is essentially based on the so-called K-transform. Technically, the main feature of this transform is its purely combinatorial nature, independent of any measure under consideration, showing many similarities with the classical Fourier transform. These characteristics of the K-transform are at the origin of the so-called combinatorial harmonic analysis on configuration spaces. Let us mention that the operator nature of the K-transform was first recognized by Lenard [Len73,Len75a,Len75b], in relation with some statistical mechanics problems. However, Lenard did not explored the indistinguishability of "particles" on finite configuration spaces.

Poissonian white noise analysis is essentially based on the chaos decomposition of an L^2-space with respect to a Poisson measure π by using an orthogonal system of Charlier polynomials. Such a chaos decomposition can be obtained in a similar way to the Wiener–Itô–Segal chaos decomposition in terms of Hermite polynomials in Gaussian white noise analysis (see e.g. [HI67,HØ99,Ito88,IK88]). This point of view may be used to study Poissonian white noise analysis and its related topics in an analogous way to the Gaussian case. In particular, this approach interprets the Poisson measures as those on a linear space (e.g. Schwartz distribution space). As Gaussian and Poissonian measures are treated on the same footing, special properties of the Poissonian ones tend to be hidden. Special aspects of Poissonian measures are related with their support properties on the space of locally finite configurations Γ. This is a subset but not a linear subspace of the Schwartz distribution space. Results related to this support property are collected in [KKO02].

The aim of this work is to extend the relations between the Poissonian white noise analysis and the combinatorial harmonic analysis initiated in [KK02]. By exploiting some of its specific techniques and tools one may improve some known results in Poissonian analysis and produce new ones. More precisely, Poissonian white noise analysis yields L^2 or a.s. results, whereas we obtain L^1 or pointwise results by the harmonic analysis.

The work is organized as follows. In Section 2 we recall the structure and concepts of Poissonian white noise analysis presented in [KKO02] (see also for more detailed references) as well as the main notions and results of combinatorial harmonic analysis on configuration spaces presented in [KK02,Kun99]. In Section 2.4 we recall the first results obtained in [KK02] concerning the relation between Poissonian white noise analysis and harmonic analysis. The key result is that the chaos decomposition can be explicitly expressed using the K-transform and an additional operator. The few proofs included complement [KK02], where the results were just announced. On the one hand, the general structure of Poissonian white noise analysis is essentially based on the chaos decomposition of elements of $L^2(\Gamma, \pi)$ through an orthogonal system of Charlier monomials. On the other hand, the combinatorial harmonic analysis on configuration spaces yields a natural decomposition through monomials. The relation between these two decompositions is established at the end of Section 3 through a linear mapping similar to the C-transform in non-Gaussian analysis (see, e.g., [KSWY98]). Furthermore, the results of [KK02] are extended to a pointwise version. In Section 4 a simple algebraic application of the K-transform to the study of the Wick product on the Poisson space [KKO02] is given. As well as in the

Gaussian case, in Poissonian white noise analysis the Wick product is defined through the chaos decomposition. However, within the harmonic analysis setting, this algebraic product can be explicitly described for functions without using their kernels w.r.t. the chaos decomposition. Here again it is used that the variables only vary in the space of configurations and not in distributions.

Properties of the action of the K-transform, and also some related operators, on the spaces of test and generalized functions introduced in [KKO02] are derived in Section 5. We prove that the K-transform maps test functions into test functions (hence distributions into distributions). This shows that the harmonic analysis on configuration spaces is compatible with the generalized function theory of Poissonian white noise analysis allowing, in particular, an extension of the notion of correlation function to the more general concept of correlation generalized function (Section 6). To obtain this extension we use the notion of generalized Radon–Nikodym derivative introduced, e.g., in [BK88]. The rest of the work is devoted to the study and applications of correlation generalized functions. More precisely, we derive an explicit formula for the chaos decomposition of the correlation generalized functions. As an example of application, in the context of Gibbs measures this formula yields an alternative characterization result of Ruelle type [Rue70] for Gibbs measures (Theorem 6.8 and Proposition 6.10).

From the technical point of view, the relations between Poissonian white noise analysis and the combinatorial harmonic analysis on configuration spaces have shown powerful properties to study a special class of functionals introduced by Bogoliubov [Bog46] to study statistical mechanics systems. These functionals, called Bogoliubov or generating functionals, are at the origin of an alternative method to study measure theory problems by using standard functional analysis techniques. Due to the special character of our approach in the study of Bogoliubov functionals, this particular application is subject of a series of forthcoming publications [KK04,KKO04,KO03].

2. Preliminaries

Throughout this work we consider a measure space $(X, \mathscr{B}(X), \sigma)$, where X is a geodesically complete connected oriented (non-compact) Riemannian C^∞-manifold, $\mathscr{B}(X)$ is the Borel σ-algebra on X and σ is a Radon measure on $(X, \mathscr{B}(X))$. In addition, we assume that σ is non-degenerate (i.e., $\sigma(O) > 0$ for all non-empty open sets $O \subset X$) and non-atomic (i.e., $\sigma(\{x\}) = 0$ for every $x \in X$). Having in mind the most interesting applications, we also assume that $\sigma(X) = \infty$.

2.1. Configuration spaces and Poisson measures

The *configuration space* $\Gamma := \Gamma_X$ over X is defined as the set of all locally finite subsets (configurations) of X,

$$\Gamma := \{\gamma \subset X \colon |\gamma \cap K| < \infty \text{ for every compact } K \subset X\},$$

where $|\cdot|$ denotes the cardinality of a set. We identify each configuration $\gamma\in\Gamma$ with the non-negative integer-valued Radon measure $\sum_{x\in\gamma}\varepsilon_x\in\mathscr{M}^+(X)$, where ε_x is the Dirac measure with mass at x and $\mathscr{M}^+(X)$ denotes the space of all non-negative Radon measures on $\mathscr{B}(X)$. In this way, Γ can be endowed with the topology induced by the vague topology on $\mathscr{M}^+(X)$. We denote by $\mathscr{B}(\Gamma)$ the corresponding Borel σ-algebra on Γ. We define the *Poisson measure* π_σ (with intensity σ) as the unique probability measure on Γ w.r.t. which the following equality holds:

$$\int_\Gamma \exp\left(\sum_{x\in\gamma}\varphi(x)\right)d\pi_\sigma(\gamma)=\exp\left(\int_X (e^{\varphi(x)}-1)d\sigma(x)\right)$$

for all $\varphi\in\mathscr{D}$. Here $\mathscr{D}$ denotes the Schwartz space of all infinitely differentiable real-valued functions on X with compact support. In the sequel, the space $L^2(\Gamma,\mathscr{B}(\Gamma),\pi_\sigma)$ of all complex-valued square integrable functions w.r.t. π_σ is shortly denoted by $L^2(\pi_\sigma)$.

Remark 2.1. Introducing the Poisson measure by this approach yields, through the Minlos theorem, a probability measure π_σ defined on $(\mathscr{D}',\mathscr{C}_\sigma(\mathscr{D}'))$, where $\mathscr{D}'$ is the dual space of $\mathscr{D}$ w.r.t. the space of real-valued functions $L^2_{\mathrm{Re}}(\sigma)\subset L^2(\sigma):=L^2(X,\mathscr{B}(X),\sigma)$ and $\mathscr{C}_\sigma(\mathscr{D}')$ is the σ-algebra generated by the cylinder sets

$$\{\omega\in\mathscr{D}':\ (\langle\omega,\varphi_1\rangle,\dots,\langle\omega,\varphi_n\rangle)\in B\},\quad \varphi_i\in\mathscr{D},\ \ B\in\mathscr{B}(\mathbb{R}^n),\ \ n\in\mathbb{N}.$$

An additional analysis shows that this measure is actually supported on generalized functions of the form $\sum_{x\in\gamma}\varepsilon_x$, $\gamma\in\Gamma$. Hence π_σ can be considered as a measure on Γ. For more details see e.g. [KKO02] and also the references therein. To exploit more effectively this support property is one of the aims of this work.

For each $Y\in\mathscr{B}(X)$ let us consider the space Γ_Y of all configurations contained in Y, $\Gamma_Y:=\{\gamma\in\Gamma:\ |\gamma\cap(X\backslash Y)|=0\}$, and the *space* $\Gamma_Y^{(n)}$ *of n-point configurations*, $\Gamma_Y^{(n)}:=\{\gamma\in\Gamma_Y:\ |\gamma|=n\}, n\in\mathbb{N}$, $\Gamma_Y^{(0)}:=\{\emptyset\}$. For $\widetilde{Y^n}:=\{(x_1,\dots,x_n):\ x_i\in Y, x_i\neq x_j \text{ if } i\neq j\}$ we introduce the mapping

$$\mathrm{sym}_Y^n:\widetilde{Y^n}\to\Gamma_Y^{(n)},$$

$$(x_1,\dots,x_n)\mapsto\{x_1,\dots,x_n\}.$$

This mapping defines a natural bijection between $\Gamma_Y^{(n)}$ and the symmetrization $\widetilde{Y^n}/S_n$ of $\widetilde{Y^n}$, where S_n is the permutation group over $\{1,\dots,n\}$. Thus, sym_Y^n induces a metric on $\Gamma_Y^{(n)}$ and then the corresponding Borel σ-algebra on $\Gamma_Y^{(n)}$ which we denote by $\mathscr{B}(\Gamma_Y^{(n)})$. For $\Lambda\in\mathscr{B}(X)$ with compact closure ($\Lambda\in\mathscr{B}_{\mathrm{c}}(X)$), it is clear that $\Gamma_\Lambda=$

$\bigsqcup_{n=0}^{\infty} \Gamma_{\Lambda}^{(n)}$. In this case we define the σ-algebra $\mathscr{B}(\Gamma_\Lambda)$ by the disjoint union of the σ-algebras $\mathscr{B}(\Gamma_{\Lambda}^{(n)})$, $n \in \mathbb{N}_0$.

Among the subsets of Γ we also distinguish the *space of finite configurations*

$$\Gamma_0 := \bigsqcup_{n=0}^{\infty} \Gamma_X^{(n)}.$$

We endow Γ_0 with the topology of disjoint union of topological spaces and with the corresponding Borel σ-algebra denoted by $\mathscr{B}(\Gamma_0)$. For the construction of a measure on Γ_0 we consider the product measure $\sigma^{\otimes n}$ on $(X^n, \mathscr{B}(X^n))$ restricted to $\widetilde{X^n}$. Note that $\sigma^{\otimes n}(X^n \backslash \widetilde{X^n}) = 0$. In the sequel, we denote by $\sigma^{(n)} := \sigma^{\otimes n} \circ (\mathrm{sym}_X^n)^{-1}$ the corresponding image measure on the space $\Gamma_X^{(n)}$ under the mapping sym_X^n $(n \in \mathbb{N})$ and we set $\sigma^{(0)}(\{\emptyset\}) := 1$. Then on $(\Gamma_0, \mathscr{B}(\Gamma_0))$ we define the so-called *Lebesgue–Poisson measure* λ_σ *with intensity measure* σ by $\lambda_\sigma := \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{(n)}$. We denote by $L^2(\lambda_\sigma)$ the corresponding complex space $L^2(\Gamma_0, \mathscr{B}(\Gamma_0), \lambda_\sigma)$. Note that λ_σ is not a finite measure if and only if $\sigma(X) = \infty$, and in this case $\pi_\sigma(\Gamma_0) = 0$.

2.2. Some aspects of Poissonian white noise analysis

The description of elements of the space $L^2(\pi_\sigma)$ by the corresponding chaos decomposition provides a unitary isomorphism between the spaces $L^2(\pi_\sigma)$ and $L^2(\lambda_\sigma)$. This fact is recalled here (see the presentation in [KKO02] and the references therein for more details and proofs).

As we mentioned in Section 2.1, the Poisson measure π_σ can be either considered on $(\Gamma, \mathscr{B}(\Gamma))$ or on $(\mathscr{D}', \mathscr{C}_\sigma(\mathscr{D}'))$, where, in contrast to Γ, $\mathscr{D}' \supset \Gamma$ is a linear space. Since $\pi_\sigma(\Gamma) = 1$, the measure space $(\mathscr{D}', \mathscr{C}_\sigma(\mathscr{D}'), \pi_\sigma)$ can be, in this way, regarded as a linear extension of the space $(\Gamma, \mathscr{B}(\Gamma), \pi_\sigma)$. In what follows we shall always keep in mind the embeddings $\Gamma \subset \mathscr{M}^+(X) \subset \mathscr{D}'$.

Given a $-1 < \varphi \in \mathscr{D}$ we define the *Poissonian exponential* $e_\pi(\varphi)$ by

$$e_\pi(\varphi, \omega) := \exp\left(\langle \omega, \log(1 + \varphi) \rangle - \int_X \varphi(x)\, d\sigma(x) \right) \tag{2.1}$$

for $\omega \in \mathscr{D}'$. The holomorphy of $e_\pi(\cdot, \omega)$, $\omega \in \mathscr{D}'$, on a neighborhood of zero allows to consider its Taylor expansion which, by the Cauchy formula, the polarization identity, and the kernel theorem (see, e.g., [BK88,KSS97,KSWY98]), provides the decomposition

$$e_\pi(\varphi, \omega) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle C_n^\sigma(\omega), \varphi^{\otimes n} \rangle, \quad \omega \in \mathscr{D}',$$

where $C_n^\sigma : \mathscr{D}' \to \mathscr{D}'^{\hat{\otimes} n}$, $n \in \mathbb{N}$, are the so-called *generalized Charlier kernels*. For $\varphi^{(n)} \in \mathscr{D}_{\mathbb{C}}^{\hat{\otimes} n}$, $n \in \mathbb{N}$ ($\mathscr{D}_{\mathbb{C}} :=$ the complexification of the space $\mathscr{D}$), we can define the corresponding *smooth Charlier monomial of order n* by $\langle C_n^\sigma(\omega), \varphi^{(n)} \rangle$, $\omega \in \mathcal{D}'$. The following orthogonality relation holds:

$$(\langle C_n^\sigma, \varphi^{(n)} \rangle, \langle C_m^\sigma, \phi^{(m)} \rangle)_{L^2(\pi_\sigma)} = \delta_{n,m} n! (\varphi^{(n)}, \phi^{(n)})_{L^2(\sigma^{\otimes n})}$$

which allows the use of an approximation procedure to extend the class of smooth Charlier monomials to measurable monomials $\langle C_n^\sigma, f^{(n)} \rangle$ with symmetric kernels $f^{(n)} \in L^2(X^n, \sigma^{\otimes n})$ ($f^{(n)} \in \hat{L}^2(X^n, \sigma^{\otimes n})$) in such a way that the above orthogonality property still holds for this extension. In other words, for each $f^{(n)} \in \hat{L}^2(X^n, \sigma^{\otimes n})$, $n \in \mathbb{N}$,

$$\langle C_n^\sigma, f^{(n)} \rangle := L^2(\pi_\sigma) - \lim_{m \to \infty} \langle C_n^\sigma, \varphi_m^{(n)} \rangle, \tag{2.2}$$

where $(\varphi_m^{(n)})_{m \in \mathbb{N}}$ is a sequence of elements of $\mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}$ converging to $f^{(n)}$ in $L^2(X^n, \sigma^{\otimes n})$.

Let us now consider the space $\mathcal{P}(\mathcal{D}')$ of *smooth continuous polynomials on* $\mathcal{D}'$

$$\mathcal{P}(\mathcal{D}') := \left\{ \Phi\colon\ \Phi(\omega) = \sum_{n=0}^{N} \langle C_n^\sigma(\omega), \varphi^{(n)} \rangle, \varphi^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}, \omega \in \mathcal{D}', N \in \mathbb{N}_0 \right\}. \tag{2.3}$$

Since the space $\mathcal{P}(\mathcal{D}')$ is densely embedded into $L^2(\pi_\sigma)$ [Sko74, Section 10, Theorem 1], it follows that for any $F \in L^2(\pi_\sigma)$ there exists a sequence $(f^{(n)})_{n=0}^\infty$, $f^{(n)} \in \hat{L}^2(X^n, \sigma^{\otimes n})$, $\sum_{n=0}^\infty n! |f^{(n)}|^2_{L^2(\sigma^{\otimes n})} < \infty$ such that

$$F = \sum_{n=0}^{\infty} \langle C_n^\sigma, f^{(n)} \rangle \tag{2.4}$$

and, moreover, by the orthogonality property,

$$||F||^2_{L^2(\pi_\sigma)} = \sum_{n=0}^{\infty} n! |f^{(n)}|^2_{L^2(\sigma^{\otimes n})}.$$

And vice versa, any series of the form (2.4) with $f^{(n)} \in \hat{L}^2(X^n, \sigma^{\otimes n})$ and $\sum_{n=0}^\infty n! |f^{(n)}|^2_{L^2(\sigma^{\otimes n})} < \infty$ defines a function from $L^2(\pi_\sigma)$. As a result, we have the so-called *chaos decomposition* (2.4).

As a consequence, the chaos decomposition (2.4) provides the unitary isomorphism $I_{\pi\lambda} : L^2(\pi_\sigma) \to L^2(\lambda_\sigma)$,

$$I_{\pi\lambda}\left(\sum_{n=0}^{\infty} \langle C_n^\sigma, f^{(n)} \rangle \right) = G,$$
$$G(\{x_1, \ldots, x_n\}) := n! f^{(n)}(x_1, \ldots, x_n), \quad n \in \mathbb{N}, \ G(\emptyset) := f^{(0)}.$$

The inverse unitary isomorphism $I_{\lambda\pi}: L^2(\lambda_\sigma) \to L^2(\pi_\sigma)$ is defined in a similar way, i.e.,

$$I_{\lambda\pi}(G) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle C_n^\sigma, G^{(n)} \rangle, \quad G^{(n)} := G\restriction_{\Gamma_X^{(n)}}, \; n \in \mathbb{N}_0.$$

Remark 2.2. Since the measure σ is non-atomic, each kernel $f^{(n)} := \frac{1}{n!} G^{(n)}$ is well-defined as an element of the space $\hat{L}^2(X^n, \sigma^{\otimes n})$.

Note that given a $-1 < \varphi \in \mathcal{D}$, for each $\gamma \in \Gamma$ we have for (2.1)

$$e_\pi(\varphi, \gamma) = \exp\left(- \int_X \varphi(x) \, d\sigma(x) \right) \prod_{x \in \gamma} (1 + \varphi(x)).$$

For $f \in B_{\mathrm{bs}}(X)$ (:= the space of bounded $\mathcal{B}(X)$-measurable functions with bounded support) we define the Poissonian exponential $e_\pi(f, \gamma)$ for $\gamma \in \Gamma \subset \mathcal{D}'(X)$ by the same expression. Through the chaos decomposition one can extend this definition to functions $f \in L^2(\sigma)$. Indeed, since the sum $\sum_{n=0}^\infty \frac{1}{n!} \langle C_n^\sigma, f^{\otimes n} \rangle$ converges in $L^2(\pi_\sigma)$, we define the Poissonian exponential $e_\pi(f) \in L^2(\pi_\sigma)$ by this sum. Its image under the isomorphism $I_{\pi\lambda}$ is the so-called (*Lebesgue–Poisson*) *coherent state* $e_\lambda(f): \Gamma_0 \to \mathbb{C}$ *corresponding to the one-particle vector* f. By definition, for any $\mathcal{B}(X)$-measurable function f,

$$e_\lambda(f, \eta) := \prod_{x \in \eta} f(x), \quad \eta \in \Gamma_0,$$

where we have set $e_\lambda(f, \emptyset) := 1$. We observe that if $f \in L^p(X, \sigma)$ for some $p \geq 1$, then $e_\lambda(f) \in L^p(\Gamma_0, \lambda_\sigma)$, and a simple computation yields

$$||e_\lambda(f)||_{L^p(\lambda_\sigma)} = \exp\left(\frac{1}{p} ||f||_{L^p(\sigma)}^p \right). \tag{2.5}$$

For what follows, it is also important to note that if $\mathcal{L} \subset L^2(\sigma)$ is a dense subspace, then the set $\{e_\lambda(f): f \in \mathcal{L}\}$ is total in $L^2(\lambda_\sigma)$.

2.3. *Harmonic analysis*

Apart from the unitary isomorphism $I_{\lambda\pi}$, the K-transform is a mapping which also maps functions on Γ_0 into functions on Γ. The definition and the main properties of the K-transform are recalled below. For more details see [KK02].

We start by defining some spaces of functions on Γ_0 and on Γ.

In the space $L^0(\Gamma_0, \mathcal{B}(\Gamma_0))$ of all (complex-valued) $\mathcal{B}(\Gamma_0)$-measurable functions let us consider the subset $L^0_{\mathrm{ls}}(\Gamma_0)$ of all functions $G \in L^0(\Gamma_0, \mathcal{B}(\Gamma_0))$ with local support, i.e., $G\restriction_{\Gamma_0 \setminus \Gamma_\Lambda} \equiv 0$ for some $\Lambda \in \mathcal{B}_{\mathrm{c}}(X)$. We also distinguish the subspace $B_{\mathrm{bs}}(\Gamma_0)$ of all bounded functions $G\in L^0_{\mathrm{ls}}(\Gamma_0)$ with bounded support, i.e., $G\restriction_{\Gamma_0\setminus\left(\bigsqcup_{n=0}^{N}\Gamma_\Lambda^{(n)}\right)}\equiv 0$ for some $N\in\mathbb{N}_0, \Lambda\in\mathscr{B}_{\mathrm{c}}(X)$. With respect to the space $L^2(\lambda_\sigma)$, $B_{\mathrm{bs}}(\Gamma_0)$ is a dense subset.

Concerning functions defined on Γ, let us denote by $L^0(\Gamma,\mathscr{B}(\Gamma))$ the space of all (complex-valued) $\mathscr{B}(\Gamma)$-measurable functions. We denote by $\mathscr{F}L^0(\Gamma)$ the class of cylinder functions, i.e., functions $F\in L^0(\Gamma,\mathscr{B}(\Gamma))$ such that for some $\Lambda\in\mathscr{B}_{\mathrm{c}}(X)$ one has $F\restriction_{\Gamma_\Lambda}\in L^0(\Gamma_\Lambda,\mathscr{B}(\Gamma_\Lambda))$ and $F(\gamma)=F\restriction_{\Gamma_\Lambda}(\gamma_\Lambda)$, $\gamma_\Lambda:=\gamma\cap\Lambda$, for all $\gamma\in\Gamma$. A cylinder function F of such a form is called *polynomially bounded*, shortly $F\in\mathscr{F}L^0_{\mathrm{pb}}(\Gamma)$, whenever there exists a polynomial P on $\mathbb{R}$ such that $|F(\gamma_\Lambda)|\leqslant P(|\gamma_\Lambda|)$ for all $\gamma\in\Gamma$.

Definition 2.3. Given a $G\in L^0_{\mathrm{ls}}(\Gamma_0)$ we define a function KG on Γ by

$$(KG)(\gamma):=\sum_{\eta\Subset\gamma}G(\eta),\quad \gamma\in\Gamma, \tag{2.6}$$

the sum being over all finite subconfigurations η from γ ($\eta\Subset\gamma$). The mapping K is called the K-transform.

Note that for any $G\in L^0_{\mathrm{ls}}(\Gamma_0)$ the sum in (2.6) has only a finite number of summands different from zero and thus KG is a well-defined function on Γ. In particular, for coherent states on the Lebesgue–Poisson space, the K-transform has an especially simple form. For all functions $f\in B_{\mathrm{bs}}(X)$,

$$(Ke_\lambda(f))(\gamma)=\prod_{x\in\gamma}(1+f(x)),\quad \gamma\in\Gamma.$$

The next result collects some properties of the K-transform.

Proposition 2.4. (i) *The K-transform is a linear and positivity preserving mapping.*

(ii) *The mapping $K:L^0_{\mathrm{ls}}(\Gamma_0)\to\mathscr{F}L^0(\Gamma)$ is invertible and its inverse mapping is defined on $\mathscr{F}L^0(\Gamma)$ by*

$$(K^{-1}F)(\eta):=\sum_{\xi\subset\eta}(-1)^{|\eta\setminus\xi|}F(\xi),\quad \eta\in\Gamma_0.$$

(iii) *For all functions $G\in B_{\mathrm{bs}}(\Gamma_0)$ one has $KG\in\mathscr{F}L^0_{\mathrm{pb}}(\Gamma)$.*

As well as the K-transform, its dual operator K^* will also play an essential role in our setting. In the sequel, we denote by $\mathscr{M}^1_{\mathrm{fm}}(\Gamma)$ the set of all probability measures μ on $(\Gamma,\mathscr{B}(\Gamma))$ with finite moments of all orders, i.e.,

$$\int_\Gamma|\gamma_\Lambda|^n\,d\mu(\gamma)<\infty\quad\text{for all } n\in\mathbb{N}\text{ and all }\Lambda\in\mathscr{B}_{\mathrm{c}}(X).$$

By the definition of a dual operator, given a $\mu \in \mathscr{M}^1_{\mathrm{fm}}(\Gamma)$, $K^*\mu =: \rho_\mu$ is a measure on $(\Gamma_0, \mathscr{B}(\Gamma_0))$ defined for each $G \in B_{\mathrm{bs}}(\Gamma_0)$ by

$$\int_{\Gamma_0} G(\eta)\, d\rho_\mu(\eta) = \int_\Gamma (KG)(\gamma)\, d\mu(\gamma). \tag{2.7}$$

Observe that the sum in KG is finite and $K|G|$ is integrable under the above assumptions. Following the terminology used in the Gibbsian case (Section 6), we call ρ_μ the *correlation measure corresponding to* μ. As a result, if $\mathscr{M}_{\mathrm{lf}}(\Gamma_0)$ denotes the class of all measures ρ on $(\Gamma_0, \mathscr{B}(\Gamma_0))$ such that $\int_{\Gamma_0} G(\eta)\, d\rho(\eta) < \infty$ for all non-negative $G \in B_{\mathrm{bs}}(\Gamma_0)$, then the general duality theory yields the mapping between the spaces $\mathscr{M}^1_{\mathrm{fm}}(\Gamma)$ and $\mathscr{M}_{\mathrm{lf}}(\Gamma_0)$:

$$K^* : \mathscr{M}^1_{\mathrm{fm}}(\Gamma) \to \mathscr{M}_{\mathrm{lf}}(\Gamma_0),$$

$$\mu \mapsto K^*\mu := \rho_\mu.$$

As a first example, the Lebesgue–Poisson measure is the correlation measure corresponding to the Poisson measure, i.e., $K^*\pi_\sigma = \rho_{\pi_\sigma} = \lambda_\sigma$.

All the notions described above are graphically summarized in the figure below:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
F \arrow[r, leftrightarrow, "{< F, \mu >= \int_\Gamma F(\gamma) d\mu(\gamma)}"] & \mu \arrow[d, "K^*"] \\
G \arrow[u, "K"] \arrow[r, leftrightarrow, "{< G, \rho_\mu >= \int_{\Gamma_0} G(\eta) d\rho_\mu(\eta)}"'] & \rho_\mu
\end{tikzcd} \tag{$*$}$$

2.4. Poissonian white noise analysis and harmonic analysis

Poissonian white noise analysis and harmonic analysis are related through an equality of operators involving the unitary isomorphism $I_{\lambda\pi}$ and the K-transform (Theorem 2.7). This formula is established by an extension of the K-transform to a convenient Banach space and by a bounded operator defined on a space of functions on Γ_0. These facts are recalled here [KK02].

Let us consider the linear operator D defined on $B_{\mathrm{bs}}(\Gamma_0)$ by

$$(DG)(\eta) := \int_{\Gamma_0} G(\eta \cup \xi)\, d\lambda_\sigma(\xi), \quad \eta \in \Gamma_0. \tag{2.8}$$

It is easy to check that $DG \in B_{\mathrm{bs}}(\Gamma_0)$, and, moreover, D is an isomorphism in $B_{\mathrm{bs}}(\Gamma_0)$. The inverse mapping is defined by

$$(D^{-1}G)(\eta) := \int_{\Gamma_0} (-1)^{|\xi|} G(\eta \cup \xi)\, d\lambda_\sigma(\xi), \quad \eta \in \Gamma_0. \tag{2.9}$$

Proposition 2.5. *The D and D^{-1} operators can be extended to bounded operators defined on $L^1(\lambda_{2\sigma})$ with values in $L^1(\lambda_\sigma)$. The extended operators we denote as before by D and D^{-1}. For all $G \in L^1(\lambda_{2\sigma})$ formulas* (2.8) *and* (2.9) *hold for λ_σ-a.a $\eta \in \Gamma_0$. Furthermore, $D^{-1}DG = G = DD^{-1}G$ for all $G \in L^1(\lambda_{4\sigma})$.*

Lemma 2.6 (Fichtner and Freudenberg [FF91], Kondratiev et al. [KKO02], Ruelle [Rue69]). *The following equality holds*:

$$\int_{\Gamma_0} \int_{\Gamma_0} G(\eta \cup \xi) H(\xi, \eta)\, d\lambda_\sigma(\eta)\, d\lambda_\sigma(\xi) = \int_{\Gamma_0} G(\eta) \sum_{\xi \subset \eta} H(\xi, \eta \backslash \xi)\, d\lambda_\sigma(\eta)$$

for all positive measurable functions $G : \Gamma_0 \to \mathbb{R}$ and $H : \Gamma_0 \times \Gamma_0 \to \mathbb{R}$.

Proof. As a direct consequence of Lemma 2.6, for each $G \in L^1(\lambda_{2\sigma})$ the norms $||DG||_{L^1(\lambda_\sigma)}$ and $||D^{-1}G||_{L^1(\lambda_\sigma)}$ are majorized by

$$\begin{aligned}\int_{\Gamma_0} \int_{\Gamma_0} |G(\eta \cup \xi)|\, d\lambda_\sigma(\eta)\, d\lambda_\sigma(\xi) &= \int_{\Gamma_0} 2^{|\eta|} |G(\eta)|\, d\lambda_\sigma(\eta) \\ &= ||G||_{L^1(\lambda_{2\sigma})},\end{aligned}$$

which is enough to prove the existence of the extensions. Standard measure theory techniques show that the extensions verify (2.8) and (2.9) [KK02]. The continuity of the operators D and D^{-1} combined with the fact that D is an isomorphism in $B_{\mathrm{bs}}(\Gamma_0)$ yields the last assertion. $\square$

Concerning the K-transform, observe that equality (2.7) leads to the inequality of norms $||KG||_{L^1(\Gamma,\mu)} \leqslant ||K|G|\,||_{L^1(\Gamma,\mu)} = ||G||_{L^1(\Gamma_0,\rho_\mu)}$ for all $G \in B_{\mathrm{bs}}(\Gamma_0)$. This fact allows an extension of the K-transform to a bounded operator $K : L^1(\Gamma_0, \rho_\mu) \to L^1(\Gamma, \mu)$ in such a way that relation (2.7) still holds. More important, for all $G \in L^1(\Gamma_0, \rho_\mu)$ also the explicit formula for K is preserved in the following sense:

$$(KG)(\gamma) = \sum_{\eta \Subset \gamma} G(\eta), \quad \mu - \text{a.a. } \gamma \in \Gamma.$$

We are now ready to state the result which connects the combinatorial harmonic analysis on configuration spaces and the Poissonian white noise analysis.

Theorem 2.7. *On $L^1(\lambda_{2\sigma})\cap L^2(\lambda_\sigma)$ the following relation holds*:

$$I_{\lambda\pi} = KD^{-1}.$$

As a straightforward consequence of Theorem 2.7, one may derive an explicit form (of an integral type) for the unitary isomorphism $I_{\lambda\pi}$.

Corollary 2.8. *For all $G\in L^1(\lambda_{2\sigma})\cap L^2(\lambda_\sigma)$ we have*

$$(I_{\lambda\pi}G)(\gamma) = \sum_{\eta\Subset\gamma}\int_{\Gamma_0}(-1)^{|\xi|}G(\xi\cup\eta)\,d\lambda_\sigma(\xi), \quad \pi_\sigma-\text{a.a. }\gamma\in\Gamma.$$

As another direct consequence of Theorem 2.7, one can deduce an explicit formula for the Charlier polynomials, already obtained in [Sur84].

Corollary 2.9. *Given a $f^{(n)}\in\mathscr{D}^{\hat{\otimes}n}$, $n\in\mathbb{N}$, for π_σ-a.a. $\gamma\in\Gamma$ we have*

$$\begin{aligned}\langle C_n^\sigma(\gamma),f^{(n)}\rangle &= \sum_{k=0}^{n}\sum_{\{x_1,\dots,x_k\}\subset\gamma}(-1)^{n-k}\frac{n!}{(n-k)!}\\ &\quad\times\int_{X^{n-k}}f^{(n)}(x_1,\dots,x_k,y_1,\dots,y_{n-k})\,d\sigma^{\otimes n-k}(y_1,\dots,y_{n-k}).\end{aligned}\tag{2.10}$$

Proof. Given a $f^{(n)}\in\mathscr{D}^{\hat{\otimes}n}$ ($n\in\mathbb{N}$), let us consider $\langle C_n^\sigma,f^{(n)}\rangle\in L^2(\pi_\sigma)$. Observe that $\langle C_n^\sigma,f^{(n)}\rangle = I_{\lambda\pi}G^{(n)}$ for

$$G^{(n)}(\eta) = \begin{cases} n!f^{(n)}(x_1,\dots,x_n) & \text{if } \eta=\{x_1,\dots,x_n\}\in\Gamma_X^{(n)},\\ 0 & \text{otherwise.}\end{cases}$$

Hence, by Corollary 2.8, it follows that for π_σ-a.a. $\gamma\in\Gamma$

$$\begin{aligned}\langle C_n^\sigma(\gamma),f^{(n)}\rangle &= (I_{\lambda\pi}G^{(n)})(\gamma) = \sum_{\eta\Subset\gamma}\int_{\Gamma_0}(-1)^{|\xi|}G^{(n)}(\xi\cup\eta)\,d\lambda_\sigma(\xi)\\ &= \sum_{k=0}^{n}\sum_{\eta\subset\gamma,|\eta|=k}\frac{(-1)^{n-k}}{(n-k)!}\int_{\Gamma_X^{(n-k)}}G^{(n)}(\xi\cup\eta)\,d\sigma^{(n-k)}(\xi). \qquad\square\end{aligned}$$

Conversely, Theorem 2.7 also allows writing the K-transform in terms of its chaos decomposition. In the sequel $B_{\text{bs}}(\Gamma_X^{(n)})$ ($n\in\mathbb{N}$) denotes the space of bounded $\mathscr{B}(\Gamma_X^{(n)})$-measurable functions with bounded support.

Corollary 2.10. *For all* $G^{(n)} \in B_{\mathrm{bs}}(\Gamma_X^{(n)})$ *we have for* π_σ*-a.a.* $\gamma \in \Gamma$

$$(KG^{(n)})(\gamma) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\Bigg\langle C_k^\sigma(\gamma), \int_{\Gamma_X^{(n-k)}} G^{(n)}(\{\cdot,\dots,\cdot,y_1,\dots,y_{n-k}\}) \times d\sigma^{(n-k)}(\{y_1,\dots,y_{n-k}\})\Bigg\rangle.$$

Proof. By Theorem 2.7, for any $G^{(n)}$ under the above conditions we find

$$KG^{(n)} = I_{\lambda\pi}(DG^{(n)}).$$

Since

$$(DG^{(n)})(\eta) = \begin{cases} \frac{1}{(n-k)!}\int_{\Gamma_X^{(n-k)}} G^{(n)}(\eta\cup\xi)\, d\sigma^{(n-k)}(\xi) & \text{if } \eta\in\Gamma_X^{(k)}, 0\leqslant k\leqslant n, \\ 0 & \text{if } \eta\in\Gamma_X^{(k)}, k>n, \end{cases} \tag{2.11}$$

the definition of the isomorphism $I_{\lambda\pi}$ yields that for π_σ-a.a. $\gamma\in\Gamma$

$$(KG^{(n)})(\gamma) = \sum_{k=0}^{n}\langle C_k^\sigma(\gamma), g^{(k)}\rangle$$

with $g^{(k)}(x_1,\dots,x_k) = \frac{1}{k!}(DG^{(n)})(\{x_1,\dots,x_k\})$. $\square$

3. Monomials

Let us denote by $\hat{B}_{\mathrm{bs}}(X^n)$ $(n\in\mathbb{N})$ the space of (complex-valued) symmetric bounded $\mathscr{B}(X^n)$-measurable functions with bounded support. According to Section 2.2, a pointwise description of the Charlier monomials is possible for kernels in $\mathscr{D}^{\hat{\otimes} n}$. By the L^2-approximation procedure described in (2.2), the monomials $\langle C_n^\sigma, f^{(n)}\rangle$ can then be extended to kernels $f^{(n)}\in\hat{L}^2(X^n,\sigma^{\otimes n})$. In particular, to $f^{(n)}\in\hat{B}_{\mathrm{bs}}(X^n)$. However, according to this L^2-approximation, the monomials obtained in this way, as functions of γ, are only defined π_σ-a.e. On the other hand, Corollary 2.9 gives the explicit formula (2.10) for the Charlier monomials. Observe that the right-hand side of (2.10) is also well-defined for $f^{(n)}\in\hat{B}_{\mathrm{bs}}(X^n)$ and, moreover, it can be extended to all $\gamma\in\Gamma$. As expression (2.10) is polynomially bounded this extension coincides with the L^2-extension discussed before. In other words, harmonic analysis gives, through equality (2.10), another way to extend the pointwise definition of the Charlier monomials $\langle C_n^\sigma, f^{(n)}\rangle$, for example, to kernels $f^{(n)}\in\hat{B}_{\mathrm{bs}}(X^n)$. So, by definition, for all $\gamma \in \Gamma$

$$\begin{aligned}\langle C_n^\sigma(\gamma), f^{(n)} \rangle &:= ((KD^{-1})G^{(n)})(\gamma) \\ &= \sum_{k=0}^{n} \sum_{\{x_1,\ldots,x_k\}\subset\gamma} (-1)^{n-k} \frac{n!}{(n-k)!} \\ &\quad \times \int_{X^{n-k}} f^{(n)}(x_1,\ldots,x_k,y_1,\ldots,y_{n-k})\, d\sigma^{\otimes n-k}(y_1,\ldots,y_{n-k}) \qquad (3.1)\end{aligned}$$

with

$$G^{(n)}(\eta) = \begin{cases} n! f^{(n)}(x_1,\ldots,x_n) & \text{if } \eta = \{x_1,\ldots,x_n\} \in \Gamma_X^{(n)}, \\ 0 & \text{otherwise.} \end{cases} \qquad (3.2)$$

Thus, additionally to Proposition 2.4, the K-transform has the following property.

Proposition 3.1. *Let $G^{(n)} \in B_{\mathrm{bs}}(\Gamma_X^{(n)})$ be given. Then for all $\gamma \in \Gamma$ (not only π_σ-a.a.) we have*

$$\begin{aligned}(KG^{(n)})(\gamma) = &\frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \Bigg\langle C_k^\sigma(\gamma), \int_{\Gamma_X^{(n-k)}} G^{(n)}(\{\cdot,\ldots,\cdot,y_1,\ldots,y_{n-k}\}) \\ &\times d\sigma^{(n-k)}(\{y_1,\ldots,y_{n-k}\}) \Bigg\rangle. \qquad (3.3)\end{aligned}$$

As a consequence, $K: B_{\mathrm{bs}}(\Gamma_0) \to \mathscr{FP}_{\mathrm{bc}}(\Gamma)$ is a linear isomorphism where $\mathscr{FP}_{\mathrm{bc}}(\Gamma)$ is the space defined by

$$\mathscr{FP}_{\mathrm{bc}}(\Gamma) := \left\{ \sum_{n=0}^{N} \langle C_n^\sigma, f^{(n)} \rangle : f^{(n)} \in \hat{B}_{\mathrm{bs}}(X^n), n = 1,\ldots,N \in \mathbb{N} \right\}.$$

Proof. For all $\gamma \in \Gamma$ it is

$$(KG^{(n)})(\gamma) = (KD^{-1}(DG^{(n)}))(\gamma).$$

According to (2.11) and Definition (3.1), this implies (3.3). $\square$

The K-transform yields a natural decomposition of functions on Γ in monomials

$$(KG^{(n)})(\gamma) = \sum_{\{x_1,\ldots,x_n\}\subset\gamma} G(\{x_1,\ldots,x_n\}) =: \langle \gamma^{\odot n}, G^{(n)} \rangle.$$

In order to establish the relation between this decomposition and the chaos decomposition, let us again consider the space $\mathscr{P}(\mathscr{D}')$ of smooth continuous polynomials on $\mathscr{D}'$ described in (2.3), and the linear operator C defined on $\mathscr{P}(\mathscr{D}')$ by

$$(C\Phi)(\gamma) := \int_\Gamma \Phi(\gamma \cup \gamma')\, d\pi_\sigma(\gamma'), \quad \Phi \in \mathscr{P}(\mathscr{D}'),\ \gamma \in \Gamma.$$

We observe that the operator C is defined in a way similar to the C-transform in non-Gaussian analysis (see e.g. [KSWY98]).

Proposition 3.2. *On $\mathscr{P}(\mathscr{D}')$ the following equality holds*:

$$KDK^{-1} = C. \tag{3.4}$$

Proof. By the definition of the space $\mathscr{P}(\mathscr{D}')$, it is enough to prove equality (3.4) for the smooth Charlier monomials $\langle C_n^\sigma, f^{(n)} \rangle$, $f^{(n)} \in \mathscr{D}_{\mathbb{C}}^{\hat{\otimes} n}$, $n \in \mathbb{N}$. According to Definition (2.1) of the Poissonian exponential, for all $-1 < \varphi \in \mathscr{D}$ we find

$$e_\pi(\varphi, \omega + \omega') = e_\pi(\varphi, \omega) e_\pi(\varphi, \omega') \exp\left(\int_X \varphi(x)\, d\sigma(x) \right),$$

leading, by comparison of coefficients, to

$$C_n^\sigma(\omega + \omega') = \sum_{k+l+m=n} \frac{n!}{k!l!m!} C_k^\sigma(\omega) \hat{\otimes} C_l^\sigma(\omega') \hat{\otimes} \mathscr{I}_m,$$

where $\mathscr{I}_m \in {\mathscr{D}_{\mathbb{C}}}'^{\hat{\otimes} m}$ denotes the distribution corresponding to the function identically equal to 1. Therefore,

$$\int_\Gamma \langle C_n^\sigma(\gamma \cup \gamma'), f^{(n)} \rangle \, d\pi_\sigma(\gamma') = \sum_{k+m=n} \binom{n}{k} \langle C_k^\sigma(\gamma) \hat{\otimes} \mathscr{I}_m, f^{(n)} \rangle,$$

where, by equality (3.3), the above expression coincides with $(KG^{(n)})(\gamma)$ for $G^{(n)}$ given as in (3.2). By definition (3.1) we have $G^{(n)} = (DK^{-1})(\langle C_n^\sigma, f^{(n)} \rangle)$ and thus

$$\int_\Gamma \langle C_n^\sigma(\gamma \cup \gamma'), f^{(n)} \rangle \, d\pi_\sigma(\gamma') = (KG^{(n)})(\gamma) = (KDK^{-1})(\langle C_n^\sigma(\gamma), f^{(n)} \rangle). \qquad \square$$

Note that the definitions of the operators D and C are similar. Apart from this fact, Proposition 3.2 states that one can obtain the operator C as the image of the operator D under the K-transform. Furthermore, if we use the notation

$$(KG^{(n)})(\gamma) := \langle \gamma^{\odot n}, G^{(n)} \rangle,$$

we then obtain the relation

$$C(\langle C_n^\sigma, G^{(n)} \rangle) = \langle \cdot^{\odot n}, G^{(n)} \rangle,$$

according to Proposition 3.2 and Theorem 2.7. In other words, the operator C transforms each chaos decomposition $\sum_n \langle C_n^\sigma(\gamma), G^{(n)} \rangle$ into the decomposition $\sum_n \langle \gamma^{\odot n}, G^{(n)} \rangle$.

4. The Wick product and the K-transform

Before proceeding further, let us illustrate with a simple algebraic example few implications of Theorem 2.7.

Let us consider the Ruelle convolution or $*$-convolution defined on G_1, $G_2 \in L^0(\Gamma_0, \mathscr{B}(\Gamma_0))$ by

$$(G_1 * G_2)(\eta) := \sum_{\xi \subset \eta} G_1(\xi) G_2(\eta \backslash \xi), \quad \eta \in \Gamma_0.$$

The space $L^0(\Gamma_0, \mathscr{B}(\Gamma_0))$ endowed with this product has a structure of a commutative algebra with unit element $e_\lambda(0)$. Observe that given G_1, $G_2 \in B_{\mathrm{bs}}(\Gamma_0)$ with support contained in $\Gamma_{\Lambda_1}, \Gamma_{\Lambda_2}$ for some $\Lambda_1, \Lambda_2 \in \mathscr{B}_{\mathrm{c}}(X)$, respectively, $G_1 * G_2$ defines a bounded $\mathscr{B}(\Gamma_0)$-measurable function with support contained in $\Gamma_{\Lambda_1 \cup \Lambda_2}$. This means that the space $B_{\mathrm{bs}}(\Gamma_0)$ is a subalgebra of $L^0(\Gamma_0, \mathscr{B}(\Gamma_0))$. Moreover, a straightforward application of Lemma 2.6 shows that the space $L^1(\Gamma_0, \lambda_\sigma)$ is also a subalgebra of $L^0(\Gamma_0, \mathscr{B}(\Gamma_0))$ and

$$\int_{\Gamma_0} (G_1 * G_2)(\eta)\, d\lambda_\sigma(\eta) = \left(\int_{\Gamma_0} G_1(\eta)\, d\lambda_\sigma(\eta) \right) \left(\int_{\Gamma_0} G_2(\eta)\, d\lambda_\sigma(\eta) \right). \tag{4.1}$$

Proposition 4.1 (Kondratiev et al. [KKO02]). *For all $G_1, G_2 \in L^1(\lambda_{2\sigma})$ one has*

$$D(G_1 * G_2) = (DG_1) * (DG_2), \quad D^{-1}(G_1 * G_2) = (D^{-1} G_1) * (D^{-1} G_2).$$

The Wick product in Gaussian analysis can be extended to the Poissonian case. For simplicity, we present here its definition on the dense subset $\mathscr{F}\mathscr{P}_{\mathrm{bc}}(\Gamma)$ in $L^2(\pi_\sigma)$ defined in Proposition 3.1. For more details and proofs see [KKO02].

Definition 4.2. For each F_1, $F_2 \in \mathscr{F}\mathscr{P}_{\mathrm{bc}}(\Gamma)$ of the form

$$F_i = \sum_{n=0}^{N_i} \langle C_n^\sigma, \varphi_i^{(n)} \rangle, \quad \varphi_i^{(n)} \in \hat{B}_{\mathrm{bs}}(X^n), \quad N_i \in \mathbb{N}_0, \ i = 1, 2,$$

the Wick product $F_1 \diamondsuit F_2 \in \mathscr{F}\mathscr{P}_{\mathrm{bc}}(\Gamma)$ is defined by

$$F_1 \diamondsuit F_2 := \sum_{n=0}^{N_1+N_2} \left\langle C_n^\sigma, \sum_{k=0}^{n} \varphi_1^{(k)} \hat{\otimes} \varphi_2^{(n-k)} \right\rangle.$$

The space $\mathscr{F}\mathscr{P}_{\mathrm{bc}}(\Gamma)$ endowed with the Wick product is a commutative algebra with unit element $e_\pi(0) \equiv 1$.

Proposition 4.3 (Kondratiev et al. [KKO02]). *For all G_1, $G_2 \in B_{\mathrm{bs}}(\Gamma_0)$ we have*

$$I_{\lambda\pi}(G_1 * G_2) = (I_{\lambda\pi} G_1) \diamondsuit (I_{\lambda\pi} G_2).$$

Proposition 4.4. *Let G_1, $G_2 \in B_{\mathrm{bs}}(\Gamma_0)$ be given. Then*

$$K(G_1 * G_2) = (KG_1) \diamondsuit (KG_2).$$

Proof. By an application of Theorem 2.7 and Propositions 4.1 and 4.3, we find

$$\begin{aligned} K(G_1 * G_2) &= I_{\lambda\pi}((DG_1) * (DG_2)) = (I_{\lambda\pi}(DG_1)) \diamondsuit (I_{\lambda\pi}(DG_2)) \\ &= (KG_1) \diamondsuit (KG_2), \end{aligned}$$

where we have used the fact that $D(B_{\mathrm{bs}}(\Gamma_0)) = B_{\mathrm{bs}}(\Gamma_0)$. $\square$

The above identity can be used to derive a new explicit formula for the Wick product.

Proposition 4.5. *Given $F_1, F_2 \in \mathscr{F}\mathscr{P}_{\mathrm{bc}}(\Gamma) \subset \mathscr{F}L^0(\Gamma)$, let $\Lambda \in \mathscr{B}_{\mathrm{c}}(X)$ and f_1, $f_2 \in L^0(\Gamma_\Lambda, \mathscr{B}(\Gamma_\Lambda))$ be such that $F_1(\gamma) = f_1(\gamma_\Lambda)$, $F_2(\gamma) = f_2(\gamma_\Lambda)$ for all $\gamma \in \Gamma$, cf. Section* 2.3. *Then, for π_σ-a.a. $\gamma \in \Gamma$ the following equality holds*:

$$(F_1 \diamondsuit F_2)(\gamma) = \sum_{\substack{\eta_1, \eta_2 \subset \gamma_\Lambda \\ \eta_1 \cap \eta_2 = \emptyset}} (-1)^{|\gamma_\Lambda \setminus (\eta_1 \cup \eta_2)|} F_1(\eta_1) F_2(\eta_2).$$

Proof. According to Proposition 4.4, for π_σ-a.a. $\gamma \in \Gamma$ we have

$$\begin{aligned} (F_1 \diamondsuit F_2)(\gamma) &= K((K^{-1}F_1) * (K^{-1}F_2))(\gamma) \\ &= K(e_\lambda(-2) * f_1 * f_2)(\gamma_\Lambda). \end{aligned}$$

By the definitions of the K-transform and the $*$-convolution, for all $\gamma \in \Gamma$ the latter expression is equal to

$$\sum_{\xi \subset \gamma_\Lambda} \sum_{\eta_1 \subset \xi} \sum_{\eta_2 \subset \xi \setminus \eta_1} (-2)^{|\xi \setminus (\eta_1 \cup \eta_2)|} f_1(\eta_1) f_2(\eta_2)$$
$$= \sum_{\eta_1 \subset \gamma_\Lambda} f_1(\eta_1) \sum_{\eta_2 \subset \gamma_\Lambda \setminus \eta_1} f_2(\eta_2) \sum_{\substack{\xi \\ \eta_1 \cup \eta_2 \subset \xi \subset \gamma_\Lambda}} (-2)^{|\xi \setminus (\eta_1 \cup \eta_2)|}$$
$$= \sum_{\eta_1 \subset \gamma_\Lambda} F_1(\eta_1) \sum_{\eta_2 \subset \gamma_\Lambda \setminus \eta_1} F_2(\eta_2)(-1)^{|\gamma_\Lambda \setminus (\eta_1 \cup \eta_2)|},$$

completing the proof. $\square$

Other algebraic results may be found in [Oli02].

5. Some operators in spaces of test and generalized functions

In order to study the action of the K-transform and the operator D on the spaces of test and generalized functions introduced in [KKO02], first we briefly recall the definition of these spaces. For simplicity, let us consider the space $X = \mathbb{R}^d$ $(d \geqslant 1)$ and the corresponding Lebesgue measure m. The unitary isomorphism $I_{\pi\lambda}$ between the spaces $L^2(\pi_m)$ and $L^2(\lambda_m)$ leads to a natural construction of spaces of test and generalized functions on the Lebesgue–Poisson space [KKO02].

Let us consider the Schwartz space $\mathscr{S} := \mathscr{S}(\mathbb{R}^d)$ of all rapidly decreasing infinitely differentiable real-valued functions on $\mathbb{R}^d$. It is a nuclear Fréchet space for the family of Hilbert spaces $\mathscr{H}^+_{k,p}$ defined by the Hilbertian norms

$$|f|^2_{k,p} := \int_{\mathbb{R}^d} \sum_{\substack{|\alpha| \leqslant k \\ \alpha \in \mathbb{N}_0^d}} |D^\alpha f(x)|^2 (1 + ||x||)^p \, dm(x), \quad f \in \mathscr{S}, k, p \in \mathbb{N}_0,$$

the sum being over all d-tuples $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ such that $|\alpha| := \alpha_1 + \cdots + \alpha_d \leqslant k$. By $\mathscr{H}^-_{k,p}$ we denote the dual space of $\mathscr{H}^+_{k,p}$ w.r.t. $L^2_{\mathrm{Re}}(m)$. Since there is no risk of confusion we preserve the notation $|\cdot|_{k,p}$ for the norm on the symmetric tensor power $(\mathscr{H}^+_{k,p})^{\hat{\otimes} n}$ and we denote by $(\mathscr{H}^-_{k,p})^{\hat{\otimes} n}$ the corresponding dual space w.r.t. $(L^2_{\mathrm{Re}}(m))^{\hat{\otimes} n}$, $n \in \mathbb{N}$.

In Poissonian analysis a space of test functions $(\mathscr{S})^1_\pi$ can be defined by the projective limit of the Hilbert spaces

$$(\mathscr{H}^+_{k,p})^1_{q,\pi} := \{F \in L^2(\pi_m) \colon ||F||_{k,p,q,\pi} < \infty\}, \quad k, p, q \in \mathbb{N}_0,$$

where, for each $F = \sum_{n=0}^{\infty} \langle C_n^\sigma, f^{(n)} \rangle \in L^2(\pi_m)$,

$$||F||_{k,p,q,\pi}^2 := \sum_{n=0}^{\infty} 2^{nq}(n!)^2 |f^{(n)}|_{k,p}^2.$$

Therefore, by the general duality theory, the dual space $(\mathcal{S})_\pi^{-1}$ of $(\mathcal{S})_\pi^1$ w.r.t. $L^2(\pi_m)$ is given by the inductive limit of the dual spaces $(\mathcal{H}_{k,p}^-)_{-q,\pi}^{-1}$ of $(\mathcal{H}_{k,p}^+)_{q,\pi}^1$ (w.r.t. $L^2(\pi_m)$).

The chaos decomposition provides a natural decomposition of the elements of the dual space $\mathcal{P}_\pi'(\mathcal{D}')$ of $\mathcal{P}(\mathcal{D}')$ (w.r.t. $L^2(\pi_m)$). For each $\psi^{(n)} \in \mathcal{D}_{\mathbb{C}}'^{\hat{\otimes} n}$ there exists a unique distribution in $\mathcal{P}_\pi'(\mathcal{D}')$, denoted by $\langle \psi^{(n)}, C_n^\sigma \rangle$, acting on smooth continuous polynomials $\Phi = \sum_{n=0}^{N} \langle C_n^\sigma, \varphi^{(n)} \rangle$ by

$$\langle\langle \langle \psi^{(n)}, C_n^\sigma \rangle, \Phi \rangle\rangle_\pi = n! \langle \psi^{(n)}, \varphi^{(n)} \rangle.$$

Here the dual pairing $\langle \cdot, \cdot \rangle$ is the bilinear extension of the inner product on $L^2(\mathbb{R}^{dn}, m^{\otimes n})$ and $\langle\langle \cdot, \cdot \rangle\rangle_\pi$ denotes the bilinear dual pairing between $\mathcal{P}_\pi'(\mathcal{D}')$ and $\mathcal{P}(\mathcal{D}')$ which extends the sesquilinear inner product on $L^2(\pi_m)$:

$$\langle\langle F, \Phi \rangle\rangle_\pi = (F, \bar{\Phi})_{L^2(\pi_m)}, \quad F \in L^2(\pi_m), \Phi \in \mathcal{P}(\mathcal{D}'), \tag{5.1}$$

where $\bar{\Phi}$ is the complex conjugate function of Φ.

Therefore, any element $\Psi \in \mathcal{P}_\pi'(\mathcal{D}')$ has a unique decomposition of the form

$$\Psi = \sum_{n=0}^{\infty} \langle \psi^{(n)}, C_n^\sigma \rangle$$

and we have

$$\langle\langle \Psi, \Phi \rangle\rangle_\pi = \sum_{n=0}^{\infty} n! \langle \psi^{(n)}, \varphi^{(n)} \rangle,$$

for all $\Phi \in \mathcal{P}(\mathcal{D}')$ (see [KSS97,KSWY98] for more details and proofs). Thus, we obtain $\mathcal{P}(\mathcal{D}') \subset L^2(\pi_m) \subset \mathcal{P}_\pi'(\mathcal{D}')$ and the elements of $\mathcal{P}_\pi'(\mathcal{D}')$ are generalized functions on $\mathcal{D}'$. Moreover, since $\mathcal{P}(\mathcal{D}') \subset (\mathcal{S})_\pi^1$, the space $(\mathcal{S})_\pi^{-1}$ may be regarded as a subspace of $\mathcal{P}_\pi'(\mathcal{D}')$ and hence we obtain the following extended chain of spaces:

$$\mathcal{P}(\mathcal{D}') \subset (\mathcal{S})_\pi^1 \subset L^2(\pi_m) \subset (\mathcal{S})_\pi^{-1} \subset \mathcal{P}_\pi'(\mathcal{D}').$$

The corresponding bilinear dual pairing is given by $\langle\langle \cdot, \cdot \rangle\rangle_\pi$ as described in (5.1).

To introduce spaces of test and generalized functions in Lebesgue–Poisson analysis, let us consider the family of Hilbert spaces $(\mathcal{H}_{k,p}^+)_{q,\lambda}^1 := I_{\pi\lambda}((\mathcal{H}_{k,p}^+)_{q,\pi}^1)$ with the natural Hilbertian norms given by

$$||G||^2_{k,p,q,\lambda} := \sum_{n=0}^{\infty} 2^{nq} |G^{(n)}|^2_{k,p}, \quad G^{(n)} = G\restriction_{\Gamma_X^{(n)}}.$$

A space of test functions $(\mathscr{S})^1_\lambda$ is defined by the projective limit of $(\mathscr{H}^+_{k,p})^1_{q,\lambda}$ and the general duality theory yields that the dual space $(\mathscr{S})^{-1}_\lambda$ of $(\mathscr{S})^1_\lambda$ (w.r.t. $L^2(\lambda_m)$) is given by the inductive limit of the dual spaces $(\mathscr{H}^-_{k,p})^{-1}_{-q,\lambda}$ of $(\mathscr{H}^+_{k,p})^1_{q,\lambda}$ (w.r.t. $L^2(\lambda_m)$).

Note that as a direct consequence of this construction, for any $k,p,q \in \mathbb{N}_0$, $I_{\pi\lambda} : (\mathscr{H}^+_{k,p})^1_{q,\pi} \to (\mathscr{H}^+_{k,p})^1_{q,\lambda}$ is a unitary isomorphism. Therefore, the mapping $I_{\pi\lambda}$ can be extended to the space $(\mathscr{H}^-_{k,p})^{-1}_{-q,\pi}$ and the extended mapping, also denoted by $I_{\pi\lambda}$, maps $(\mathscr{H}^-_{k,p})^{-1}_{-q,\pi}$ onto $(\mathscr{H}^-_{k,p})^{-1}_{-q,\lambda}$. The dual pairing between $(\mathscr{S})^{-1}_\lambda$ and $(\mathscr{S})^1_\lambda$ is realized as an extension of the inner product on $L^2(\lambda_m)$, i.e.,

$$\langle\langle G, F \rangle\rangle_\lambda = (G, \bar{F})_{L^2(\lambda_m)}, \quad G \in (\mathscr{S})^1_\lambda, F \in L^2(\lambda_m).$$

Furthermore, we have a natural decomposition $(\Psi^{(n)})^\infty_{n=0}$ for generalized functions $\Psi \in (\mathscr{S})^{-1}_\lambda$ in such a way that for all $G \in (\mathscr{S})^1_\lambda$ the following equality holds:

$$\langle\langle \Psi, G \rangle\rangle_\lambda = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \Psi^{(n)}, G^{(n)} \rangle.$$

See [KKO02] for more details.

Theorem 5.1. *We have*

$$K : (\mathscr{S})^1_\lambda \to (\mathscr{S})^1_\pi$$

continuously.

The proof of this result follows as a consequence of the next lemma.

Lemma 5.2. *For each k, $q \in \mathbb{N}_0$ and $p \geqslant d+1$, the following estimate of norms holds*:

$$||KG||^2_{k,p,q,\pi} \leqslant ||e_\lambda((1 + ||\cdot||)^{-p})||_{L^1(\lambda_m)} ||G||^2_{k,p,q+1,\lambda}, \quad G \in (\mathscr{S})^1_\lambda.$$

Proof. For all $G \in B_{\mathrm{bs}}(\Gamma_0)$ such that $G^{(n)} \in \mathscr{D}_{\mathbb{C}}^{\hat{\otimes} n}$, $n \in \mathbb{N}$, we have $(KG)(\gamma) = \sum_{n=0}^\infty (KG^{(n)})(\gamma)$. Thus, by Corollary 2.10, KG has the chaos decomposition given by

$$KG = \sum_{n=0}^{\infty} \left\langle C_n^\sigma, \sum_{j=0}^{\infty} \frac{1}{n!j!} \int_{\Gamma^{(j)}_{\mathbb{R}^d}} G^{(n+j)}(\{\cdot, \dots, \cdot, y_1, \dots, y_j\})\, dm^{(j)}(\{y_1, \dots, y_j\}) \right\rangle.$$

In order to estimate the norm $\|\cdot\|_{k,p,q,\pi}$ of KG, let us first compute the integral which appears in the definition of this norm. For simplicity, in the sequel we use the notation $dm(\{x_i\}_{i=1}^n)$ for $dm^{(n)}(\{x_1,\dots,x_n\})$, and we set $dm(\{x_i\}_{i=1}^0) := 1$. By the Cauchy–Schwarz inequality, for each $k\in\mathbb{N}_0$, $n\in\mathbb{N}$, and $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}_0^{dn}$, $|\alpha|\leqslant k$, we obtain

$$\begin{aligned}
&\int_{\Gamma_{\mathbb{R}^d}^{(n)}}\left|\sum_{j=0}^\infty \frac{1}{n!j!}\int_{\Gamma_{\mathbb{R}^d}^{(j)}}(D_x^\alpha G^{(n+j)})(\{x_1,\dots,x_n,y_1,\dots,y_j\})\,dm(\{y_i\}_{i=1}^j)\right|^2\\
&\quad\times\prod_{i=1}^n(1+\|x_i\|)^p\,dm(\{x_i\}_{i=1}^n)\\
&\quad\leqslant\int_{\Gamma_{\mathbb{R}^d}^{(n)}}\left(\sum_{j=0}^\infty\frac{1}{n!j!}\int_{\Gamma_{\mathbb{R}^d}^{(j)}}|(D_x^\alpha G^{(n+j)})(\{x_1,\dots,x_n,y_1,\dots,y_j\}|\right.\\
&\qquad\left.\times\prod_{i=1}^j(1+\|y_i\|)^{\frac{p}{2}}\prod_{i=1}^j(1+\|y_i\|)^{-\frac{p}{2}}\,dm(\{y_i\}_{i=1}^j)\right)^2\prod_{i=1}^n(1+\|x_i\|)^p\,dm(\{x_i\}_{i=1}^n)\\
&\quad\leqslant\frac{1}{(n!)^2}\exp\left(\int_{\mathbb{R}^d}(1+\|x\|)^{-p}\,dm(x)\right)\\
&\qquad\times\sum_{j=0}^\infty\frac{1}{j!}\int_{\Gamma_{\mathbb{R}^d}^{(n+j)}}|(D^{\alpha'}G^{(n+j)})(\{x_1,\dots,x_{n+j}\})|^2\\
&\qquad\times\prod_{i=1}^{n+j}(1+\|x_i\|)^p\,dm(\{x_i\}_{i=1}^{n+j}),
\end{aligned}$$

where $\alpha':=(\alpha_1,\dots,\alpha_n,0,\dots,0)\in\mathbb{N}_0^{d(n+j)}$ and the integral $\int_{\mathbb{R}^d}(1+\|x\|)^{-p}\,dm(x)$ is finite if and only if $p\geqslant d+1$. Then, the exponential of this integral coincides with the $L^1(\lambda_m)$-norm of the coherent state corresponding to the function $(1+\|\cdot\|)^{-p}$ (cf. (2.5)).

Therefore for $p\geqslant d+1$ we have

$$\|KG\|_{k,p,q,\pi}^2\leqslant\|e_\lambda((1+\|\cdot\|)^{-p})\|_{L^1(\lambda_m)}\sum_{n=0}^\infty 2^{nq}\sum_{j=0}^\infty\frac{1}{j!}|G^{(n+j)}|_{k,p}^2.$$

To estimate the above sums, note that

$$\begin{aligned}
\sum_{m=0}^\infty|G^{(m)}|_{k,p}^2\sum_{n=0}^m 2^{nq}\frac{1}{(m-n)!}&\leqslant\sum_{m=0}^\infty 2^{mq}|G^{(m)}|_{k,p}^2\sum_{n=0}^m\frac{1}{(m-n)!}\\
&\leqslant\sum_{m=0}^\infty 2^{m(q+1)}|G^{(m)}|_{k,p}^2=\|G\|_{k,p,q+1,\lambda}^2.\qquad\square
\end{aligned}$$

Remark 5.3. For each $l \in \mathbb{N}$, a similar computation yields

$$||G||_{L^1(\lambda_{lm})} \leqslant ||e_\lambda(l(1+||\cdot||)^{-p})||^{1/2}_{L^1(\lambda_m)} ||G||_{0,p,q+l-1,\lambda},$$

that is, the condition $p \geqslant d+1$ postulated in Lemma 5.2 is an integrability condition for test functions in $(\mathscr{S})^1_\lambda$.

Corollary 5.4. *If $k, q \in \mathbb{N}_0$ and $p \geqslant d+1$, then*

$$(\mathscr{H}^+_{k,p})^1_{q+2,\lambda} \subset D((\mathscr{H}^+_{k,p})^1_{q+1,\lambda}) \subset (\mathscr{H}^+_{k,p})^1_{q,\lambda}.$$

As a consequence, D continuously maps the space $(\mathscr{S})^1_\lambda$ into itself.

Proof. In view of Theorem 2.7, for all $G \in B_{\mathrm{bs}}(\Gamma_0)$ one has

$$||DG||_{k,p,q,\lambda} = ||I_{\lambda\pi}(DG)||_{k,p,q,\pi} = ||KG||_{k,p,q,\pi},$$

proving the inclusion

$$D((\mathscr{H}^+_{k,p})^1_{q+1,\lambda}) \subset (\mathscr{H}^+_{k,p})^1_{q,\lambda}$$

by an application of Lemma 5.2. On the other hand, we have

$$|(D^{-1}G)^{(n)}(\{x_1,\dots,x_n\})| \leqslant (D|G|)^{(n)}(\{x_1,\dots,x_n\}).$$

Thus the same estimates as before yield $D^{-1}((\mathscr{H}^+_{k,p})^1_{q+1,\lambda}) \subset (\mathscr{H}^+_{k,p})^1_{q,\lambda}$. According to Proposition 2.5 and Remark 5.3, this implies $(\mathscr{H}^+_{k,p})^1_{q+1,\lambda} \subset D((\mathscr{H}^+_{k,p})^1_{q,\lambda})$. $\square$

Remark 5.5. This proof also shows that the operator D^{-1} continuously maps $(\mathscr{S})^1_\lambda$ into itself.

Corollary 5.4 allows us to define the adjoint operator D^* of D on the space of generalized functions $(\mathscr{S})^{-1}_\lambda$. Similarly Theorem 5.1 leads to the following definition of the adjoint operator K^+ of the K-transform on generalized functions. Note that also heuristically $K^+ \neq K^*$, cf. Section 6.

Definition 5.6. (i) For each $\Psi \in (\mathscr{S})^{-1}_\lambda$, $D^*\Psi$ is the unique element of $(\mathscr{S})^{-1}_\lambda$ such that

$$\forall G \in (\mathscr{S})^1_\lambda, \quad \langle\langle D^*\Psi, G\rangle\rangle_\lambda := \langle\langle \Psi, DG\rangle\rangle_\lambda.$$

(ii) For each $\Psi \in (\mathscr{S})^{-1}_\pi$, $K^+\Psi$ is the unique element of $(\mathscr{S})^{-1}_\lambda$ such that

$$\forall G \in (\mathscr{S})^1_\lambda, \quad \langle\langle K^+\Psi, G\rangle\rangle_\lambda := \langle\langle \Psi, KG\rangle\rangle_\pi.$$

Moreover, by Theorem 5.1 and its corollary, we can state the following result for generalized functions.

Corollary 5.7. *The operators*

$$D^* : (\mathscr{S})_\lambda^{-1} \to (\mathscr{S})_\lambda^{-1} \quad \textit{and} \quad K^+ : (\mathscr{S})_\pi^{-1} \to (\mathscr{S})_\lambda^{-1}$$

are continuous.

Remark 5.8. The estimates of norms in the proofs of Lemma 5.2 and Corollary 5.4 show, in particular, that with $r := (1 + \|\cdot\|)^{-p}$ $(p \geqslant d+1)$ and $k = q = 0$ one has

$$D^* : L^2(\Gamma_0, \lambda_{2rm}) \to L^2(\Gamma_0, \lambda_{rm})$$

continuously. The action of D^* on elements $G \in L^2(\Gamma_0, \lambda_{2rm})$ is given by

$$(D^* G)(\eta) = \sum_{\xi \subset \eta} G(\xi) = (G * e_\lambda(1))(\eta), \quad \lambda_m - \text{a.a. } \eta \in \Gamma_0,$$

which coincides with $(KG)\upharpoonright_{\Gamma_0}(\eta)$ for $G \in B_{\rm bs}(\Gamma_0)$ [Oli02].

Remark 5.9. In a similar way, for each $\Psi \in (\mathscr{S})_\lambda^{-1}$ we can also define $D^{*^{-1}} \Psi \in (\mathscr{S})_\lambda^{-1}$:

$$\forall G \in (\mathscr{S})_\lambda^1, \quad \langle\langle D^{*^{-1}} \Psi, G \rangle\rangle_\lambda := \langle\langle \Psi, D^{-1} G \rangle\rangle_\lambda.$$

The linear operator $D^{*^{-1}} : (\mathscr{S})_\lambda^{-1} \to (\mathscr{S})_\lambda^{-1}$ is continuous. Furthermore, under the conditions of Remark 5.8 one has $D^{*^{-1}} : L^2(\Gamma_0, \lambda_{2rm}) \to L^2(\Gamma_0, \lambda_{rm})$ and for each $G \in L^2(\Gamma_0, \lambda_{2rm})$

$$(D^{*^{-1}} G)(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \backslash \xi|} G(\xi) = (G * e_\lambda(-1))(\eta), \quad \lambda_m - \text{a.a. } \eta \in \Gamma_0.$$

Remark 5.10. The operators D^* and $D^{*^{-1}}$ can be also described by creation operators $a_\lambda^+(h)$ on the Lebesgue–Poisson space, see e.g. [KKO02, Section 6.4], namely, Theorem 6.19 therein for more details and proofs. For each $G \in B_{\rm bs}(\Gamma_0)$,

$$D^* G = e^{a_\lambda^+(1)} G, \quad D^{*^{-1}} G = e^{a_\lambda^+(-1)} G,$$

where for each $h \in L^2_{\rm Re}(m)$ and all $G \in B_{\rm bs}(\Gamma_0)$

$$(a_\lambda^+(h) G)(\eta) := \sum_{x \in \eta} G(\eta \backslash \{x\}) h(x), \quad \lambda_m - \text{a.a. } \eta \in \Gamma_0.$$

This exponential representation using the operator $a_\lambda^+(1)$ was already used in statistical mechanics for the study of the BBGKY hierarchy, see e.g. [GMP89].

6. Correlation functions and the chaos decomposition

This section begins by recalling two formulas well-known in statistical mechanics, see e.g. [Rue70], and in the theory of point processes, see e.g. [DVJ88]. Often they are considered as definitions. In the sequel, given a probability measure τ on Γ, we denote by $\tau \circ p_\Lambda^{-1}$ the image measure on the space Γ_Λ under the mapping $p_\Lambda : \Gamma \to \Gamma_\Lambda$ defined by $p_\Lambda(\gamma) := \gamma_\Lambda$, $\gamma \in \Gamma$, i.e., the projection of τ onto Γ_Λ.

Proposition 6.1. (i) *Let $\mu \in \mathscr{M}^1_{\mathrm{fm}}(\Gamma)$ be a measure which is locally absolutely continuous w.r.t. π_m, i.e., for all $\Lambda \in \mathscr{B}_{\mathrm{c}}(\mathbb{R}^d)$ the measure $\mu^\Lambda := \mu \circ p_\Lambda^{-1}$ is absolutely continuous w.r.t. $\pi_m^\Lambda := \pi_m \circ p_\Lambda^{-1}$. Then, $\rho_\mu = K^*\mu$ is absolutely continuous w.r.t. λ_m. Furthermore, for all $\Lambda \in \mathscr{B}_{\mathrm{c}}(\mathbb{R}^d)$ we have*

$$k_\mu(\eta) := \frac{d\rho_\mu}{d\lambda_m}(\eta) = \int_{\Gamma_\Lambda} \frac{d\mu^\Lambda}{d\pi_m^\Lambda}(\gamma \cup \eta)\, d\pi_m^\Lambda(\gamma) \quad \text{for } \lambda_m - a.a.\ \eta \in \Gamma_\Lambda.$$

The density k_μ is called the correlation function corresponding to the measure μ.

(ii) *Let $\mu \in \mathscr{M}^1_{\mathrm{fm}}(\Gamma)$ be given. Assume that $\int_{\Gamma_\Lambda} 2^{|\eta|}\, d\rho_\mu(\eta) < \infty$ for all $\Lambda \in \mathscr{B}_{\mathrm{c}}(\mathbb{R}^d)$. If ρ_μ is absolutely continuous w.r.t. λ_m, then μ is locally absolutely continuous w.r.t. π_m and for all $\Lambda \in \mathscr{B}_{\mathrm{c}}(\mathbb{R}^d)$*

$$\frac{d\mu^\Lambda}{d\pi_m^\Lambda}(\gamma) = e^{m(\Lambda)} \int_{\Gamma_\Lambda} (-1)^{|\eta|} k_\mu(\gamma \cup \eta)\, d\lambda_m(\eta) \quad \text{for } \pi_m^\Lambda - a.a.\ \gamma \in \Gamma_\Lambda.$$

For the proof see e.g. [KK02].

Our aim now is to derive an explicit formula for the chaos decomposition of the correlation function k_μ defined in Proposition 6.1 using Theorem 2.7. In view of Section 5, more generally, one may obtain such a formula for the so-called correlation generalized functions (Definition 6.2).

Let us consider a measure $\mu \in \mathscr{M}^1_{\mathrm{fm}}(\Gamma)$ such that the linear functional

$$(\mathscr{S})^1_\pi \ni \Phi \mapsto \int_\Gamma \Phi(\gamma)\, d\mu(\gamma) \in \mathbb{C}$$

is continuous. In this case, there exists a (unique) generalized function $R_\mu \in (\mathscr{S})^{-1}_\pi$ such that

$$\forall \Phi \in (\mathscr{S})^1_\pi, \quad \langle\langle R_\mu, \Phi \rangle\rangle_\pi = \int_\Gamma \Phi(\gamma)\, d\mu(\gamma).$$

In the sequel, we use the notation $d\mu = R_\mu\, d\pi_m$ or $R_\mu = \frac{d\mu}{d\pi_m}$ in the sense of generalized Radon–Nikodym derivatives introduced, e.g., in [BK88, Vol. I].

Definition 6.2. Let $\mu \in \mathscr{M}^1_{\mathrm{fm}}(\Gamma)$ be given. If $d\mu = R_\mu \, d\pi_m$ for some $R_\mu \in (\mathscr{S})^{-1}_\pi$, then the generalized function $k_\mu := K^+ R_\mu \in (\mathscr{S})^{-1}_\lambda$ is called the correlation generalized function corresponding to μ.

Remark 6.3. Under the conditions of Definition 6.2, the correlation measure ρ_μ corresponding to μ is given by $d\rho_\mu = k_\mu \, d\lambda_m$ in the sense of generalized functions. Indeed, by the definitions of K^+ and k_μ, for all $G \in (\mathscr{S})^1_\lambda$ we have

$$\int_{\Gamma_0} (KG)(\gamma) \, d\mu(\gamma) = \langle\langle R_\mu, KG \rangle\rangle_\pi = \langle\langle k_\mu, G \rangle\rangle_\lambda$$

and the assertion follows by the definition of the correlation measure ρ_μ.

It is then clear that the generalized functions R_μ and k_μ are a generalization of the densities $\frac{d\mu^\Lambda}{d\pi^\Lambda_m}$ and k_μ postulated in Proposition 6.1.

Given a $\mu \in \mathscr{M}^1_{\mathrm{fm}}(\Gamma)$ such that $d\mu = R_\mu \, d\pi_m$ with $R_\mu \in (\mathscr{S})^{-1}_\pi$, let us now consider the generalized function $r_\mu \in (\mathscr{S})^{-1}_\lambda$ defined by

$$r_\mu := I_{\pi\lambda} R_\mu. \tag{6.1}$$

A straightforward computation using Theorem 2.7 yields

$$k_\mu = K^+ R_\mu = D^*(I_{\pi\lambda} R_\mu) = D^* r_\mu.$$

Furthermore, we can deduce an explicit formula for the chaos decomposition of k_μ and r_μ.

Theorem 6.4. *For each $n \in \mathbb{N}$ we have the following equalities*:

$$k^{(n)}_\mu = \sum_{k \leqslant n} \binom{n}{k} r^{(k)}_\mu \tag{6.2}$$

and

$$r^{(n)}_\mu = \sum_{k \leqslant n} (-1)^{n-k} \binom{n}{k} k^{(k)}_\mu \tag{6.3}$$

in the sense of generalized functions, i.e., for all $G \in (\mathscr{S})^1_\lambda$

$$\langle k^{(n)}_\mu, G^{(n)} \rangle = \left\langle \sum_{k \leqslant n} \binom{n}{k} r^{(k)}_\mu \hat{\otimes} \mathscr{I}_{n-k}, G^{(n)} \right\rangle \tag{6.4}$$

and

$$\langle r_\mu^{(n)}, G^{(n)} \rangle = \left\langle \sum_{k \leqslant n} (-1)^{n-k} \binom{n}{k} k_\mu^{(k)} \hat{\otimes} \mathscr{I}_{n-k}, G^{(n)} \right\rangle, \tag{6.5}$$

where $\mathscr{I}_{n-k} \in \mathscr{D}_{\mathbb{C}}'^{\hat{\otimes} n-k}$ *is the distribution corresponding to the function identically equal to* 1.

Proof. Since $k_\mu = D^* r_\mu$, for each $n \in \mathbb{N}$ one has $k_\mu^{(n)} = (D^* r_\mu)^{(n)}$. Therefore, for every $G \in (\mathscr{S})_\lambda^1$ we have

$$\langle k_\mu^{(n)}, G^{(n)} \rangle = \langle (D^* r_\mu)^{(n)}, G^{(n)} \rangle,$$

where the right-hand side (up to a factor $n!$) is given by

$$\langle \langle (D^* r_\mu)^{(n)}, G \rangle \rangle_\lambda = \langle \langle r_\mu, DG^{(n)} \rangle \rangle_\lambda,$$

which is equal to

$$\sum_{k=0}^{n} \frac{1}{k!} \left\langle r_\mu^{(k)}, \frac{1}{(n-k)!} \int_{\Gamma_{\mathbb{R}^d}^{(n-k)}} G^{(n)}(\{\cdot, \ldots, \cdot, y_1, \ldots, y_{n-k}\})\, dm^{(n-k)}(\{y_1, \ldots, y_{n-k}\}) \right\rangle$$

according to (2.11). In this way equality (6.4) is proved. The proof of equality (6.5) is done in a similar way by using the fact that $r_\mu = D^{*^{-1}} k_\mu$. □

This result leads to an alternative description of Gibbs measures using r_μ. For this purpose, we have first to introduce the framework of Gibbs measures.

Given a symmetric measurable function $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, called a *pair potential*, let us consider the *energy functional* $E : \Gamma_0 \to \mathbb{R} \cup \{+\infty\}$ defined by

$$E(\eta) := \sum_{\{x,y\} \subset \eta} \phi(x, y), \quad E(\emptyset) := E(\{x\}) := 0$$

and the *interaction energy* $W(\eta, \gamma)$ *between* $\eta \in \Gamma_0$ *and* $\gamma \in \Gamma$ given by

$$W(\eta, \gamma) := \begin{cases} \sum_{x \in \eta, y \in \gamma} \phi(x, y) & \text{if } \sum_{x \in \eta, y \in \gamma} |\phi(x, y)| < \infty \\ +\infty & \text{otherwise} \end{cases},$$

$W(\emptyset, \gamma) := W(\eta, \emptyset) := 0.$

Definition 6.5. A probability measure μ on Γ is called a Gibbs measure corresponding to the potential ϕ, the intensity measure m, and the inverse temperature $\beta > 0$ if μ fulfills the Ruelle equation, i.e., for all positive functions

$F \in L^0(\Gamma, \mathscr{B}(\Gamma))$ and all $\Lambda \in \mathscr{B}_c(\mathbb{R}^d)$,

$$\int_\Gamma F(\gamma)\, d\mu(\gamma) = \int_{\Gamma_\Lambda} \int_{\Gamma_{\mathbb{R}^d \setminus \Lambda}} F(\eta \cup \gamma) \exp(-\beta E(\eta) - \beta W(\eta, \gamma))\, d\mu(\gamma)\, d\lambda_m(\eta).$$

For translation invariant pair potentials ϕ verifying the standard Ruelle conditions (RC) of translation invariance, stability (S), regularity (I), and lower regularity (which may be found in [Rue70]), Ruelle proved in [Rue70] the next result for Gibbs measures. For non-translation invariant pair potentials the same result was proved in [Kun99].

Proposition 6.6. *Let $\mu \in \mathscr{M}^1_{\mathrm{fm}}(\Gamma)$ be a measure locally absolutely continuous w.r.t. the Poisson measure π_m and such that the corresponding correlation function k_μ fulfills the Ruelle bound, i.e.,*

$$\exists C_{\mathrm{R}} > 0, \quad k_\mu(\eta) \leqslant (C_{\mathrm{R}})^{|\eta|}, \quad \lambda_m - a.a.\ \eta \in \Gamma_0.$$

Then, μ is a Gibbs measure corresponding to a potential ϕ fulfilling the (RC) conditions if and only if k_μ is a solution of the Mayer–Montroll equation, i.e.,

$$k_\mu(\eta \cup \xi) = e^{-\beta E(\xi) - \beta W(\xi, \eta)} \int_{\Gamma_0} e_\lambda(e^{-\beta W(\xi, \cdot)} - 1, \zeta) k_\mu(\eta \cup \zeta)\, d\lambda_m(\zeta),$$

$\lambda_m \otimes \lambda_m$-a.e.

Remark 6.7. The assumption that the correlation function k_μ fulfills the Ruelle bound is already needed to insure that the Mayer–Montroll equation is well-defined.

Theorem 6.8. *Let ϕ be a pair potential fulfilling (S) and (I), and $\mu \in \mathscr{M}^1_{\mathrm{fm}}(\Gamma)$ be a measure locally absolutely continuous w.r.t. the Poisson measure π_m. Then, k_μ fulfills the Ruelle bound if and only if r_μ as described in (6.1) fulfills the Ruelle bound. Under these conditions, k_μ is a solution of the Mayer–Montroll equation if and only if r_μ is a solution of the following equation:*

$$\begin{aligned} r(\eta \cup \xi) = {} & \sum_{\xi_1 \subset \xi} \exp\left(\int_{\mathbb{R}^d} (e^{-\beta W(\xi_1, \{x\})} - 1)\, dm(x) \right) (-1)^{|\xi \setminus \xi_1|} e^{-\beta E(\xi_1)} \\ & \times \sum_{\eta_1 \subset \eta} e^{-\beta W(\xi_1, \eta_1)} \int_{\Gamma_0} e_\lambda(e^{-\beta W(\xi_1, \cdot)} - 1, (\eta \setminus \eta_1) \cup \zeta) r(\eta_1 \cup \zeta)\, d\lambda_m(\zeta), \end{aligned} \tag{6.6}$$

$\lambda_m \otimes \lambda_m$-a.e.

Proof. The first part of the proof directly follows from (6.2) and (6.3).

According to equality (6.2), the Mayer–Montroll equation may be rewritten as

$$k_\mu(\eta \cup \xi) = e^{-\beta E(\xi) - \beta W(\xi,\eta)} \int_{\Gamma_0} e_\lambda(e^{-\beta W(\xi,\cdot)} - 1, \zeta) \sum_{\varsigma \subset \eta \cup \zeta} r_\mu(\varsigma)\, d\lambda_m(\zeta),$$

where the above integral is equal to

$$\begin{aligned}
&\sum_{\eta_1 \subset \eta} \int_{\Gamma_0} e_\lambda(e^{-\beta W(\xi,\cdot)} - 1, \zeta) \sum_{\zeta_1 \subset \zeta} r_\mu(\eta_1 \cup \zeta_1)\, d\lambda_m(\zeta) \\
&\quad = \sum_{\eta_1 \subset \eta} \int_{\Gamma_0} e_\lambda(e^{-\beta W(\xi,\cdot)} - 1, \zeta)(D^* r_\mu(\eta_1 \cup \cdot))(\zeta)\, d\lambda_m(\zeta) \\
&\quad = \sum_{\eta_1 \subset \eta} \int_{\Gamma_0} (De_\lambda(e^{-\beta W(\xi,\cdot)} - 1))(\zeta) r_\mu(\eta_1 \cup \zeta)\, d\lambda_m(\zeta) \\
&\quad = \int_{\Gamma_0} e_\lambda(e^{-\beta W(\xi,\cdot)} - 1, \varsigma)\, d\lambda_m(\varsigma) \\
&\qquad \times \sum_{\eta_1 \subset \eta} \int_{\Gamma_0} e_\lambda(e^{-\beta W(\xi,\cdot)} - 1, \zeta) r_\mu(\eta_1 \cup \zeta)\, d\lambda_m(\zeta).
\end{aligned}$$

On the other hand, relation (6.3) leads to

$$\begin{aligned}
r_\mu(\eta \cup \xi) &= \sum_{\varsigma \subset \eta \cup \xi} (-1)^{|(\eta \cup \xi) \setminus \varsigma|} k_\mu(\varsigma) \\
&= \sum_{\xi_1 \subset \xi} \sum_{\eta_1 \subset \eta} (-1)^{|\eta \setminus \eta_1| + |\xi \setminus \xi_1|} k_\mu(\eta_1 \cup \xi_1).
\end{aligned}$$

This equality together with the previous calculation gives

$$\begin{aligned}
r_\mu(\eta \cup \xi) = &\sum_{\xi_1 \subset \xi} \exp\left(\int_{\mathbb{R}^d} (e^{-\beta W(\xi_1, \{x\})} - 1)\, dm(x) \right) (-1)^{|\xi \setminus \xi_1|} e^{-\beta E(\xi_1)} \\
&\times \sum_{\eta_1 \subset \eta} (-1)^{|\eta \setminus \eta_1|} e^{-\beta W(\xi_1, \eta_1)} \sum_{\eta_2 \subset \eta_1} \int_{\Gamma_0} e_\lambda(e^{-\beta W(\xi_1,\cdot)} - 1, \zeta) r_\mu(\eta_2 \cup \zeta)\, d\lambda_m(\zeta),
\end{aligned}$$

where

$$\begin{aligned}
&\sum_{\eta_1 \subset \eta} (-1)^{|\eta \setminus \eta_1|} e^{-\beta W(\xi_1, \eta_1)} \sum_{\eta_2 \subset \eta_1} \int_{\Gamma_0} e_\lambda(e^{-\beta W(\xi_1,\cdot)} - 1, \zeta) r_\mu(\eta_2 \cup \zeta)\, d\lambda_m(\zeta) \\
&\quad = \sum_{\eta_1 \subset \eta} (-1)^{|\eta \setminus \eta_1|} \sum_{\eta_2 \subset \eta_1} e^{-\beta W(\xi_1, \eta_1 \setminus \eta_2)} e^{-\beta W(\xi_1, \eta_2)} \\
&\qquad \times \int_{\Gamma_0} e_\lambda(e^{-\beta W(\xi_1,\cdot)} - 1, \zeta) r_\mu(\eta_2 \cup \zeta)\, d\lambda_m(\zeta)
\end{aligned}$$

$$= \sum_{\eta_2 \subset \eta} e^{-\beta W(\xi_1,\eta_2)} \left(\int_{\Gamma_0} e_\lambda(e^{-\beta W(\xi_1,\cdot)} - 1, \zeta) r_\mu(\eta_2 \cup \zeta)\, d\lambda_m(\zeta) \right)$$

$$\times \sum_{\eta_1 \subset \eta \setminus \eta_2} (-1)^{|(\eta \setminus \eta_2) \setminus \eta_1|} e^{-\beta W(\xi_1,\eta_1)}$$

$$= \sum_{\eta_2 \subset \eta} e^{-\beta W(\xi_1,\eta_2)} \left(\int_{\Gamma_0} e_\lambda(e^{-\beta W(\xi_1,\cdot)} - 1, \zeta) r_\mu(\eta_2 \cup \zeta)\, d\lambda_m(\zeta) \right)$$

$$\times\ e_\lambda(e^{-\beta W(\xi_1,\cdot)} - 1, \eta \setminus \eta_2)$$

$$= \sum_{\eta_2 \subset \eta} e^{-\beta W(\xi_1,\eta_2)} \int_{\Gamma_0} e_\lambda(e^{-\beta W(\xi_1,\cdot)} - 1, (\eta \setminus \eta_2) \cup \zeta) r_\mu(\eta_2 \cup \zeta)\, d\lambda_m(\zeta).$$

In this way we have proved that if k_μ is a solution of the Mayer–Montroll equation, then r_μ is a solution of Eq. (6.6). The assertion follows by an application of a similar procedure to Eq. (6.6). $\square$

Remark 6.9. In particular, if $\xi = \{x\}$ ($x \in \mathbb{R}^d$) then, under the conditions of Theorem 6.8, we obtain the equivalence between the equation

$$r_\mu(\eta \cup \{x\}) + r_\mu(\eta) = \exp\left(\int_{\mathbb{R}^d} (e^{-\beta\phi(x,y)} - 1)\, dm(y) \right)$$

$$\times \sum_{\xi \subset \eta} e^{-\beta W(\{x\},\xi)} \int_{\Gamma_0} e_\lambda(e^{-\beta\phi(x,\cdot)} - 1, (\eta \setminus \xi) \cup \zeta) r_\mu(\xi \cup \zeta)\, d\lambda_m(\zeta),$$

$$\lambda_m \otimes m - \text{a.e.}$$

and the so-called Kirkwood–Salsburg equation, i.e.,

$$k_\mu(\eta \cup \{x\}) = e^{-\beta W(\{x\},\eta)} \int_{\Gamma_0} e_\lambda(e^{-\beta\phi(x,\cdot)} - 1, \zeta) k_\mu(\eta \cup \zeta)\, d\lambda_m(\zeta), \quad \lambda_m \otimes m - \text{a.e.}$$

Thus, Proposition 6.6 and Theorem 6.8 combined yield the following characterization result for Gibbs measures.

Proposition 6.10. *Let ϕ be a pair potential fulfilling (RC). Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be a measure locally absolutely continuous w.r.t. the Poisson measure π_m and such that r_μ fulfills the Ruelle bound. Then, μ is a Gibbs measure corresponding to the potential ϕ, the intensity measure m, and the inverse temperature β if and only if r_μ is a solution of Eq.* (6.6).

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