v^*-ALGEBRAS, INDEPENDENCE ALGEBRAS AND LOGIC

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Abstract. Independence algebras were introduced in the early 1990s by specialists in semigroup theory, as a tool to explain similarities between the transformation monoid on a set and the endomorphism monoid of a vector space. It turned out that these algebras had already been defined and studied in the 1960s, under the name of v^*-algebras, by specialists in universal algebra (and statistics). Our goal is to complete this picture by discussing how, during the middle period, independence algebras began to play a very important role in logic.

1. Introduction

There is an impressive body of deep results on independence algebras produced by experts in logic, semigroup theory and universal algebra. The aim of this paper is to survey some of the applications of the classification of independence algebras to questions in logic, and conversely, some applications of results in logic to the classification of independence algebras. Our motivation is to make better known the interactions between the two subjects, which have largely been ignored by both sides. Finally, we explain how independence algebras have recently attracted the attention of people working in both logic and group theory, and also in semigroups and rings. At the heart of these interactions are different approaches to extending van der Waerden's notion of independence. We begin with some motivational remarks about independence algebras and explain some connections with other branches of algebra.

The monoid of all endomorphisms of a vector space V (that is to say, the monoid of all homomorphisms of V into itself) and the monoid of all endomorphisms of a set X (where X is construed as an algebra with no operations and the monoid consists of all mappings of X into itself) have much in common. To describe some of their similarities, it is helpful to use a bit of (common) notation and terminology. Specifically, if S is one of these two monoids, and if a and b are in S, then ab denotes the (functional) composition of a and b. The *range* of a is, by definition, the set of values ia $(= a(i))$, where i ranges over the domain of a (the space V or the set X). The *rank* of a is the dimension of its range in the case of a vector space V, and the cardinality of its range in the case of a set X. (The dimension of a vector space is the cardinality of some — any — basis, and the dimension of a set is the cardinality of the set.) The *kernel* of a is the set of pairs (i, j) such that $ia = ja$. The function a is *idempotent* if $aa = a$. We write

$$aS = \{ac : c \in S\} \qquad \text{and} \qquad Sb = \{cb : c \in S\},$$

and we write U for the group of units in S, that is to say, the set of mappings in S that have a (compositional) inverse. If T is a subset of S, then $\langle T\rangle$ denotes the

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subsemigroup of S generated by T (the smallest subsemigroup of S that includes T).

If S is one of the two endomorphism monoids mentioned above, then the following properties hold for all $a, b \in S$ (where functions are written on the right of their arguments):

(1) $aS = bS$ if and only if a and b have the same kernel;
(2) $Sa = Sb$ if and only if a and b have the same range;
(3) $SaS = SbS$ if and only if a and b have the same rank;
(4) $SaS = SbS$ if and only if there exists a c in S such that $aS = cS$ and $Sc = Sb$;
(5) the semigroup $S \setminus U$ is generated by the set of its idempotent elements when the dimension of the underlying algebra (V or X) is finite (see [24, 48] and also [4, 7, 17, 23, 61]);
(6) if $a \in S \setminus U$, then $\langle \{a\} \cup U \rangle \setminus U$ is generated by the set of its idempotent elements when the dimension of the underlying algebra is finite (see [3, 6, 9, 10]);
(7) there exists an injective mapping a and a surjective mapping b in S such that $S = \langle U \cup \{a, b\} \rangle$ when the dimension of the underlying algebra is infinite (see [2, 8, 46, 47]);
(8) the subsemigroup of S consisting of all elements of finite rank is a completely semisimple semigroup, that is to say, it is regular and its principal factors are all completely 0-simple or completely simple (see [41]).

These examples (and many other) were floating around when Sullivan [66] wrote down a list of similarities and dissimilarities of the two monoids, and posed the problem of finding a general theory that could explain both the similarities and the dissimilarities (see also [27]). About the same time, Fountain, Gould and Lewin (see [31, 32, 41]) developed a general class of algebras for which the corresponding endomorphism monoids have properties (1)–(5) and (8) above. They called these algebras *independence algebras*, as the definition relies upon a universal algebraic notion of independence. (For a detailed account on the origins of independence algebras see [5].)

As Fountain, Gould and Lewin later learned, their discovery of independence algebras was in fact a rediscovery. Already in the late 1950s, Edward Marczewski and his collaborators investigated general notions of independence that contain, as special cases, many of the well-known independence notions occurring throughout mathematics. This eventually led them to the introduction of independence algebras under the name of v^*-algebras. In the 1990s, when Fountain, Gould and Lewin rediscovered v^*-algebras, there were already quite a number of papers written about them. In particular they had been described, up to (term) definitional equivalence, in a series of papers, the most important of which is Urbanik [67].

Therefore the usual *official* history of independence algebras (or v^*-algebras — the algebras are the same, so we will use the two names interchangeably) goes as follows.

(1) They were introduced and thoroughly studied in the 1960s by experts in universal algebra; they appeared as a class of algebras closely linked to some abstract notions of independence.

(2) Many years later, they were rediscovered (and successfully used) by experts in semigroup theory to explain the similarities (and later the dissimilarities) between the endomorphism monoids of vector space and sets.

The purpose of this paper is to complete this picture by explaining how and why v^*-algebras attracted the attention of experts in logic.

In Section 2, we discuss algebraic notions of independence, and in particular we give several characterizations of the notion of an independence algebra. In Section 3, we review some of the basic notions of logic that will be involved in our discussion. In Section 4, we explain how logicians in the 1970s became interested in independence algebras and we state their main results. In Section 5, we explain how independence algebras have recently attracted the attention of people working in logic and group theory.

2. Independence algebras

One of the important general notions of independence is defined in terms of closure systems. A *closure system* is a pair $\langle A, \mathrm{cl}\rangle$, consisting of a set A and a (unary) function cl, the *closure operator*, mapping the power set of A (the class of all subsets of A) into itself, that satisfies the following conditions for all subsets X and Y of A: (1) $X \subseteq \mathrm{cl}(X)$; (2) $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$; (3) if $X \subseteq Y$, then $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$. Subsets X with the property that $\mathrm{cl}(X) = X$ are said to be *closed*. The closure system is *algebraic* if, whenever $x \in \mathrm{cl}(X)$, then $x \in \mathrm{cl}(Y)$ for some finite subset Y of X.

A subset X of (the universe of) a closure system (A, cl) is called *C-independent* if $a \notin \mathrm{cl}(X \setminus \{a\})$ for all a in X. The closure system itself is said to satisfy the *exchange property* if, for every subset X of A and for all elements a, b of A, whenever $a \in \mathrm{cl}(X \cup \{b\})$, but $a \notin \mathrm{cl}(X)$, then $b \in \mathrm{cl}(X \cup \{a\})$. (In certain logical contexts, a closure system satisfying the exchange property is called a *pre-geometry*.) The exchange property can actually be formulated in a number of different ways, as the following lemma indicates.

Lemma 2.1. *(*[53] *p.50, Exercise 6) For a closure system* (A, cl)*, the following conditions are equivalent:*

(1) *the exchange property holds;*
(2) *for every subset X of A and every element a in A, if X is C-independent and a is not in* $\mathrm{cl}(X)$*, then $X \cup \{a\}$ is C-independent;*
(3) *for every subset X of A, if Y is a maximal C-independent subset of X, then* $\mathrm{cl}(X) = \mathrm{cl}(Y)$*;*
(4) *for subsets X, Y of A with $Y \subseteq X$, if Y is C-independent, then there is a C-independent set Z with $Y \subseteq Z \subseteq X$ and* $\mathrm{cl}(Z) = \mathrm{cl}(X)$*.*

It follows from part (4) of the lemma, that when the exchange property holds, the closure system has a maximal C-independent set. Such a set is called a *basis*. Moreover, every basis has the same cardinality, and this cardinality is called the *dimension* of the system.

Every algebra naturally gives rise to a *subalgebra closure system*. We conceive of an algebra A as a nonempty set (the universe), together with a collection of finitary operations, and we use the same symbol A to denote both the algebra and its universe. For each subset X of A, the (subalgebra) closure of X is the subalgebra of A generated by X, that is to say, the smallest subalgebra of A that includes X.

It is denoted by $\langle X\rangle$. The system $(A, \langle\ \rangle)$ is a closure system and in fact it is an algebraic closure system.

Here are some examples. If A is a vector space and X is a subset of A, then $\langle X\rangle$ is the subspace generated by X. If G is a group, then a G-*set* is an algebra $(A, f_g)_{g\in G}$, with one unary operation f_g for each element g in the group, that satisfies the equations $af_e = a$ and $(af_g)f_h = af_{gh}$ for all a in A and all g, h in G (where e is the identity element of the group, and gh is the composition of g with h). These unary operations form a group of permutations of A, and the closure $\langle X\rangle$ of a subset X of A is the orbit of X under these permutations. If the group G is trivial (so that A is essentially a set without operations, except for the identity function), then the closure of X is always just X, and the dimension of X is its cardinality.

A *matroid algebra* is an algebra such that the exchange property holds for its subalgebra closure system. G-sets with a subalgebra of distinguished constants that contain all fixed points, and vectors spaces over skew fields with a subspace of distinguished constants are examples of matroid algebras.

There is another notion of independence for algebras A that was introduced by Marczewski in [50]. A subset X of A is said to be *M-independent* if every map from X to A can be extended to some homomorphism from $\langle X\rangle$ to A. Although the letter "M" stands for Marczewski, it can also be thought of as standing for *mapping*, since the definition of M-independence involves extensions of mappings. It is not difficult to check that M-independence implies C-independence among subsets of A, but the converse does not hold in general (see [40]). An algebra A is said to be an *MC-algebra* if in A the notions of C-independence and M-independence coincide, that is to say, for every subset X of A, we have

$$X \text{ is C-independent} \qquad \text{if and only if} \qquad X \text{ is M-independent.}$$

MC-algebras appear in the literature under the name v^{**}-algebras (see [56] and [68]). Every absolutely free algebra is an example of an MC-algebra. (An algebra is said to be *absolutely free* if it is free in the class of all algebras of the same similarity type. See [68] for this and many other examples.)

A matroid algebra that is simultaneously an MC-algebra is called an *independence algebra* or a v^*-*algebra*. Familiar examples include free G-sets with a subalgebra of distinguished constants that include all fixed points (sets without operations are a particular instance of this) and vector spaces over skew fields, with a subspace of distinguished constants (see [20, 68]). Traditionally (see [31, 32, 41]), independence algebras are defined as algebras A satisfying the exchange property and the property

[F] for every subset X of A, if no element x in X belongs to $\langle X \setminus \{x\}\rangle$, then every mapping from X into A can be extended to a homomorphism from $\langle X\rangle$ into A.

Condition $[F]$ just says that every C-independent set X is also M-independent. As the converse holds in arbitrary algebras, the equivalence of our definition with the traditional one is clear. (For other notions of independence, and for a discussion of the induced classes of algebras defined by the identification of two different notions of independence, see [11].)

Since an independence algebra is a matroid algebra, we can attach to any subalgebra a *dimension* (or *rank*) which is the cardinality of some (any) basis, that is to say, any C-independent generating set. Observe that the bases of an independence algebra are just the minimal generating sets, or, equivalently, the maximal C-independent sets.

Interestingly, in any MC-algebra two bases always have the same cardinality (see [56]). The exchange property is therefore not needed in the definition of an independence algebra to ensure that bases have the same cardinality. Rather, it is needed to guarantee that bases really do exist. In fact, there are MC-algebras without bases (see [44, §32, example after the proof of Theorem 4]). G. H. Wenzel (see [44, p. 219, Exercise 30]) introduced the notion of a v'-algebra as an MC-algebra in which every maximal C-independent set is a generating set. Clearly, each v'-algebra has the key properties of v^*-algebras, namely it has a basis and all bases have the same cardinal. Moreover, the assumption that a maximal C-independent set is set of generators seems weaker than the exchange property. However, as Wenzel (and Urbanik) soon realized, every v'-algebra is a v^*-algebra, and of course vice versa (see [44, p. 219, Exercise 34] or Lemma 2.1 above).

The class of *based* MC-algebras — that is, MC-algebras with a C-independent generating set — *is* strictly larger than the class of v^*-algebras, since it contains every absolutely free algebra (and these are, in general, not v^*-algebras); for other examples see [11, Section 9]. It might be of some interest to experts in logic, semigroup theory, and universal algebra to investigate this broader class further.

Narkiewicz [55] gave two different characterizations of independence algebras. Here is the first.

Proposition 2.2. *An algebra A is a v^*-algebra if and only if it satisfies the following two conditions.*

(I) *The set $\{a\}$ is M-independent in A whenever a is an element of A that is not a constant.*

(II) *If a set $\{a_1, \ldots, a_n\}$ is M-independent in A, but $\{a_1, \ldots, a_{n+1}\}$ is not, then the element a_{n+1} is generated by the set $\{a_1, \ldots, a_n\}$.*

The second characterization was suggested to Narkiewicz by Świerezkowski.

Proposition 2.3. *An algebra A is a v^*-algebra if and only if it satisfies the following conditions.*

(I) *A is an MC-algebra.*

(II) *In each subalgebra of A with a finite basis (i.e., a C-independent generating set) consisting of k elements, every M-independent set of k elements forms a basis for the subalgebra.*

The striking similarities (and dissimilarities) between the endomorphism monoids of vector spaces and sets suggested that there should be a general theory that could explain the similarities, and this led to the discovery of independence algebras. Palfy [58] introduced a class of algebras and proved a classification theorem for them that is strikingly similar to Urbanik's classification of v^*-algebras. Jan Mycielski, in a private communication, has suggested the problem of finding a general class of algebras that contains both v^*-algebras and Palfy's algebras, and that admits a classification theorem which includes the theorems of Urbanik and Palfy theorems as special cases. Each algebra A in such a class of algebras should satisfy the following property: if t is a non-constant unary term of A and there exists $b \in A$ such that $t(b)$ is not a constant, then t is a bijection. (A term is said to be *constant* in

A if it assumes the same value on every argument; an element in A is said to be a *constant* if it belongs to the subalgebra of A generated by the empty set.) Palfy's algebras have this property by definition; and we now show that independence algebras also have this property. Suppose A is an independence algebra of rank at least one and consider any non-constant term t. If b is an element of A that is not a constant, then $\{b\}$ is M-independent (by Proposition 2.2) and so any map $f : \{b\} \to A$ extends to a homomorphism from $\langle b \rangle$ into A. In particular, $t(f(b)) = f(t(b))$ and consequentely $t(b)$ is not a constant since t is non-constant. Hence $\langle b \rangle = \langle t(b) \rangle$ so that $b = q(t(b))$ for some unary term q. It follows that $f(b) = qt(f(b))$ for all maps $f : \{b\} \to A$ so that $x = qt(x)$, for all elements $x \in A$, and hence $qt = 1$. Clearly q is non-constant and so the same argument shows that $pq = 1$ for some unary term p. It follows that $p = t$ and thus t is a bijection, as claimed.

Before closing this section, it is worth pointing out that many papers have recently appeared in which the ideas that led specialists in semigroup theory to introduce independence algebras in the first place have been extended. The motivation for these extensions is the general problem of describing the rings R such that all the non-invertible matrices with entries in R are products of idempotents. The idea has been to consider a notion of independence less general than C-independence, called PC-independence, and then to study matroid MPC-algebras (algebras in which M = PC), and in particular to study the endomorphism monoid of these algebras. As PC-independence is less general than C-independence the class of MPC-algebras contains the class of MC-algebras. Although some techniques from independence algebras can be transferred, this new situation involves significant additional complications. For generalizations of independence algebras prompted by this motivation see for example [28, 29, 30, 42, 43].

3. Logic

The aim of this section is to introduce some concepts from logic that we shall need, and to fix our notation. A first-order language consists of a set of symbols, and sets of expressions (strings of these symbols) called terms, formulas, and sentences. The *logical symbols* are: parentheses (which are used to ensure unique readability of expressions); the usual sentential connectives $\neg$ (read "not"), $\vee$ (read "or"), $\wedge$ (read "and"), $\to$ (read "implies"), and $\leftrightarrow$ (read "if and only if"); quantifiers $\forall$ (read "for all") and $\exists$ (read "there exists"); the symbol $=$ (read "equals"); and a countably infinite sequence of variables. The *non-logical symbols* are of two types: *operation symbols*, each of some finite rank (constants are identified with operation symbols of rank 0), and *relation symbols*, each of some finite rank. The set of *terms* is defined recursively: every variable and every constant is a term; inductively, if f is an operation symbol of rank n, and if $t_1, \ldots, t_n$ are terms, then $ft_1 \ldots t_n$ is a term. The set of *formulas* is also defined recursively: *equations* are expressions of the form $t_1 = t_2$; *atomic formulas* are expressions that are either equations or expressions of the form $Rt_1 \ldots t_n$, where $t_1, \ldots, t_n$ are terms and R is a relation symbol of rank n; inductively, if ϕ_1 and ϕ_2 are formulas, then so are

$$(\neg\phi_1), \quad (\phi_1 \vee \phi_2), \quad (\phi_1 \wedge \phi_2), \quad (\phi_1 \to \phi_2), \quad (\phi_1 \leftrightarrow \phi_2), \quad (\exists x\phi_1), \quad (\forall x\phi_1),$$

where x is any variable of the language. (When writing formulas below, we shall not feel compelled to write explicitly every parenthesis.)

Associated with a first-order language is also a set of rules (called *rules of inference*) for proving statements. A *theory* in the language is a set of formulas that is closed under the rules of inference. If the theory has a set of axioms of a particular form, then the theory is often said to be of that form. For instance a theory is *equational* if it has a set of axioms that are equations, and it is *quasi-equational* if if has a set of axioms of the form

$$(\phi_1 \wedge \cdots \wedge \phi_n) \to \phi_{n+1},$$

where the formulas ϕ_i are all equations.

A mathematical structure for a first-order language consists of a non-empty set D, together with a collection of operations and relations corresponding to the operation and relation symbols of the language: if f is an operation symbol of rank n, then f^D is an n-ary operation on D; if R is a relation symbol of rank n, then R^D is an n-ary relation on D. Such a structure is said to be a *model* of a given theory in the language if every formula (or, equivalently, every axiom) of the theory is true in the structure. The class of models of a given type of theory is often called by the same name as the theory. For instance, the class of models of an equational theory is called an *equational class*, or a *variety*, and the class of models of a quasi-equational theory is called a *quasi-equational class*, or a *quasi-variety*. With every structure for a first-order language one can associate a *theory in that language*, namely the set of all formulas that are true in the structure. It is called the theory of the structure.

Suppose D is a structure for a given first-order language. A subset E of the universe of D is (first-order) *definable* in D from parameters (elements) $b_1, \ldots, b_n$ if there is a formula $\phi(x, y_1, \ldots, y_n)$ in the language whose free variables are included among $x, y_1, \ldots, y_n$, and such that

$$E = \{a : \phi(a, b_1, \ldots, b_n) \text{ is true in } D\}$$

(where $\phi(a, b_1, \ldots, b_n)$ is the formula obtained from ϕ by replacing all free occurrences of the variables $x, y_1, \ldots, y_n$ with $a, b_1, \ldots, b_n$ respectively). When the definable subset E contains just one element, say a, then we say that the element a is definable in D from the parameters $b_1, \ldots, b_n$. The notions of a definable subset and a definable element lead naturally to two closure operators on D (more precisely, on the class of all subsets of the universe of D). The *algebraic closure operator* acl is defined on each subset X of D as follows: $\mathrm{acl}(X)$ is the union of the finite sets that are definable in D using parameters from X; in other words, it consists of those elements a in D such that a belongs to a finite subset of D that is definable using parameters from X. For instance, in the field of complex numbers, the algebraic closure of a set X is the set of complex roots of polynomials with coefficients in X. The *definable closure operator* dcl is defined on each subset X of D as follows: $\mathrm{dcl}(X)$ is the set of elements definable in D using parameters from X. In the field of complex numbers, the definable closure of a set X is just the subfield generated by X.

A notion related to definability is that of an elementary extension of D. A structure D' is an *elementary extension* of D if D is a substructure of D' (that is to say, the universe of D is a non-empty subset of the universe of D' that is closed under the operations of D', and the operations and relations of D are the restrictions to the universe of D of the corresponding operations and relations of D') with the following additional property: for each formula $\phi(x, y_1, \ldots, y_m)$ and parameters $b_1, \ldots, b_m$ in D, if $\phi(x, b_1, \ldots, b_m)$ defines a non-empty subset of D', then it defines a non-empty subset of D.

We now turn to more specialized notions of logic related to the study of categorical theories. Recall that a consistent mathematical theory (in any language) is said to be (absolutely) *categorical* if all its models are isomorphic. For example, Euclidean geometry as axiomatized by Hilbert, and arithmetic as axiomatized by Dedekind and Peano, are both categorical (second order) theories. For first-order theories, the notion of absolute categoricity does not play an important role. The reason is that any first-order theory with an infinite model has models of arbitrarily large cardinalities, and therefore cannot be categorical. For this reason, Łoś [49] proposed the following weaker notion: a theory is *categorical in power* κ if it has a model of power κ, and if all models of power κ are isomorphic. For instance, the theory of vector spaces over a fixed finite field is categorical in every infinite power, while the theory of vector spaces over the rational numbers is categorical in every uncountable power, but not in power $\aleph_0$. (Any two vector spaces over the rational numbers that have the same uncountable cardinality κ must have dimension κ and therefore must be isomorphic; on the other hand, for each non-zero cardinal number $\kappa \leq \aleph_0$, there is a countably infinite vector space of dimension κ over the rational numbers, and these vector spaces are not isomorphic to one another.) The theory of atomless Boolean algebras is an example of a theory that is categorical in power $\aleph_0$, but not in uncountable powers. Inspired by Steinitz's theorem that every algebraically closed field is determined by the cardinality of its transcendence base over the prime field, Łoś [49] conjectured, and Morley [54] proved, that a countable first-order theory (that is, a theory in a first-order language with countably many symbols) is categorical in one uncountable power if and only if it is categorical in every uncountable power. Shelah [62] proved an analogous result for uncountable first-order theories.

A notion that has proved very useful in the study of first-order theories categorical in uncountable powers is that of a strongly minimal structure. A structure D is said to be *strongly minimal* if, in every elementary extension of D, every subset that is definable without using parameters is either finite or cofinite. This notion was introduced by Marsh [51], and is related to Morley's notion of a set of rank 1 and degree 1. The reason such structures are of interest to us is that, in this case, the closure system (D, acl) satisfies the exchange property (with cl replaced by acl) and is therefore a pre-geometry.

A *geometry* is a pre-geometry (A, cl) (a closure system satisfying the exchange property) with the following additional properties: $\mathrm{cl}(\varnothing) = \varnothing$, and $\mathrm{cl}(\{a\}) = \{a\}$ for each a in A. With each strongly minimal structure D, one can associate a geometry (A, cl) as follows:

$$A = \{\mathrm{acl}(\{a\}) : a \in D \setminus \mathrm{cl}(\varnothing)\},$$

and if X is any subset of A, say $X = \{\mathrm{acl}(\{a\}) : a \in Y\}$ (where Y is a subset of D), then

$$\mathrm{cl}(X) = \{\mathrm{acl}(\{a\}) : a \in \mathrm{acl}(Y)\}.$$

The following example illustrates some of the notions that have been discussed in this section. Consider the theory Φ of algebraically closed fields of characteristic zero. (It is understood below that the characteristic is always zero.) The language of Φ is that of rings. There are four non-logical constants, all operation symbols:

the nullary operation symbols 0 and 1, and the binary operation symbols + and $\cdot$. Any ring — for example, the ring $\mathbb{Z}$ of integers or the field $\mathbb{R}$ of real numbers — is an appropriate structure for this language. The axioms of Φ state that a model is a field of characteristic zero, and that every non-zero polynomial has a root (see [18]). An example of a model of Φ is the field $\mathbb{C}$ of complex numbers. In fact, the theory of $\mathbb{C}$ coincides with Φ: a formula is in Φ (and hence is true of all algebraically closed fields) if and only if it is true in $\mathbb{C}$. Note that $\mathbb{R}$ is not a model of Φ. In fact, the formula $x^2 + 1 = 0$ has a solution in $\mathbb{C}$ but no solution in $\mathbb{R}$.

The theory Φ is categorical in power κ for any uncountable cardinal κ. Indeed, if C is a model of Φ of power κ, then C has a transcendental basis (an independent set of transcendental numbers) over the subfield of rational numbers, and C is determined up to isomorphism by the cardinality of this basis.

Also, as was proved by Tarski, the theory Φ admits elimination of quantifiers in the sense that every formula ϕ is provably equivalent to a (finite) Boolean combination of equations of the form $t = 0$, where t is a term — that is to say, a polynomial — in the language of rings, and the variables that occur in t occur freely in ϕ (see [18]). This property implies that every model of Φ — and in particular, the field $\mathbb{C}$ of complex numbers — is strongly minimal. Indeed, by quantifier elimination, every formula with one free variable is equivalent to a Boolean combination of polynomial equations in one variable. Since non-zero polynomial equations on one variable have only finitely many solutions, it follows that every definable subset of a model of Φ is a (set-theoretic) Boolean combination of finite sets, and is therefore either finite or cofinite. Another consequence of quantifier elimination is that in algebraically closed fields, definable sets are Boolean combinations of algebraic varieties. A third consequence is that, if C and C' are algebraically closed fields, then C' is an elementary extension of C if and only if C is a subfield of C'.

For every subset X of an algebraically closed field C, the definable closure $\mathrm{dcl}(X)$ is the smallest subfield of C that includes X, that is to say, it is the subfield of C generated by X; the algebraic closure $\mathrm{acl}(X)$ is the smallest algebraically closed subfield of C that includes X, that is to say, it is the algebraically closed subfield of C generated by X. The dimension associated with the algebraic closure system (C, acl) is closely related to the dimension of algebraic varieties.

Other examples of strongly minimal structures are: G-sets with a subalgebra of distinguished constants that includes all fixed points, and vector spaces over skew fields with a subspace of distinguished constants. In the latter case, the operators acl and dcl coincide, and the dimension of the associated algebraic closure system coincides with the vector space dimension over the subspace.

4. Quasi-varieties Categorical in Power

In logic, v^*-algebras have played an important role in the study of first-order theories categorical in power. In particular, Urbanik's description in [67] of the v^*-algebras of dimension at least three was used by Givant in the early 1970s to give a complete description of all varieties and quasi-varieties categorical in power (see [38], [39] and the cited abstracts). A different, but related description was discovered independently, and about the same time, by Palyutin (see [59]).

As the examples discussed in the preceding section make clear, for countable first-order theories, categoricity in power $\aleph_0$ does not imply, and is not implied by, categoricity in uncountable cardinalities. In 1971, Alfred Tarski, pondering this lack of symmetry, asked if, for varieties (equational classes), this distinction between $\aleph_0$ and uncountable cardinalities would disappear if one replaced the notion of categoricity in power by a notion of freeness in power. (Recall that every variety, and, more generally, every quasi-variety, contains free algebras with κ free generators, for every cardinal $\kappa > 0$.) A quasi-variety is said to be *free in power* κ if it has a model of power κ, and if all models of power κ are free (over the variety). The relationship between the notions of categoricity and freeness in power is easy to describe: in cardinalities larger than the size of the language, the two notions are equivalent; as regards the cardinality of the language, a quasi-variety categorical in that cardinality is necessarily free in that cardinality, but the reverse implication fails.

Tarski observed that the variety of vector spaces over the rational numbers is free in every infinite power (although it is not $\aleph_0$-categorical), and he asked whether a variety free in some infinite power is necessarily free in every infinite power (in which it has a model). For countable languages, Baldwin, Lachlan, and McKenzie (see [16]), and independently, Palyutin (see [1]) proved that a quasi-variety categorical in power $\aleph_0$ is categorical (and hence free) in all higher powers. Baldwin and Lachlan, and also Palyutin and Taitslin, proved that for quasi-varieties with a finite model of power at least two, categoricity in power $\aleph_1$ implies categoricity (and hence freeness) in power $\aleph_0$ and in each finite cardinality in which there is a model. Givant [33], [35] (see also [38]) proved (for languages of arbitrary cardinality) that if a quasi-variety is free, but not categorical, in some infinite power, then it is categorical in all higher powers and in fact each of its models is a v^*-algebra. Similarly, he showed that if a quasi-variety has a finite, non-trivial model, and if all its finite models are free, then it is categorical in all infinite powers and each of its models is again a v^*-algebra. With the essential help of Urbanik's theorem, Givant [34] (see also [38]) gave a complete description of these quasi-varieties.

Theorem 4.1. *If K is a quasi-variety that is free, but not categorical, in some infinite power, or else if it has a finite non-trivial model and all its finite models are free, then K is* (*term*) *definitionally equivalent to one of the following:*

(1) *a quasi-variety of vector spaces over a fixed skew field, with a fixed subspace of distinguished constants;*
(2) *a quasi-variety of affine spaces over a fixed skew field, with a fixed subspace of distinguished unary translation functions;*
(3) *a quasi-variety of G-sets over a fixed group G with a fixed subalgebra of distinguished constants that contains all fixed points, and such that each unary operation has at most one fixed point.*

For varieties, the description assumes a simpler form.

Corollary 4.2. *If K is a variety that satisfies the hypotheses of the preceding theorem, then K is definitionally equivalent to one of the following:*

(1) *a variety of vector spaces over a fixed skew field;*
(2) *a variety of affine spaces over a fixed skew field;*
(3) *the variety of sets;*
(4) *the variety of sets with a distinguished constant.*

It turns out that, contrary to Tarski's hope, not every variety categorical in some infinite power is necessarily free in all infinite powers in which there are models.

For a concrete example, consider a variety or quasi-variety K of v^*-algebras, for example a variety of affine spaces over a fixed skew field, or the variety of sets. For a given positive integer n, form the n-th Cartesian power of each algebra in K (the product of each algebra with itself n times), and adjoin to the basic operations of this power an n-ary decomposition function and n unary projection functions. (These additional functions permit one to axiomatize the class of n-th powers of algebras in K by means of a set of equations.) The resulting variety or quasi-variety K_n of n-th powers of algebras in K is categorical in suitably large infinite powers, but when $n \geq 2$, it will not be free in all infinite powers (or in all finite powers) in which there is a model.

In the case of the n-th powers of affine spaces over a fixed skew field, it is possible to extend the preceding construction somewhat further by adjoining to each space a unary operation that equationally determines a distinguished hyperplane of fixed dimension k, where $0 \leq k \leq n$. These classes are definitionally equivalent forms of the class of affine spaces over the ring of n-by-n matrices with entries from the skew field, and with a distinguished idempotent n-by-n matrix operation δ_k whose first k diagonal entries are 1 and all other entries are 0. When $k = 0$, the resulting class is definitionally equivalent to the class of n-th powers of vector spaces over the skew field, or, equivalently, to the variety of modules over the ring of n-by-n matrices with entries from the skew field.

Givant [36] (see also [38]) showed that for each quasi-variety categorical in some infinite power, there is, roughly speaking, an equation that defines a v^*-algebra inside each model, and the models of the quasi-variety are obtained by the n-th power construction (for some positive integer n) from a quasi-variety of v^*-algebras. In fact, he proved the following theorem (see [37] and [39]).

Theorem 4.3. *A quasi-variety is categorical in some infinite power if and only if it is definitionally equivalent to one of the following, for some positive integer n and some integer k with $0 \leq k \leq n$:*

(1) *a quasi-variety of affine spaces over the ring of n-by-n matrices with entries from a fixed skew field, with a fixed subspace of distinguished unary translation functions, and with a distinguished unary n-by-n idempotent matrix operation δ_k as described above;*
(2) *a quasi-variety of the n-th powers of G-sets over a fixed group G with a fixed subalgebra of distinguished constants that contains all fixed points, and such that each unary operation has at most one fixed point.*

Again, the description simplifies for varieties.

Corollary 4.4. *A variety is categorical in some infinite power if and only if it is definitionally equivalent to one of the following, for some positive integer n and some integer k with $0 \leq k \leq n$:*

(1) *a variety of affine spaces over the ring of n-by-n matrices with entries from a fixed skew field, with a distinguished unary n-by-n idempotent matrix operation δ_k as described above;*
(2) *the variety of n-th powers of sets;*
(3) *the variety of n-powers of sets with a distinguished constant.*

5. The Classification of Independence Algebras

According to Urbanik's theorem, every independence algebra (or v^*-algebra) of dimension at least 3 is, up to (term) definitional equivalence, either a free G-set with a distinguished subalgebra of constants that contains all fixed points, or a vector space over a skew field, with a distinguished subspace of constants, or an affine space over a skew field, with a distinguished subspace of unary translation functions. The case of dimension less than 3 was also considered by Grätzer and Urbanik ([44] and [68]). Another perspective on the representation of independence algebras is provided by a recent result of Cameron and Szabó [20] that we now explain. Note however that the Cameron-Szabó theorem below (Theorem 5.1) is not contained in Urbanik's theorems, and vice versa. First, the theorem below applies only to finite independence algebras, whereas Urbanik's theorems apply to all independence algebras. Second, the representation below is in terms of a notion of "endomorphism" equivalence that will be defined below, whereas Urbanik's representation is in terms of the notion of term definitional equivalence. Third, the theorem below covers all finite dimensional finite algebras, whereas the main theorem of Urbanik is for algebras of dimension at least three — but, as pointed out, there are other results, similar to Urbanik's, for algebras of dimension less than three.

An independence algebra A determines (and, in some sense, is determined by) the collection of its subalgebras and the monoid S of its endomorphisms. Following the terminology of [20], we say that two independence algebras A_1 and A_2 are *equivalent* if there is a bijection $\theta : A_1 \to A_2$ such that: (1) both θ and θ^{-1} map subalgebras to subalgebras; (2) if $f_i : A_i \to A_i$ (for $i = 1, 2$) are maps satisfying $f_1\theta = \theta f_2$, then f_1 is an endomorphism of A_1 if and only if f_2 is an endomorphism of A_2.

The classification of finite independence algebras up to this equivalence is the following.

Theorem 5.1 ([20]). *Any finite algebra A is an independence algebra if and only if it is equivalent to one of the following three types.*

1. *Trivial finite independence algebras: $A = (X \times G) \cup C$, where X is a set, G a group, C a left G-set, and with nullary operations ν_c ($c \in C$) with value c, and unary operations λ_g ($g \in G$) given by*

$$\lambda_g((x, h)) = (x, gh) \text{ for } x \in X,\ h \in G,$$

$$\lambda_g(c) = g(c) \text{ for } c \in C.$$

2. *Non-trivial finite independence algebras with $\langle\emptyset\rangle \neq \emptyset$: $A = V[W]$, where V a vector space, W a subspace of V, and $V[W]$ is the vector space V with a distinguished constant w adjoined for each $w \in W$.*
3. *Non-trivial finite independence algebras with $\langle\emptyset\rangle = \emptyset$: A is either $\mathrm{Aff}(V)[+W]$ for some vector space V, subspace W, and $\mathrm{Aff}(V)[+W]$ is the affine space $\mathrm{Aff}(V)$ with a distinguished unary translation function $\tau_w(x) = x + w$ adjoined for each $w \in W$, or else A is an affine near field algebra.*

(An independence algebra is trivial if the associated pre-geometry is trivial i.e. the closure operator satisfies $\mathrm{cl}(X) = \bigcup_{x\in X} \mathrm{cl}(\{x\})$ for every subset X of the pre-geometry. A near field is a structure that satisfies all of the field axioms except commutative law for multiplication and the right distributivity law.)

This result for the finite independence algebras relies on the classification of finite simple groups, and more precisely on the work of Maund [52] classifying the geometric groups of rank at least 2 (see [20]). A geometric group is a permutation group such that the pointwise stabilizer of any sequence of points acts transitively on the points it does not fix (if any). The relevance of geometric groups to the classification problem is due to the fact that the automorphism group of an independence algebra is a geometric group ([20] Theorem 6.1).

The following theorem is related to the preceding one.

Theorem 5.2 ([20]). *The subalgebra lattice of a finite dimensional independence algebra is a Boolean lattice or a projective or affine geometry.*

The same result, but for dimensions ≥ 8, was obtained earlier by Zilber [76], using his independent determination of the geometric groups of rank at least 7. His approach does not use the classification of finite simple groups, but rather some combinatorial results concerning the study and classification of strongly minimal structures whose pre-geometry is non trivial and locally modular, i.e.,

$$\dim \mathrm{cl}_Z(X \cup Y) + \dim \mathrm{cl}_Z(X \cap Y) = \dim \mathrm{cl}_Z(X) + \dim \mathrm{cl}_Z(Y)$$

for all subsets X, Y and Z where (for any subset W) $cl_Z(W)$ is defined to be $cl(W \cup Z)$, and dim $cl_Z(W)$ is the dimension associated with the closure operator. See [75] for the finite dimensional case which implies the infinite dimensional one, and also the earlier works [70], [71], [73], [74] on the infinite dimensional case. All of this is reproduced in monograph [78]. The work of Evans [25] is also a very good source on arguments similar to Zilber's.

Theorem 5.1 applies to finite independence algebras. The classification of infinite independence algebras that are locally finite is as follows.

Theorem 5.3 ([20]). *A non-trivial, locally finite, infinite independence algebra is equivalent to $V[W]$ or to $\mathrm{Aff}(V)[+W]$ for some infinite vector space V over a finite field, and some finite-dimensional subspace W.*

This theorem is already implicit in Zilber's work from the 1980's, which applies to a more general context than that of algebras where dcl = acl. The proof uses his determination of the geometric groups of rank at least 7, not Maund's work.

The following is a related result due to Zilber [72] and, independently, to Cherlin, Mills, and Neumann (who used the classification of finite simple groups in their proof - see [60] for details).

Theorem 5.4. *The geometry associated with an $\aleph_0$-categorical, strongly minimal structure is either trivial or a projective or affine geometry of infinite dimension over some finite field.*

Let A be a non-trivial (i.e. whose pre-geometry is non trivial) infinite independence algebra that is locally finite. From A we construct a structure D in the following way: the universe of D is the set A; there are no operations; the relations of D are, for each natural number n, the $n+1$-ary relation R_n on the universe defined by

$$R_n(x_1, \dots, x_n, x) \qquad \text{if and only if} \qquad x \in \langle \{x_1, \dots, x_n\} \rangle.$$

It is easy to verify that the theory of D is categorical in power $\aleph_0$ and admits elimination of quantifiers, that D itself is a strongly minimal structure, and that the pre-geometry (D, acl) coincides with $(A, \langle\ \rangle)$ (see [60] page 99 for details). It follows from Theorem 5.4 that the geometry associated with D (and hence with A) is equivalent to a projective or an affine geometry of infinite dimension over some finite field.

For other important are related results connecting strongly minimal structures and independence algebras see [76] and [77].

Strongly minimal structures and their associated pre-geometries play an important role in logic. Here are three examples. (1) They were used to give a new proof of Morley's theorem that if a countable theory is categorical in some uncountable cardinal, then it is categorical in every uncountable cardinal (see [15]). (2) Variants of strongly minimal structures and their associated pre-geometries were used to prove the Manin-Mumford and Mordel-Lang conjectures from number theory (see [18]). (3) Zilber's theorem (Theorem 5.4) was proved to show that an infinite structure D whose theory is categorical in every infinite cardinal is not finitely axiomatizable (see [60]).

It is worth mentioning here the development of geometric stability theory, an important branch of logic. Much of this development is only tangentially related to the study of independence algebras, but there are connections. The notion of a stable theory was introduced by Shelah in his analysis of uncountable theories categorical in power. Roughly speaking, a theory is *unstable* (not stable) if in some model it is possible to define a linear order on some infinite set of n-tuples. Thus, any theory of linear orderings (with infinite models), any extension of the theory of Boolean algebras, and the theory of Peano arithmetic are all examples of unstable theories. Shelah proved that an unstable (countable) theory has the maximum possible number of non-isomorphic models in every uncountable cardinality λ, namely 2^λ. Within the class of stable theories Shelah isolated the subclass of superstable theories, and its subclass of ω-stable theories. The latter class was already introduced by Morley under the name of *totally transcendental* theories. Shelah proved that, in fact, a (countable) non-superstable theory has 2^λ models of cardinality λ for every uncountable cardinal λ. Countable theories categorical in uncountable powers are examples of ω-stable theories, but there are other interesting examples as well; for instance, the theory of differentially closed fields of characteristic zero, and the theory of an abelian group that is a direct sum of a divisible abelian group and an abelian group of bounded exponent (for example, $(\mathbb{Z}/3\mathbb{Z})^\omega \oplus (\mathbb{Z}/2\mathbb{Z})^\omega$). An example of a superstable theory that is not ω-stable is the theory of $(\mathbb{Z}, +)$ of the integers under addition, and an example of a stable theory that is not superstable is the theory of $(\mathbb{Z}^\omega, +)$.

One of the original problems in stability theory was the determination of the number of non-isomorphic models of each appropriately large cardinality of a given stable theory. One of Shelah's key ideas was to analyze the models of such a theory using an independence relation called nonforking. Shelah's more general independence relation, which can be introduced axiomatically, does not possess the exchange property, but in some suitable cases it permits one to re-construct the models from sets of realizations of "regular types" for which an exchange principle holds. For details on all of this and much more see [63], [13], [19] and [60]; the references [21] and [22] are also very useful.

Two related lines of research in mathematical logic that are worth mentioning are: the solution of Vaught's conjecture ([69]) for varieties by Hart, Starchenko and

Valeriote [45], which depends very much on representation theorems and makes use of some tame congruence theory; and Zilber's generalization in [79] of the notion of strongly minimality in order to study sentences in $L_{\omega_1,\omega}$ that have as models certain combinatorial geometries (matroids) with additional properties, one of which turns out to be a special case of what Shelah ([64], [65]) calls excellence. For the latter, see also [14] where it is shown that the excellence condition yields directly something like an M-independence (where, however, the morphisms are not just homomorphisms).

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References

[1] A. I. Abakumov, E. A. Palyutin, M. A. Taitslin and Yu. E. Shishmarev, *Kategorichnye kvazimnogoobrazija*, **Algebra i Logika**, **11** (1972), 3–38.
[2] J. Araújo, *Generators for the Semigroup of Endomorphisms of an Independence Algebra*, **Algebra Colloq.**, **9** (4) (2002), 375–382
[3] J. Araújo, *Normal semigroups of endomorphisms of proper independence algebras are idempotent generated*, **Proc. Edinburgh Math. Soc.**, **45** (2002), 205–217.
[4] J. Araújo, *Idempotent generated endomorphisms of an independence algebra*, **Semigroup Forum**, **67** (2003), 464–467.
[5] J. Araújo and J. Fountain, *The Origins of Independence Algebras*, *Proceedings of the Workshop on Semigroups and Languages (Lisbon 2002)*, World Scientific, (2004), 54–67
[6] J. Araújo and J. Fountain, *A description of normal semigroups of endomorphisms of proper independence algebras*, **Comm. Algebra**, **3** (2005), 2705–2711.
[7] J. Araújo and J. M. Mitchell, *An elementary proof that every singular matrix is a product of idempotent matrices*, **Amer. Math. Monthly**, **112** (7) (2005), 641–645.
[8] J. Araújo and J. M. Mitchell, *Relative ranks in the monoid of endomorphisms of an independence algebra*, **Monatsh. Math.**, **151** (2007), 1–10.
[9] J. Araújo and F. C. Silva, *Semigroups of linear endomorphisms closed under conjugation*, **Comm. Algebra**, **28** (2000), 3679–3689.
[10] J. Araújo and F. C. Silva, *Semigroups of matrices closed under conjugation by normal linear groups*, **JP J. Algebra Number Theory Appl.**, **5** (2005), 353–345.
[11] J. Araújo and F. Wehrung, *Embedding properties of endomorphism semigroups*, **Fundam. Math., 202** (2009), 125–146.
[12] J. T. Baldwin, *First order theories of abstract dependence relations*, **Ann. Pure Appl. Logic**, **26** (1984), 215–243.
[13] J. T. Baldwin, **Fundamentals of Stability Theory**, Perspective in Mathematical Logic. Springer-Verlag 1988.
[14] J. T. Baldwin, *Notes on quasiminimality and excellence*, **Bull. Symbolic Logic**, **10** (2004) 334–367.
[15] J. T. Baldwin and A. H. Lachlan, *Almost strongly minimal theories*, **J. Symbolic Logic**, **37** (1972), 487–493.
[16] J. T. Baldwin and A. H. Lachlan, *On universal Horn classes categorical in some infinite power*, **Algebra Universalis**, **3** (1973), 98–111.
[17] C. S. Ballantine, *Products of idempotent matrices*, **Linear Algebra Appl. 19** (1978), 81–86.
[18] E. Bouscaren (Ed.), **Model theory and algebraic geometry**, Lecture Notes in Mathematics 1696 Springer-Verlag 1998.
[19] S. Buechler, **Essential Stability Theory**, Perspective in Mathematical Logic. Springer-Verlag 1996.
[20] P. J. Cameron and C. Szabó, *Independence algebras*, **J. London Math. Soc.**, **61** (2000), 321–334.

[21] G. L. Cherlin, *Combinatorial problems connected with finite homogeneity*, **Contemporary Mathematics**, **131** (1992) 3–30.
[22] G. L. Cherlin and E. Hrushovski, *Finite Structures with Few Types*, **Annals of Mathematics Studies**, Princeton University Press 2003.
[23] R. J. H. Dawlings, *Products of idempotents in the semigroup of singular endomorphisms of a finite-dimensional vector space*, **Proc. Roy. Soc. Edinburgh Sect. A**, **91** (1981/82), 123–133.
[24] J. A. Erdos, *On products of idempotent matrices*, **Glasgow Math. J. 8** (1967), 118–122.
[25] D. Evans, *Homogeneous geometries*, **Proc. London Math. Soc.** , **52** (1986), 305–327.
[26] C.-A. Faure and A. Frölicher, **Modern projective geometry**, Kluwer, Dordrecht, 2000.
[27] J. Fountain, *The depth of the semigroup of balanced endomorphisms*, **Mathematika 41** (1994), 199–208.
[28] J. Fountain and V. Gould, *Relatively free algebras with weak exchange properties*, **J. Austral. Math. Soc. 75** (2003), 355–384.
[29] J. Fountain and V. Gould, *Endomorphisms of relatively free algebras with weak exchange properties*, **Algebra Universalis, 51** (2004), 257–285.
[30] J. Fountain and V. Gould, *Products of idempotent endomorphisms of relatively free algebras with weak exchange properties*, **Proc. Edinburgh Math. Soc.**, **50** (2007), 343–362.
[31] J. Fountain and A. Lewin, *Products of idempotent endomorphisms of an independence algebra of finite rank*, **Proc. Edinburgh Math. Soc. 35** (1992), 493–500.
[32] J. Fountain and A. Lewin, *Products of idempotent endomorphisms of an independence algebra of infinite rank*, **Math. Proc. Cambridge Philos. Soc. 114** (1993), 303–319.
[33] S. R. Givant, *Universal classes of algebras free in power*, **Notices Amer. Math. Soc.**, **19** (1972), p. A-717.
[34] S. R. Givant, *A representation theorem for universal classes of algebras in which all members are free*, **Notices Amer. Math. Soc.**. **19** (1972), p. A-767.
[35] S. R. Givant, *Universal class of algebras which are free in some infinite power*, **Notices Amer. Math. Soc.**, **20** (1973), p. A-338.
[36] S. R. Givant, *A representation theorem for quasivarieties categorical in power*, **Notices Amer. Math. Soc.**, **20** (1973), p. A-461.
[37] S. R. Givant, *A complete representation theorem for varieties categorical in power*, **Notices Amer. Math. Soc.**, **22** (1975), A-33–A-34.
[38] S. R. Givant, *Universal Horn classes categorical or free in power*, **Ann. Math. Logic**, **15** (1978), 1–53.
[39] S. R. Givant, *A representation theorem for universal Horn classes categorical in power*, **Ann. Math. Logic**, **17** (1979), 91–116.
[40] K. Głazek, *Some old and new problems in the independence theory*, **Colloq. Math. 42** (1979), 127–189.
[41] V. Gould, *Independence algebras*, **Algebra Universalis 33** (1995), 294–318.
[42] V. Gould, *Independence structures*, **Proceedings of the International Conference 'Semigroups and their applications, including semigroup rings'**, ed. P. Higgins, S. Kublovsky, A. Mikhalev and J. Ponizovskii, St Petersburg, June 19–30, 1995. Available at `http://www-users.york.ac.uk/~varg1/indstruc.ps`
[43] V. Gould, *Independence algebras, basis algebras and semigroups of quotients*, *to appear.*
[44] G. Grätzer, **Universal Algebra**, Springer-Verlag, New York, 1979.
[45] B. Hart, S. Starchenko and M. Valeriote, *Vaught's conjecture for varieties*, **Trans. Amer. Math. Soc.**, **342** (1994) 173–196.
[46] P. M. Higgins, J. M. Howie and N. Ruškuc, *Generators and factorisations of transformation semigroups*, **Proc. Roy. Soc. Edinburgh Sect. A 128** (1998), 1355–1369.
[47] P. M. Higgins, J. M. Howie, J. D. Mitchell and N. Ruškuc, *Countable versus uncountable ranks in infinite semigroups of transformations and relations*, **Proc. Edinburgh Math. Soc.**, **46** (2003), 531–544.
[48] J. M. Howie, *The subsemigroup generated by the idempotents of a full transformation semigroup*, **J. London Math. Soc. 41** (1966), 707–716.
[49] J. Łoś, *On the categoricity in power of elementary deductive systems and some related problems*, **Colloq. Math.**, **3** (1954), 58–62.
[50] E. Marczewski, *A general scheme of the notions of independence in mathematics*, **Bull. Acad. Polon. Sci. 6** (1958), 731–736.

[51] W. E. Marsh, *On ω_1-categorical and not ω-categorical theories*, Dartmouth College, 1966. Dissertation.

[52] T. Maund, *D. Phil. Thesis*, Oxford University, 1989.

[53] R. N. McKenzie, G. F. McNulty and W. F. Taylor, *Algebra, lattices, varieties,* Vol. I (Wadsworth, Monterey, 1983).

[54] M. Morley, *Categoricity in power*, **Trans. Amer. Math. Soc.**, **114** (1965), 514–538.

[55] W. Narkiewicz, *Independence in a certain class of abstract algebras*, **Fund. Math. 50** (1961/62), 333–340.

[56] W. Narkiewicz, *On a certain class of abstract algebras*, **Fund. Math. 54**, 115–124 (1964).

[57] J. G. Oxley, *Infinite matroids*, pp. 73–90 in **Matroid Applications**, (Cambridge University Press, 1992), Ed. N. White.

[58] P. Palfy, *Unary polynomials in algebras. I.*, **Algebra Universalis 18** (3) (1984), 262–273.

[59] E. A. Palyutin, *Opisanye kategorichnyh kvazimnogoobrazii*, **Algebra i Logika**, **14** (1975), 145–185.

[60] A. Pillay, **Geometric stability theory**, Oxford Science Publications 1996.

[61] M. A. Reynolds and R. P. Sullivan, *Products of idempotent linear transformations*, **Proc. Roy. Soc. Edinburgh A 100** (1985), 123–138.

[62] S. Shelah, *Categoricity of uncountable theories*, **Proceedings of the Tarski Symposium, Proceedings of Symposia in Pure Mathematics**, vol. 25, ed. L. Henkin et al., American Mathematical Society, Providence R.I., 1974, 187–204.

[63] S. Shelah, **Classification Theory and the Number of Nonisomorphic Models**, North-Holland. 1978.

[64] S. Shelah, *Classification theory for nonelementary classes. I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$ part A*, **Israel J. Math. 46** (3) (1983) 212–240.

[65] S. Shelah, *Classification theory for nonelementary classes. I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$ part B*, **Israel J. Math. 46** (3) (1983) 241–271.

[66] R. P. Sullivan, *Transformation semigroups and linear algebra*, pp. 290–295 in **Proc. Monash Conference on Semigroup Theory**, (World Scientific, 1991), Ed. T. E. Hall, P. R. Jones and J. C. Meakin.

[67] K. Urbanik, *A representation theorem for v^*-algebras*, **Fund. Math.**, **52** (1963), 291–317.

[68] K. Urbanik, *Linear independence in abstract algebras*, **Colloq. Math. 14** (1966), 233–255.

[69] R. Vaught, *Denumerable models of complete theories*, pp. 303–321 in **Infinitistic Methods (Proc. Symp. Foundations Math.)**, (Warsaw 1959), Panstwowe Wydawnictwo Nauk. Warsaw/Pergamon Press (1961).

[70] B. I. Zilber *Strongly minimal countably categorical theories* (Russian), **Siberian Math. J. 21** (2) (1980), pp. 98–112.

[71] B. I. Zilber *On the finite axiomatizability problem of theories categorical in all infinite powers* (Russian), pp. 69–74 in **Investigations in Theoretical Programming**. Kazakh. Gos. University, (Alma-Ata 1981).

[72] B. I. Zilber *The structure of models of uncountably categorical theories*, pp. 359–368 in **Proc. Internat. Congr. Math.** (Warsaw 1983).

[73] B. I. Zilber *Strongly minimal countably categorical theories II* (Russian), **Siberian Math. J. 25** (3) (1984), pp. 71–88.

[74] B. I. Zilber *Strongly minimal countably categorical theories III* (Russian), **Siberian Math. J. 25** (4) (1984), pp. 63–77.

[75] B. I. Zilber *Finite homogeneous geometries*, pp. 186–208 in **Proc. 6th Easter Conference in Model Theory**, (Berlin: Humboldt University 1988), Ed. B. I. Dahn et al..

[76] B. I. Zilber *Hereditary transitive groups and quasi-Urbanik structures* (Russian), pp. 58–77 in **Model Theory and its Application**, **Proc. Math. Inst. Sib. Branch Ac. Sci. USSR**, (Novosibirsk 1988), Ed. Yu. Ershov. (English translation: Amer. Math. Soc. Transl. **195** (2) (1999), pp. 165–180.)

[77] B. I. Zilber *Quasi-Urbanik structures* (Russian), pp. 50–67 in **Model-theoretic algebra**, **Collect. Sci. Works**, (Alma-Ata 1989).

[78] B. I. Zilber *Uncontably categorical theories* **Transl. Math. Monographs 117**, Providence, RI: AMS, 1993.

[79] B. I. Zilber *A categoricity theorem for quasiminimal excellent classes*, 297–306 in **Logica and its Applications**, **Contemporary Mathematics**, Providence, RI: AMS, 2005.

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