

Generators for the Semigroup of Endomorphisms of an Independence Algebra

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Abstract

Given an independence algebra \mathcal{A} of infinite rank, we denote the endomorphism monoid and the automorphism group of \mathcal{A} by $End(\mathcal{A})$ and $Aut(\mathcal{A})$ respectively. This paper is concerned with finding minimal subsets R of $End(\mathcal{A})$ such that $Aut(\mathcal{A}) \cup E(End(\mathcal{A})) \cup R$ is a generating set for $End(\mathcal{A})$, where $E(End(\mathcal{A}))$ denotes its set of idempotents.

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1 Introduction

We assume the reader to have a basic knowledge of the theory of independence algebras. We recommend [1], [2] and [3] as references.

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The first step in the definition of independence algebras is the introduction of a notion of independence valid for universal algebras. A subset X of an algebra is said to be *independent* if $X = \emptyset$ or if for every element $x \in X$, we have $x \notin \langle X \setminus \{x\} \rangle$; a set is *dependent* if it is not independent.

Lemma 1 *For an algebra \mathcal{A} , the following conditions are equivalent:*

- (1) *For every subset X of A and all elements u, v of A , if the element $u \in \langle X \cup \{v\} \rangle$ and $u \notin \langle X \rangle$, then $v \in \langle X \cup \{u\} \rangle$.*
- (2) *For every subset X of A and every element $u \in A$, if X is independent and $u \notin \langle X \rangle$, then $X \cup \{u\}$ is independent.*
- (3) *For every subset X of A , if Y is a maximal independent subset of X , then $\langle X \rangle = \langle Y \rangle$.*
- (4) *For subsets X, Y of A with $Y \subseteq X$, if Y is independent, then there is an independent set Z with $Y \subseteq Z \subseteq X$ and $\langle Z \rangle = \langle X \rangle$.*

Proof: [7] (p.50, Exercise 6). ■

Let \mathcal{A} be an algebra with universe A . \mathcal{A} is said to have the *exchange property* or to satisfy [EP] if it satisfies the equivalent conditions of Lemma 1. A basis for \mathcal{A} is a subset of A which generates A and is independent. It is clear from Lemma 1 that any algebra with [EP] has a basis. Furthermore, for such an algebra, bases may be characterised as minimal generating sets or maximal independent sets, and all bases for \mathcal{A} have the same cardinality ([3], Proposition 3.3). This cardinal is called the rank of \mathcal{A} and is written $rank(\mathcal{A})$.

A partial endomorphism a of \mathcal{A} is a morphism from a subalgebra \mathcal{B} of \mathcal{A} into \mathcal{A} . The domain of a partial endomorphism a is denoted by $\Delta(a)$ and the image of a is denoted by $\nabla(a)$. Clearly, both $\Delta(a)$ and $\nabla(a)$ are subalgebras of \mathcal{A} . Obviously, an endomorphism $a \in End(\mathcal{A})$ is a partial endomorphism such that $\Delta(a) = A$. If \mathcal{A} is an algebra satisfying [EP] and $a \in End(\mathcal{A})$, then $rank(a)$ is the rank of the image of a , that is, $rank(a) = rank(\nabla(a))$.

We observe that (4) of Lemma 1 tells us that any independent subset of A can be extended to a basis for \mathcal{A} . We also remark that if \mathcal{A} satisfies [EP], then so does any subalgebra of \mathcal{A} . The next two lemmas can be found in [1].

Lemma 2 *Let \mathcal{A} be an algebra which satisfies [EP]. If X, Y, Z are subsets of A such that $X \cup Y$ and $X \cup Z$ are bases for \mathcal{A} and $X \cap Y = X \cap Z = \emptyset$, then $|Y| = |Z|$.*

As a consequence of this lemma we can define the corank of a subalgebra \mathcal{B} of \mathcal{A} as follows. Let X be a basis for \mathcal{B} and extend X to obtain a basis $X \cup Y$ for \mathcal{A} . Then $\text{corank}(\mathcal{B}) = |Y|$.

Lemma 3 *Let \mathcal{A} be an algebra which satisfies [EP]. If \mathcal{B}, \mathcal{C} are subalgebras of \mathcal{A} with $\mathcal{B} \subseteq \mathcal{C}$, then $\text{co-rank}(\mathcal{C}) \leq \text{co-rank}(\mathcal{B})$.*

We now define an *independence algebra* to be an algebra \mathcal{A} which satisfies [EP] and also satisfies:

[F] For any basis X of \mathcal{A} and $\alpha : X \rightarrow A$, there is an endomorphism $\bar{\alpha}$ of \mathcal{A} such that $\bar{\alpha}|_X = \alpha$.

In general we use the same letter α to represent the mapping $\alpha : X \rightarrow A$ and the endomorphism induced by that mapping.

When working with independence algebras a very useful tool is that of *preimage basis*. Let $a \in \text{End}(\mathcal{A})$ and let Y be a basis for $\nabla(a)$. A preimage basis is a set Y_0 such that $a|_{Y_0}$ is one-one and $(Y_0)a = Y$.

For every semigroup S , let $E(S)$ be the set of idempotents of S . The study of generators for $\text{End}(\mathcal{A})$ has been considered by various authors in different cases. When the independence algebra \mathcal{A} is an algebra without operations, that is, \mathcal{A} is just an infinite set, say A , then $\text{End}(\mathcal{A})$ coincides with $T(A)$, the transformation semigroup on A , and $\text{Aut}(\mathcal{A})$ with the symmetric group on A , written $\text{Sym}(A)$. In this case Howie [5] described the semigroup generated by $E(T(A))$ and, in particular, proved that $\langle \text{Sym}(A) \cup E(T(A)) \rangle$ is strictly contained in $T(A)$.

When \mathcal{A} is an infinite dimensional vector space, then $\text{End}(\mathcal{A})$ is the semigroup of all linear transformations from \mathcal{A} into itself. Reynolds and Sullivan [8] described $\langle E(\text{End}(\mathcal{A})) \rangle$, the semigroup generated by the idempotents of $\text{End}(\mathcal{A})$, and proved that in this case $\langle \text{Aut}(\mathcal{A}) \cup E(\text{End}(\mathcal{A})) \rangle$ is also strictly contained in $\text{End}(\mathcal{A})$.

Fountain and Lewin [2] generalized these results for a particular class of independence algebras (namely, *strong independence algebras*) describing the semigroup $\langle E(\text{End}(\mathcal{A})) \rangle$ and showing that the semigroup $\langle \text{Aut}(\mathcal{A}) \cup E(\text{End}(\mathcal{A})) \rangle$ is strictly contained in $\text{End}(\mathcal{A})$.

Recently Higgins, Howie and Ruškuc [4] have proved that, given an infinite set X , the set $Sym(X) \cup E(T(X))$ can be extended in a minimal way to become a generating set of $T(X)$. In fact they found a minimal set R such that $T(X) = \langle Sym(X) \cup E(T(X)) \cup R \rangle$. Our aim is to generalize this result for arbitrary independence algebras, that is, we shall find a minimal set $R \subseteq End(\mathcal{A})$ such that

$$End(\mathcal{A}) = \langle Aut(\mathcal{A}) \cup E(End(\mathcal{A})) \cup R \rangle.$$

It is easy to prove that this set R must contain both an injection and a surjection (see Theorem 14). Hence we define a particular injection, α , and a particular surjection, α^* , such that $Aut(\mathcal{A}) \cup E(End(\mathcal{A})) \cup \{\alpha, \alpha^*\}$ is a generating set of $End(\mathcal{A})$.

2 The Main Results

In what follows \mathcal{A} is a fixed independence algebra of infinite rank, B is a fixed basis of \mathcal{A} . We also fix an endomorphism $\alpha \in End(\mathcal{A})$ such that $\alpha : B \rightarrow Y$ is a one-one mapping and $|B| = |Y| = corank(\alpha)$. Note that we have $rank(\alpha) = corank(\alpha) = rank(\mathcal{A})$. Finally fix a set W such that $Y \cap W = \emptyset$ and $Y \cup W$ is a basis of \mathcal{A} .

Next, let $\alpha^{-1} : Y \rightarrow B$ be the inverse mapping of $\alpha : B \rightarrow Y$. Define

$$\begin{aligned} \alpha^* : Y \cup W &\rightarrow B \\ y \in Y &\mapsto y\alpha^{-1} \\ x \in W &\mapsto x. \end{aligned}$$

Thus $\alpha^*|_W = id|_W$ and $\alpha^*|_Y = \alpha^{-1}$.

Let $G^{-1} = Aut(\mathcal{A}) \cup \{\alpha, \alpha^{-1}\}$ and let $G^* = Aut(\mathcal{A}) \cup \{\alpha, \alpha^*\}$. Let $In(\mathcal{A})$ be the set of total one-one endomorphisms of \mathcal{A} . In $In(\mathcal{A})$ define a relation ρ as follows: for all $\alpha, \beta \in In(\mathcal{A})$,

$$(\alpha, \beta) \in \rho \Leftrightarrow corank(\alpha) = corank(\beta).$$

Clearly, ρ is an equivalence relation in $In(\mathcal{A})$. Observe that for all $\alpha \in In(\mathcal{A})$ we have $rank(\alpha) = rank(\mathcal{A})$.

We start by proving the following lemma.

Lemma 4 *If β is in $\langle G^{-1} \rangle \cap \text{In}(\mathcal{A})$, then $\beta\rho \subseteq \langle G^{-1} \rangle$.*

Proof: Let B be the fixed basis of \mathcal{A} and take $\gamma \in \beta\rho$. Let $B\gamma \cup B_2$ and $B\beta \cup C_2$ be two bases of \mathcal{A} . As $\gamma \in \beta\rho$, we have $\text{corank}(\gamma) = \text{corank}(\beta)$ and hence $|B_2| = |C_2|$. Thus there is a bijection $g_2 : C_2 \rightarrow B_2$. It is easy to see that $g_1 : B\beta \rightarrow B\gamma$, defined by $(b\beta)g_1 = b\gamma$, is a bijection. Thus we can consider the bijection $g = g_1 \cup g_2 : B\beta \cup C_2 \rightarrow B\gamma \cup B_2$ which induces an automorphism of \mathcal{A} and satisfies $\beta g = \gamma$. Hence $\gamma \in \langle G^{-1} \rangle$. ■
Similarly, we can prove the next lemma.

Lemma 5 *Let $\beta \in \langle G^* \rangle \cap \text{In}(\mathcal{A})$. Then $\beta\rho \subseteq \langle G^* \rangle$.*

Our first aim is to prove the following result.

Theorem 6 *$\text{In}(\mathcal{A}) \subseteq \langle G^{-1} \rangle$.*

We start by proving the following lemma.

Lemma 7 *Let $\lambda \leq \text{rank}(\mathcal{A})$ be a cardinal number. Then there is an element $\beta \in \langle G^{-1} \rangle \cap \text{In}(\mathcal{A})$ such that $\text{corank}(\beta) = \lambda$.*

Proof: If $\lambda = \text{rank}(\mathcal{A})$ we can take $\beta = \alpha$, the element defined at the beginning of this section. Hence we assume that $\lambda < \text{rank}(\mathcal{A})$. Let B be the basis of \mathcal{A} associated to α and $Z \subseteq B$ such that $|Z| = \lambda$. Clearly, $(B \setminus Z)\alpha = Y \setminus Z\alpha$. But $|Z| = |Z\alpha|$ implies $|Y| > |Z\alpha|$ (as $|Z| = \lambda < \text{rank}(\mathcal{A}) = |Y|$), and hence $|Y| = |Y \setminus Z\alpha|$. Thus we can take a bijection $g_1 : Y \rightarrow Y \setminus Z\alpha$.

In the same way, $|Z| = |Z\alpha|$ implies $|W| > |Z\alpha|$ (as $|Z| = \lambda < \text{rank}(\mathcal{A}) = |W|$), and hence we have $|W \cup Z\alpha| = |W|$. So we can take a bijection $g_2 : W \rightarrow W \cup Z\alpha$.

Now, $g = g_1 \cup g_2 : Y \cup W \rightarrow (Y \setminus Z\alpha) \cup (W \cup Z\alpha)$ is a bijection and so induces an automorphism of \mathcal{A} . This automorphism will be represented by the same letter g . Now,

$$(B)\alpha g\alpha^{-1} = (Y)g\alpha^{-1} = (Y \setminus Z\alpha)\alpha^{-1} = B \setminus Z.$$

Thus $\alpha g\alpha^{-1} \in \text{In}(\mathcal{A})$ and $\text{corank}(\alpha g\alpha^{-1}) = |Z| = \lambda$ and the lemma follows. ■

Observe that $B\alpha g \subseteq Y$ and so, as $\alpha^{-1}|_Y = \alpha^*|_Y$, we have

$$\beta = \alpha g \alpha^{-1} = \alpha g \alpha^* \in \langle G^* \rangle.$$

Now we can easily prove Theorem 6.

Proof: Let $\zeta \in \text{In}(\mathcal{A})$. Let $\lambda = \text{corank}(\zeta)$. By the previous lemma, there is an element $\beta \in \langle G^{-1} \rangle \cap \text{In}(\mathcal{A})$ such that $\text{corank}(\beta) = \text{corank}(\zeta)$. Thus $\zeta \in \beta\rho$ and hence, by Lemma 4, $\zeta \in \langle G^{-1} \rangle$. Therefore $\text{In}(\mathcal{A}) \subseteq \langle G^{-1} \rangle$. ■

Let $\zeta \in \text{In}(\mathcal{A})$. The element β , which was constructed in Lemma 7, belongs to $\langle G^* \rangle$ as observed after the proof of Lemma 7. Hence, for every $\lambda \leq \text{rank}(\mathcal{A})$, there is an element $\beta \in \langle G^* \rangle \cap \text{In}(\mathcal{A})$ such that $\text{corank}(\beta) = \lambda$. In particular $\text{corank}(\beta) = \text{corank}(\zeta)$, for some $\beta \in \langle G^* \rangle \cap \text{In}(\mathcal{A})$, and hence $\zeta \in \beta\rho$. On the other hand, from Lemma 5, we have that if $\beta \in \langle G^* \rangle \cap \text{In}(\mathcal{A})$ then $\beta\rho \subseteq \langle G^* \rangle$. Thus $\text{In}(\mathcal{A}) \subseteq \langle G^* \rangle$.

Next we aim to prove the following lemma.

Lemma 8 *Let Y_0 and Z be two independent subsets of \mathcal{A} such that we have $|Y_0| = |Z| = \text{rank}(\mathcal{A})$. Let $\beta : \langle Y_0 \rangle \rightarrow \langle Z \rangle$ be an isomorphism. Then there is an element $\bar{\beta} \in \langle G^* \cup E(\text{End}(\mathcal{A})) \rangle$ such that $\bar{\beta}|_{\langle Y_0 \rangle} = \beta$.*

To prove this lemma we start with an observation, introduce a definition and then prove three lemmas.

Let $w = \mu_1 \dots \mu_n$, where all the elements μ_i belong to $\text{Aut}(G) \cup \{\alpha\}$. Let T be a subset of A and suppose that $\nabla(w|_T) \subseteq Y$. Then $(w\alpha^{-1})|_T = (w\alpha^*)|_T$. In fact, this follows immediately from the equality $\alpha^{-1}|_Y = \alpha^*|_Y$.

Inspired by the proof of Lemma 7 we define some objects that will be useful in the next results. For a set $Z \subset B$ such that $|Z| < |B|$, fix an element $g_Z \in \text{Aut}(\mathcal{A})$ such that

$$g_Z|_Y : \langle Y \rangle \rightarrow \langle Y \setminus Z\alpha \rangle$$

is an isomorphism. Moreover, let $\beta_Z = \alpha g_Z \alpha^{-1}$. We have $\beta_Z : \langle B \rangle \rightarrow \langle B \setminus Z \rangle$ and $\beta_Z^{-1} : \langle B \setminus Z \rangle \rightarrow \langle B \rangle$.

Lemma 9 *Let Z be contained in the fixed basis B and suppose that we have $|Z| < |B|$. Then there exists $\gamma \in \langle G^* \rangle$ such that $\gamma|_{B \setminus Z} = \beta_Z^{-1}$.*

Proof: β_Z^{-1} was defined as

$$\begin{aligned} \beta_Z^{-1} : \langle B \setminus Z \rangle &\rightarrow \langle B \rangle \\ x\beta_Z &\mapsto x \end{aligned}$$

where $\beta_Z = \alpha g_Z \alpha^{-1}$. So, $\beta_Z^{-1} = \alpha g_Z^{-1} \alpha^{-1}|_{B \setminus Z}$.

But $(B \setminus Z)\alpha g_Z^{-1}$ is equal to $(B\alpha \setminus Z\alpha)g_Z^{-1}$ (since α is one-one on B) and hence, as $B\alpha = Y$, it follows that

$$(B\alpha \setminus Z\alpha)g_Z^{-1} = (Y \setminus Z\alpha)g_Z^{-1} = Y.$$

Thus $\alpha g_Z^{-1} \alpha^{-1}|_{B \setminus Z} = \alpha g_Z^{-1} \alpha^*|_{B \setminus Z}$, by the observation after Lemma 8. ■

Lemma 10 *Let B be the fixed basis of \mathcal{A} , $Z \subseteq B$ such that $|Z| < |B|$, and let $\gamma : \langle B \rangle \rightarrow \langle U \rangle$ be an isomorphism belonging to $\beta_Z \rho$. Then*

$$\begin{aligned} \gamma^{-1} : \langle U \rangle &\rightarrow \langle B \rangle \\ x\gamma &\mapsto x \end{aligned}$$

can be extended to an element of $\langle G^ \rangle$.*

Proof: If $\gamma \in \beta_Z \rho$, then there is $h \in \text{Aut}(\mathcal{A})$ such that $h : \langle B \setminus Z \rangle \rightarrow \langle U \rangle$ and $\beta_Z h = \gamma$. (See the proof of Lemma 4).

Now, $\beta_Z = \alpha g_Z \alpha^{-1}$ implies $\gamma = \alpha g_Z \alpha^{-1} h$ and hence we have that $\gamma^{-1} = h^{-1} \alpha g_Z^{-1} \alpha^{-1}|_{\langle U \rangle}$. Moreover, $(U)h^{-1} \alpha g_Z^{-1} = (B \setminus Z)\alpha g_Z^{-1} = (Y \setminus Z\alpha)g_Z^{-1} = Y$. Thus $h^{-1} \alpha g_Z^{-1} \alpha^{-1}|_{\langle U \rangle} = h^{-1} \alpha g_Z^{-1} \alpha^*|_{\langle U \rangle}$, by the observation after Lemma 8. ■

Lemma 11 *Let Y_0 be an independent set such that $|Y_0| = \text{rank}(\mathcal{A})$. Let $\gamma : \langle Y_0 \rangle \rightarrow \mathcal{A}$ be an isomorphism. Then there exists $\bar{\gamma} \in \langle G^* \rangle$ such that $\bar{\gamma}|_{\langle Y_0 \rangle} = \gamma$.*

Proof: Suppose, first, that $\text{corank}\langle Y_0 \rangle = \text{rank}(\mathcal{A})$. Then there is a set W_0 such that $Y_0 \cup W_0$ is a basis for \mathcal{A} and $|Y_0| = |Y| = |W| = |W_0|$. Now take $g \in \text{Aut}(\mathcal{A})$ such that $Y_0 g = Y$ and $W_0 g = W$. Then $(Y_0)g\alpha^* = B$. Moreover, $Y_0 \gamma$ is a basis for \mathcal{A} too. Thus there is $h \in \text{Aut}(\mathcal{A})$ such that for all $y \in Y_0$ we have $y\gamma h = yg\alpha^*$. Hence, as Y_0 is a basis of $\Delta(\gamma)$, it follows

that $\gamma h = g\alpha_{|_{Y_0}}^*$ and so $\gamma = g\alpha_{|_{Y_0}}^* h_{|_{Y_0}}^{-1}$. The result is proved for the special case of γ such that $\text{corank}(Y_0) = \text{rank}(\mathcal{A})$.

Suppose now that $\lambda = \text{corank}(Y_0) < \text{rank}(\mathcal{A})$ and let $\gamma^{-1} : \mathcal{A} \rightarrow \langle Y_0 \rangle$ be the inverse of γ . This mapping is an isomorphism such that $\text{rank}(\gamma^{-1}) = \text{rank}(\mathcal{A})$ and $\text{corank}(\gamma^{-1}) = \lambda$. Thus, for $Z \subseteq B$ such that $|Z| = \lambda$, we have $\gamma^{-1} \in \beta_Z \rho$ (recall that $\text{rank}(\beta_Z) = \text{rank}(\mathcal{A})$ and $\text{corank}(\beta_Z) = |Z|$). Then, by Lemma 10, $\gamma = (\gamma^{-1})^{-1}$ can be extended to an element of $\langle G^* \rangle$ and the result follows. ■

We can now prove Lemma 8.

Proof: As $|Y_0| = \text{rank}(\mathcal{A})$, we have a bijection $g_1 : Y_0 \rightarrow B$, where B is the fixed basis of \mathcal{A} . The isomorphism $g_1 : \langle Y_0 \rangle \rightarrow \langle B \rangle$ satisfies the conditions of the previous lemma. Thus there is an element $\gamma \in \langle G^* \rangle$ such that $\gamma_{|_{\langle Y_0 \rangle}} = g_1$.

Now, consider $g_2 : Y_0 g_1 \rightarrow Z$ defined by $(y g_1) g_2 = y \beta$. Clearly, g_2 is a bijection and so we can use Theorem 6 on page 5 concluding that the mapping $g_2 : \langle Y_0 g_1 \rangle = \langle B \rangle \rightarrow \langle Z \rangle$ is an element of $\langle G^{-1} \rangle$. In fact, as was seen after Theorem 6, we can assume that $g_2 \in \langle G^* \rangle$. Thus $\beta = g_1 g_2 = \gamma g_{2|_{\langle Y_0 \rangle}}$ and $\gamma g_2 \in \langle G^* \rangle$. So, there is $\bar{\beta} = \gamma g_2 \in \langle G^* \rangle$ such that $\bar{\beta}_{|_{\langle Y_0 \rangle}} = \beta$. ■

Lemma 12 *Let Y_0 and Z be two independent subsets of \mathcal{A} such that $|Y_0| = |Z|$. Let $\beta : \langle Y_0 \rangle \rightarrow \langle Z \rangle$ be an isomorphism. Then there is $\bar{\beta} \in \langle G^* \cup E(\text{End}(\mathcal{A})) \rangle$ such that $\bar{\beta}_{|_{\langle Y_0 \rangle}} = \beta$.*

Proof: By Lemma 8 the result holds if $|Y_0| = \text{rank}(\mathcal{A})$ and so we can suppose that $|Y_0| = |Z| < \text{rank}(\mathcal{A})$. Consider two bases B, C of \mathcal{A} , such that $Y_0 \subseteq B$ and $Z \subseteq C$. Then we have $|B| = |B \setminus Y_0| = |C \setminus Z| = |C|$ and so there exists a bijection $\pi_1 : B \setminus Y_0 \rightarrow C \setminus Z$. Moreover let x_0 be a fixed element in Y_0 and consider the following idempotent:

$$\begin{aligned} \epsilon : B &\rightarrow Y_0 \\ x &\mapsto x \text{ if } x \in Y_0 \\ x &\mapsto x_0 \text{ otherwise.} \end{aligned}$$

Finally, let

$$\begin{aligned} \pi = \beta \cup \pi_1 : Y_0 \cup (B \setminus Y_0) &\rightarrow Z \cup (C \setminus Z) \\ y \in Y_0 &\mapsto y \beta \in Z \\ b \in B \setminus Y_0 &\mapsto b \pi_1 \in C \setminus Z. \end{aligned}$$

Clearly, π is a bijection. Now, let $y \in Y_0$. We have $y\epsilon\pi = y\pi = y\beta$ and hence $\epsilon\pi|_{\langle Y_0 \rangle} = \beta$ and $\epsilon\pi \in \langle G^* \cup E(\text{End}(\mathcal{A})) \rangle$. ■

We are now in a position to prove our main results.

Theorem 13 *The semigroup $\langle G^* \cup E(\text{End}(\mathcal{A})) \rangle = \text{End}(\mathcal{A})$.*

Proof: Let $\beta \in \text{End}(\mathcal{A})$. Let Z be a basis of $\nabla\beta$ and Y_0 be a preimage basis of Z . Then $\beta_0 : \langle Y_0 \rangle \rightarrow \langle Z \rangle$, where $y\beta_0 = y\beta$ is an isomorphism. By the previous lemma, there is an element $\bar{\beta} \in \langle G^* \cup E(\text{End}(\mathcal{A})) \rangle$ such that $\bar{\beta}|_{\langle Y_0 \rangle} = \beta_0$. In the same way there is an element $\bar{\gamma} \in \langle G^* \cup E(\text{End}(\mathcal{A})) \rangle$ such that $\bar{\gamma}|_{\langle Z \rangle} = \beta_0^{-1}$. Now, let $B = Y_0 \cup W_0$ be a basis of \mathcal{A} . For all $y \in Y_0$, we have $y\beta\bar{\gamma}\beta = y\beta$. On the other hand, let $w \in W_0$. As $w\beta \in \nabla\beta$ and Z is a basis of the image of β , we have $w\beta = t(z_1, \dots, z_n) = t(y_1\beta, \dots, y_n\beta)$, for some $y_1, \dots, y_n \in Y_0$. Thus,

$$\begin{aligned} w\beta\bar{\gamma}\beta &= t(z_1, \dots, z_n)\bar{\gamma}\beta \\ &= t(y_1\beta, \dots, y_n\beta)\bar{\gamma}\beta \\ &= t(y_1\beta\bar{\gamma}\beta, \dots, y_n\beta\bar{\gamma}\beta) \\ &= t(y_1\beta, \dots, y_n\beta) \\ &= w\beta, \end{aligned}$$

and consequently $\beta\bar{\gamma}\beta = \beta$.

Moreover, the image of $\beta\bar{\gamma}$ is $\langle Y_0 \rangle$ and $\beta|_{\langle Y_0 \rangle} = \beta_0 = \bar{\beta}|_{\langle Y_0 \rangle}$ and hence $\beta\bar{\gamma}\bar{\beta} = \beta\bar{\gamma}\beta$. Finally, $\beta\bar{\gamma}\beta = \beta$ implies $\beta\bar{\gamma}$ is an idempotent. Thus $\beta\bar{\gamma} \in E(\text{End}(\mathcal{A}))$ and $\bar{\beta} \in \langle G^* \cup E(\text{End}(\mathcal{A})) \rangle$ so that we have proved that $\beta\bar{\gamma}\bar{\beta} \in \langle G^* \cup E(\text{End}(\mathcal{A})) \rangle$. ■

We will now prove a theorem which generalizes Theorem 5.15 of [4] to independence algebras, through the use of a similar technique.

Theorem 14 *Let \mathcal{A} be an independence algebra of infinite rank and let T be a subset of $\text{End}(\mathcal{A})$ such that $T \cap G = \emptyset$ and*

$$\langle G, E(\text{End}(\mathcal{A})), T \rangle = \text{End}(\mathcal{A}).$$

Then T must contain at least one injection and at least one surjection.

Proof: We start with some easy observations. Let $p, q \in \text{End}(\mathcal{A})$. Then

- (1) if both p and q are injective (surjective), then so is pq ;
(2) if pq is injective, then so is p since

$$ap = bp \Rightarrow apq = bpq \Rightarrow a = b;$$

- (3) if pq is surjective, then so is q since

$$A = (A)pq \subseteq (A)q;$$

- (4) if p is injective or surjective and $p \in E(\text{End}(\mathcal{A}))$, then $p = \text{id}_A$.

Let $\delta \in \text{End}(\mathcal{A})$ be an injective endomorphism which is not surjective and suppose that $\delta = p_1 p_2 \dots p_n$, where p_1, \dots, p_n belong to $G \cup E(\text{End}(\mathcal{A})) \cup T$. We may assume that none of p_1, \dots, p_n is the identity mapping. Not all p_1, \dots, p_n are in G since $\delta \notin G$. Let i be the first number in $[n]$ such that $p_i \notin G$. As $p_1 \dots p_{i-1}$ is a composition of bijections, it is a bijection. Thus, $(p_1 \dots p_{i-1})^{-1} \delta = p_i \dots p_n$ is injective by (1), so p_i is injective by (2) and so $p_i \in T$ by (4).

To show that T must also contain a surjection, let σ be a surjective element in $\text{End}(\mathcal{A}) \setminus G$ and suppose that $\sigma = p_1 \dots p_n$ where $p_i \in E(\text{End}(\mathcal{A})) \cup T \cup G$ and none of the p_i is the identity id_A . Then at least one of the p_k ($k \in [n]$) belongs to $T \cup E(\text{End}(\mathcal{A}))$. In fact, if all the p_k were in G then σ would be in G which contradicts our assumption about σ . Thus some p_k must belong to $E(\text{End}(\mathcal{A})) \cup T$. Let p_i be the last element of $E(\text{End}(\mathcal{A})) \cup T$ which occurs in the factors of σ . Then $\sigma(p_{i+1} \dots p_n)^{-1} = p_1 \dots p_i$ is surjective by (1), so p_i is surjective by (3) and so $p_i \in T$. ■

This theorem proves that the set $R = \{\alpha, \alpha^*\}$ is minimal among the sets T such that $\langle G \cup E(\text{End}(\mathcal{A})) \cup T \rangle = \text{End}(\mathcal{A})$.

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