The Lefschetz coincidence theorem in o-minimal expansions of fields

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June 29, 2009

Abstract

In this paper we prove the Lefschetz coincidence theorem in o-minimal expansions of fields using the o-minimal singular homology and cohomology.

*With partial support from the FCT (Fundação para a Ciência e Tecnologia), program POCTI (Portugal/FEDER-EU). *MSC: 03C64, 20E99. Keywords and phrases:* O-minimal structures, Lefschetz coincidence theorem, o-minimal (co)homology.

1 Introduction

We work over an o-minimal expansion $\mathcal{N} = (N, 0, 1, <, +, \cdot, \dots)$ of a real closed field N. Definable means $\mathcal{N}$-definable (possibly with parameters). As it is well known, o-minimal structures are a wide ranging generalization of semi-algebraic and sub-analytic geometry. Good references on o-minimality are, for example, the book [8] by van den Dries and the notes [5] by Michel Coste. For semi-algebraic geometry relevant to this paper the reader should consult the work by Delfs and Knebusch such as [6] and the book [3] by Bochnak, Coste and M-F. Roy.

The goal of the paper is to present the proof of the following o-minimal version of the Lefschetz coincidence theorem.

Theorem 1.1 (Lefschetz coincidence theorem) *Let X and Y be orientable, definably compact definable manifolds of dimension n. Suppose that $f, g : X \longrightarrow Y$ are continuous definable maps whose Lefschetz coincidence number is nonzero. Then there is $x \in X$ such that $f(x) = g(x)$.*

This result implies an o-minimal Lefschetz fixed point theorem for definable continuous maps on orientable, definably compact definable manifolds as in [1]. For a more general o-minimal Lefschetz-Hopf fixed point theorem generalizing Brumfiel's Hopf fixed point theorem for semi-algebraic maps in [4] see [10].

Our proof of o-minimal Lefschetz coincidence theorem above follows the proofs of its topological analogue ([14], [15]). The only difficulty is the o-minimal relative Poincaré duality theorem for orientable definable manifolds (Theorem 3.5) which is handled by replacing classical arguments such as compactness ([7]) by the definable triangulation theorem ([8]) and the existence of coverings by definable proper sub-balls ([1], [11], [18]). With this result available in the o-minimal setting, we develop in Subsection 4.1 the o-minimal analogue of part of the classical theory of Thom, Lefschetz and Euler classes as in ([7]) and prove in a rather classical and algebraic way the Lefschetz coincidence theorem in Subsection 4.2.

2 O-minimal (co)homology

For o-minimal expansions of real closed fields, Woerheide ([17]) constructs o-minimal singular homology (H_*, d_*) with coefficients in $\mathbb{Z}$ satisfying o-minimal Eilenberg-Steenrod homology axioms (the analogues of the classical Eilenberg-Steenrod axioms for the category of definable sets with definable continuous maps).

The definition of o-minimal singular homology is quite easy, but the verification of the axioms is very difficult as we now explain. Given a definable set X we consider, for each $m \geq 0$, the abelian group $S_m(X)$

freely generated by the singular definable simplices $\sigma : \Delta^m \longrightarrow X$, where $\Delta^m = \{(t_0, \dots, t_m) \in N^{m+1} : \sum_i t_i = 1, t_i \geq 0\}$ is the standard m-dimensional simplex. The boundary operator $\partial_{m+1} : S_{m+1}(X) \longrightarrow S_m(X)$ (morphism of degree -1) is defined as in the classical case making $S_*(X)$ a free chain complex. Also, a definable continuous map $f : X \longrightarrow Y$ induces a chain map $f_\sharp : S_*(X) \longrightarrow S_*(Y)$ (i.e., a morphism of degree zero satisfying $f_\sharp \circ \partial_* = \partial_* \circ f_\sharp$). Similarly one defines the definable singular chain complex of pairs of definable sets $A \subseteq X$ by $S_*(X, A) = S_*(X)/S_*(A)$. The graded group $H_*(X)$ is defined as the homology of the chain complex $S_*(X)$. Similarly $H_*(X, A)$ is the homology of $S_*(X, A)$. A definable continuous map $f : X \longrightarrow Y$ induces a homomorphism $f_* : H_*(X) \longrightarrow H_*(Y)$ of graded groups (via $f_\sharp$). In the same way, a definable continuous map $f : (X, A) \longrightarrow (Y, B)$ (i.e., a definable continuous map $f : X \longrightarrow Y$ such that $f(A) \subseteq B$) induces a homomorphism $f_* : H_*(X, A) \longrightarrow H_*(Y, B)$ of graded groups.

Theorem 2.1 (Woerheide [17]) *The o-minimal homology functor H_* constructed above satisfies the o-minimal Eilenberg-Steenrod axioms:*

Homotopy Axiom. *If $f, g : (X, A) \longrightarrow (Y, B)$ are definable maps and there is a definable homotopy between f and g, then*

$$f_* = g_* : H_n(X, A) \longrightarrow H_n(Y, B)$$

for all $n \in \mathbb{N}$.

Exactness Axiom. *For $A \subseteq X$ definable sets if $i : (A, \emptyset) \longrightarrow (X, \emptyset)$ and $j : (X, \emptyset) \longrightarrow (X, A)$ are the inclusions, then we have a natural exact sequence*

$$\cdots \longrightarrow H_n(A, \emptyset) \xrightarrow{i_*} H_n(X, \emptyset) \xrightarrow{j_*} H_n(X, A) \xrightarrow{d_n} H_{n-1}(A, \emptyset) \longrightarrow \cdots .$$

Excision Axiom. *For every pair $A \subseteq X$ of definable sets and every definable open subset U of X such that $\overline{U} \subseteq \mathring{A}$, the inclusion $(X - U, A - U) \longrightarrow (X, A)$ induces isomorphisms*

$$H_n(X - U, A - U) \longrightarrow H_n(X, A)$$

for all $n \in \mathbb{N}$.

Dimension Axiom. *If X is a one point set, then $H_n(X, \emptyset) = 0$ for all $n \neq 0$ and $H_0(X) = \mathbb{Z}$.*

Woerheide's result is based on a definable triangulation theorem ([8]) and on the method of acyclic models from homological algebra and is rather complicated due to the fact that, in arbitrary o-minimal expansions of fields, the classical simplicial approximation theorem, the method of repeated barycentric subdivisions and the Lebesgue number property for a standard simplex Δ^n fail.

We make now a few comments comparing the classical proof of the excision axiom and Woerhiede proof of the o-minimal excision axiom.

For $z \in \widetilde{S}_*(X)$ with $z = \sum_{j=1}^{l} a_j \alpha_j$ we have a chain map

$$z_\sharp : \widetilde{S}_*(\Delta^n) \longrightarrow \widetilde{S}_*(X) : \beta \mapsto z_\sharp \beta = \sum_{i,j} a_j b_i (\alpha_j \circ \beta_i)$$

where $\beta = \sum_{i=1}^{k} b_i \beta_i$.

Let X be a definable set. The *barycentric subdivision*

$$\mathrm{Sd}_n : \widetilde{S}_n(X) \longrightarrow \widetilde{S}_n(X)$$

is defined as follows: for $n \leq -1$, Sd_n is the trivial homomorphism, Sd_{-1} is the identity and, for $n \geq 0$, we set

$$\mathrm{Sd}_n(z) = z_\sharp(b_n.\mathrm{Sd}_{n-1}\partial 1_{\Delta^n})$$

where b_n is the barycentre of Δ^n. Here we use the cone construction which is defined in the following way. Let $X \subseteq N^m$ be a convex definable set and let $p \in X$. The *cone construction over p in X* is a sequence of homomorphisms $z \mapsto p.z$: $\widetilde{S}_*(X) \longrightarrow \widetilde{S}_{*+1}(X)$ defined as follows: For $n < -1$, $p.$ is defined as the trivial homomorphism and for $n \geq -1$ and a basis element σ, we set

$$p.\sigma(\sum_{i=0}^{n+1} t_i e_i) = \begin{cases} p \quad \text{if} \quad t_0 = 1 \\ \\ t_0 p + (1-t_0)\sigma(\sum_{i=1}^{n+1} \frac{t_i}{1-t_0} e_i) \quad \text{if} \quad t_0 \neq 1. \end{cases}$$

In the classical case we apply the Lebesgue number property to the repeated barycentric subdivision operator

$$\mathrm{Sd}^k = (\mathrm{Sd}_n^k)_{n \in \mathbb{Z}} : \widetilde{S}_*(X) \longrightarrow \widetilde{S}_*(X)$$

where Sd^k is the composition of Sd with itself k times, to prove the following lemma.

Lemma 2.2 *Suppose that X is a topological space and let U and V be open subsets of X such that $X = U \cup V$. If $z \in \widetilde{S}_n(X)$, then there is a sufficiently large $k \in \mathbb{N}$ such that $\mathrm{Sd}_n^k(z) \in \widetilde{S}_n(U) + \widetilde{S}_n(V)$.*

This lemma implies the excision axiom. In the o-minimal case Woerheide replaces Sd^k by the *subdivision operator*

$$\mathrm{Sd}_i^K : \widetilde{S}_i(X) \longrightarrow \widetilde{S}_i(X)$$

where (Φ, K) is a definable triangulation of X. The subdivision operator is defined by

$$\mathrm{Sd}_i^K(z) = (\mathrm{Sd}z)_\sharp(\gamma_i^n)_\sharp(\Phi^{-1})_\sharp \tau_K F_n\langle e_{n-i}, \ldots, e_n\rangle$$

where $F_n : \widetilde{C}_*(E^n) \longrightarrow \widetilde{C}_*(K)$ is the o-minimal simplicial chain map induced by $\Phi : E^n \longrightarrow K$, $\gamma_i^n : \Delta^n \longrightarrow \Delta^i$ is defined by

$$\gamma_i^n(\sum_{j=0}^n a_j e_j) = \sum_{j=0}^i (a_{n-i+j} + \frac{\sum_{k=0}^{n-i-1} a_k}{i+1}) e_j$$

and E^n is the standard simplicial complex such that $|E^n| = \Delta^n$.

Woerheide proves the following lemma which, as in the classical case, implies the o-minimal excision axiom.

Lemma 2.3 ([17]) *Suppose that X is a definable set and let U and V be open definable subsets of X such that $X = U \cup V$. If $z \in \widetilde{S}_n(X)$, then there is a definable triangulation (Φ, K) of Δ^n compatible with E^n such that $\mathrm{Sd}_n^K(z) \in \widetilde{S}_n(U) + \widetilde{S}_n(V)$.*

Woerheide's construction easily gives, as in the classical case ([7] Chapter VI, Section 7), o-minimal singular homology with coefficients in $\mathbb{Q}$. Indeed, if $f : X \longrightarrow Y$ is a definable continuous map, one defines *o-minimal singular homology with coefficients in* $\mathbb{Q}$ by

$$H_m(X; \mathbb{Q}) = H_m(S_*(X) \otimes \mathbb{Q})$$

and $f_* : H_m(X; \mathbb{Q}) \longrightarrow H_m(Y; \mathbb{Q})$ is the homomorphism induced by $f_\sharp \otimes \mathrm{id}$. This o-minimal homology with coefficients in $\mathbb{Q}$ satisfies the corresponding Eilenberg-Steenrod axioms. We often apply the Universal Coefficient theorem and identify $H_m(X) \otimes \mathbb{Q}$ with $H_m(X; \mathbb{Q})$ (and the corresponding f_*'s) as $\mathbb{Q}$-vector spaces.

Similarly, as in the classical case ([7] Chapter VI, Section 7), we have the *o-minimal singular cohomology with coefficients in* $\mathbb{Q}$

$$H^m(X; \mathbb{Q}) = H_{-m}(\mathrm{Hom}(S_*(X), \mathbb{Q}))$$

with homomorphism $f^* : H^m(Y; \mathbb{Q}) \longrightarrow H^m(X; \mathbb{Q})$ induced by $\mathrm{Hom}(f_\sharp, \mathbb{Q})$. This o-minimal cohomology with coefficients in $\mathbb{Q}$ satisfies the corresponding corresponding Eilenberg-Steenrod axioms. We often apply the Universal Coefficients theorem and identify $\mathrm{Hom}(H_m(X), \mathbb{Q})$ with $H^m(X; \mathbb{Q})$ (and the corresponding f^*'s) as $\mathbb{Q}$-vector spaces.

By construction of (H_*, d_*) and (H^*, d^*) one can also develop the theory of products for the o-minimal singular homology and cohomology in the same purely algebraic way as in the classical case ([7] Chapter VI and VII). For completeness we recall this in the Appendix (Section 5 below) since it will be used in the proof of our main result.

For further details on o-minimal singular homology the reader should see the paper [13] by the authors.

3 O-minimal relative Poincaré duality

Before we prove the o-minimal relative Poincaré duality we introduce some notation and recall orientation theory for definable manifolds.

In this paper, by a definable manifold we always mean an affine Hausdorff definable manifold, i.e., a definable subset X of N^k with a cover by relatively open definable subsets $U_1, \dots, U_l$ such that, for each $i = 1, \dots, l$, there is a definable homeomorphism $\phi_i : U_i \longrightarrow V_i$ where V_i is an open definable subset of N^n and, for all $j = 1, \dots, l$, the map $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$ is a definable homeomorphism. A definable manifold (or a definable set) X is definably compact if it is a closed and bounded subset of N^k (see [16]) and X is definably connected if and only if it is not the union of two disjoint clopen definable subsets.

Let X be a definable manifold of dimension n. We call a finite collection $(W_l, h_l)_{l \in L}$ of open definable subsets W_l of X together with the definable homeomorphisms $h_l : W_l \longrightarrow B_n(0, \epsilon_l) \subseteq R^n$ *definable charts of X by open balls.* In this context it is natural to call each W_l a *definable sub-ball of X* and a definable subset U of X of the form $h_l^{-1}(B_n(0, \delta))$ with $0 < \delta < \epsilon_l$ a *definable proper sub-ball of W_l (or of X))* since we will have a definable homeomorphism from the closure $\overline{U}$ of U in X into the closed unit ball in R^n sending $\overline{U} - U$ into the unit $(n-1)$-sphere.

In this context we have the following fundamental result:

Theorem 3.1 ([11], [1], [18]) *If X is a definable manifold of dimension n, then X can be covered by finitely many definable sub-balls of X. In particular, if $A \subseteq X$ is a definably compact definable subset of X, then A can be covered by finitely many definable proper sub-balls of X.*

This theorem is used to define orientation theory for definable manifolds:

Definition 3.2 ([1], [2]) An *orientation* on a definable manifold X of dimension n is a map

$$s : X \longrightarrow \sqcup_{x \in X} H_n(X, X - x; \mathbb{Z})$$

which assigns to each $x \in X$ a generator $s(x)$ of $H_n(X, X - x; \mathbb{Z}) \simeq \mathbb{Z}$ and is such that for every definable proper sub-ball U of X there is a class $\alpha_U \in H_n(X, X - U; \mathbb{Z})$ such that $s(u) = j_u^U(\alpha_U)$ for each $u \in U$, where $j_u^U : H_n(X, X - U; \mathbb{Z}) \longrightarrow H_n(X, X - u; \mathbb{Z})$ is the homomorphism induced by the inclusion.

Theorem 3.1 is used together with classical arguments to prove:

Theorem 3.3 ([1]) *Suppose that X is a definable manifold of dimension n with an orientation s. If A is a definably compact definable subset of X, then*

$s_{|A}$ is uniquely determined by a fundamental class ζ_A in $H_n(X, X-A;\mathbb{Z})$ as $s_{|A}(x) = j_x^A(\zeta_A)$ where $j_x^A : H_n(X, X-A;\mathbb{Z}) \longrightarrow H_n(X, X-x;\mathbb{Z})$ is the homomorphism induced by the inclusion.

We now proceed towards the proof of the o-minimal relative Poincaré duality. But we will require the following lemma.

Lemma 3.4 *Let X be a definable manifold. If $L \subseteq K$ are definably compact definable subset of X, then $H^*(K, L;\mathbb{Q})$ is isomorphic to the direct limit $\lim_{(U,V)\in\Omega(K,L)} H^*(U, V;\mathbb{Q})$ where $\Omega(K, L)$ is the set of pairs (U, V) such that U (resp., V) is an open definable neighbourhood of K (resp., L) in X directed by reversed inclusion.*

Proof. We first show that if K is a definably compact definable subset of X, then $H^*(K;\mathbb{Q})$ is isomorphic to the direct limit $\lim_{(X,V)\in\Omega(X,K)} H^*(V;\mathbb{Q})$.

In fact, if V is an open definable neighbourhood of K in X, by [8] Chapter VIII, 3.3 and 3.4, there is an open definable neigbourhood U of K in X such that $U \subseteq V$ such that K is a definable deformation retract of U. Hence, the inclusion $K \longrightarrow U$ induces an isomorphism $H^*(U;\mathbb{Q}) \longrightarrow H^*(K;\mathbb{Q})$.

The general case stated in the lemma follows from the special case together with the exactness axiom. $\square$

Theorem 3.5 *Assume that X is an orientable definable manifold of dimension n and let $B \subseteq A$ be definably compact definable subsets of X. Then, for every $q \in \mathbb{Z}$, there is an isomorphism*

$$D_{X,A} : H^q(A, B;\mathbb{Q}) \longrightarrow H_{n-q}(X - B, X - A;\mathbb{Q})$$

which is natural with respect to inclusions of pairs of definably compact definable subsets of X.

Proof. First we observe that if $K_1, K_2 \subseteq X$ are definably compact definable subsets and the theorem holds for $(K_1, \emptyset), (K_2, \emptyset)$ and $(K_1 \cap K_2, \emptyset)$, then the theorem holds for $(K_1 \cup K_2, \emptyset)$.

For $i = 1, 2$, let V_i be a definable open neighbourhood of K_i in X. Consider the diagram

$$\begin{array}{ccccc} H^{q-1}(V_1;\mathbb{Q}) \oplus H^{q-1}(V_2;\mathbb{Q}) & \longrightarrow & H^{q-1}(V;\mathbb{Q}) & \xrightarrow{d^*} & H^q(W;\mathbb{Q}) \\ \downarrow{\scriptstyle \cap\zeta_{K_1} \oplus \zeta_{K_2}} & & \downarrow{\scriptstyle \cap\zeta_K} & & \downarrow{\scriptstyle \cap\zeta_L} \\ H_{n-q+1}(K_1';\mathbb{Q}) \oplus H_{n-q+1}(K_2';\mathbb{Q}) & \longrightarrow & H_{n-q+1}(K';\mathbb{Q}) & \xrightarrow{d_*} & H_{n-q}(L';\mathbb{Q}). \end{array}$$

where $K = K_1 \cap K_2$, $L = K_1 \cup K_2$, $V = V_1 \cap V_2$, $W = V_1 \cup V_2$ and for $A \in \{K_1, K_2, K, L\}$ we use A' to denote the pair $(X, X - A)$.

In this diagram the rows are from the Mayer-Vietoris sequence ([17] or [9]) and therefore are exact. The first and the third squares are commutative by naturality of cap product (Theorem 5.7 (1)). By excision, $H_{n-q+1}(K';\mathbb{Q}) \simeq H_{n-q+1}(V, V-K;\mathbb{Q})$ and $H_{n-q}(L';\mathbb{Q}) \simeq H_{n-q}(W, W-L;\mathbb{Q})$. Hence, by Proposition 5.8 taking $X = W$, $X_i = V_i$, $Y_i = X - K_i$ and $\alpha = \zeta_L$, we see that the second square in this diagram is commutative.

The mapping from $\Omega(K_1, K_2)$ into $\Omega(X, K_i)$ (resp., $\Omega(X, K)$ and $\Omega(X, L)$) which sends (V_1, V_2) to (X, V_i) (resp., $(X, V_1 \cap V_2)$ and $(X, V_1 \cup V_2)$) is cofinal. If we pass to the limit, by Lemma 3.4, we get the diagram

$$\begin{tikzcd}
H^{q-1}(K_1;\mathbb{Q}) \oplus H^{q-1}(K_2;\mathbb{Q}) \arrow[r] \arrow[d, "D_{X,K_1} \oplus D_{X,K_2}"] & H^{q-1}(K;\mathbb{Q}) \arrow[r, "d^*"] \arrow[d, "D_{X,K}"] & H^q(L;\mathbb{Q}) \arrow[d, "D_{X,L}"] \\
H_{n-q+1}(K_1';\mathbb{Q}) \oplus H_{n-q+1}(K_2';\mathbb{Q}) \arrow[r] & H_{n-q+1}(K';\mathbb{Q}) \arrow[r, "d_*"] & H_{n-q}(L';\mathbb{Q}).
\end{tikzcd}$$

By [7] Chapter VIII, 5.21 (a purely algebraic result) this diagram is still commutative with exact rows. By assumption and the five lemma, the arrow with $D_{X,L}$ is an isomorphism as required.

We now show that the theorem holds for pairs of the form $(K, \emptyset)$ where K is a nonempty definably compact definable subset of X.

Arguing as in the proof of Case 5 in the proof of [1] Theorem 5.2, we see that there is a finite family $\{\emptyset, K_1, \dots, K_l\}$ closed under intersection of definably compact definable subsets of K such that $K = \cup\{K_i : i = 1, \dots, l\}$ and there are finitely many definable proper sub-balls $U_1, \dots, U_k$ in X such that for each i there is a j_i such that $K_i \subseteq U_{j_i}$.

The theorem holds for $(K, \emptyset)$ by induction on l. The inductive step follows from what we saw at the beginning of the proof. So suppose that $K = K_i$ and $U = U_{j_i}$. Since we are interested in the limit of the homomorphisms $-\cap\zeta_K : H^q(V;\mathbb{Q}) \longrightarrow H_{n-q}(X, X-K;\mathbb{Q})$ with $V \in \Omega(X, K)$, and by the excision axiom this limit is the same as the limit $-\cap\zeta_K : H^q(V;\mathbb{Q}) \longrightarrow H_{n-q}(U, U-K;\mathbb{Q})$ with $V \in \Omega(U, K)$, by the definable triangulation theorem ([8]), we can assume that $U = N^n$ and K is the geometric realization of a closed simplicial complex in N^n. Furthermore, as explained above, by induction on the number of closed simpleces, we can assume that K is the geometric realization of a closed simplex in N^n. The argument in the proof of Case 1 in the proof of [1] Theorem 5.2 shows that $H_{n-q}(N^n, N^n - K;\mathbb{Q})$ is zero except for $q = 0$ in which case it is $\mathbb{Q}$. On the other hand, clearly $H^q(K;\mathbb{Q})$ is zero except for $q = 0$ in which case it is $\mathbb{Q}$ and by definable retration the same holds for the cohomology of elements in a cofinal collection $\mathcal{C}$ of open definable sets in $\Omega(N^n, K)$. So the homomorphisms $-\cap\zeta_K : H^q(V;\mathbb{Q}) \longrightarrow H_{n-q}(N^n, N^n - K;\mathbb{Q})$ are isomorphisms for all $V \in \mathcal{C}$, and hence, the limit homomorphism $D_{X,K} : H^q(V;\mathbb{Q}) \longrightarrow H_{n-q}(X, X-K;\mathbb{Q})$ is an isomorphism as required.

We now prove the general case. Consider the diagram

$$
\begin{tikzcd}
H^{q-1}(K;\mathbb{Q}) \arrow[r] \arrow[d, "D_{X,K}"] & H^{q-1}(L;\mathbb{Q}) \arrow[r, "d^*"] \arrow[d, "D_{X,L}"] & H^q(K,L;\mathbb{Q}) \arrow[r] \arrow[d, "D_{X,K}"] & H^q(K;\mathbb{Q}) \arrow[d, "D_{X,K}"] \\
H_{n-q+1}(K';\mathbb{Q}) \arrow[r] & H_{n-q+1}(L';\mathbb{Q}) \arrow[r, "d_*"] & H_{n-q}(L'',K'';\mathbb{Q}) \arrow[r] & H_{n-q}(K';\mathbb{Q}).
\end{tikzcd}
$$

where $L' = (X, X - L)$, $K' = (X, X - K)$, $L'' = X - L$ and $K'' = X - K$. In this diagram, the first row is exact by exactness axiom, the second row is exact by Mayer-Vietoris ([17] or [9]), the first and the third squares are commutative by naturality of cap product (Theorem 5.7 (1)). The second square is commutative, because it is the direct limit of the corresponding squares for open definable neighbourhoods $V \subseteq V'$ of $L \subseteq K$, and each of these squares commutes by Corollary 5.10 with $X = V'$, $W = X - L$ and $U = X - K$. Therefore, the 5-lemma and the theorem for $(K, \emptyset)$ and $(L, \emptyset)$ implies that the theorem holds for (K, L). $\square$

Corollary 3.6 *Let X be an orientable, definably compact definable manifold of dimension n. Then for all $q \in \mathbb{Z}$, the homomorphism*

$$D_X : H^q(X;\mathbb{Q}) \longrightarrow H_{n-q}(X;\mathbb{Q}),\ \ D_X(\sigma) = \sigma \cap \zeta_X$$

is an isomorphism and determines a dual pairing

$$\langle -, - \rangle : H^q(X;\mathbb{Q}) \otimes H^{n-q}(X;\mathbb{Q}) \longrightarrow \mathbb{Q}$$

given by $\langle x, y \rangle = (x \cup y, \zeta_X)$.

Proof. The fact that $D_{X,X}(\sigma) = D_X(\sigma) = \sigma \cap \zeta_X$ is an isomorphism is a consequence of the proof of Theorem 3.5. Since $(x \cup y, \zeta_X) = (-1)^{\deg x \deg y}(y, x \cap \zeta_X)$, the Kronecker product $(\ ,\ )$ is a dual pairing and $- \cap \zeta_X$ is an isomorphism, it follows that $\langle -, - \rangle$ is a dual pairing. $\square$

Another consequence of Theorem 3.5 is the theory of o-minimal homology and cohomology transfers of continuous definable maps which we now present as it will be requred later.

Corollary 3.7 *Suppose that $f : X \longrightarrow Y$ is a continuous definable map of orientable, definably compact definable manifolds of dimensions n and m respectively. Then there is a homomorphism*

$$f^! : H^q(X;\mathbb{Q}) \longrightarrow H^{m-n+q}(Y;\mathbb{Q}),$$

called cohomology tranfer, which is given by $D_Y^{-1} \circ f_* \circ D_X$, and the following hold: (1) $(g \circ f)^! = g^! \circ f^!$; (2) $1_X^! = \mathrm{id}$; (3) $f^!(f^*\alpha \cup \beta) = \alpha \cup f^!\beta$.

Similarly, there is a homomorphism

$$f_! : H_q(Y;\mathbb{Q}) \longrightarrow H_{m-n+q}(X;\mathbb{Q}),$$

called homology tranfer, which is given by $D_X \circ f^* \circ D_Y^{-1}$*, and the following hold: (1)* $(g\circ f)_! = f_! \circ g_!$*; (2)* $1_{X\,!} = \mathrm{id}$*; (3)* $f_!(\alpha \cap \beta) = f^*\alpha \cap f_!\beta$*; (4)* $f_*(\alpha \cap f_!\beta) = (-1)^{(m-\mathrm{deg}\beta)(m-n)} f^!\alpha \cap \beta$.

Proof. This follows easily from the definitions. For details compare with [7] Chapter VIII, Exercise 10.14 (4). □

The remarks that follow below are also easy consequences of the definitions together with the properties of o-minimal singular (co)homology products.

Remark 3.8 Suppose that $f : X \longrightarrow Y$ and $g : Z \longrightarrow W$ are continuous definable maps of orientable, definably compact definable manifolds of dimensions n, m, l and k respectively. Then

$$(f\times g)^!(\alpha \times \beta) = (-1)^{(n+m)\mathrm{deg}\beta + m(k-l)} f^!(\alpha) \times g^!(\beta)$$

and

$$(f\times g)_!(\sigma \times \tau) = (-1)^{(n+m)(k-\mathrm{deg}\tau)} f_!(\sigma) \times g_!(\tau).$$

Remark 3.9 Suppose that $f : X \longrightarrow Y$ is a continuous definable map of orientable, definably compact definable manifolds of dimensions n. Then $f_* \circ f_! = \mathrm{deg} f = f^! \circ f^*$, $f_! \circ f_* = \mathrm{deg} f$ on the image of $f_!$ and $f^* \circ f^! = \mathrm{deg} f$ on the image of f^*. For the notion of degree $\mathrm{deg} f$ of a continuous definable map see [12].

4 Lefschetz coincidence theorem

Once we develop the theory of the Thom, Lefschetz and Euler classes below, we introduce the Lefschetz coincidence number of continuous definable maps and prove in a rather classical and algebraic way the Lefschetz coincidence theorem.

4.1 The Thom, Lefschetz and Euler classes

Let Y be an orientable, definable manifold of dimension $n+k$, X an orientable, definably compact definable manifold of dimension n and $z : X \longrightarrow Y$ a closed definable embedding. We assume that $z(X)$ is orientable with the induced orientation.

Let A be a definably compact definable subset of X. By Theorem 3.5, for all $q \in \mathbb{Z}$, we have an isomorphism

$$D_{X,A} \circ z^* \circ D^{-1}_{Y,z(A)} : H_q(Y, Y - z(A);\mathbb{Q}) \longrightarrow H_{q-k}(X, X-A;\mathbb{Q}).$$

In particular, we have that

$$H_q(Y, Y - z(A); \mathbb{Q}) = 0 \quad \text{for} \quad q < k$$

and

$$H_k(Y, Y - z(A); \mathbb{Q}) \simeq H_0(X, X - A; \mathbb{Q}) \simeq \mathbb{Q}^l$$

where l is the number of definably connected components of X which lie in A.

Definition 4.1 The generators $\nu_1, \ldots, \nu_l$ of $H_k(Y, Y - z(X); \mathbb{Q})$ are called the *transverse classes*. If X is definably connected, we denote the unique tranverse class by $\nu_{Y,X}$.

The unique class $\tau_{Y,X} \in H^k(Y, Y - z(X); \mathbb{Q})$ such $(\tau_{Y,X}, \nu_i) = 1$ for all $i = 1, \ldots, l$, is called the *Thom class* and its image $\Lambda_{Y,X} = j^*(\tau_{Y,X}) \in H^k(Y; \mathbb{Q})$, where $j : Y \longrightarrow (Y, Y - z(X))$ is the inclusion, is called the *Lefschtez class.* The class $\chi_{Y,X} = z^*(\Lambda_{Y,X}) \in H^n(X; \mathbb{Q})$ is called the *Euler class.*

Example 4.2 Let X be an orientable, definably compact definable manifold of dimension n and $\Delta_X : X \longrightarrow X \times X$ the diagonal map and $\Delta_X \subseteq X \times X$ the diagonal. The Thom class $\tau_{X \times X, X}$ is denoted by $\tau_X \in H^n(X \times X, X \times X - \Delta_X; \mathbb{Q})$. The Lefschtez class $\Lambda_{X \times X, X}$ is denoted by $\Lambda_X \in H^n(X \times X; \mathbb{Q})$. The Euler class $\chi_{X \times X, X}$ is denoted by $\chi_X \in H^n(X; \mathbb{Q})$.

As in the classical case, below we set

$$z_! := (-1)^{k(n+k-q)} D_{X,A} \circ z^* \circ D^{-1}_{Y, z(A)}.$$

Proposition 4.3 *Let Y be an orientable, definable manifold of dimension $n+k$. Suppose X is an orientable, definably compact definable manifold of dimension n, $z : X \longrightarrow Y$ a closed definable embedding and $z(X)$ is orientable with the induced orientation. Let A be a definably compact definable subset of X and W an open definable subset of Y such that $z(X) - z(A) \subseteq W \subseteq Y - z(A)$. Then $z_*(\zeta_{X,A}) = \tau_{Y,X} \cap \zeta_{Y,z(A)}$ where $z : (X, X - A) \longrightarrow (Y, W)$ is the inclusion.*

Proof. First observe that by Theorem 3.7, we have:

Claim 4.4 $z_!(\alpha \cap \zeta_{Y,z(A)}) = (-1)^{k\deg\alpha}(z^*\alpha) \cap \zeta_{X,A}$ *for all $\alpha \in H^*(Y, W; \mathbb{Q})$.*

Proof. If λ is the image of α under the composition

$$H^*(Y, W; \mathbb{Q}) \to H^*(z(X), z(X) - z(A); \mathbb{Q}) \to H^*(z(X); \mathbb{Q})$$

where the last arrow is induced by the isomorphism of Lemma 3.4, then $\alpha \cap \zeta_{Y,z(A)} = D_{Y,z(X)}(\lambda)$ and $(z^*\alpha) \cap \zeta_{X,A} = D_X(z^*\lambda)$ by the definition of the right hand sides. Since $z_!(D_{Y,z(X)}(\lambda)) = (-1)^{k\deg\alpha} D_X(z^*\lambda)$, the claim holds. $\square$

For U an open definable subset of Y such that $z(X) \subseteq W \subseteq U$ and $z(X)$ is closed in U, let $\tau_{U,X}$ and $\zeta_{U,z(A)}$ be the classes obtained from $\tau_{Y,X}$ and $\zeta_{Y,z(A)}$ by excision isomorphisms.

Claim 4.5 *If $r : (U, V) \longrightarrow (X, X - A)$ is a definable retraction, i.e., $r \circ z = 1_X$, where V is an open definable subset of W such that $V \subseteq U - z(A)$, then $r_*(\tau_{U,X} \cap \zeta_{U,z(A)}) = \zeta_{X,A}$.*

Proof. We start by proving the claim for A a point x. Let $\mu \in H^n(X, X-x; \mathbb{Q})$ be such that $(\mu, \zeta_{X,x}) = 1$. Then $(r^*\mu) \cap \zeta_{U,z(x)} = (-1)^{kn}\nu_{U,x}$ where $\nu_{U,x}$ is the transverse class of x. Indeed, by Claim 4.4,

$$\begin{aligned}
(-1)^{kn} z_!((r^*\mu) \cap \zeta_{U,z(x)}) &= ((z^* \circ r^*)\mu) \cap \zeta_{X,x} \\
&= \mu \cap \zeta_{X,x}.
\end{aligned}$$

But $\mu \cap \zeta_{X,x}$ equals $(\mu, \zeta_{X,x}) = 1$ times the homology class of x. Hence, by definition of transverse classes, $(r^*\mu) \cap \zeta_{U,z(x)} = (-1)^{kn}\nu_{U,x}$ as required.

We have

$$\begin{aligned}
(\mu, r_*(\tau_{U,X} \cap \zeta_{U,z(x)})) &= (r^*\mu, \tau_{U,X} \cap \zeta_{U,z(x)}) \\
&= (r^*\mu \cup \tau_{U,X}, \zeta_{U,z(x)}) \\
&= (-1)^{kn}(\tau_{U,X}, (r^*\mu) \cap \zeta_{U,z(x)}) \\
&= (\tau_{U,X}, \nu_{U,x}) \\
&= 1.
\end{aligned}$$

Thus, since $r_*(\tau_{U,X} \cap \zeta_{U,z(x)})$ is a multiple of $\zeta_{X,x}$, this proves that $r_*(\tau_{U,X} \cap \zeta_{U,x}) = \zeta_{X,x}$.

For the general case, let $x \in A$ and consider the inclusions $l : (U, V) \longrightarrow (U, r^{-1}(z(X) - z(x)))$ and $i : (X, X - A) \longrightarrow (X, X - x)$. Then we have a commutative diagram

$$\begin{tikzcd}
H_*(U, V; \mathbb{Q}) \arrow[r, "l_*"] \arrow[d, "r_*"] & H_*(U, r^{-1}(X - x); \mathbb{Q}) \arrow[d, "r_*"] \\
H_*(X, X - A; \mathbb{Q}) \arrow[r, "i_*"] & H_*(X, X - x; \mathbb{Q}).
\end{tikzcd}$$

Then

$$\begin{aligned}
i_*(r_*(\tau_{U,X} \cap \zeta_{U,z(A)})) &= r_* \circ l_*(\tau_{U,X} \cap \zeta_{U,z(A)}) \\
&= r_*(\tau_{U,X} \cap \zeta_{U,z(x)}) \\
&= \zeta_{X,x}
\end{aligned}$$

using naturality of cap products and the first part of the proof. But then, by definition of $\zeta_{X,A}$, we have $r_*(\tau_{U,X} \cap \zeta_{U,z(A)}) = \zeta_{X,A}$. $\Box$

Since $z(X)$ is closed in Y, by [8] Chapter VIII, 3.3, there is a definable retraction $r : U \longrightarrow X$ where U is an open definable subset of Y such that $z(X) \subseteq U$ and $z(X)$ is closed in U. Let V be a definable neighbourhood of $z(X) - z(A)$ in $r^{-1}(X - A) \cap W$. Then by [8] Chapter VIII, 3.4, the composition $(U, V) \xrightarrow{r} (X, X - A) \xrightarrow{z} (Y, W)$ is definably homotopic to the inclusion $i : (U, V) \longrightarrow (Y, W)$. Hence, $i_* = z_* \circ r_*$ and, by Claim 4.5 and naturality of cap products,

$$\begin{aligned} z_*(\zeta_{X,A}) &= z_* \circ r_*(\tau_{U,X} \cap \zeta_{U,z(A)}) \\ &= i_*(i^* \tau_{Y,X} \cap \zeta_{U,z(A)}) \\ &= \tau_{Y,X} \cap i_* \zeta_{U,z(A)} \\ &= \tau_{Y,X} \cap \zeta_{Y,z(A)} \end{aligned}$$

as required. $\Box$

The proof of our next result is purely algebraic using Proposition 4.3.

Proposition 4.6 *Suppose that X is an orientable, definably compact definable manifold of dimension n. Let $\{b_i : i \in I\}$ be a basis of $H^*(X; \mathbb{Q})$ and $\{\widehat{b}_i : i \in I\}$ the dual basis of $H^*(X; \mathbb{Q})$, i.e., $\langle \widehat{b}_i, b_j \rangle = \delta_{ij}$ for all $i, j \in I$. Then*

$$\Lambda_X = \sum_{i \in I} (-1)^{\deg b_i} \widehat{b}_i \times b_i \text{ and } \chi_X = \sum_{i \in I} (-1)^{\deg b_i} \widehat{b}_i \cup b_i.$$

Furthermore, $(\chi_X, \zeta_X) = \chi(X)$, the o-minimal Euler-Poincaré characteristic of X, and, $\Delta_{X*}(\zeta_X) = \Lambda_X \cap \zeta_{X \times X}$.

Proof. By Proposition 4.3 and naturality of cap products, we have

$$\begin{aligned} \Delta_{X*}(\zeta_X) &= \tau_X \cap \zeta_{X \times X, \Delta_X} \\ &= \tau_X \cap j_*(\zeta_{X \times X}) \\ &= j^*(\tau_X) \cap \zeta_{X \times X}. \end{aligned}$$

Thus $\Delta_{X*}(\zeta_X) = \Lambda_X \cap \zeta_{X \times X}$.

We start by proving the following claim, where we are using here the Künneth formula for o-minimal singular cohomology to express elements of $H^*(X \times X; \mathbb{Q})$.

Claim 4.7 *Suppose that $\sigma = \sum_{l,k \in I} A_{l,k} \widehat{b}_l \times b_k$ is an element of $H^*(X \times X; \mathbb{Q})$. Then $\langle b_i \times \widehat{b}_j, \sigma \rangle$ equals $(-1)^{-n \deg b_i} A_{i,j}$.*

Proof. We have that $\langle b_i \times \widehat{b}_j, \sigma\rangle$ equals $\sum A_{l,k}\langle b_i \times \widehat{b}_j, \widehat{b}_l \times b_k\rangle$. But

$$\begin{aligned}\langle b_i \times \widehat{b}_j, \widehat{b}_l \times b_k\rangle &= ((b_i \times \widehat{b}_j) \cup (\widehat{b}_l \times b_k), \zeta_{X\times X})\\ &= (-1)^{(n-\mathrm{deg}b_j)(n-\mathrm{deg}b_l)}((b_i \cup \widehat{b}_l) \times (\widehat{b}_j \cup b_k), \zeta_{X\times X})\\ &= ((b_i \cup \widehat{b}_l) \times (\widehat{b}_j \cup b_k), \zeta_{X\times X})\\ &= (-1)^{n((n-\mathrm{deg}b_j)+\mathrm{deg}b_k)}(b_i \cup \widehat{b}_l, \zeta_X)(\widehat{b}_j \cup b_k, \zeta_X).\end{aligned}$$

(Where in these equalities we used: definition of duality pairing, multiplicativity of cup and cross products, $\zeta_{X\times X} = \zeta_X \times \zeta_X$ and duality of cross products respectively.) Finally, $(b_i \cup \widehat{b}_l, \zeta_X) = (-1)^{\mathrm{deg}b_i(n-\mathrm{deg}b_l)}(\widehat{b}_l \cup b_i, \zeta_X) = (-1)^{\mathrm{deg}b_i(n-\mathrm{deg}b_l)}\delta_{l,i}$ and $(\widehat{b}_j \cup b_k, \zeta_X) = \delta_{j,k}$. Putting all of this together and using the fact that $\mathrm{deg}b_l = \mathrm{deg}b_k$, we see that $\langle b_i \times \widehat{b}_j, \sigma\rangle$ equals $(-1)^{-n\mathrm{deg}b_i}A_{i,j}$. $\square$

Suppose that $\Lambda_X = \sum_{l,k\in I} A_{l,k}\widehat{b}_l \times b_k$. We are going to compute $\langle b_i \times \widehat{b}_j, \Lambda_X\rangle$ in two ways. By Claim 4.7, $\langle b_i \times \widehat{b}_j, \Lambda_X\rangle = (-1)^{-n\mathrm{deg}b_i}A_{i,j}$.

On the other hand, by definition, $\langle b_i \times \widehat{b}_j, \Lambda_X\rangle$ is $((b_i \times \widehat{b}_j) \cup \Lambda_X, \zeta_{X\times X})$ which equals $(b_i \times \widehat{b}_j, \Lambda_X \cap \zeta_{X\times X})$. Using the definition of Λ_X, the last expression is equal to $(b_i \times \widehat{b}_j, \Delta_{X*}(\zeta_X))$. By naturality of Kronecker product, this is $(\Delta_X^*(b_i \times \widehat{b}_j), \zeta_X) = (b_i \cup \widehat{b}_j, \zeta_X) = (-1)^{\mathrm{deg}b_i(n-\mathrm{deg}b_j)}(\widehat{b}_j \cup b_i, \zeta_X)$ by the relation between cup and cross product and the skew commutativity of cup products. But by definition the last expression is $(-1)^{\mathrm{deg}b_i(n-\mathrm{deg}b_j)}\langle \widehat{b}_j, b_i\rangle = (-1)^{\mathrm{deg}b_i(n-\mathrm{deg}b_j)}\delta_{ij}$. Thus, $A_{i,j} = (-1)^{\mathrm{deg}b_i}\delta_{i,j}$ as required.

Since $\chi_X = \Delta_X^*(\Lambda_X)$, the description of χ_X follows from the relation between cup and cross product. Also, $(\chi_X, \zeta_X) = \sum(-1)^{\mathrm{deg}b_i}(\widehat{b}_i \cup b_i, \zeta_X) = \sum(-1)^{\mathrm{deg}b_i}\delta_{i,j} = \chi(X)$. $\square$

4.2 Lefschetz coincidence theorem

In this subsection, X, Y and Z will be definably connected, definably compact, orientable definable manifolds of dimension n, m and k respectively. Also, $\Delta_X : X \longrightarrow X \times X$ will denote the natural inclusion of X into its diagonal Δ_X.

Definition 4.8 Let $L^p(X;\mathbb{Q}) = \mathrm{Hom}(H^{n-p}(X;\mathbb{Q}), H^{n-p}(X;\mathbb{Q}))$ and let

$$L^*(X;\mathbb{Q}) = \sum_{p=0}^{n} L^p(X;\mathbb{Q}).$$

For each p, let $\{b_i : i \in I_p\}$ be a basis of $H^{n-p}(X;\mathbb{Q})$ and let $\{\widehat{b}_i : i \in I_p\}$ be the dual basis on $H^p(X;\mathbb{Q})$. Then we have a canonical isomorphism

$$k^p : H^p(X;\mathbb{Q}) \otimes H_p(X;\mathbb{Q}) \longrightarrow L^p(X;\mathbb{Q})$$

which sends $\sum_{i,j\in I_p} A_{i,j}\widehat{b}_i \otimes D_X(b_j)$ into the element of $L^p(X;\mathbb{Q})$ whose matrix relative to the fixed basis is $(A_{i,j})_{i,j\in I_p}$. The isomorphisms k^p induce a canonical isomorphism

$$k : \sum_{p=0}^{n} H^p(X;\mathbb{Q}) \otimes H_p(X;\mathbb{Q}) \longrightarrow L^*(X;\mathbb{Q})$$

given by $k = \sum_{p=0}^{n}(-1)^p k^p$. The *Lefschetz isomorphism for* X is the isomorphism of $\mathbb{Q}$-modules

$$\lambda_X : L^*(X;\mathbb{Q}) \longrightarrow H^n(X \times X;\mathbb{Q})$$

given by $\lambda_X = \alpha' \circ (1_X^* \otimes D_X^{-1}) \circ k^{-1}$ where α' is the Künneth isomorphism for o-minimal singular cohomology and D_X^{-1} is the inverse of the Poincaré duality isomorphism (see Theorem 3.5).

Remark 4.9 Note that by Proposition 4.6 and the definition of λ_X, we have $\Lambda_X = \lambda_X(1_X^*)$.

Lemma 4.10 *Let* $\mathrm{Tr} : L^*(X;\mathbb{Q}) \longrightarrow \mathbb{Q}$ *be the linear map given by* $\mathrm{Tr}\sigma = \sum_{p=0}^{n}(-1)^p \mathrm{tr}_p \sigma^p$ *where* $\sigma = \sum_{p=0}^{n} \sigma^p$, $\sigma^p \in L^p(X;\mathbb{Q})$. *Then*

$$\mathrm{Tr}\sigma = (\Delta_X^* \lambda_X(\sigma), \zeta_X).$$

Proof. It is enough to consider $\sigma = k(\beta \otimes D_X\gamma)$ with $\beta \in H^p(X;\mathbb{Q})$ and $\gamma \in H^{n-p}(X;\mathbb{Q})$. Then, by ordinary linear algebra

$$\begin{aligned}
\mathrm{Tr}\sigma &= (-1)^{p+p}(\beta, D_X\gamma) \\
&= (\beta, D_X\gamma) \\
&= (\beta, \gamma \cap \zeta_X) \\
&= (\beta \cup \gamma, \zeta_X) \\
&= (\Delta_X^* \alpha'(\beta \otimes \gamma), \zeta_X) \\
&= (\Delta_X^* \lambda_X(\sigma), \zeta_X).
\end{aligned}$$

□

Lemma 4.11 *Let* $\sigma \in L^*(Y;\mathbb{Q})$ *and let* $f, g : X \longrightarrow Y$ *be continuous definable maps where* $\dim X = \dim Y$. *Then*

$$(f \times g)^*(\lambda_Y(\sigma)) = \lambda_X(f^* \circ \sigma \circ g^!).$$

Proof. It is enough to take $\sigma = k(\alpha \otimes D_Y\beta)$ with $\alpha \in H^p(Y;\mathbb{Q})$ and $\beta \in H^{n-p}(Y;\mathbb{Q})$. We have

$$\begin{aligned}
(f^* \circ \sigma \circ g^!)(\gamma) &= (-1)^p (g^!(\gamma), D_Y\beta) f^*(\alpha) \\
&= (-1)^p (\gamma, D_X g^*(\beta)) f^*(\alpha) \\
&= [k(f^*\alpha \otimes D_X g^*\beta)](\gamma).
\end{aligned}$$

for all $\gamma \in H^p(X;\mathbb{Q})$. Therefore, $\lambda_X(f^* \circ \sigma \circ g^!) = (f \times g)^* \circ \alpha'(\alpha \otimes \beta) = (f \times g)^*(\lambda_Y(\sigma))$. $\square$

Definition 4.12 Let $f, g : X \longrightarrow Y$ be continuous definable maps and suppose that $\dim X = \dim Y$. The *Lefschetz coincidence number of f and g* is defined by

$$\lambda(f, g;\mathbb{Q}) = \sum_{p=0}^{n} (-1)^p \mathrm{tr}_p(f^* \circ g^!).$$

Note that if $X = Y$, then $\lambda(f, 1_X;\mathbb{Q})$ is denoted by $\lambda(f;\mathbb{Q})$.

Remark 4.13 Remark 4.9 and Lemmas 4.10, 4.11 imply that

$$\lambda(f, g;\mathbb{Q}) = (\Delta_X^* \circ (f \times g)^*(\Lambda_Y), \zeta_X).$$

Thus $\lambda(f, g;\mathbb{Q}) = (-1)^n \lambda(g, f;\mathbb{Q})$. Since $\mathrm{tr}_p(AB) = \mathrm{tr}_p(BA)$, we also have $\lambda(f, g;\mathbb{Q}) = \sum_{p=0}^{n} (-1)^p \mathrm{tr}_p(f^! \circ g^*)$. Clearly, $f^* \circ g^! = D_X^{-1} \circ f_! \circ g_* \circ D_X$. So, $\lambda(f, g;\mathbb{Q}) = \sum_{p=0}^{n} (-1)^p \mathrm{tr}_p(f_! \circ g_*)$. Similarly, $g^! \circ f^* = D_Y^{-1} \circ f_! \circ g_* \circ D_Y$ and so $\lambda(f, g;\mathbb{Q}) = \sum_{p=0}^{n} (-1)^p \mathrm{tr}_p(f_* \circ g_!)$.

If $h : Z \longrightarrow X$ is a third continuous definable map and $\dim Z = \dim X$, then by Remark 3.9, $\lambda(f \circ h, g \circ h;\mathbb{Q}) = (\deg h)\lambda(f, g;\mathbb{Q})$. In particular, $\lambda(f, f;\mathbb{Q}) = (\deg f)E(X)$ (where $E(X)$ is the o-minimal Euler characteristic of X, see [8] and [1]).

We are now ready to prove the main theorem of the paper.

Proof of Theorem 1.1: We have $\lambda(f, g;\mathbb{Q}) = (\Delta_X^* \circ (f \times g)^*(\Lambda_Y), \zeta_X)$. If there is no $x \in X$ such that $f(x) = g(x)$, then we have a factorisation

$$\begin{tikzcd} X \arrow[r, "f\times g"] \arrow[d, "\Delta_X"] & Y \times Y \\ X \times X \arrow[r, "f\times g"] & Y \times Y - \Delta_Y . \arrow[u, "i"'] \end{tikzcd}$$

where i is the inclusion. Since $H^n(i)H^n(j) = 0$ and $\Lambda_Y = H^n(j)(\tau_Y)$, we have $0 = \Delta_X^* \circ (f \times g)^* \circ i^*(\Lambda_Y) = (f \times g)^*(\Lambda_Y)$ and therefore $\lambda(f, g;\mathbb{Q}) = 0$. $\square$

We end the subsection with another characterization of the Lefschetz coincidence number and yet another classical proof of Theorem 1.1.

Proposition 4.14 *Let X be an orientable, definably compact definable manifold of dimension n. Then there is a graded bilinear map (called the intersection product)*

$$\cdot : H_p(X;\mathbb{Q}) \otimes H_q(X;\mathbb{Q}) \longrightarrow H_{p+q-n}(X;\mathbb{Q})$$

defined by

$$\alpha \cdot \beta = D_X(D_X^{-1}(\alpha) \cup D_X^{-1}(\beta))$$

and such that the following hold:

(1) **Naturality.** $f_!(\alpha \cdot \beta) = f_!(\alpha) \cdot f_!(\beta)$.

(2) **Skew commutativity.** $\alpha \cdot \beta = (-1)^{(n-\deg\alpha)(n-\deg\beta)}\beta \cdot \alpha$.

(3) **Associativity.** $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.

(4) **Units.** $\zeta_X \cdot \beta = \beta = \beta \cdot \zeta_X$.

(5) **Multiplicativity.** $(\alpha \cdot \beta) \times (\sigma \cdot \gamma) = (-1)^{(n-\deg\alpha)(m-\deg\gamma)}(\alpha \times \sigma) \cdot (\beta \times \gamma)$.

Proof. The properties of the intersection product follow easily from the definition and the properties of cap and cup products. $\square$

Using the relationship between cup and cross product, it is easy to prove the following remark.

Remark 4.15 If X and Y are orientable, definably compact definable manifolds of dimension n and m respectively, then the intersection product satisfies:

$$\alpha \cdot \beta = (-1)^{n(n-\deg\beta)}\Delta_{X!}(\alpha \times \beta)$$

and

$$\alpha \times \sigma = (-1)^{n(m-\deg\sigma)}(p_{X!}\alpha) \cdot (p_{Y!}\sigma)$$

where $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$ are the projections.

Theorem 4.16 *Let $f, g : X \longrightarrow Y$ be continuous definable maps between orientable, definably compact definable manifolds of dimension n. If $\zeta_f = (1_X \times f)_* \circ \Delta_{X*}(\zeta_X)$ and $\zeta_g = (1_X \times g)_* \circ \Delta_{X*}(\zeta_X)$, then*

$$\lambda(f, g; \mathbb{Q}) = \epsilon_*(\zeta_f \cdot \zeta_g)$$

where $\epsilon_* : H_0(X \times Y; \mathbb{Q}) \longrightarrow \mathbb{Q}$ is the augmentation. In particular, if $\lambda(f, g; \mathbb{Q}) \neq 0$, then there is $x \in X$ such that $f(x) = g(x)$.

Proof. Let γ_f and γ_g be elements in $H^n(X\times Y;\mathbb{Q})$ such that $\gamma_f \cap \zeta_{X\times Y} = \zeta_f$ and $\gamma_g \cap \zeta_{X\times Y} = \zeta_g$. Then

$$\begin{aligned}
\epsilon_*(\zeta_f \cdot \zeta_g) &= \epsilon_*((\gamma_g \cup \gamma_f) \cap \zeta_{X\times Y}) \\
&= \epsilon_*(\gamma_g \cap (\gamma_f \cap \zeta_{X\times Y})) \\
&= \epsilon_*(\gamma_g \cap \zeta_f) \\
&= (\gamma_g, \zeta_f) \text{ (by definition of augmentation)} \\
&= (D^{-1}_{X\times Y} \circ (1_X \times g)_*(\Delta_{X*}(\zeta_X)), (1_X \times f)_*(\Delta_{X*}(\zeta_X))) \\
&= (D^{-1}_{X\times Y} \circ (1_X \times g)_* \circ D_{X\times X}(\Lambda_X), (1_X \times f)_*(\Delta_{X*}(\zeta_X))) \\
&= ((1^!_X \times g^!)(\Lambda_X), (1_X \times f)_*(\Delta_{X*}(\zeta_X))) \qquad \text{(i)} \\
&= ((1_X \times f)^* \circ (1^!_X \times g^!)(\Lambda_X), \Delta_{X*}(\zeta_X)) \qquad \text{(ii)} \\
&= ((1^*_X \circ 1^!_X \times f^* \circ g^!)(\Lambda_X), \Delta_{X*}(\zeta_X)) \qquad \text{(iii)} \\
&= \sum (-1)^{\deg b_i} (\widehat{b}_i \times f^* \circ g^!(b_i), \Delta_{X*}(\zeta_X)) \qquad \text{(iv)} \\
&= \sum (-1)^{\deg b_i} (\widehat{b}_i \cup f^* \circ g^!(b_i), \zeta_X) \qquad \text{(v)} \\
&= \sum (-1)^{\deg b_i} \langle \widehat{b}_i, f^* \circ g^!(b_i) \rangle \\
&= \sum_{p=0}^{n} (-1)^p \mathrm{tr}_p(f^* \circ g^!)
\end{aligned}$$

where: (i) $D_{X\times Y} = D_X \times D_Y$ and $D_{X\times X} = D_X \times D_X$; (ii) the naturality of the Kronecker product and Proposition 4.6; (iii) the naturality of cross product and Remark 3.8; (iv) Proposition 4.6; (v) the naturality of the Kronecker product.

Let Γ_f (resp., Γ_g) be the graph of f (resp., g). Then there are $\sigma_f \in H^n(X \times Y, X \times Y - \Gamma_f; \mathbb{Q})$ and $\sigma_g \in H^n(X \times Y, X \times Y - \Gamma_g; \mathbb{Q})$ such that γ_f (resp., γ_g) is the image of σ_f (resp., σ_g) by the homomorphism induced by inclusion. If f and g have no coincidence, then $\Gamma_f \cap \Gamma_g = \emptyset$, and so, $\sigma_f \cup \sigma_g \in H^{2n}(X \times Y, X \times Y; \mathbb{Q}) = 0$. Therefore, by naturality of cup products, $\gamma_f \cup \gamma_g = 0$ and $\lambda(f, g; \mathbb{Q}) = \epsilon_*(\zeta_f \cdot \zeta_g) = 0$. $\square$

5 Appendix: O-minimal ring (co)homology theory

By construction of (H_*, d_*) and (H^*, d^*) one can also develop the theory of products for the o-minimal singular homology and cohomology in the same purely algebraic way as in the classical case ([7] Chapter VI and VII). For completeness we include here this theory.

First recall that if $(X, A), (X, B)$ are pairs of definable sets with $A \subseteq X$ and $B \subseteq X$, then we call $(X; A, B)$ a *definable triad.* We say that a definable triad $(X; A, B)$ is an *excisive triad (with respect to (H_*, d_*))* if the inclusion

$(A, A \cap B) \longrightarrow (A \cup B, B)$ induces isomorphisms $H_*(A, A \cap B) \simeq H_*(A \cup B, B)$.

Let $(X, A), (Y, B)$ be pairs of definable sets with $A \subseteq X$ and $B \subseteq Y$. Then we will write $(X, A) \times (Y, B)$ for $(X \times Y, A \times Y \cup X \times B)$.

As we pointed out in [10], the o-minimal version of the Eilenberg-Zilber theorem (Proposition 3.2 in [12]) gives, as in [7] Chapter VI, Section 12 and Chapter VII, Section 2 respectively, the following two theorems:

Theorem 5.1 (Künneth Formula for Homology, [10]) *Let (X, A) and (Y, B) be pairs of definable sets with $A \subseteq X$ and $B \subseteq Y$ such that $(X \times Y; A \times Y, X \times B)$ is an excisive triad. Then, for all $n \in \mathbb{Z}$, there is an isomorphism*

$$\alpha'' : \sum_{i+j=n} H_i(X, A; \mathbb{Q}) \otimes H_j(Y, B; \mathbb{Q}) \longrightarrow H_n((X, A) \times (Y, B); \mathbb{Q}).$$

The homomorphism α'' from Theorem 5.1 is called the *homology (external) cross product* and $\alpha''(a \otimes b)$ is denoted $a \times b$.

Theorem 5.2 ([10]) *The homology cross product satisfies the following properties:*

(1) **Naturality**. $(f \times g)_*(\alpha \times \beta) = (f_*\alpha) \times (g_*\beta)$.

(2) **Skew-commutativity**. $t_*(\alpha \times \beta) = (-1)^{\deg\alpha \deg\beta} \beta \times \alpha$ *where* $t : X \times Y \longrightarrow Y \times X$ *commutes factors.*

(3) **Associativity**. $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$.

(4) **Units**. $1 \times \alpha = \alpha \times 1 = \alpha$.

(5) **Stability**. $d_*(\alpha \times \beta) = i_{1*}(d_*\alpha \times \beta) + i_{2*}((-1)^{\deg\alpha} \alpha \times d_*\beta)$ *where* $\alpha \in H_i(X, A; \mathbb{Q})$, $\beta \in H_j(Y, B; \mathbb{Q})$ *and* $i_1 : (A \times Y, A \times B) \longrightarrow (A \times Y \cup X \times B, A \times B)$ *and* $i_2 : (X \times B, A \times B) \longrightarrow (A \times Y \cup X \times B, A \times B)$ *are the inclusions.*

By dualizing the Eilenberg-Zilber maps from the o-minimal version of the Eilenberg-Zilber theorem (Proposition 3.2 in [12]) gives, as in [7] Chapter VI, Section 10, the following:

Theorem 5.3 (Künneth Formula for Cohomology) *For pairs of definable sets (X, A) and (Y, B) with $A \subseteq X$ and $B \subseteq Y$ such that $(X \times Y; A \times Y, X \times B)$ is an excisive triad, we have that, for all $n \in \mathbb{Z}$, there is an isomorphism*

$$\alpha' : \sum_{i+j=n} H^i(X, A; \mathbb{Q}) \otimes H^j(Y, B; \mathbb{Q}) \longrightarrow H^n((X, A) \times (Y, B); \mathbb{Q}).$$

Let (X,A) be a pair of definable sets with $A \subseteq X$. The *Kronecker product*

$$(\ ,\ ) : H^*(X,A;\mathbb{Q}) \otimes H_*(X,A;\mathbb{Q}) \longrightarrow \mathbb{Q}$$

is the homomorphism $(e \otimes \mathrm{id})_* \circ \alpha$ where

$$\alpha : H^*(X,A;\mathbb{Q}) \otimes H_*(X,A;\mathbb{Q}) \longrightarrow H_*(\mathrm{Hom}(S_*(X,A) \otimes \mathbb{Q}, \mathbb{Q}) \otimes (S_*(X,A) \otimes \mathbb{Q}))$$

is the Künneth homomorphism from homological algebra (see [7] Chapter VI, Theorem 9.13) and

$$(e \otimes \mathrm{id}) : \mathrm{Hom}(S_*(X,A) \otimes \mathbb{Q}, \mathbb{Q}) \otimes (S_*(X,A) \otimes \mathbb{Q}) \longrightarrow \mathbb{Q} \otimes \mathbb{Q} \simeq \mathbb{Q}$$

is the evaluation chain map given by $(e \otimes \mathrm{id})(\sigma \otimes (a \otimes m)) = \sigma(a) \otimes m$.

By purely algebraic arguments, compare with [7] Chapter VII, 1.8 and 1.12, we see that the Kronecker product is a dual pairing satisfying:

$$(f^*\alpha, \beta) = (\alpha, f_*\beta).$$

The homomorphism α' from Theorem 5.3 is called the *cohomology (external) cross product* and $\alpha'(a \otimes b)$ is denoted $a \times b$.

Theorem 5.4 *The cohomology cross product satisfies the following properties:*

(1) **Naturality**. $(f \times g)^*(\alpha \times \beta) = (f^*\alpha) \times (g^*\beta)$.

(2) **Skew-commutativity**. $t^*(\alpha \times \beta) = (-1)^{\deg\alpha \deg\beta} \beta \times \alpha$ *where* $t : X \times Y \longrightarrow Y \times X$ *commutes factors.*

(3) **Associativity**. $(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma)$.

(4) **Units**. $1 \times \alpha = \alpha \times 1 = \alpha$.

(5) **Duality**. $(\alpha \times \beta, \sigma \times \tau) = (-1)^{\deg\tau \deg\alpha} (\alpha, \sigma) \otimes (\beta, \tau)$ *where* $\sigma \in H_*(X,A;\mathbb{Q})$, $\tau \in H_*(Y,B;\mathbb{Q})$, $\alpha \in H^*(X,A;\mathbb{Q})$ *and* $\beta \in H^*(Y,B;\mathbb{Q})$.

(6) **Stability**. $d^*(\alpha \times \beta) = (i_{1*} \oplus i_{2*})^{-1}((d_*\alpha \times \beta) + ((-1)^{\deg\alpha} \alpha \times d_*\beta))$ *where* $\alpha \in H^i(A;\mathbb{Q})$, $\beta \in H^j(B;\mathbb{Q})$ *and* $i_1 : (A \times Y, A \times B) \longrightarrow (A \times Y \cup X \times B, A \times B)$ *and* $i_2 : (X \times B, A \times B) \longrightarrow (A \times Y \cup X \times B, A \times B)$ *are the inclusions.*

Proof. This is obtained from the proof of Theorem 5.2 by applying the functor $\mathrm{Hom}(-, \mathbb{Q} \otimes \mathbb{Q})$ and using the dual of the Eilenberg-Zilber map from the proof of Theorem 5.3 (compare with [7] Chapter VII, Section 7 for the details). $\square$

We now introduce the cup products. Although these are equivalent to the cross products, they are usually more convenient.

Theorem 5.5 *Suppose $(X; A_1, A_2)$ is an excisive triad of definable sets. Then we have a canonical graded bilinear map (called cup product)*

$$\cup : H^*(X, A_1; \mathbb{Q}) \otimes H^*(X, A_2; \mathbb{Q}) \longrightarrow H^*(X, A_1 \cup A_2; \mathbb{Q})$$

such that:

(1) **Naturality**. $f^*(\alpha \cup \beta) = (f^*\alpha) \cup (f^*\beta)$.

(2) **Skew-commutativity**. $\alpha \cup \beta = (-1)^{\deg\alpha \deg\beta}\ \beta \cup \alpha$.

(3) **Associativity**. $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$.

(4) **Units**. $1 \cup \alpha = \alpha \cup 1 = \alpha$.

(5) **Multiplicativity**. $(\alpha \times \beta) \cup (\sigma \times \tau) = (-1)^{\deg\beta \deg\sigma}(\alpha \cup \sigma) \times (\beta \cup \tau)$.

(6) **Stability**. $d^* \circ (j^*)^{-1}(\alpha \cup i * \beta) = (d^*\alpha) \cup \beta$ *where* $i : (A_1, A_1 \cap A_2) \longrightarrow (X, A_2)$ *and* $j : (A_1, A_1 \cap A_2) \longrightarrow (A_1 \cup A_2, A_2)$ *are the inclusions.*

Proof. The cup product $\cup$ is the graded bilinear map given by $j'^* \circ D'^* \circ \gamma_* \circ \alpha$ where: α is the map from the Künneth formula for cochain complexes (see [7] Chapter VI, Theorem 9.13); γ is the chain map from the proof of Theorem 5.3 (the dual Eilenberg-Zilber map); $D' = \mathrm{Hom}(D, \mathbb{Q})$ is the cochain map from

$$\mathrm{Hom}(S_*(X, A_1), \mathbb{Q}) \otimes \mathrm{Hom}(S_*(X, A_2), \mathbb{Q}) \longrightarrow \mathrm{Hom}(\frac{S_*(X)}{S_*(A_1) + S_*(A_2)}, \mathbb{Q})$$

where D is the natural chain map (called diagonal map) given by

$$D = \zeta \circ \Delta : \frac{S_*(X)}{S_*(A_1) + S_*(A_2)} \longrightarrow S_*(X, A_1) \otimes S_*(X, A_2),$$

here $\Delta : S_*(X) \longrightarrow S_*(X) \longrightarrow S_*(X \times X)$ is the natural chain map induced by the diagonal map $X \longrightarrow X \times X$ and $\zeta : S_*(X \times X) \longrightarrow S_*(X) \otimes S_*(X)$ is a Eilenberg-Zilber chain map; and j' is the homotopy equivalence

$$\mathrm{Hom}(\frac{S_*(X)}{S_*(A_1) + S_*(A_2)}, \mathbb{Q}) \longrightarrow \mathrm{Hom}(S_*(X, A_1 \cup A_2), \mathbb{Q})$$

which exists since $(X; A_1, A_2)$ is an excisive triad of definable sets.

The properties of the cup product listed above, follow from corresponding properties for the Eilenberg-Zilber chain equivalence. Since these are purely algebraic, we refer the reader to [7] Chapter VII, Section 8 for the details. $\square$

Similarly to the classical case ([7] Chapter VII, Section 8) we also have:

Remark 5.6 The cohomology cross product is related to the cup product by:

(1) $\alpha \times \beta = p^*\alpha \cup q^*\beta$ where $p : (X \times Y, A \times Y) \longrightarrow (X, A)$ and $q : (X \times Y, X \times B) \longrightarrow (Y, B)$ are the projections (and we assume here that $(X \times Y; A \times Y, X \times B)$ is excisive)

(2) $\alpha \cup \beta = \Delta_X^*(\alpha \times \beta)$ where $\Delta_X : (X, A_1 \cup A_2) \longrightarrow (X \times X, A_1 \times X \cup X \times A_2)$ is the diagonal map (and we assume here that $(X \times X; A_1 \times X, X \times A_2)$ is excisive).

Theorem 5.5 implies that $H^*(X;\mathbb{Q})$ is a graded $\mathbb{Q}$-algebra under cup product, $H^*(\ ;\mathbb{Q})$ is a functor from the category definable sets into the category of graded skew-commutative (associative) $\mathbb{Q}$-algebra with unit element and $H^*(X, A;\mathbb{Q})$ is a graded $H^*(X;\mathbb{Q})$-module with respect to the cup product $\cup : H^*(X;\mathbb{Q}) \otimes H^*(X, A;\mathbb{Q}) \longrightarrow H^*(X, A;\mathbb{Q})$. Moreover, the cross product $\times : H^*(X;\mathbb{Q}) \otimes H^*(Y;\mathbb{Q}) \longrightarrow H^*(X \times Y;\mathbb{Q})$ is a homomorphism of graded skew-commutative associative $\mathbb{Q}$-algebras.

Another useful product is the cap product. This is in some sense dual to the cup product.

Theorem 5.7 *Suppose that $(X; A_1, A_2)$ is an excisive triad of definable sets. Then we have a canonical graded bilinear map (called cap product)*

$$\cap : H^p(X, A_2;\mathbb{Q}) \otimes H_{p+q}(X, A_1 \cup A_2;\mathbb{Q}) \longrightarrow H_q(X, A_1;\mathbb{Q})$$

such that:

(1) **Naturality**. $f_*((f^*\alpha) \cap \beta) = \alpha \cap (f_*\beta)$.

(2) **Associativity**. $(\alpha \cup \beta) \cap \gamma = \alpha \cap (\beta \cap \gamma)$.

(3) **Units**. $1 \cap \alpha = \alpha \cap 1 = \alpha$.

(4) **Duality**. $(\alpha \cup \beta, \sigma) = (\alpha, \beta \cap \sigma)$.

(5) **Multiplicativity**. $(\alpha \times \beta) \cap (\sigma \times \tau) = (-1)^{\deg\beta\deg\sigma}(\alpha \cap \sigma) \times (\beta \cap \tau)$.

(6) **Stability**. $d_*(\alpha \cap \beta) = (-1)^{\deg\alpha}(i^*(\alpha) \cap (j^*)^{-1} \circ d_*(\beta)$ *and* $d^*(\alpha) \cap \beta + (-1)^{\deg\alpha} i_*(\alpha \cap (j^*)^{-1} \circ d_*(\beta)) = 0$ *where* $i : (A_1, A_1 \cap A_2) \longrightarrow (X, A_2)$ *and* $j : (A_1, A_1 \cap A_2) \longrightarrow (X, A_2)$ *are the inclusions.*

Proof. The cap product is the bilinear map given by $E_* \circ (\mathrm{id} \otimes D)_* \circ (\mathrm{id} \otimes k)_* \circ \alpha$ where D and α are as before,

$$\mathrm{id} : \mathrm{Hom}(S_*(X, A_2), \mathbb{Q}) \longrightarrow \mathrm{Hom}(S_*(X, A_2), \mathbb{Q})$$

is the identity map, k is the chain equivalence

$$k : S_*(X, A_1 \cup A_2) \otimes \mathbb{Q} \longrightarrow \frac{S_*(X)}{S_*(A_1) + S_*(A_2)} \otimes \mathbb{Q}$$

and E is the natural evaluation map

$$\mathrm{Hom}(S_*(X, A_2), \mathbb{Q}) \otimes (S_*(X, A_1) \otimes S_*(X, A_2) \otimes \mathbb{Q}) \longrightarrow S_*(X, A_1) \otimes \mathbb{Q}$$

given by $E(\sigma\otimes a\otimes b\otimes m)=(-1)^{\mathrm{deg}\sigma\mathrm{deg}a}a\otimes m\otimes\sigma(b)$. The proofs of the properties of the cap product are simple computations as before (compare with [7] Chapter VII, Section 12 for the details). $\square$

The next result is proved as in the classical case ([7] Chapter VII, Section 12, Proposition 12.20) with the following change: one replaces the use of [7] Chapter III, 7.3 by its o-minimal analogue given by Lemma 2.3.

Below, we denote by $Z^n(X,A;\mathbb{Q})$ the kernel of $\partial^n : S^n(X,A;\mathbb{Q}) \longrightarrow S^{n+1}(X,A;\mathbb{Q})$.

Proposition 5.8 *Suppose that X_1, X_2, Y_1, Y_2 are open definable subsets of a definable set X such that $X_1\cup Y_1 = X_2\cup Y_2 = X_1\cup X_2 = X$. Let $X' = X_1\cap X_2$, $Y' = Y_1\cap Y_2$, $Y = Y_1\cup Y_2$ and let $j : (X', X'\cap Y) \longrightarrow (X,Y)$ denote the inclusion. For $\alpha\in H_*(X,Y';\mathbb{Q})$, let $\alpha'\in H_*(X',X'\cap Y;\mathbb{Q})$ denote its image under the composition*

$$H_*(X,Y';\mathbb{Q})\to H_*(X,Y;\mathbb{Q}) \stackrel{(j_*)^{-1}}{\to} H_*(X',X'\cap Y;\mathbb{Q}).$$

Then the following diagram

$$\begin{tikzcd} H^*(X';\mathbb{Q}) \arrow[r, "j_*\circ(\ \cap\alpha')"] \arrow[d, "\delta^*"] & H_*(X,Y;\mathbb{Q}) \arrow[d, "\delta^*"] \\ H^*(X;\mathbb{Q}) \arrow[r, "\cap\alpha"] & H_*(X,Y';\mathbb{Q}). \end{tikzcd}$$

where δ_* and δ^* are the Mayer-Vietoris boundaries, is commutative.

Proof. By the proof of the cohomology Mayer-Vietoris sequence ([17] or [9]), we have that δ^* is the composition

$$H^*(X';\mathbb{Q}) \stackrel{d^*}{\to} H^*(X_1,X';\mathbb{Q}) \stackrel{(l^*)^{-1}}{\to} H^*(X,X_2;\mathbb{Q})\to H^*(X;\mathbb{Q})$$

where $l : (X_1,X') \longrightarrow (X,X_2)$ is the inclusion. Similarly, by the proof of the homology Mayer-Vietoris sequence ([17] or [9]), we have that δ_* is the composition

$$H_*(X',Y;\mathbb{Q}) \stackrel{c_*\circ d_*}{\to} H_*(Y,Y_1;\mathbb{Q}) \stackrel{(k_*)^{-1}}{\to} H_*(Y_2,Y';\mathbb{Q})\to H_*(X,Y';\mathbb{Q})$$

where $a : (Y,\emptyset) \longrightarrow (Y,Y_1)$ and $k : (Y_2,Y') \longrightarrow (Y,Y_1)$ are the inclusions.

Let $b : (Y_2, X_2\cap Y_2) \longrightarrow (X,X_2)$ be the inclusion and let $\alpha_1\in H_*(Y_2,Y'\cup(X_2\cap Y_2);\mathbb{Q})$ denote the image of α under the composition

$$H_*(X,Y';\mathbb{Q})\to H_*(X,Y'\cup X_2;\mathbb{Q}) \stackrel{(m_*)^{-1}}{\to} H_*(Y_2,Y'\cup(X_2\cap Y_2);\mathbb{Q})$$

where $m : (Y_2, Y'\cup(X_2\cap Y_2)) \longrightarrow (X, Y'\cup X_2)$ is the inclusion. Clearly, the proposition is a consequence of the following claim.

Claim 5.9 *The diagrams*

$$\begin{array}{ccc} H^*(X';\mathbb{Q}) & \xrightarrow{j_*\circ(\ \cap\alpha')} & H_*(X,Y;\mathbb{Q}) \\ \Big\downarrow{\scriptstyle (l^*)^{-1}\circ d^*} & & \Big\downarrow{\scriptstyle (k_*)^{-1}\circ(c_*\circ d_*)} \\ H^*(X,X_2;\mathbb{Q}) & \xrightarrow{\cap\alpha_1\circ b^*} & H_*(Y_2,Y';\mathbb{Q}). \end{array}$$

and

$$\begin{array}{ccc} H^*(X,X_2;\mathbb{Q}) & \xrightarrow{\cap\alpha_1\circ b^*} & H_*(Y_2,Y';\mathbb{Q}) \\ \Big\downarrow & & \Big\downarrow \\ H^*(X;\mathbb{Q}) & \xrightarrow{\cap\alpha} & H_*(X,Y';\mathbb{Q}). \end{array}$$

are commutative.

Proof of Claim 5.9: Below, we also denote by $\cap$ the chain map which induces the cap product.We will as well identify cycles (resp., cocycles) with their images under chain maps (resp., cochain maps) induced by certain inclusion maps.

Let $\beta \in H^*(X';\mathbb{Q})$ and take $z \in Z^*(X';\mathbb{Q})$ a representative of β and extend it (by zero outside $S_*(X';\mathbb{Q})$) to $z' \in S^*(X;\mathbb{Q})$. Then $\partial^* z'_{|S_*(X_1;\mathbb{Q})}$ represents $d^*\beta$. By excision axiom, there is $w \in Z^*(X,X_2;\mathbb{Q})$ such that $w_{|S_*(X_1;\mathbb{Q})} = \partial^* z'_{|S_*(X_1;\mathbb{Q})} + \partial^* w'$ where $w' \in S^*(X_1,X';\mathbb{Q})$. Extend w' (by zero outside $S_*(X_1;\mathbb{Q})$) to $w'' \in S^*(X,X_2;\mathbb{Q})$, and replace w by $w - \partial^* w''$. The new cocycle w then satisfies $w_{|S_*(X_1;\mathbb{Q})} = \partial^* z'_{|S_*(X_1;\mathbb{Q})}$ and represents the image of β in $H^*(X,X_2;\mathbb{Q})$ and $H^*(X;\mathbb{Q})$.

Because $X_1 \cap Y_2$, $X_2 \cap Y_1$ and X' are open definable subsets of X which cover X, by Lemma 2.3, we can find a representative a of α such that $a = a_1 + a_2 + a'$ with $a_1 \in S_*(X_1 \cap Y_2;\mathbb{Q})$, $a_2 \in S_*(X_2 \cap Y_1;\mathbb{Q})$, $a' \in S_*(X';\mathbb{Q})$ and, of course, $\partial_* a \in S_*(Y';\mathbb{Q})$. Then a' represents α' and a_1 represents α_1. It follows that the image of β in $H^*(Y,Y_1;\mathbb{Q})$ along the two ways of the first diagram of the claim has representative $w \cap a_1$ and $\partial_*(z \cap a') = (-1)^{\deg z} z \cap \partial_* a'$ (by Theorem 5.7 (6)). We claim that these elements determine the same homology class. Note that, since $z'_{|S_*(X_1 \cap Y_2;\mathbb{Q})} = 0$ we have $z' \cap a_1 = 0$. Hence, by Theorem 5.7 (6), we have $\partial_*(z' \cap a_1) = \partial^* z' \cap a_1 + (-1)^{\deg z} z' \cap \partial_* a_1$ $= \partial^* z' \cap a_1 + (-1)^{\deg z} z' \cap \partial_* a - (-1)^{\deg z} z' \cap \partial_* a_2 - (-1)^{\deg z} z' \cap \partial_* a'$. But we have (i) $\partial^*(z' \cap a_1) = w \cap a_1$ since $a_1 \in S_*(X_1;\mathbb{Q})$ and $w_{|S_*(X_1;\mathbb{Q})} = \partial^* z'_{|S_*(X_1;\mathbb{Q})}$; (ii) $z' \cap \partial_* a' = z \cap \partial_* a'$ since $\partial_* a' \in S_*(X';\mathbb{Q})$ and $z'_{|S(X';\mathbb{Q})} = z$ and (iii) $z' \cap \partial_* a - z' \cap \partial_* a_2 \in S_*(Y_1;\mathbb{Q})$ because $\partial_* a$ and a_2 are in $S_*(Y_1;\mathbb{Q})$. Hence, $\partial_*(z' \cap a_1) - [w \cap a_1 - (-1)^{\deg z} z \cap \partial_* a'] \in S_*(Y_1;\mathbb{Q})$ as required.

It remains to show the commutativity of the second diagram of the claim. Let $\gamma \in H^*(X,X_2;\mathbb{Q})$ and take $u \in Z^*(X,X_2;\mathbb{Q})$ a representative of γ. Then we have $u \cap (a_2 + a') = 0$ because $a_2 + a' \in S_*(X_2;\mathbb{Q})$ and $u_{|S_*(X_2;\mathbb{Q})} = 0$. Hence, $u \cap a = u \cap a_1$ as required. $\square$

□

Corollary 5.10 *If we take $X_1 = X$, $X_2 = V$, $Y_1 = U$ and $Y_2 = W$ in Proposition 5.8, then we get the following commutative diagram*

$$\begin{tikzcd}
H^*(V;\mathbb{Q}) \arrow[r, "d^*"] \arrow[d, "\cap\alpha'"] & H^*(X,V;\mathbb{Q}) \arrow[r, "b^*\circ(l^*)^{-1}"] & H^*(W,V\cap W;\mathbb{Q}) \arrow[d, "k_*\circ(\ \cap\alpha_1)"] \\
H_*(V,V\cap W;\mathbb{Q}) \arrow[r, "j_*"] & H_*(X,W;\mathbb{Q}) \arrow[r, "d_*"] & H_*(W,U;\mathbb{Q}).
\end{tikzcd}$$

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