

## Abstract

Let  $\mathcal{C}$  be a class of mathematical structures of some type (with a well defined concept of substructure and isomorphism) and let  $\mathcal{F}$  be a subset of  $\mathcal{C}$ . Denote by  $[[\mathcal{C} \mid \mathcal{F}]]$  the class of structures  $S$  in  $\mathcal{C}$  such that no structure in  $\mathcal{F}$  is isomorphic to a substructure of  $S$ . We can look at  $\mathcal{F}$  as a list of forbidden substructures and to  $[[\mathcal{C} \mid \mathcal{F}]]$  as the class of all structures in  $\mathcal{C}$  that avoid the banned substructures.

A *forbidden substructure theorem* characterizes a class  $\mathcal{C}_1 \subseteq \mathcal{C}$  as

$$\mathcal{C}_1 = [[\mathcal{C} \mid \mathcal{F}]],$$

for some family of structures  $\mathcal{F}$ . This theorem can be read as *a structure  $S$  belongs to  $\mathcal{C}_1$  if and only if  $S$  belongs to  $\mathcal{C}$  and does not contain a substructure isomorphic to a model in  $\mathcal{F}$*  or, in short, *a structure belongs to  $\mathcal{C}_1$  if and only if it belongs to  $\mathcal{C}$  and avoids the models in  $\mathcal{F}$ .*

Probably, the paramount forbidden substructure theorem for graphs is the following:

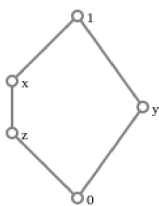
$$\mathcal{B} = [[\mathcal{G} \mid \mathcal{O}]],$$

where  $\mathcal{B}$  denotes the bipartite graphs,  $\mathcal{G}$  denotes all the graphs, and  $\mathcal{O}$  denotes the cycles of odd length.

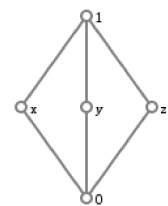
Regarding algebra, the most famous forbidden substructure theorem is probably Birkhoff's Theorem [21]:

$$\mathcal{D} = [[\mathcal{L} \mid \{p, d\}]],$$

where  $\mathcal{D}$  denotes the distributive lattices,  $\mathcal{L}$  denotes all the lattices, and  $\{p, d\}$  is the set containing the pentagon  $p$  and the diamond  $d$ .



(a) Pentagon



(b) Diamond

Figure 5: The two models in Birkhoff's Theorem

The two theorems above are examples of forbidden substructure theorems (FST) for relational algebras. The goal of this thesis is to produce a computational tool that autonomously conjectures and proves FST for relational algebras.

This tool has been made available in <http://fst.educa.pt/> and was able to prove many known and previously unknown theorems. It starts by finding models that belong to the larger class  $\mathcal{C}$ , but do not belong to the smaller class  $\mathcal{C}_1$ , and when no more models are found, it formulates the conjecture that  $\mathcal{C}_1 = [[\mathcal{C} \mid \mathcal{F}]]$ . If a proof is found, we have a theorem; if a counter-example is found, we add it to the

conjectured family of models and repeat the process. Room for improvement is on the case in which neither a proof nor a counter-example are found.

**Keywords:** Prover9, Mace4, Automated Theorem Proving, ProverX, Forbidden Substructures Theorems, Algebra.