Abstract

Let $\mathcal{C}$ be a class of mathematical structures of some type (with a well defined concept of substructure and isomorphism) and let $\mathcal{F}$ be a subset of $\mathcal{C}$. Denote by $[\![\mathcal{C} \mid \mathcal{F}]\!]$ the class of structures S in $\mathcal{C}$ such that no structure in $\mathcal{F}$ is isomorphic to a substructure of S. We can look at $\mathcal{F}$ as a list of forbidden substructures and to $[\![\mathcal{C} \mid \mathcal{F}]\!]$ as the class of all structures in $\mathcal{C}$ that avoid the banned substructures.

A *forbidden substructure theorem* characterizes a class $\mathcal{C}_1 \subseteq \mathcal{C}$ as

$$\mathcal{C}_1 = [\![\mathcal{C} \mid \mathcal{F}]\!],$$

for some family of structures $\mathcal{F}$. This theorem can be read as *a structure S belongs to $\mathcal{C}_1$ if and only if S belongs to $\mathcal{C}$ and does not contain a substructure isomorphic to a model in $\mathcal{F}$* or, in short, *a structure belongs to $\mathcal{C}_1$ if and only if it belongs to $\mathcal{C}$ and avoids the models in $\mathcal{F}$.*

Probably, the paramount forbidden substructure theorem for graphs is the following:

$$\mathcal{B} = [\![\mathcal{G} \mid \mathcal{O}]\!],$$

where $\mathcal{B}$ denotes the bipartite graphs, $\mathcal{G}$ denotes all the graphs, and $\mathcal{O}$ denotes the cycles of odd length.

Regarding algebra, the most famous forbidden substructure theorem is probably Birkhoff's Theorem [21]:

$$\mathcal{D} = [\![\mathcal{L} \mid \{p, d\}]\!],$$

where $\mathcal{D}$ denotes the distributive lattices, $\mathcal{L}$ denotes all the lattices, and $\{p, d\}$ is the set containing the pentagon p and the diamond d.

(a) Pentagon (b) Diamond

Figure 5: The two models in Birkhoff's Theorem

The two theorems above are examples of forbidden substructure theorems (FST) for relational algebras. The goal of this thesis is to produce a computational tool that autonomously conjectures and proves FST for relational algebras.

This tool has been made available in http://fst.educa.pt/ and was able to prove many known and previously unknown theorems. It starts by finding models that belong to the larger class $\mathcal{C}$, but do not belong to the smaller class $\mathcal{C}_1$, and when no more models are found, it formulates the conjecture that $\mathcal{C}_1 = [\![\mathcal{C} \mid \mathcal{F}]\!]$. If a proof is found, we have a theorem; if a counter-example is found, we add it to the conjectured family of models and repeat the process. Room for improvement is on the case in which neither a proof nor a counter-example are found.

Keywords: Prover9, Mace4, Automated Theorem Proving, ProverX, Forbidden Substructures Theorems, Algebra.