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OPEN DESCENDANTS OF NON-DIAGONAL INVARIANTS

L.R Huiszoon, A.N. Schellekens, N.Sousa

*NIKHEF Theory Group
P.O. Box 41882, 1009 DB Amsterdam, The Netherlands*

Abstract

The open descendants of simple current automorphism invariants are constructed. We consider the case where the order of the current is two or odd. We prove that our solutions satisfy the completeness conditions, positivity and integrality of the open and closed sectors and the Klein bottle constraint (apart from an interesting exception). In order to do this, we derive some new relations between the tensor Y and the fixed point conformal field theory. Some non-standard Klein bottle projections are considered as well.

1 Introduction

After a ten year period of neglect, open strings have received more interest recently for a variety of reasons: their rôle in the duality picture, the discovery of D-branes and the appearance of non-commutative geometry, and recent developments in phenomenological string theory, such as brane world scenarios and large extra dimensions. It is therefore worthwhile to explore open string vacua in more detail.

A first step in the classification of open string theories is a classification of closed string theories. This amounts to classifying all possible modular invariants. There exists a systematic way of building modular invariants using simple currents [2]. This turns out to be very powerful: apart from the charge conjugation invariant and a few sporadic exceptions, all modular invariants can be built with the use of simple currents. This suggests the possibility that also in the open string case simple currents should play a crucial rôle in a systematic approach. Furthermore simple current properties are not specific for one particular type of conformal field theories (such as WZW models), but they are generic. It is precisely a generic CFT description of open strings that we are after.

A modular invariant can be of ‘automorphism’ type or ‘extension’ type or products thereof. The first ones are permutations of the primary fields that leave the fusion rules invariant. Among them are the diagonal and charge conjugation invariant. Theories described by an extension invariant always contain extra currents that extend the chiral algebra. The modular invariant of this extension is then again of automorphism type. This extension procedure does not raise any new problem in open string theories. Therefore we only need to consider pure automorphism invariants.

A second ingredient in the construction of open descendants is the classification of boundary conditions for a given bulk theory. A bulk theory is specified by a chiral algebra and a particular modular invariant that describes the pairing between left- and rightmoving representations of this algebra. The boundary conditions are encoded in the ‘boundary coefficients’ $B_{b\alpha}$. These have the following intuitive interpretation. When a field b approaches a boundary α , it gets reflected to its charge conjugate b^c with a strength proportional to $B_{b\alpha}$. Throughout this letter, we will assume that the boundaries leave the bulk symmetry invariant.

In the case that all fields couple to their charge conjugate, Cardy [3] con-

jectured the boundary coefficients for a generic conformal field theory. We will refer to this as the C-diagonal case. In general, the boundary coefficients are constrained by “sewing constraints” [4] [5]. Unfortunately, these constraints require knowledge of OPE-coefficients and duality (fusing and braiding) matrices, which are only known in a limited number of cases, such as $SU(2)$ WZW models. The authors of [5] [6] found the boundary coefficients for the “ D_{odd} ” automorphism invariants of $SU(2)$ WZW models by solving these sewing constraints. Furthermore, they postulated very restrictive conditions for the annulus coefficients (open string state multiplicities) known as the ‘completeness conditions’. These conditions have the advantage that they do not involve duality matrices, so that they are applicable in all cases. Fuchs and Schweigert [7] generalized this to the boundary coefficients for an arbitrary order 2, half-integer spin simple current invariant. They constructed the corresponding annulus coefficients and showed that these were integer and satisfied the completeness conditions.

A third ingredient in the construction of open descendants are the Klein bottle and Möbius strip amplitudes. In short, the Klein bottle projects the closed string spectrum, described by the torus, to an unoriented one and the Möbius strip does the same with the open spectrum, which is described by the annulus. The description of Klein Bottle and Möbius strip requires the introduction of another set of quantities, the ‘crosscap coefficients’ Γ_b , which together with the boundary coefficients $B_{b\alpha}$ form the set of data needed for a complete description of open string spectra. The introduction of crosscaps to the string worldsheet leads to extra sewing constraints, the so-called ‘crosscap constraints’ [8]. These again require knowledge of the – in general unknown – duality matrices. Since general formulas for duality matrices are not likely to be available soon, any attempt to arrive at a generic CFT description will have to be less ambitious. Instead of trying to solve all consistency conditions we will focus here on a non-trivial but accessible subset. This is somewhat analogous to the situation with closed string construction. Although in principle one would have to check duality of the four-point function on the sphere, in practice it is usually sufficient, and much simpler, to check modular invariance on the torus. The conditions we will consider in this paper are the aforementioned completeness conditions, positivity and integrality of the open and closed string partition functions, and a ‘Klein bottle constraint’, formulated in [9], as an alternative to the crosscap constraint. We require these constraints to be satisfied for any CFT, thus focusing on those solutions

that are most likely to survive the full set of sewing constraints. Since our solutions are simple current related there may in fact be a chance that simple current relations among the duality matrices are sufficient to prove that all sewing conditions are satisfied, if they are satisfied in the C-diagonal case.

Until now, only in the C-diagonal case there exists a general prescription to construct open descendants. This is based on the conjecture of Cardy, extended with a conjecture for the crosscap. It was generalized to a class of simple current related crosscap coefficients, and it was shown that all consistency conditions mentioned in the previous paragraph are satisfied. In this letter we will provide the prescription for the construction of open descendants in case of pure automorphism modular invariants built from odd order and order 2 simple currents.

The rest of this paper is organized as follows: In section 2 we review the method of open descendants. In section 3 we present solutions of the consistency conditions for automorphism invariants built from odd order simple currents. This case is relatively simple because of the absence of short orbits. Non-standard Klein bottle projections are discussed in a subsection. Section 4 is devoted to the construction of open descendants of order 2 simple current automorphism invariants. Due to the possibility of fixed points, this case is harder to deal with and we find that the correct crosscap coefficient requires a subtle sign choice. In some rare cases, the naive version of the Klein bottle constraint is violated and we comment on a possible solution. Some essential relations involving various types of generalized fusion coefficients are derived in appendix A.

2 Open descendants

In this section we review the method of open descendants, based on the original work of [1] [3] [9]. The closed sector is described by a modular invariant torus partition function

$$T = \sum_{ij} Z_{ij} \chi_i \bar{\chi}_j \quad , \quad (1)$$

where the index i labels the (primary) fields and the matrix Z_{ij} is the modular invariant that commutes with the modular transformations S and T . We will only be interested in the case where Z_{ij} is a pure automorphism, so that

it has entries that are 0 or 1. A bar on the characters χ_i denotes “anti-holomorphic”. We will introduce a name for the fields that couple to their charge conjugate in the torus: *transverse channel fields*, or transverse fields for short, and denote them by a, b . This name is justified since only these fields can survive near a crosscap or boundary, and therefore only these fields can propagate in the transverse channels.

The closed oriented theory is projected to an unoriented theory by the Klein bottle partition function K . This partition function, or direct channel amplitude, is related by a channel transformation S to the transverse channel \tilde{K} :

$$K = \sum_i K_i \chi_i \xrightarrow{S} \tilde{K} = \sum_i \Gamma_i^2 \chi_i \quad \text{where} \quad K_i = \sum_b S_{bi} \Gamma_b \Gamma_b . \quad (2)$$

The Klein bottle has to satisfy two consistency conditions. The first condition guarantees that the closed string sector $(T + K)/2$ has positive, integral state multiplicities:

$$K_i = \epsilon_i Z_{ii} , \quad (3)$$

where $\epsilon_i = \pm 1$ for $Z_{ii} = 1$ and $\epsilon_i = 0$ for $Z_{ii} = 0$. These signs have to satisfy an additional constraint, namely

$$\epsilon_i \epsilon_j \epsilon_k N_{ij}^k \geq 0 , \quad (4)$$

where N_{ij}^k are the fusion coefficients given by Verlinde’s formula [11]. This condition, which we will refer to as the Klein bottle constraint, guarantees that states that are projected out by the Klein bottle cannot re-appear as intermediate states in tree diagrams.

The open sector is described by the direct annulus channel A and direct Möbius channel M . They are related to the transverse channels \tilde{A} and \tilde{M} by a channel transformation, which is S in case of the annulus and $P = \sqrt{T}ST^2S\sqrt{T}$ for the Möbius strip:

$$A_{\alpha\beta} = \sum_i A_{i\alpha\beta} \chi_i \xrightarrow{S} \tilde{A}_{\alpha\beta} = \sum_b B_{b\alpha} B_{b\beta} \chi_b \quad (5)$$

$$\text{where} \quad A_{i\alpha\beta} = \sum_b S_{bi} B_{b\alpha} B_{b\beta} ,$$

$$M_\alpha = \pm \sum_i M_{i\alpha} \hat{\chi}_i \xrightarrow{P} \tilde{M}_\alpha = \pm \sum_b \Gamma_b B_b \hat{\chi}_b \quad (6)$$

$$\text{where} \quad M_{i\alpha} = \sum_b P_{bi} B_{b\alpha} \Gamma_b ,$$

where the hatted characters are defined by $\hat{\chi}_i = (\sqrt{T_i})^{-1}\chi_i$. The indices α, β label the boundary conditions. To ensure that the open sector, given by $(A + M)/2$ has non-negative, integer state degeneracies, the annulus and Möbius strip coefficients have to satisfy

$$|M_{i\alpha}| \leq A_{i\alpha\alpha} \text{ and } M_{i\alpha} = A_{i\alpha\alpha} \pmod{2} . \quad (7)$$

We will refer to this condition as the positivity and integrality condition. Furthermore, the annulus coefficients have to satisfy the completeness conditions of [5]. This is automatically the case if the reflection coefficients $R_{b\alpha} = B_{b\alpha}\sqrt{S_{b0}}$ satisfy [12]

$$\sum_b R_{b\alpha} R_{b\beta}^* = \delta_{\alpha\beta} ; \quad \sum_\alpha R_{b\alpha} R_{c\alpha}^* = \delta_{bc} . \quad (8)$$

These conditions will not be considered anymore, since they do not involve the crosscap coefficients. That they are satisfied follows from the results of [7], from which we obtain our boundary coefficients.

To summarize: In the construction of open descendants, for a given modular invariant and consistent Klein bottle projection, we have to find the correct annulus and Möbius coefficients. This amounts to finding a set of boundary labels α and coefficients $B_{b\alpha}$ and Γ_b such that equations (3), (4), (7) and (8) are satisfied. Let us present the only model-independent solution that is presently known, in its most general form. It holds for charge conjugation invariants, the so-called C-diagonal [3] case. The relevant coefficients are in this case [10]

$$B_{b\alpha} = \frac{S_{b\alpha}}{\sqrt{S_{bL}}} , \quad \Gamma_b = \frac{P_{bL}}{\sqrt{S_{bL}}} , \quad (9)$$

$$K_i = Y_{iLL^c} , \quad A_{i\alpha\beta} = N_{L^c i, \alpha\beta} , \quad M_{i\alpha} = Y_{L^c \alpha, Li} , \quad (10)$$

where L is a simple current and the tensor Y is defined as

$$Y_{ij}{}^k = \sum_m \frac{S_{mi} P_{mj} P_m^k}{S_{m0}} . \quad (11)$$

A crucial relation, derived in [10] using results from [13], is

$$|Y_{i0}{}^k| \leq N_{ii}^k \text{ and } Y_{i0}{}^k = N_{ii}^k \pmod{2} . \quad (12)$$

In [10] it was shown that this relation implies that all constraints are satisfied.

3 Z_{odd} simple current invariants

First we review some facts about simple currents [2] [14]. An important quantity is the (monodromy) charge of a field i with respect to J

$$Q_J(i) = h_J + h_i - h_{J \times i} \pmod{1} , \quad (13)$$

where the h_i are the conformal weights of i . The order of a simple current is the smallest integer N for which $J^N = 0$. The charge of a simple current with respect to itself is $Q_J(J) = r/N \pmod{1}$ where r is the monodromy parameter. In [2] it is explained how we can construct modular invariant partition functions with simple currents¹. The result for odd N can be summarized as follows: the modular invariant is of pure automorphism type if and only if the monodromy parameter r and the order N do not have common factors.

All primaries in a theory with such a current are organized in orbits of length N . The charges of these fields with respect to J are multiples of r/N , and every charge appears precisely once on a given orbit. We will denote the charge-zero fields by i_0 .

The following torus is modular invariant [2]:

$$T = \sum_{i_0} \sum_{n=1}^N \chi_{[J^n i_0]} \bar{\chi}_{[J^n i_0]} . \quad (14)$$

Note that the charge-zero fields are the transverse fields. The diagonal part of this partition function is

$$Z_{ii} = C_{i_0 i_0} , \quad (15)$$

where C denotes the charge conjugation matrix. So a field i appears diagonally in the torus when the charge-zero field on the orbit it lies on is self-conjugate.

As an ansatz for the boundary and crosscap coefficients we take ²

$$B_{b\alpha} = \sqrt{N} \frac{S_{b\alpha}}{\sqrt{S_{b0}}} , \quad \Gamma_b = \sqrt{N} \frac{P_{b0}}{\sqrt{S_{b0}}} , \quad (16)$$

¹We multiply the modular invariants found in [2] by the charge conjugation matrix C_{ij} . The result is of course a modular invariant as well.

²This is a straightforward generalization of the results of [7]. The classifying algebra is in our case just the charge-zero subalgebra of the fusion rules.

where b labels the transverse fields and the index α the boundary conditions, which are in one-to-one correspondence with the orbits. Note that the boundary coefficient is independent of a representative, since b is chargeless and so $S_{b,J^t\alpha} = S_{b\alpha}$ for any t . It is straightforward to compute the other coefficients

$$K_i = \sum_{n=1}^N Y_{J^n i, 00} \quad , \quad A_{i\alpha\beta} = \sum_{n=1}^N N_{J^n i, \alpha\beta} \quad , \quad M_{i\alpha} = \sum_{n=1}^N Y_{J^n \alpha, 0i} \quad . \quad (17)$$

Note that in all three expressions only one term contributes. For instance the Klein bottle coefficient can be written as $K_i = Y_{i_0 00}$, so it equals the Frobenius-Schur indicator of the orbit to which i belongs. We will postulate that nonzero Frobenius-Schur indicators are conserved in fusion. Possible violations of this postulate and possible consequences will be discussed in section 4.3. If the postulate holds the Klein bottle constraint (4) is satisfied.

With equation (12) it is straightforward to check positivity and integrality of the open sector, equation (7). Furthermore it may be shown that the annulus coefficient respects the completeness conditions. This is strong evidence for the correctness of our ansatz (16).

3.1 Non-standard Klein bottles

In [10] we developed a method for constructing non-trivial Klein bottles with the use of simple currents in the C-diagonal case. Simple current Klein bottles can be constructed for Z_{odd} simple current invariants as well. Let L be an order M simple current with arbitrary spin. Then

$$K_i^{[L]} = e^{2\pi i Q_L(i_0)} Y_{i_0 00} = Y_{i_0 L L^c} \quad (18)$$

is a consistent Klein bottle. Reality of this Klein bottle follows from the observation that $Q_L(i_0)$ is (half-)integer when i_0 is real. When i_0 is complex, a possible imaginary phase factor is killed by the Frobenius-Schur indicator. The Klein bottle constraint (4) follows from the conservation of Q_L in fusion. The other coefficients are

$$B_{b\alpha}^{[L]} = \sqrt{N} \frac{S_{b\alpha}}{\sqrt{S_{bL}}} \quad , \quad \Gamma_b^{[L]} = \sqrt{N} \frac{P_{bL}}{\sqrt{S_{bL}}} \quad , \quad (19)$$

$$A_{i\alpha\beta}^{[L]} = \sum_{n=1}^N N_{[J^n L^c i], \alpha, \beta} \quad , \quad M_{i\alpha}^{[L]} = \sum_{n=1}^N Y_{[J^n L^c \alpha], L, i} \quad . \quad (20)$$

It is straightforward to show that the various consistency conditions are satisfied.

4 Z_2 simple current invariants

In this section we discuss the open descendants of modular invariants built from order 2 simple currents. Only when the spin of the current is half-integer is the resulting invariant a pure automorphism [2]. There are three kinds of orbits: length 2 orbits that contain charge-zero fields i_0 , length 2 orbits that contain charge-one-half fields i_1 and length 1 orbits (fixed points) that contain charge-one-half fields f . Let J denote such a half-integer spin, order 2 simple current. The following torus is modular invariant [2]:

$$T = \sum_{i_0} \chi_{i_0} \bar{\chi}_{i_0^c} + \sum_{i_1} \chi_{i_1} \bar{\chi}_{(Ji_1)^c} + \sum_f \chi_f \bar{\chi}_{f^c} . \quad (21)$$

The authors of [7] were able to find the boundary coefficients:

$$B_{b\alpha} = \sqrt{2} \left(\frac{S_{b\alpha}}{\sqrt{S_{b0}}} \right) \quad B_{b,f^\pm} = \frac{1}{\sqrt{2}} \left(\frac{S_{bf}}{\sqrt{S_{b0}}} \right) \quad \text{for } Q(b) = 0 , \quad (22)$$

$$B_{b\alpha} = 0 \quad B_{b,f^\pm} = \pm \frac{1}{\sqrt{2}} \left(\frac{\check{S}_{bf}}{\sqrt{S_{b0}}} \right) \quad \text{for } Jb = b . \quad (23)$$

where, in the case of WZW-models, \check{S} is the S -matrix of the fixed point conformal field theory (FCFT) [2], or, more generally, the orbit Lie-algebra [17]. In section 4.2 we will comment on its definition in general conformal field theories. Note that there is one boundary coefficient α for every length 2 orbit and two boundary coefficients f^\pm for every fixed point. As an ansatz for the crosscap coefficient we take

$$\Gamma_b = \frac{1}{\sqrt{2}} \frac{[P_{b0} + \epsilon_{J,m} P_{bJ}]}{\sqrt{S_{b0}}} , \quad (24)$$

where $\epsilon_{J,m} = e^{\pi i [h_J + \frac{m+2}{2}]} = \pm 1$, whose appearance will become clear later. The odd integer m is defined in section 4.2. The Klein bottle becomes

$$K_i = \frac{1}{2} [Y_{i00} + Y_{iJJ} + 2\epsilon_{J,m} Y_{i0J}] . \quad (25)$$

Using relations from the appendix of [10] one can show that for chargeless fields the Klein bottle is Y_{i00} and for charged fields (including fixed points) it is $\epsilon_{J,m}Y_{i0J}$. There are cases (although rare) where (25) violates the Klein bottle constraint (4). In section 4.3 we will comment on this issue and propose a slightly weaker version of the Klein bottle condition that is satisfied by all models. The annulus [7] and Möbius coefficients become

$$A_{i\alpha\beta} = N_{i\alpha\beta} + N_{Ji,\alpha\beta} \quad , \quad A_{if\pm g\pm} = \frac{1}{2}[N_{ifg} + \check{N}_{ifg}] \quad , \quad (26)$$

$$A_{i\alpha f\pm} = N_{i\alpha f} \quad , \quad A_{if\pm g\mp} = \frac{1}{2}[N_{ifg} - \check{N}_{ifg}] \quad , \quad (27)$$

$$M_{i\alpha} = Y_{\alpha 0i} + Y_{\alpha Ji} \quad , \quad M_{if\pm} = \frac{1}{2}[Y_{f0i} + \epsilon_{J,m}Y_{fJi}] \quad , \quad (28)$$

where \check{N}_{ifg} is defined in equation (59). The annulus coefficients satisfy [7] the completeness conditions of [5]. The positivity and integrality condition for boundary coefficients that are not fixed points follows from equation (12). For boundary coefficients that are fixed points, we have to prove

$$\frac{1}{2}|Y_{f0i} + \epsilon_{J,m}Y_{fJi}| \leq \frac{1}{2}[N_{iff} + \check{N}_{iff}] \quad , \quad (29)$$

$$\frac{1}{2}[Y_{f0i} + \epsilon_{J,m}Y_{fJi}] = \frac{1}{2}[N_{iff} + \check{N}_{iff}] \pmod{2} \quad . \quad (30)$$

Due to the appearance of the \check{N} , equation (12) is not applicable in this case. However, note that for $i = (i_1, f)$ the above relations are easily satisfied, since both sides vanish by charge conservation. In appendix A we prove the above positivity and integrality condition for $i = i_0$. Note that without the sign $\epsilon_{J,m}$ in equation (24), the positivity and integrality condition will be violated for theories with $\epsilon_{J,m} = -1$.

4.1 A non-standard Klein bottle

In this section we include a non-standard Klein bottle to our solution. Let L denote an arbitrary simple current. Consider

$$K_i^{[L]} = e^{2\pi i Q_L(i)} K_i \quad . \quad (31)$$

Note that not every L is allowed. We have to require that $2Q_L(i) = 0 \pmod{1}$ when $K_i \neq 0$, otherwise the Klein bottle coefficient is not real. Recall [2] that

the charge of a field with respect to any simple current is (half-)integer when the field is self-conjugate. So the Klein bottle coefficient of a charge-zero field i_0 and a fixed point f is a sign for any L , since these fields only appear in the direct Klein bottle when they are self-conjugate. From the torus (21), we see that charge-one-half fields i_1 propagate in the direct Klein bottle when they satisfy $i_1 = Ji_1^c$, which implies, by charge conservation, $Q_L(J) = 2Q_L(i_1)$. We conclude that equation (31) is a consistent Klein bottle if $Q_L(J) = 0 \pmod{1}$. Note that there is at least one L that satisfies this requirement, namely $L = J$. Let us focus on this case. Then equation (31) becomes

$$K_i^{[J]} = \frac{1}{2}[Y_{i00} + Y_{iJJ} - 2\epsilon_{J,m}Y_{i0J}] \quad . \quad (32)$$

The crosscap coefficient is

$$\Gamma_b^{[J]} = \frac{1}{\sqrt{2}} \frac{[P_{b0} - \epsilon_{J,m}P_{bJ}]}{\sqrt{S_{b0}}} \quad . \quad (33)$$

As an ansatz for the boundary coefficients, we take equations (22) and (23) and replace $\sqrt{S_{b0}}$ by $\sqrt{S_{bJ}}$ as in [10]. The annulus and Möbius coefficients that change relative to the standard Klein bottle projection are

$$A_{if^\pm g^\pm}^{[J]} = \frac{1}{2}[N_{ifg} - \check{N}_{ifg}] \quad , \quad A_{if^\pm g^\mp}^{[J]} = \frac{1}{2}[N_{ifg} + \check{N}_{ifg}] \quad , \quad (34)$$

$$M_{if^\pm}^{[J]} = \frac{1}{2}[Y_{f0i} - \epsilon_{J,m}Y_{fJi}] \quad . \quad (35)$$

Miraculously, the positivity and integrality condition in the open sector for fixed point boundary indices now becomes

$$\frac{1}{2}|Y_{f0i} - \epsilon_{J,m}Y_{fJi}| \leq \frac{1}{2}[N_{iff} - \check{N}_{iff}] \quad , \quad (36)$$

$$\frac{1}{2}[Y_{f0i} - \epsilon_{J,m}Y_{fJi}] = \frac{1}{2}[N_{iff} - \check{N}_{iff}] \pmod{2} \quad , \quad (37)$$

which we prove in appendix A.

4.2 The matrix \check{S}

In WZW-models the matrix \check{S} is the modular transformation matrix of the orbit Lie algebra, and it differs by a known phase from the fixed point resolution matrix S^J . The fixed point resolution matrices are explicitly known

for WZW-models [15] and extended WZW models [16]. The generalization of \check{S} beyond WZW-models is not straightforward, since orbit Lie algebras are related to foldings of Dynking diagrams, a concept that has no obvious CFT generalization. One could try to define \check{S} as the transformation matrix of the twining characters (as defined in [17]), but their definition does not straightforwardly generalize either. Even if such a generalization is possible, we do not know if the twining characters are always well-behaved under modular transformations, nor the relation between \check{S} and S^J . We will postulate here that \check{S} exists and that it is related to S^J in a similar way as in WZW-models. Thus we take as the definition of \check{S}

$$\check{S} = e^{\delta\pi i \frac{m}{24}} S^J . \quad (38)$$

For (half)-integer spin currents of WZW-models the number m (which is defined modulo 24 and is related to a phase-shift in the matrix T) is known, and it is an even (odd) integer for integer (half-integer) spin currents. This is what we will assume in general. Note that for half-integer spin currents there are twelve possible values for m , resulting in four possible phases in the definition of \check{S} . Without further information \check{S} is then known up to a factor ± 1 or $\pm i$. In our case a sign change is irrelevant, since this only determines the choice between f^+ and f^- . A change by a factor i changes the sign of $\epsilon_{J,m}$ and \check{N} . These sign changes can be flipped by choosing the opposite Klein bottle projection, as explained above, so that finally there is no genuine ambiguity left. This definition of \check{S} appears to be adequate in all cases with the possible exception of the Klein bottle constraint violations mentioned in the next subsection.

4.3 The Klein bottle constraint

In this section we discuss the Klein bottle constraint for open descendants of order 2 automorphism invariants. This condition reads

$$N_{ij}{}^k K_i K_j K_k \geq 0 \quad , \quad (39)$$

where the Klein bottle for the various fields is

$$K_{i_0} = Y_{i_0 00} \quad , \quad K_{i_1} = \epsilon_{m,J} Y_{i_1 0J} \quad , \quad K_f = \epsilon_{m,J} Y_{f 0J} = \eta_f Y_{f 00} \quad , \quad (40)$$

where in the last step (63) was used.

Since the charges with respect to any simple current are conserved (mod 1) in fusion, there are four sectors where the fusion coefficients are non-zero. We will discuss the Klein bottle constraint in these sectors one by one.

- **The coupling between three charge-zero fields.**

Since the Klein bottle in this sector is just the Frobenius-Schur indicator (in the original theory), the Klein bottle constraint is satisfied trivially.

- **The coupling between a charge-zero field and two fixed points.**

From equation (40) we find that the Klein bottle coefficient for a fixed point is equal to its Frobenius-Schur indicator times a sign, η_f . If η_f does not depend on f the Klein bottle constraint is satisfied (assuming it holds in the C-diagonal case). In case when two self-conjugate fixed points f and g with $\eta_f \neq \eta_g$ have a nonzero coupling with a charge-zero field i_0 , the Klein bottle constraint is violated! We will comment on this below.

For the remaining two cases we make use of the extended tensor product method of appendix A. We tensor the theory under consideration with a second one, also with a half-integer spin current, and extend the chiral algebra by the product of the half-integer spin currents of the two theories. It is convenient to tensor with a series of theories, namely $SO(2N+1)$ level 1. In these theories $Y_{f\bar{0}J} = (-1)^N Y_{f\bar{0}0} \equiv \epsilon_N$, which is a sign. Then we derive from equations (52), (53) and (54)

$$Y_{i_e 0_e 0_e} = Y_{i_0 00} \tag{41}$$

$$Y_{j_e 0_e 0_e} = \epsilon_N [Y_{j_1 00} + (-1)^N \epsilon_J Y_{j_1 0J}] \tag{42}$$

$$Y_{f_{e,\mu} 0_e 0_e} = \frac{1}{2} \epsilon_N [Y_{f00} + (-1)^N \epsilon_J Y_{f0J}] = \frac{1}{2} \epsilon_N K_f [\eta_f + \epsilon_J (-1)^N \epsilon_{J,m}] \tag{43}$$

where $\epsilon_J = e^{\pi i [h_J + \frac{1}{2}]}$. Furthermore we have

$$N_{i_e j_e}^{k_e} = N_{i_0 j_1}^{k_1} + N_{J i_0, j_1}^{k_1} \quad , \quad N_{i_e j_e}^{f_{e,\mu}} = N_{i_0 j_1}^f \tag{44}$$

Now we use the Klein bottle constraint in the extended tensor product theory to deal with the last two cases.

- **The coupling between a charge-zero field and two charge-one-half fields.**

We wish to prove

$$N_{i_0 j_1}^{k_1} K_{i_0} K_{j_1} K_{k_1} \geq 0 \quad . \quad (45)$$

From the torus (21), we know that a charge-one-half field j_1 propagates in the direct Klein bottle when $j_1^c = Jj_1$. So these fields must be complex, *i.e.*, have a vanishing Frobenius-Schur indicator $Y_{j_1 00}$. From equations (42) and (40), we see that the Klein bottle coefficient for these fields becomes $K_{j_1} = (-1)^N \epsilon_{J,m} \epsilon_N \epsilon_J Y_{j_e 0_e 0_e}$. However, the signs are irrelevant, since they do not depend on j_1 and cancel between the two charge-one-half fields. Then the Klein bottle constraint follows from the (proposed) conservation of Frobenius-Schur indicators in the extended tensor theory

$$N_{i_e j_e}^{k_e} Y_{i_e 0_e 0_e} Y_{j_e 0_e 0_e} Y_{k_e 0_e 0_e} \geq 0 \quad (46)$$

and the fact that $N_{i_e j_e}^{k_e} \geq N_{i_0 j_1}^{k_1}$, as a consequence of equation (44).

- **The coupling between a charge-zero field, a charge-one-half field and a fixed point.**

We can always choose N in (43) such that the two terms do not cancel. For that choice of N we have then $K_f = (-1)^N \epsilon_{J,m} \epsilon_N \epsilon_J Y_{f_e, \mu 0_e 0_e}$, exactly the relation we found in the previous case. Then the fixed point behaves like an ordinary charge-one-half field, and the rest of the argument is completely analogous to the previous case.

4.4 Violations of the Klein bottle constraint

The violation of the Klein bottle constraint noted above may be resolved in several ways. First of all it is possible that either such models must be rejected, or that there is something wrong with the general formalism developed here. In the examples where the violation occurs we always have to use the conjectured relation (38) to compute \check{S} . This may cast some suspicion on this conjecture. However, there is a more interesting possibility. The violation is due to different signs η_f in the relation $\check{S}_{fg}^2 = \eta_f C_{fg}$. This splits the set of fixed points into two sets, one with $\eta_f = 1$ and one with

$\eta_f = -1$. Conjugation closes on each set. It is easy to show that \check{S}_{fg} must have vanishing matrix elements between fields of different sets. This then implies that $\check{N}_{ifg} = 0$. This immediately implies that N_{ifg} must be even, since the average of both coefficients must be an integer [7]. This may point towards a possible solution of the puzzle.

It is instructive to consider the postulate of section 3 regarding conservation of the Frobenius-Schur indicator in fusion. The FS-indicator is the generalization to CFT of the notion of real (R, FS=1), pseudo-real (P, FS=-1) or complex representations (C, FS=0) [13]. This property is preserved in simple Lie-algebra tensor products: schematically $R \times R = R$, $P \times P = R$ and $P \times R = P$, where on the left-hand side we consider two irreducible representations and on the right-hand side we have a direct sum of irreducible representations, which may consist out of complex conjugate pairs. Strictly speaking the latter already violate FS-conservation, but since their indicator is 0 such a violation does not affect the Klein bottle constraint. By analogy with FS-indicator of simple Lie-algebra representations, we expect the FS-indicator to be preserved in fusion unless the fusion produces a pair of complex conjugate representations (which together can form a real or pseudo-real representation) or an even number of real or pseudo-real representations. The latter possibilities are logically possible (for example for two pseudo-real representations one can choose a real basis), but they never occur in WZW-fusion or simple Lie-algebra tensor products because there always exists a conserved simple current charge which is 0 on real representations, 1/2 on pseudo-real ones and anything on complex representations. The conservation of this charge then enforces conservation of the FS-indicator in the sense described above. However, in general CFT's we may expect violations of the conservation rule, leading to a violation of the Klein bottle constraint (4) by negative, *even* integers. We do not know of any CFT where such a violation occurs for the FS-indicators (*i.e.* for the Klein bottles of the C-diagonal models), but remarkably we find precisely such a violation for the Klein bottles of certain off-diagonal models.

This leads one to suspect that perhaps violations by even negative integers are allowed. Consider for example two fields with Klein bottle coefficients $K_1 = K_2 = 1$ and a third with $K_3 = -1$. The existence of a closed string coupling of the first two fields to the third implies a coupling of two symmetrically projected unoriented string fields to the symmetric projection of the third one. But if $K_3 = -1$ the symmetric projection is an unphysical state,

since the Klein bottle produces an anti-symmetric projection. However, if the coupling has multiplicity two (or even) it is possible for fields 1 and 2 to couple to a symmetric combination of two anti-symmetrically projected (hence physical) states 3. This is precisely the reason why a violation of FS-conservation was argued to be possible in principle, and is presumably how these Klein bottle constraint violations should be interpreted as well. If this interpretation is correct there is no inconsistency, and the formulation of the Klein bottle constraint must be weakened as explained above.

5 Conclusions and outlook

In this paper we have presented natural and general candidates for crosscap coefficients belonging to theories with a non-trivial automorphism invariants in the bulk. Although we do not pretend to have proved that the resulting open string theories are fully consistent, at least we have demonstrated that our solution passes a number of non-trivial checks. In rare cases there may be a problem with the Klein bottle constraint, but we have argued that the fact that the violation is always by even integers may provide a way out of the problem, and that perhaps a slightly weaker form of the constraint might be acceptable.

The crosscap coefficients are an essential ingredient in the tadpole cancellation conditions, which can now be studied for open descendants of a much larger class of bulk theories.

Nevertheless there are still several classes of automorphism invariants for which the crosscap coefficients remain unknown. Obvious extensions of our results can be made to automorphisms generated by Z_{2^n} simple currents and to non-cyclic simple current groups. One may in fact hope for one general formula covering all cases. We expect to address this question in the near future. Another interesting direction is the study of diagonal invariants (*i.e.* a charge conjugation invariant of a C-diagonal theory) and exceptional invariants such as the “ E_7 ” invariant of $SU(2)$ level 16. Furthermore the entire formalism should be extended to the case of boundaries that break bulk symmetries.

A Inequalities from tensoring

In this appendix we obtain some useful relations among the quantities N, \check{N} and Y , which are needed for proving integrality, positivity and the Klein bottle constraint.

The strategy is as follows. We will show that the Klein bottle coefficient of section 4 can be related to the Frobenius-Schur indicator of a different theory. The conservation of this indicator will translate into the Klein bottle constraint. Similarly, equation (12), which holds in this theory, turns out to prove positivity and integrality in the open sector.

More explicitly, we tensor the theory of interest with another theory that contains a half-integer spin simple current. We then extend the chiral algebra of this tensor theory with the products of the half-integer spin simple currents. We label the fields in the tensor product by $i_t = (i, \bar{i})$ and the fields in the extended tensor product by i_e . In particular, $J_t = (J, \bar{J})$ is a simple current and when f is a fixed point of J and \bar{f} a fixed point of \bar{J} , $f_t = (f, \bar{f})$ is a fixed point of J_t .

The matrices S, T and P and the coefficients N_{ijk} and Y_{ijk} of the tensor theory are related to those of its components in a straightforward way. To deal with the extension we make use of the results of [15] to resolve the fixed points of J_t . This then gives us the matrix S of the extended theory in terms of the matrices S and S^J (the fixed point resolution matrix) of the components. We then compute P, N and Y .

In the extended theory only fields $i_t = (i, \bar{i})$ that are chargeless under J_t exist, so $Q_J(i) = Q_{\bar{J}}(\bar{i})$. Furthermore, these fields are grouped by the action of the simple current J_t . For every fixed point in the tensor theory there are two fields in the extension [2], which we label by $\mu, \nu = (1, 2)$.

The matrices S, T and P of this extension are related to those of the unextended theory, which is the tensor theory. For the matrix S we have for instance

$$S_{i_e j_e} = 2S_{i_t j_t} \quad , \quad S_{i_e f_e, \mu} = S_{i_t f_t} \quad . \quad (47)$$

When the indices are two fixed points the result is

$$S_{f_e, \mu g_e, \nu} = S_{f_t g_t} E_{\mu\nu} + S_{f_t g_t}^J F_{\mu\nu} \quad , \quad (48)$$

where the matrices E and F are given by

$$E = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad , \quad F = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad . \quad (49)$$

The matrix P in the extension, is given by

$$P_{i_e j_e} = 2\bar{P}_{i_t j_t} \quad , \quad P_{f_e, \mu j_e} = \bar{P}_{f_t j_t} \quad , \quad P_{f_e, \mu g_{e,v}} = 0 \quad . \quad (50)$$

where the ‘‘orbit averaged P -matrix’’ is ($x = i_t, f_t$)

$$\bar{P}_{x j_t} = \frac{1}{2} \left[P_{x j_t} + \sqrt{\frac{T_{j_t j_t}}{T_{J_{j_t}, J_{j_t}}}} P_{x J_{j_t}} \right] \quad . \quad (51)$$

We can now relate the tensor Y of the extended theory to the tensors Y of the components of the tensor product. We will be particularly interested in the Frobenius-Schur indicators of $i_e = (i_0, \bar{0}) + (J i_0, \bar{J})$, $j_e = (j_1, \bar{f}) + (J j_1, \bar{f})$ and $f_e = (f, \bar{f})$. The subscripts 0 and 1 were introduced in section 4, and refer to the charges of the fields. A straightforward calculation shows

$$Y_{i_e 0_e 0_e} = Y_{i_0 0_0} Y_{\bar{0} \bar{0}} + \epsilon_{J, \bar{J}} Y_{i_0 0_J} Y_{\bar{0} \bar{0} \bar{J}} = Y_{i_0 0_0} \quad , \quad (52)$$

$$Y_{j_e 0_e 0_e} = Y_{j_1 0_0} Y_{\bar{f} \bar{0}} + \epsilon_{J, \bar{J}} Y_{j_1 0_J} Y_{\bar{f} \bar{0} \bar{J}} \quad , \quad (53)$$

$$Y_{f_e, \mu 0_e 0_e} = \frac{1}{2} [Y_{f 0_0} Y_{\bar{f} \bar{0}} + \epsilon_{J, \bar{J}} Y_{f 0_J} Y_{\bar{f} \bar{0} \bar{J}}] \quad , \quad (54)$$

where $\epsilon_{J, \bar{J}} = e^{\pi i [h_J + h_{\bar{J}}]} = \pm 1$. In the first line we used $|Y_{\bar{0} \bar{0} \bar{J}}| \leq N_{\bar{0} \bar{0}}^{\bar{J}} = 0$ (see equation (12)). The fusion coefficients in the extended theory can also be related to those of the components of the tensor theory. We have for example:

$$N_{i_e j_e}^{k_e} = N_{i_0 j_1}^{k_1} N_{\bar{0} \bar{f}}^{\bar{f}} + N_{J i_0 j_1}^{k_1} N_{\bar{J} \bar{f}}^{\bar{f}} \quad , \quad (55)$$

$$N_{i_e j_e}^{f_e, \mu} = N_{i_0 j_1}^f N_{\bar{0} \bar{f}}^{\bar{f}} \quad , \quad (56)$$

where $k_e = (k_1, \bar{f}) + (J k_1, \bar{f})$.

One can also show that

$$Y_{f_e, \mu 0_e i_e} = \frac{1}{2} [Y_{f 0_{i_0}} Y_{\bar{f} \bar{0}} + \epsilon_{J, \bar{J}} Y_{f J_{i_0}} Y_{\bar{f} \bar{0} \bar{J}}] \quad (57)$$

and

$$N_{iefe,\mu ge,\mu} = \frac{1}{2}[N_{iofg}N_{\bar{0}\bar{f}\bar{g}} + e^{-\frac{1}{2}\pi i(m+\bar{m})}\check{N}_{iofg}\check{N}_{\bar{0}\bar{f}\bar{g}}] \quad , \quad (58)$$

where

$$\check{N}_{iofg} = \sum_{h,Jh=h} \frac{S_{hi}\check{S}_{hf}\check{S}_{hg}}{S_{h0}} \quad , \quad \check{N}_{ofg} = \eta_f C_{fg} \quad , \quad (59)$$

where η_f is a sign. The phase factor in (58) is due to the fact that we have expressed the fixed point resolution matrices S^J (which appear naturally in fixed point resolutions for integer spin currents) in terms of \check{S} using (38). In the tensor theory m_t is an even integer, which decomposes as $m_t = m + \bar{m}$, where m and \bar{m} are odd.

In WZW-models this sign is in fact independent of f , but in extended WZW-models examples are known where it is not [16].³ In the extended tensor theory, equation (12) must hold. So for instance

$$|Y_{fe,\mu 0eie}| \leq N_{iefe,\mu fe,\mu} \quad . \quad (60)$$

We now make a specific choice for the “bar-theory”, namely the Ising model with fields $\bar{0} = 0$, $\bar{f} = \sigma$ and $\bar{J} = \psi$. Equation (60) then becomes

$$\frac{1}{2}|Y_{f0i0} \pm \epsilon_{J,m} Y_{fJi0}| \leq \frac{1}{2}[N_{ioff} \pm \check{N}_{ioff}] \quad , \quad (61)$$

where $\epsilon_{J,m}$ is defined in the main text. Note that the above equations are the positivity conditions in the open sector where the boundary labels are fixed points for the two Klein bottle projections of section 4. Integrality can be proved in a similar way. It is not hard to convince oneself that the above bound is independent of the chosen “bar-theory”. Another consequence of this inequality can be seen by taking $i = 0$. With the use of equation (59) we get

$$\frac{1}{2}|Y_{f00} \pm \epsilon_{J,m} Y_{fJ0}| \leq \frac{1}{2}C_{ff} \pm \eta_f C_{ff} \quad . \quad (62)$$

Note that this implies

$$Y_{f00} = \eta_f \epsilon_{m,J} Y_{f0J} \quad . \quad (63)$$

³Note that η is defined here in terms of the square of \check{S} , not the square of S^J as in [15]. If (38) holds, $\eta_f = \pm 1$.

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