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Ruler, Compass, and Nail: Constructing a Total Spherical Perspective

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We obtain a construction of a total spherical perspective with ruler, compass, and nail. This is a generalization of the spherical perspective of Barre and Flocon to a 360-degree field of view. Since the 1960s, several generalizations of this perspective have been proposed, but they were either works of a computational nature, inadequate for drawing with simple instruments, or lacked a general method for solving all vanishing points. We establish a general setup for anamorphosis and central perspective, define the total spherical perspective within this framework, study its topology, and show how to solve it with simple instruments. We consider its uses both in freehand drawing and in computer visualization, and its relation with the problem of reflection on a sphere.

Keywords: visual arts; geometry; anamorphosis; spherical perspective; ruler and compass

AMS Subject Classification: 51N05 ; 68U05

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1. Introduction

The main purpose of this work is to offer a method the artist can use for drawing total spherical perspectives like those of Figures 1, 11 and 14, by simple ruler and compass constructions.

A well-known 1968 work by Barre and Flocon [4] describes a method for drawing a spherical perspective with ruler and compass. Spherical perspective is however a misnomer, as the method is limited to a hemisphere, obtaining a 180 degree view around an axis.

Later authors attempted to generalize this work to a 360 degree view. In a 1983 paper in *Leonardo* [5], Casas attempted the construction of a flat-sphere perspective, but the lack of a bijection at the *blowup* led him to believe that a mathematically well-defined flattening did not exist. Not only does a flattening exist, there is an infinite number of them (two are described in the present work), and one must be chosen to specify a perspective. Not having done this, Casas was limited to a discussion of qualitative properties. In [12] Moose implemented the program of Casas through an ad-hoc gridding scheme that specified an actual perspective, though not one that generalizes [4].

There is in these latter works a misconception of the problem. Barre and Flocon's contribution was not to create a flattening of the hemisphere. They chose one well known to cartographers as acknowledged in [3]: the azimuthal equidistant projection, known to the French as the Postel projection. Their contribution was to fit it into the framework of a perspective (often called the *fisheye* perspective), providing a classification of all lines and vanishing points and a way to plot them by elementary means. It is these methods that must be extended to generalize their work.

This is not solved by the methods that today make spherical perspectives common in computers, digital cameras and smartphones. Pixel-by-pixel rendering allows for curvilinear perspectives with wide angles of view or even with ajustable projection surfaces [6, 8], but circumvents rather than solves the original problem for either geometer or artist. For the geometer, the point of perspective is to tame infinity through compactification - by contrast, the computer sweeps infinity under the rug, by sampling the scene a few million times while ignoring completely line classifications or vanishing points. For the artist wishing to draw with his own hand and eye, these algorithms are useless. One does not wish to draw with calculator in hand, but to extend the ruler and compass constructions to the full sphere.

As I argued in [1], it is unlikely that Barre and Flocon ignored that their flattening could be extended to 360 degrees, after their work with Bouligand [3] in which the cartographic options were surveyed. My impression is that they stopped at 180 degrees for two reasons: their stated purpose of keeping linear deformations within reasonable bounds, and the difficulty of plotting line projections beyond the equator, where they stop being well approximated by arcs of circles. We shall see below how lines can be projected in a simple way beyond the 180 degree mark.

Some of the results in the present paper were anticipated implicitly in the artistic work of the Belgian architect Gérard Michel. While circulating the first draft of the current paper I was contacted by Gérard, a fellow urban sketcher, who showed me several works in spherical perspective he had done in the 1970s, drawing from urban scenes. These drawings, along with brief hints on the artist's process, where published only recently [11]. Although the paper does not describe the artist's method in detail, the drawings and grids make clear that some of the results here presented were known in some form by Michel (though not all - for instance, his grids clearly make use of Proposition 2, but not Proposition 4). The present paper can be seen as a systematization and extension of these methods. As far as I know the present paper is the first systematic presentation of the total spherical perspective that clearly formalizes and solves it; that is, provides a classification of all lines and vanishing points, and a complete method to find and draw them with simple instruments, from either orthographic plans or from nature.

This paper is structured as follows: In section 2 a general definition of central perspective is set up, based on the notion of anamorphosis. I propose that this is the proper way of establishing a perspective as a well defined mathematical object in a way that solves the ambiguities regarding the blowup of the flattening map. This section caters to the mathematician and may be at first glossed over by the artist who just wants to learn how to draw in spherical perspective. In section 3 our spherical perspective is established within the framework of the previous section, and in section 4 it is finally shown how to plot it by hand - this is the section of most interest to the practical artist, in which our ruler and compass method is thoroughly explained. Then some basic examples of constructions are presented. We end in section 6 by comparing spherical perspective with reflection on a sphere.

2. Perspectives

Perspectives are representations of spatial scenes on a plane, with relation to an observer, that preserve some aspects of the observer's visual sensation. Because visual occlusion is radial, most perspectives used by artists (linear, cylindrical, spherical) are central perspectives. Ahead we will define a central perspective as a composition of two maps: an anamorphosis and a flattening.

In dictionaries and perspective manuals the term anamorphosis describes an *inverse* problem

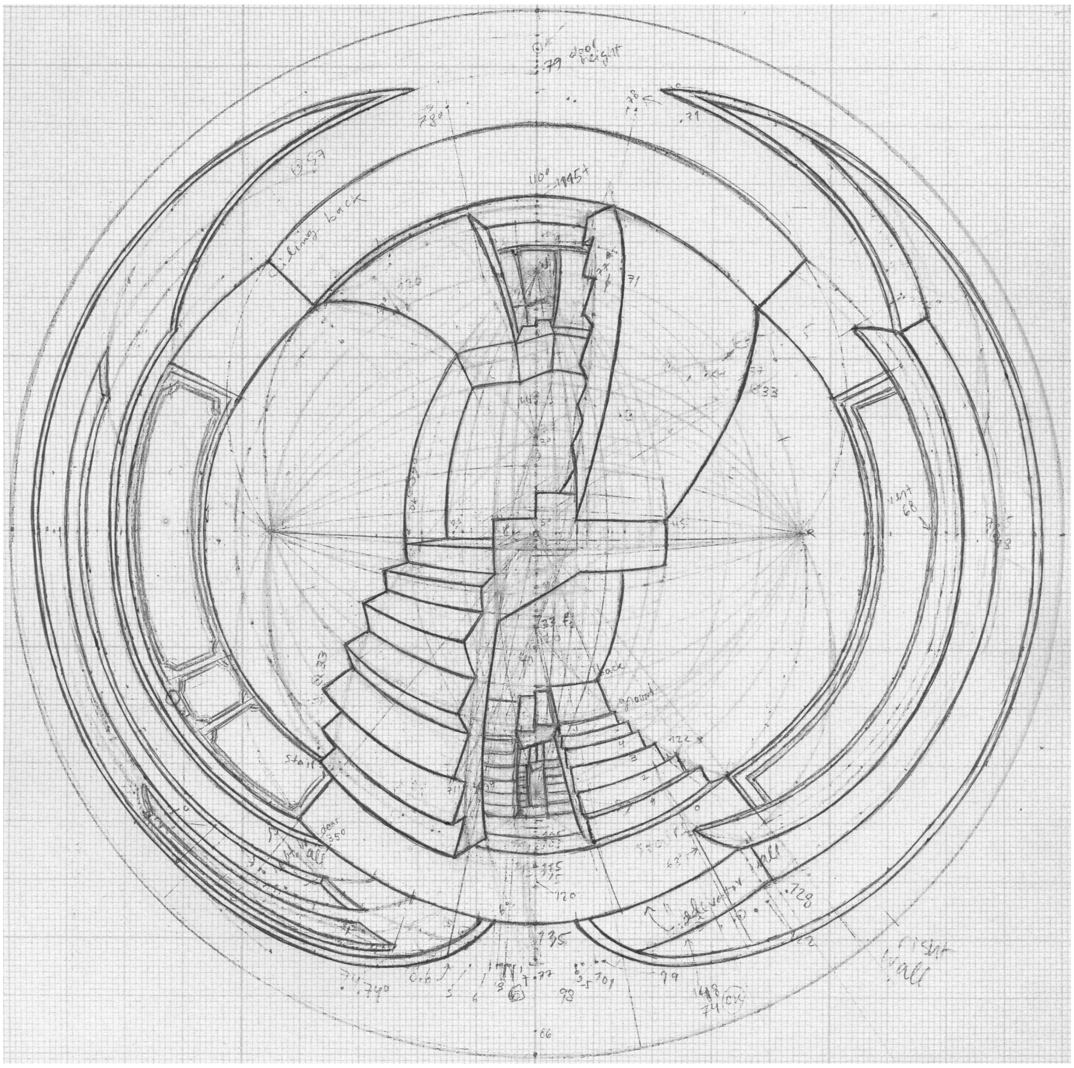

Figure 1. A 360 degree spherical perspective of a stairwell, drawn from observation. Stairs go to vanishing points at $\pm 33°$. Construction lines left in to give an idea of the steps involved. Drawing by the author.

that relates to its etymology ("to form again"): the game of finding the correct point to observe a picture. But it is more enlightening from a didactic and conceptual viewpoint (and also most in accordance with its role in the history of perspective) to define anamorphosis as a *direct* geometric construction that sets the foundation for building a perspective [2]. Vanishing points will be defined at the level of anamorphosis, even before one settles on a specific perspective.

2.1 *Anamorphosis, Topology and Vanishing points*

We shall speak of an *observer* to mean a point O in three-dimensional Euclidean space. We shall speak of a *scene* to mean a closed set in that space.

A fundamental fact about vision is that, with few and notable exceptions, occlusion is radial, i.e., points along the same ray from the viewer are seen as equivalent. Hence the draughtsman, like the astronomer, deals with rays rather than points and angles rather than lengths. This allows for a piece of trompe l'oeil to be created by the process of conic anamorphosis: a two-dimensional picture on a surface S that creates, for an observer at O, the visual illusion of a spatial scene Σ.

Let $\mathcal{R}_O$ be the set of rays from O. Let S^2_O be the unit sphere centered at O. The isomorphism $P \mapsto \overrightarrow{OP}$ endows $\mathcal{R}_O$ with the topology of the sphere. Hence we can speak of the topological closure of a set of rays from O. Let $cl(X)$ denote the closure of a set X.

A scene Σ defines a cone of rays from O, $C_O(\Sigma) = \{\overrightarrow{OP} : P \in \Sigma\}$, which we call the *cone of sight* of Σ from O. We say that a surface S is *central* relative to a point O if any ray from O intersects S at most once. We say that S is an anamorphic surface relative to O if it is a compact central surface relative to O.

Definition 1. *Let S be an anamorphic surface for O and Σ a scene. We say that $C_{O,S}(\Sigma) = cl(C_O(\Sigma) \cap S)$ is the anamorphosis of Σ on S relative to O. Let $\Lambda : \mathrm{R}^3 \setminus \{O\} \to S$ be the map $P \mapsto \overrightarrow{OP} \cap S$. We call Λ the anamorphism (or conic projection) onto S relative to O. We use the same name for the corresponding map $\Lambda : \mathcal{R}_O \to S$.*

From the point of view of the topologist, the purpose of perspective is the compactification of a spatial scene. A spatial line is closed but not bounded. Its conic projection onto a compact surface will be bounded but generally not closed. To make it closed, hence compact, we must add to it its *vanishing points*. We will define the vanishing points of a scene in an intrinsic way that does not depend on the specific perspective under consideration but only on the point O.

Definition 2. *We say that $\mathcal{V}_O(\Sigma) = cl(C_O(\Sigma)) \setminus C_O(\Sigma)$ is the set of* vanishing points *of scene Σ relative to O. We say that $\mathcal{V}_O(\Sigma) \cap S$ is the set of vanishing points of Σ in the anamorphosis $C_{O,S}(\Sigma)$.*

Hence, the anamorphosis of Σ onto S is the union of $\Lambda(\Sigma)$, the strict conic projection onto S, with its vanishing points. The following is easy to show:

Proposition 1. *Let r be a line and r_O its translation to O. Then the set of vanishing points of r in S is $r_O \cap S$. Analogously, let H be a plane and H_O its translation to O. Then the vanishing set of H in S (called its vanishing line) is $H_O \cap S$. Hence the anamorphosis of a line AB onto S is a subset of the vanishing line of the plane AOB.*

2.2 *Anamorphosis onto a sphere*

The sphere is the manifold that most naturally expresses visual data, due to the isomorphism between its points and the rays of sight. Anamorphosis onto a sphere is therefore the simplest and most symmetric: all lines project equally up to rotation and have exactly two vanishing points. All other anamorphoses in artistic practice (plane, cylinder, hemisphere) result in less elegant descriptions of vanishing points and lines. Beautiful examples of sphere anamorphoses can be seen in the work of Dick Termes [13]. His “termespheres”, though designed to be seen from the outside, are sphere anamorphoses with regard to their center, and would be visual simulacra of their spatial scenes if observed from there. To better understand these anamorphoses (and later our perspective) let’s recall some generalities about circles on spheres:

A *great circle* is a circle of maximum radius on a sphere, defined by the intersection of the sphere with a plane through its center O.

Given a point P on the sphere we call *antipode point* of P the diametrically opposite point on the sphere, and we denote it by $P^\star$.

Two non-antipodal points P and Q on the sphere define a unique great circle, the intersection of the sphere with the plane POQ. We call this the PQ great circle.

Each point P on the sphere defines a family of great circles that covers the sphere, all crossing both P and its antipode $P^\star$. We call these circles P-great circles or $PP^\star$-great circles and call P and $P^\star$ the poles of the family. A *meridian* is one connected half of a great circle. We call P-meridian or $PP^\star$-meridian a meridian whose endpoints are P and $P^\star$.

We can now construct the anamorphosis of a generic spatial line:

Let l be a line, $O \notin l$. There is a single plane H through O containing l. This plane defines a great circle C on the sphere. The cone of sight of l is $C_O(l) = \{\overrightarrow{OP} : P \in l\}$, a half-plane contained in H whose boundary is the line l_O, the translation of l to the origin. l_O is the union of two rays from O none of which is a ray of sight of an actual point of l but correspond to the limit of the directions of sight of an observer that follows l in both directions. Hence the strict conic projection of $C_O(l)$ onto the sphere is a meridian $M \subset C$ with its two antipodal endpoints missing. These two points are the intersection of l_O with the sphere, and are the vanishing points of the line. Taking the topological closure of M we get the anamorphosis of l onto S, which is a full meridian, being the union of M with the vanishing points.

In the degenerate case $O \in l$, l projects onto two antipodal points, with no vanishing points.

Analogously, we obtain the anamorphic image of a generic plane:

Let H be a plane, $O \notin H$. The cone of sight $C_O(H)$ is a half-space whose boundary is H_0, the plane through O parallel to H. The boundary is not contained in the set of rays of sight of individual points of H. The strict conic projection onto the sphere will be a hemisphere missing its boundary great circle C. Taking the closure of the conic projection we get the anamorphosis of H, a full hemisphere containing the *vanishing circle C*.

In the case $O \in H$, H projects onto a great circle with no vanishing points.

2.3 *From Anamorphosis to Perspective*

Anamorphosis onto a surface provides a 2D optical simulacrum of a 3D scene. But artists generally prefer to draw on planes. In order to work on a plane the artist must pay a price just as the cartographer who abandons the globe for the convenience of the chart. The cartographer loses isometry, and the artist must break the spell of the anamorphic *trompe l'oeil*.

Going from anamorphosis to perspective - as going from globe to chart - can be done in an infinite variety of ways. Intuitively, we would like to say a perspective is an anamorphosis onto a surface S followed by a flattening of S onto a plane, and we'd like such maps to be at least continuous; but if we try to do that naively we find that usually (e.g. in cylindrical or spherical perspective) the flattening map π will only be well defined on a dense open set of S. This led Casas to much confusion in [5]. The solution is to instead ensure that the inverse of π extends to a continuous map between compact sets. Thus we preserve the essential role of compactification.

Definition 3. *Let $\Lambda : \mathcal{R}_O \to S$ be an anamorphism. We say that $\pi : U \to \mathrm{R}^2$ is a* flattening *of S if U is an open dense subset of S, π is an homeomorphism onto $\pi(U)$, and there is a continuous map $\tilde{\pi} : cl(\pi(U)) \to S$ such that $\tilde{\pi}|_{\pi(U)} = \pi^{-1}$. We say that $p = \pi \circ \Lambda$ is the* perspective *associated to the flattening π. Let $\tilde{p} = \Lambda^{-1} \circ \tilde{\pi}$. Given a scene Σ, we say that $\tilde{p}^{-1}(\Sigma)$ is the strict perspective image of Σ, that $\tilde{p}^{-1}(\mathcal{V}_O(\Sigma))$ is the vanishing set of Σ, and that the perspective image of Σ is the union of its strict perspective image with its vanishing set.*

We find that the fundamental maps are not so much p and π but $\tilde{\pi}$ and $\tilde{p}$. That is, functional arrows are exactly the reverse of the naive view when we consider the topology. We resist the temptation to do away with tradition altogether and will still call p the perspective.

Apart from these formalities, a perspective should follow two informal but crucial requirements: First, it should be evocative of the visual experience, i.e., preserve at least some aspects of the spatial illusion that anamorphosis affords. Second, it must be *solvable*. By solving a perspective we mean finding and plotting the images of the basic idealized objects - points, lines and planes - out of which more complex scenes are constructed. It follows from Proposition 1 that the image of a line AB is a subset of the vanishing set of plane AOB. Hence solving a perspective reduces to solving its vanishing points. Whether a perspective is solvable depends on what tools we allow to solve it. It may be solvable by a computer but not by the unaided human artist. In this work we insist for our perspective to be solvable by ruler and compass.

Among the infinite flattenings available for each surface S, a dense set will preserve nothing of visual interest, or will be too hard to solve. Considering the classical examples of perspective we see that the flattenings are chosen in order to relate naturally to their anamorphic surface, and to satisfy our two requirements: In classical perspective the anamorphic surface is already a plane, so the natural flattening is the identity map (modulo scaling). Straight lines are preserved.

In cylindrical perspective the anamorphic surface is a cylinder, which is a developable surface, so it can be cut and unfolded isometrically. Spatial lines become ellipses by anamorphosis and sinusoidals upon flattening. These can be plotted in good approximation by ruler and compass. In (hemi)spherical perspective the anamorphosis turns lines into arcs of great circles. There is no isometric flattening of a sphere (the curse of cartography) so Barre and Flocon chose a flattening that preserves lengths along key meridians and that, crucially, turns meridians into circular arcs, being thus solvable by ruler and compass.

There is an interesting symmetry between spherical and plane perspective. In classical perspective the flattening is trivial but the anamorphosis is not. In spherical perspective the opposite is true. This is because in classical perspective the plane of the anamorphosis can be identified with the plane of the perspective, while in the spherical perspective the anamorphic sphere can be identified with the set of directions, so the flattening in the former case and the anamorphosis in the latter can be identified with the identity map. This gives classical perspective its special status: since the flattening is trivial, anamorphosis is preserved. So called "perspective deformation" is a misnomer, resulting from the failure of the observer to stand at point O. The distortion of linear measurements (the so-called "paradox" of Leonardo) is a necessary consequence of the preservation of solid angles from O, and a feature, not a bug, of an effective *trompe l'oeil* [1].

3. Total spherical perspective: Flattening a sphere

We now define our total spherical perspective within the scheme outlined above: an anamorphosis followed by a flattening. The anamorphosis is fully determined by the choice of the surface and the place of the observer (the unit sphere S^2, with the observer $\mathbf{O}$ at its center). We have studied it above, so it remains to define the flattening.

We start by defining an observer-centered reference frame. We consider a ray stemming from $\mathbf{O}$, representing a privileged direction of sight. We call it the *central ray of sight* and its axis we call the *central axis of sight* . We place an orthonormal right-handed coordinate system xyz in $\mathbf{O}$, such that the positive side of the y axis coincides with the central ray of sight. For easy reference we name the points where the three axes cut the sphere: we call **F**ront the intersection of the central ray of sight with the sphere and **B**ack its antipode point; **R**ight the point where the x axis touches the sphere and **L**eft it's antipode; **U**p the point where the positive z axis touches the sphere and **D**own its antipode, and we represent these points by their initials written in bold.

From now on we will simplify notations with the following convention: a spatial point and its plane projection will be denoted by the same letter, the spatial point in bold font and the projection in italic font. Hence, $P = p(\mathbf{P})$ will be the perspective of a spatial point $\mathbf{P}$. In particular, $P = \pi(\mathbf{P})$ will be the flattening of a point on the sphere, so the perspective images of reference points $\mathbf{F}, \mathbf{B}, \mathbf{L}, \mathbf{R}, \mathbf{U}, \mathbf{D}$ will be F, B, L, R, U, D respectively.

We call the $y = 0$ plane (orthogonal to the central axis of sight) the *observer's plane.* The observer's plane intersects the sphere in a great circle we call the *equator.* We call the $x = 0$ plane the *sagittal plane*, and $z = 0$ we call the *plane of the horizon.* We call the **F**-meridians *central* meridians. We call the half-space $y > 0$ the *anterior* half-space (representing everything in front of the observer) and the half-space $y < 0$ we call the *posterior* half-space (representing all that is behind the observer).

We will now construct a flattening of the sphere. This is a construction for the *azimuthal equidistant projection*, well known to cartographers and astronomers. A restriction of this map to a single hemisphere is used in [4]. Our purpose here is to establish a derivation of this map that is adequate to our purposes and show that it fits within our definition of perspective.

Intuitively, we picture it thus: we look at the sphere as the union of its central meridians, which we think of as inextensible threads. We cut the threads free at **B**, and pull them straight along their tangents at **F**, flattening them onto the plane tangent to the sphere at **F** (Figure 2).

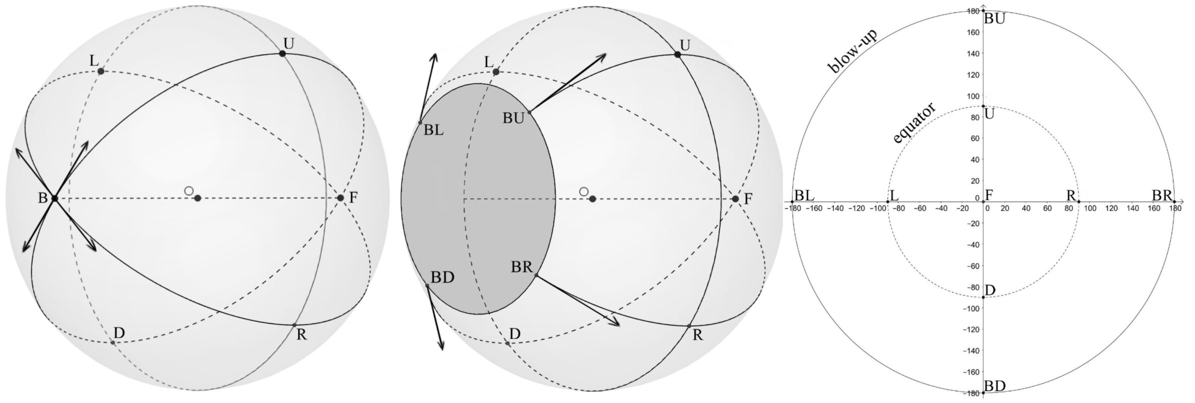

Figure 2. Point **B** is blown-up to a circle (**BD**-**BR**-**BU**-**BL**) and the punctured sphere is flattened onto the perspective disc. Distances (or angles from O) are preserved along points on each **F**-meridian. Here we see them marked along the **U**-**D** and **L**-**R** measuring lines, ranging from -180 to 180 degrees along each diameter of the perspective disc.

The straightened threads radiate from **F**, forming a disc D of radius r. We call the boundary circle of the disc the *blowup* of **B**, as we see this point as having been blown-up into the set of rays of the tangent plane of the sphere at **B**, each ray corresponding to one of the meridians from which **B** could be approached . We now formalize this construction:

Let $D = \{(x, z) \in \mathrm{R}^2 : x^2 + z^2 < r^2\}$. Let $\pi : S^2 \setminus \{\mathbf{B}\} \to D$ be the homeomorphism such that

$\mathcal{C}_0$) each central meridian maps onto a line segment.

$\mathcal{C}_1$) distances are preserved along each central meridian.

$\mathcal{C}_2$) angles between central meridians are preserved at **F**.

Extending π^{-1} to the closure of its domain we obtain the continuous map between compact sets, $\tilde{\pi} : cl(D) \to S^2$. By continuity, it verifies $\tilde{\pi}(P) = \mathbf{B}$ for all P on the blowup circle $cl(D) \setminus D$, and $p = \pi \circ \Lambda$ defines a perspective according to Definition 3.

Condition $\mathcal{C}_1$ means that the map is an isometry for each **F**-meridian separately. Since distances measured along great circles of the sphere are proportional to angles from the center, this means that if $\mathbf{P}, \mathbf{Q}$ are points on the same **F**-meridian and if P, Q are their images, then $|PQ| = \angle\mathbf{POQ}$ up to multiplication by a scale factor[1]. Conditions $\mathcal{C}_0$ and $\mathcal{C}_1$ imply that **F** will be mapped to the center of the disc with images of the **F**-meridians radiating from it as line segments.

Condition $\mathcal{C}_2$ means that the angles between these segments at F will be equal to the angles of the corresponding meridians at **F**. This ensures the central meridian images will be distributed radially preserving their tangents at **F**, that is, they will look as if orthogonally dropped onto the tangent plane of the sphere at **F**. We call *longitude* of an **F**-meridian the angle at **F** between its tangent and that of the **F**-meridian through **R**. By $\mathcal{C}_2$, The longitude of a meridian equals the angle between its image and the FR measuring line.

$\mathcal{C}_1$ and $\mathcal{C}_2$ together imply that the images of the two meridians of each great circle through **F** form a diameter of the perspective disc and that distances are preserved within each diameter. For this reason we call the diameters of the perspective disc *measuring lines*.

We will define equator of the perspective disc as the perspective image of the sphere's equator. This is a circle, with half the radius of the disc, upon which lie the images of points **R**,**L**,**U**,**D**. It divides the perspective disc into two parts: an inner disc that is the flattening of the anterior hemisphere, and an outer ring, between the equator and the blowup, that is the flattening of the posterior hemisphere (Figure 2).

In terms of the Cartesian (x, y, z) coordinates, the flattening composed with anamorphosis

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[1]For points on the images of these meridians we will freely abuse notation and write equalities between angles and linear measures such as $|\overline{XZ}| = |\overline{XY}| + 180°$ to mean that these equalities are valid modulo product by the adequate scale factors.

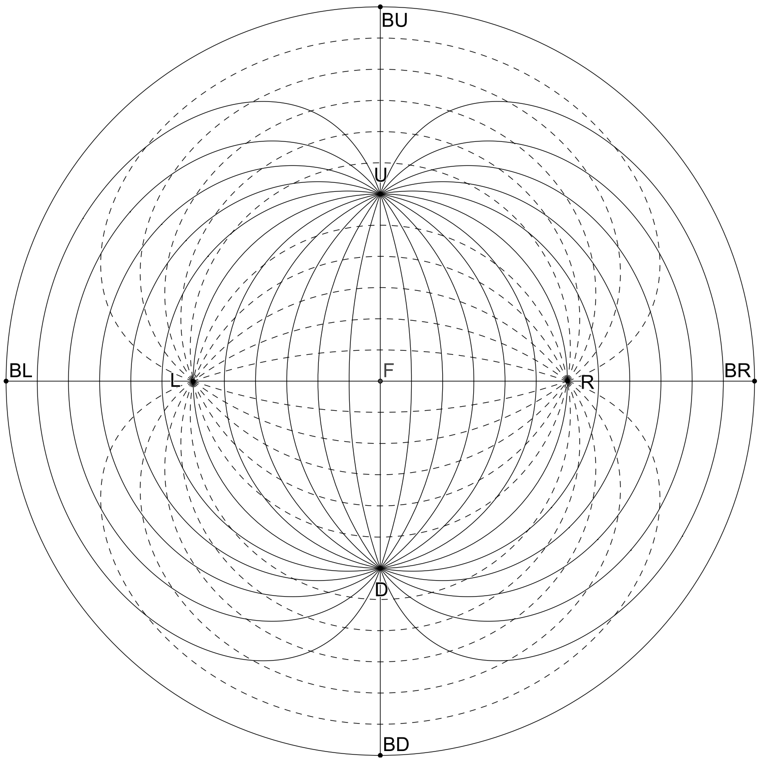


Figure 3. Flattening of **UD** (solid lines) and **LR** (dashed lines) great circles from Equation 1, at 15 degree intervals. These great circles correspond to the perspective images of vertical and frontal horizontal lines respectively.

$\mathbf{P} \mapsto \overrightarrow{\mathbf{OP}}/||\overrightarrow{\mathbf{OP}}||$ gives the perspective map $p : \mathcal{R}_\mathbf{O} \setminus \{\overrightarrow{\mathbf{OB}}\} \mapsto D$,

$$p([x, y, z]) = \frac{(x, z)}{\sqrt{x^2 + z^2}} arccos \left(\frac{y}{\sqrt{x^2 + y^2 + z^2}} \right) \tag{1}$$

this can be seen as projecting orthogonally against the $xz-$plane, taking the unit vector, and then scaling to a length equal to the value of the angle $\angle\mathbf{POF}$.

The natural set of spherical coordinates for this map is (ρ, λ, θ) with

$$\rho = |\mathbf{OP}| = \sqrt{x^2 + y^2 + z^2},\ \lambda = \angle\mathbf{POF} = arccos\left(\frac{y}{|\mathbf{OP}|}\right),\ \theta = arccos\left(\frac{x}{\sqrt{x^2 + z^2}}\right) \tag{2}$$

where one can see λ as the latitude, measured from **F**, and θ as the longitude, measured from **R**. In these coordinates the anamorphosis becomes trivial, $p = \pi \circ \Lambda$ identifies with the flattening π and we see clearly that the perspective image of P doesn't depend on ρ, which was to be expected, since Λ is a central projection:

$$p(\rho, \lambda, \theta) = \pi(\rho, \theta) = \lambda(cos(\theta), sin(\theta)) \tag{3}$$

4. Solving a scene with ruler, compass, and nail

The explicit form of the map obtained in the previous section would be enough for a pixel-by-pixel rendering of a scene on a computer. It is however useless for the human artist. A perspective that is useful for the draughtsman must stipulate how to solve a scene with simple instruments. In what follows we will show how to solve a scene in total spherical perspective with ruler and compass, with allowance for marked rulers and plotting of arbitrary angles with protractors. The addition of a further tool - a nail - will further simplify practical constructions. We assume the data for the scene as given either from direct measurements of angles (theodolite) or from Cartesian coordinates (architect's plan/orthographic views).

A common technique to solve scenes in classical perspective is to make the plane of the perspective image do double or triple duty by superposing on it various orthogonal projections. This also works in spherical perspective. We will illustrate this in our first graphical construction:

Construction 1. *Construction of the perspective image of a point on the observer's plane: Let* $\mathbf{P} \neq \mathbf{O}$ *be a point on the observer's plane. Then* $\overrightarrow{\mathbf{OP}}$ *crosses the equator of the sphere. Hence the perspective image of* $\mathbf{P}$ *will be the point* P *at the equator of the perspective disc such that* $\angle PFR = \angle\mathbf{POR}$*. If the (x,y,z) coordinates of* $\mathbf{P}$ *are given, we can construct* P *graphically thus: We make the plane of the drawing represent both the perspective disc and the back orthogonal projection view of the sphere onto the observer's plane, with* F *in the perspective disc coinciding with* $\mathbf{O}$ *in the orthogonal view and the disc scaled in such a way that the equator's perspective image coincides with its orthogonal projection image. Let the orthogonal image of* $\mathbf{P}$ *be* P^b*. Plot* P^b *from its (x,z) coordinates. Then* $\overrightarrow{FP^b}$ *is the orthogonal projection of* $\overrightarrow{\mathbf{OP}}$ *and* P *is the intersection of* $\overrightarrow{FP^b}$ *with the equator of the perspective disc.*

The problem of solving a scene can be divided into two parts: plotting points and lines in the anterior half-space and in the posterior half-space. The anterior half-space is solved in [4]. We will give here a very condensed version of that method, adapted to our needs.

4.1 *Solving the anterior hemisphere*

It is well known [3, 4] that the perspective image of lines in the anterior half-space is well approximated by arcs of circles. This is important, since an arc of circle is the next best thing to a line: first, it is easy to draw with ruler and compass; second, three points determine a single arc of circle through them (hence only one more point must be found than for straight lines).

The problem of solving the anterior hemisphere is divided in two cases: frontal and receding lines.

4.1.1 Images of frontal lines

We say that a plane is *frontal* if it is parallel to the observer's plane. We say that a line is frontal if it lies on a frontal plane. Let l be a line on a frontal plane H. First suppose that H is not the observer's plane. Translating l to $\mathbf{O}$ we find it has two vanishing points $\mathbf{V}$ and $\mathbf{V}^\star$ which define diametrically opposite points on the sphere's equator. Their images are found by drawing the translated line directly on the perspective disc, to obtain its intersection with the disc's equator (as in Construction 1, above). Next, we find a third point. If l is not vertical, it intersects the sagittal plane at some point $\mathbf{P}$. We plot the measure of the angle $\angle\mathbf{POF}$ on the vertical measuring line. If l is vertical then it crosses the plane of the horizon and we measure instead the angle with the central axis at this point, and plot it on the horizontal measuring line. The image of l is well approximated by the arc of circle $VPV^\star$ (Figure 4). If $\mathbf{P} \in \overrightarrow{\mathbf{OF}}$ then $P \equiv F$, so l projects onto a diameter of the disc.

Now suppose that H is the observer's plane. We get V and $V^\star$ as above, but P will now project on the equator of the perspective disc. The arc of circle will be one half of the equator.

Note: The natural angles to measure with a theodolite when drawing from nature are those on the horizontal and vertical measuring lines - hence our focus on those measurements.

Construction 2. *Perspective of an arbitrary point* P *on the anterior half-space. Consider the frontal plane going through* $\mathbf{P}$ *and on it a vertical line* v *and a horizontal line* h *going through* $\mathbf{P}$*. We already know how to solve these lines. The perspective image of* $\mathbf{P}$ *will be found at the intersection of the images of* v *and* h*.*

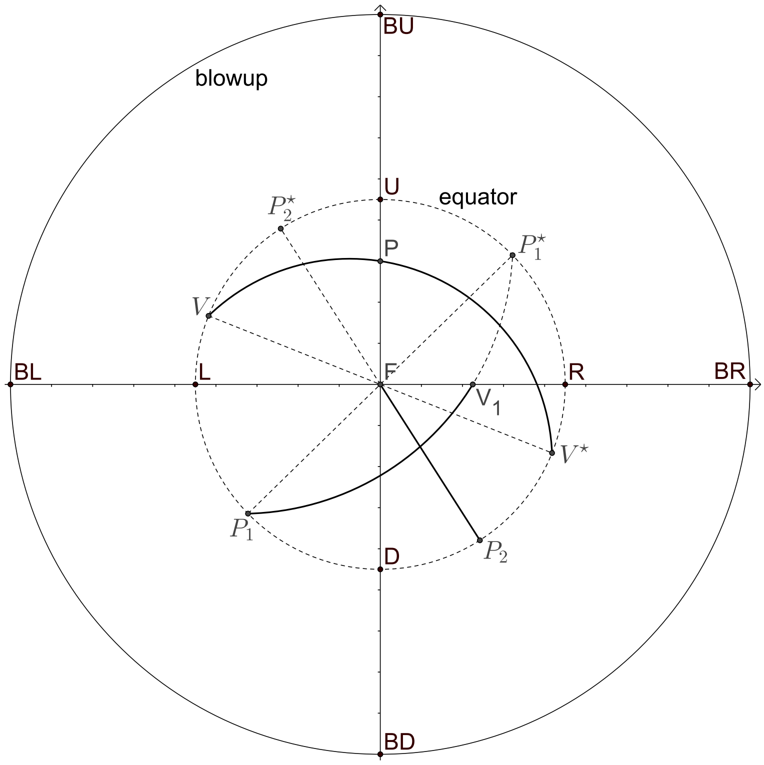


Figure 4. Perspective images of lines restricted to the anterior hemisphere. Arc $VPV^\star$ is the image of a frontal line. Arc $P_1V_1P_1^\star$ is the image of the plane of a receding line, and the arc from P_1 to V_1 is the image of the line itself. The line segment $\overline{P_2F}$ is the image of a central line. The image of its plane extends to the image of the antipode of P_2.

4.1.2 Images of receding lines

We say that a line is a *receding line* if it intersects the observer's plane at a single point. Let $\mathbf{P}$ be the point of intersection of a receding line l with the observer's plane. We plot P as in Construction 1. The plane H defined by $\mathbf{O}$ and l must also intersect the equator at the antipodal point $P^\star$. To find a third point, we translate l to $\mathbf{O}$ and intersect it with the sphere to find the two vanishing points. One of these will be on the anterior hemisphere, so we plot it by Construction 2. Let its image be V. We trace the auxiliary arc of circle $PVP^\star$ that is the image of the plane H in the anterior disc. The anterior image of l will be the part of the arc that lies between V and P.

If l lies on a plane through an $\mathbf{F}$-meridian, it will project into a diameter of the disc. A particular case is that of the central lines. We say that a line is central if it is perpendicular to the observer's plane. In this case $V \equiv F$, hence V will be between P and $P^\star$, the image of H will be the straight line segment $\overline{PP^\star}$ and the image of l will be the segment $\overline{PF}$ (Figure 4). Hence, central lines project as in classical perspective.

This ends our condensed review of (hemi)spherical perspective as presented in [4]. Outside of the anterior disc the images of lines are no longer well approximated by circles. We maintain that this is one of the reasons why Barre and Flocon limited their perspective to 180°. We will now show how to extend it to the full 360° view.

4.2 *The full* 360°

We now wish to project the full image of a generic spatial line. We know lines project onto meridians. It is best to start by solving the complete great circle and then delimit the meridian by finding its end points. Our strategy is to piggyback on the known procedure for the anterior half space and use it to obtain a plot of the full great circle. The key lies in plotting antipodal points. On what follows, let r be the (arbitrary up to scale factor) radius of the perspective disc.

4.2.1 Plotting antipodal points

Proposition 2. *Let* $\mathbf{P}$ *be a point in space such that* $\mathbf{F} \neq \mathbf{P} \neq \mathbf{O}$. *Then* $P^\star$ *is the point on* $\overrightarrow{PF}$ *such that* $|PP^\star| = r$.

Proof. Let C be the great circle through $\mathbf{F}$ and $\mathbf{P}$. Since $\mathbf{F} \in C$, the image of C is a measuring line. C contains the P-meridian $G = \mathbf{PFP^\star}$. Since $\mathbf{B} \notin G$ and the flattening is continuous, the image of C is connected and preserves the ordering of points, so G projects to the segment $\overline{PP^\star}$ and $P^\star \in \overrightarrow{PF}$. Since G is on a measuring line and $\angle\mathbf{POP^\star} = 180°$, then $|PP^\star| = r$. □

Intuitively: on the sphere, you will find $\mathbf{P^\star}$ by travelling 180 degrees from $\mathbf{P}$ along the single great circle that crosses $\mathbf{P}$ and $\mathbf{F}$. But since $\mathbf{F}$-great circles flatten onto length preserving diameters, this means you find $P^\star$ from P by following the diameter through P for half its length along the $\overrightarrow{PF}$ direction (Figure 5). This proposition allows us to easily plot the antipode of a known point P. Just draw line FP, center the compass at P and open to the radius r of the perspective disc, then intersect with FP to find $P^\star$. Or preferably, if using a marked ruler, pass the ruler through P and F with the zero mark at P, and plot $P^\star$ where the ruler marks r.

For the purposes of freehand drawing of a perspective it is often useful, when plotting points nearer to the equator than to F, to use instead the following result:

Proposition 3. *Let* $\mathbf{P}$ *be a point in space such that* $\mathbf{F} \neq \mathbf{P} \neq \mathbf{O}$. *Let* P_B *be the intersection of* $\overrightarrow{PF}$ *with the blowup of* $\mathbf{B}$. *Then* $P^\star$ *is the point on* $\overrightarrow{PF}$ *such that* $|P^\star P_B| = |FP|$. *Also,* $|P^\star F| = |P(-P_B)|$, *where* $-P_B$ *is the point on the perspective disc diametrically opposite to* P_B.

Proof. The plane $H = \mathbf{FOP}$ defines a great circle C that contains $\mathbf{P}, \mathbf{P^\star}, \mathbf{F}$, and $\mathbf{B}$. On that plane, the lines $\mathbf{PP^\star}$ and $\mathbf{FB}$ intersect at $\mathbf{O}$, and therefore we have the equalities between opposing angles $\angle\mathbf{POF} = \angle\mathbf{P^\star OB}$ and $\angle\mathbf{POB} = \angle\mathbf{P^\star OF}$ (Figure 5). Since C is a great circle through $\mathbf{F}$, $\tilde{\pi}^{-1}(C)$ is a diameter of the perspective disc. On C we have a cyclic order of points $\mathbf{P}-\mathbf{F}-\mathbf{P^\star}-\mathbf{B}$. Since $\tilde{\pi}$ is continuous, the order is preserved on the perspective image and we have $(-P_B)-P-F-P^\star-P_B$ where P_B and $(-P_B)$ are the points of the blowup corresponding to the directions of the two meridians of C at $\mathbf{B}$. Because distances are preserved along measuring lines, the two angle equalities above imply $|PF| = |P^\star P_B|$ and $|P(-P_B)| = |P^\star F|$ respectively. □

The practical interest of Proposition 3 is that for freehand drawing of lines it is often easier for the artist to transport the measurement $\left|\overline{PF}\right|$ by eye than to transport the radius of the disc without an actual compass or ruler. But, having a compass at hand, or a marked ruler, the use of Proposition 2 makes for very efficient plotting of antipodes.

Let us use Proposition 2 to plot the image of a great circle's posterior meridian from the known image of its anterior meridian. The idea is simple: we sample arbitrary points Y_i in the given meridian, plot their antipodes by Proposition 2, and then interpolate them. We start by defining a practical interpolation procedure using circular arcs:

Construction 3. *Construction of fat lines: Let* $P_i, i = 1, \ldots, n$ *be a set of* n *ordered points sampled from a curve* C. *Through each successive set of three points pass a circular arc, to get arcs* $P_1P_2P_3, P_2P_3P_4, \ldots, P_{n-2}P_{n-1}P_n$. *These overlapping arcs form a* "fat line" *that approximates* C. *The degree to which successive arcs fail to exactly overlap (how "fat" the envelope of these arcs is) indicates the amount of error in the approximation and the need to take a finer sample with larger* n. *This is also indicated by the size of the angle between tangents of overlapping arc at their common point (Figure 6).*

The *fat line* provides a systematic way to judge the need to sample more points. In drawing practice, however, we will usually just plot successive non-overlapping arcs and judge the error by how much the tangents differ at the transition between arcs. We now proceed to our purpose:

Construction 4. *Construction of fat line approximations of posterior meridians: Let* C_a *be the perspective image of the anterior meridian of a great circle* C *on the sphere. To obtain an*

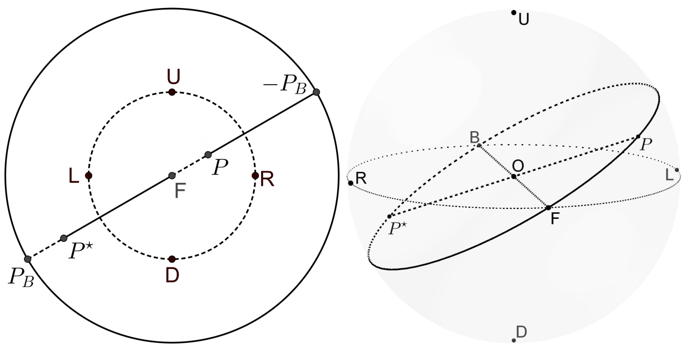


Figure 5. Construction of the antipode of P. The great circle of the plane **POF** (Right) projects onto the line PF (Left). $P^\star$ is the point on ray $\overrightarrow{PF}$ such that $|PP^\star| = r$. The equality of opposing pairs of angles $\angle\mathbf{POF} = \angle\mathbf{P^\star OB}$ and $\angle\mathbf{POB} = \angle\mathbf{P^\star OF}$ on the great circle implies that $|PF| = |P^\star P_B|$ and $|P(-P_B)| = |P^\star F|$ on the perspective plane.

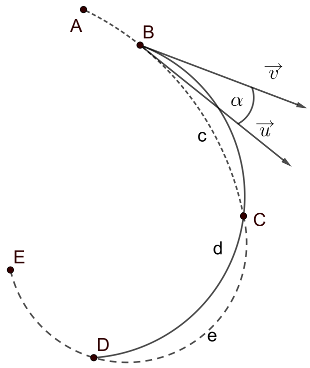


Figure 6. Fatline through points $A - E$, formed by overlapping circular arcs $c = ABC$, $d = BCD$, $e = CDE$ (rendered in dashed, solid and wide dashed lines respectively). Coarseness of interpolation is estimated by the area ("fat") between the overlapping pairs of curves (c, d) and (d, e) or by the angle between the tangents of curve pairs at their common points (e.g., the angle α at B between the tangents of c and d).

approximation of the posterior image C_p of C, trace an arbitrary number of measuring lines $m_1, \ldots, m_K$ through F. Intersect each of these lines with C_a to get points $Y_1, \ldots, Y_K$, and use Proposition 2 to obtain the antipodes $Y_i^\star$ (Figure 7). Through the points $Y_i^\star$ construct a fat line approximation of C_p according to Construction 3. Refine the sample $\{Y_i\}$ by adding more m_i as needed until the fat line is thin enough for the required drawing tolerance.[2]

The practical draughtsman can dramatically improve the efficiency of Construction 4 by adding a simple tool to his kit: a nail.

Construction 5. *Ruler, Compass and Nail: Assume as given the anterior meridian C_a. Suppose a marked ruler is available, with a zero mark and an r (perspective disc radius) mark. Stick a nail at the center F of the perspective disc. Then, if you lead the zero mark of the ruler along the curve C_a while keeping the ruler's edge sliding against the nail, Proposition 2 ensures that the r mark of the ruler will automatically move along the antipodal curve C_p (Figure 7). This allows you to mark as many points $Y_i^\star$ of C_p as desired along the path of the r mark, without having to draw and measure each line m_i in Construction 4. In this way you can easily plot a great number of antipodal points very quickly, allowing C_p to be interpolated by hand with good precision by joining each set of three successive points with arcs of constant curvature.*

A simple mechanical device would make this construction even more efficient: a ruler with a slit of length r along its length, with a spotter at one end and a pencil point on the other. As the user follows half of a meridian with the spotting end, the nail slides along the slit and the pencil traces the antipodal meridian in a continuous line, with no need for interpolation. None such refinement is needed, however, and even the nail may remain merely conceptual, although a physical one can make quite a difference in drawing speed (do try it with a thumbtack!).

We are now ready to plot arbitrary lines in full perspective. We have the following cases:

4.2.2 Images of frontal posterior lines

Let l be a line in a frontal posterior plane. Let H be the plane defined by l and **O**, and C its great circle. Suppose l is not vertical. Then l crosses the sagital plane at a point **P**, and P will be

[2] A practical note: the standard way of finding the circle through A, B, C by intersecting the perpendiculars to $\overline{AB}$ and $\overline{AC}$ to find the center is the most tiresome part of the process. For a freehand sketch, the artist can instead eyeball a line of constant curvature between three well spaced points, so that the arc of circle can be obtained without finding its center. A draughtsman should master freehand drawing lines of constant curvature (circles) almost as much as zero curvature (straight) lines.

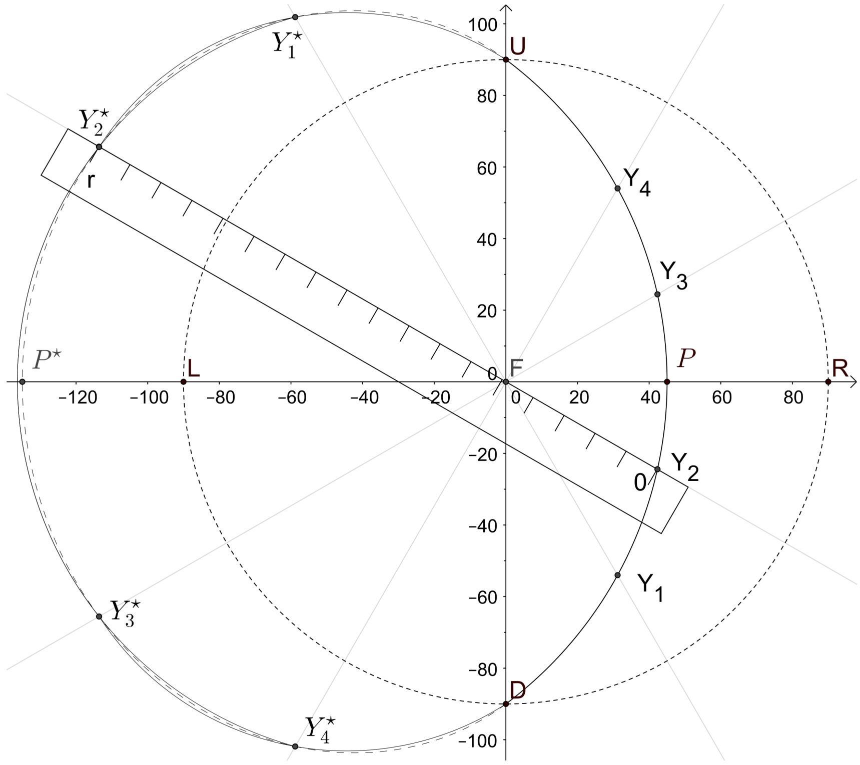


Figure 7. Perspective image of a **UD** great circle passing at 45 degrees to the observer's right. Choosing four measuring lines set at 30 and 60 degrees to the horizontal axis, we get points $Y_1 - Y_4$ on the arc UPD. We find their antipodes $Y_i^\star$ at a distance r along each measuring line (Proposition 2), and interpolate a fat line through them. On the picture, the dashed line $UP^\star D$ is the exact antipodal line of the arc of circle UPD on the anterior view, and the filled lines are a 'fat line' approximation. Even this coarse four-point approximation fails at its worst by about one degree, but a large number of points could be obtained quickly through ruler and nail: stick a nail at F; then, as you lead the 0 mark of the ruler over the anterior curve UPD, sliding the ruler along the nail, the r mark will automatically trace the posterior curve $UP^\star D$.

a point on the posterior ring of the perspective disc, such that $|FP| = \angle\mathbf{POF}$, on $\overrightarrow{FU}$ or on $\overrightarrow{FD}$ according to whether $\mathbf{P}$ is above or below the observer. By Proposition 2, the antipode of $\mathbf{P}$ will map to the point $P^\star \in \overrightarrow{PF}$ such that $|PP^\star| = r$. This point will be in the anterior perspective disc, therefore we can approximate the anterior image of C by the arc of circle $C_a = VP^\star V^\star$, where V and $V^\star$ are two vanishing points at the equator. We can now use Construction 4 to obtain the fat line approximation of the antipodal image C_p. Then the full image of C will be $C_a \cup C_p$ and the image of l will be the C_p meridian (Figures 8, 9). Note that if $\mathbf{P} \equiv \mathbf{B}$ then l flattens to two disconnected line segments: a diameter of the full disc minus its intersection with the inner disc. This line is however connected when considered in the topology induced by $\tilde{\pi}$, since the blowup of $\mathbf{B}$ - seen as a single point - connects both segments.

Construction 6. *We can now plot an arbitrary point* $\mathbf{P}$ *on the posterior half-space: pass vertical and horizontal lines through* $\mathbf{P}$*, plot them according to the procedure just described, and intersect their images to find* P*.*

4.2.3 Images of receding lines

Let l be a line that crosses the observer's plane at a single point $\mathbf{P}$. Let H be the plane defined by l and $\mathbf{O}$, and C its great circle. By Construction 1 and Proposition 2 we obtain the points P and $P^\star$ on the perspective disc's equator. Displacing l to the origin we obtain two vanishing points; one on the anterior hemisphere, Let it be $\mathbf{V}$, and its antipode $\mathbf{V}^\star$ on the posterior hemisphere. Plot V by Construction 2, then use Proposition 2 to plot $V^\star$. The arc of circle $C_a = PVP^\star$ is the anterior image of C. From C_a plot the antipodal meridian C_p by Construction 4. This plots the full image of the great circle C. To get the image of l, select within it the arc $VPV^\star$ (Figures 8, 9).

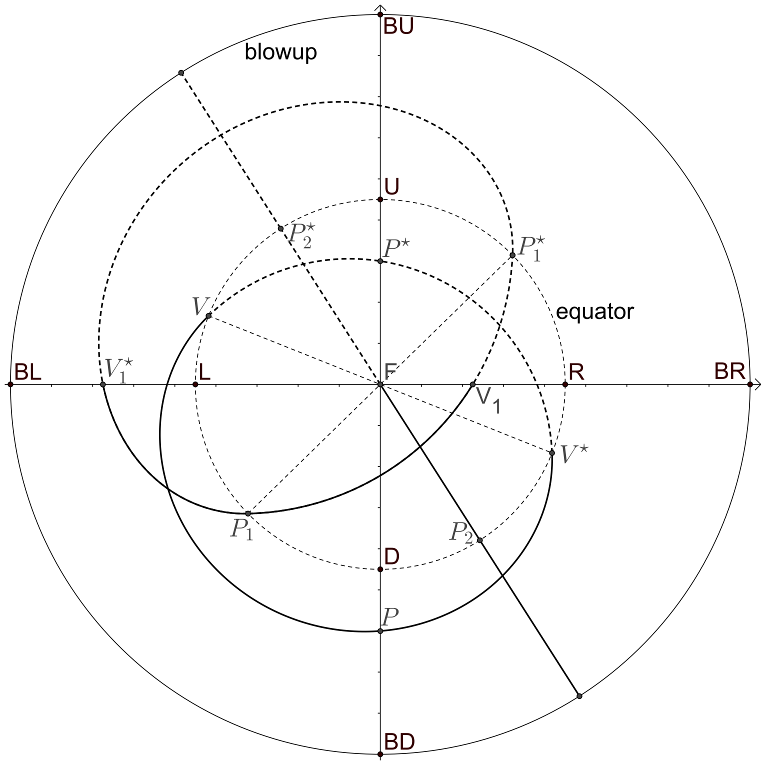


Figure 8. Total spherical perspective of frontal, receding, and central lines. Arc $VPV^\star$ is the image of a frontal posterior line. Arc $V_1P_1V_1^\star$ is the image of a receding line. The radius $\overrightarrow{FP_2}$ is the image of a central line. Note in each case the dashed curve that extend the image of each line to the image of its great circle.

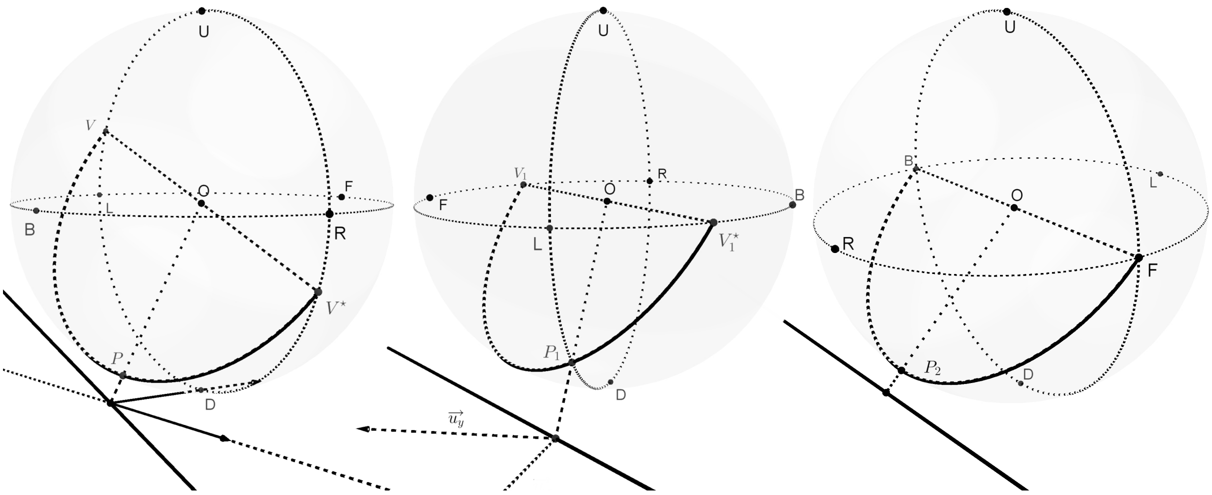


Figure 9. Spatial view of the lines in Figure 8 and their meridians. Refer to the cardinal points for orientation. Left: a sloping line in a frontal plane behind the observer. P lies on the sagittal plane, the vanishing points on the observer's plane. Middle: a receding, horizontal line, going at 45° to the observer's right. P_1 is on the observer plane, V_1 and $V_1^\star$ are on the equator. Right: a central line passing below and to the observer's right, vanishes at F and B. P_2 is on the observer's plane.

If l is on the plane of an $\mathbf{F}$-meridian, it will project into a measuring line. In the particular case in which l is a central line, then $\mathbf{V} \equiv \mathbf{F}$ and $\mathbf{V}^\star \equiv \mathbf{B}$, and the image of l will be a radius of the perspective disc, from F to a point on the blowup circle. We identify the whole blowup circle with a single vanishing point B since $\tilde{\pi}$ maps it to $\mathbf{B}$. The intersection point of $\overrightarrow{FP}$ with the blowup circle codifies both the vanishing point $\mathbf{B}$ itself and the direction (or the meridian) from which it is approached as the line of sight follows l to infinity.

4.3 *Plotting curves of constant angular elevation*

We have learned enough to solve a scene when we have the Cartesian coordinates of its points - for instance when drawing from an architectural plan. When drawing from observation, however, the artist measures only the angles subtended by objects. We have already seen what the natural spherical coordinates are for this perspective (the angles λ and θ defined above), and it is possible to construct a simple device to measure these angles directly, but the more habitual set of angles are the horizontal angular displacement ξ together with the angular elevation ζ, defined thus: ξ is the angle between the central ray $\overrightarrow{\mathbf{OF}}$ and the orthogonal projection of $\overrightarrow{\mathbf{OP}}$ against the plane of the horizon. ζ is the angle between $\overrightarrow{\mathbf{OP}}$ and its orthogonal projection on the plane of the horizon. These are the angles one measures with a standard theodolite.

Lines of constant horizontal displacement ξ are the images of vertical lines and we already know how to plot them. Lines of constant elevation ζ are circles on the anamorphic sphere obtained by intersection with horizontal planes. For short (and somewhat mixing geographical metaphors), we will call these circles and their images *parallels*.

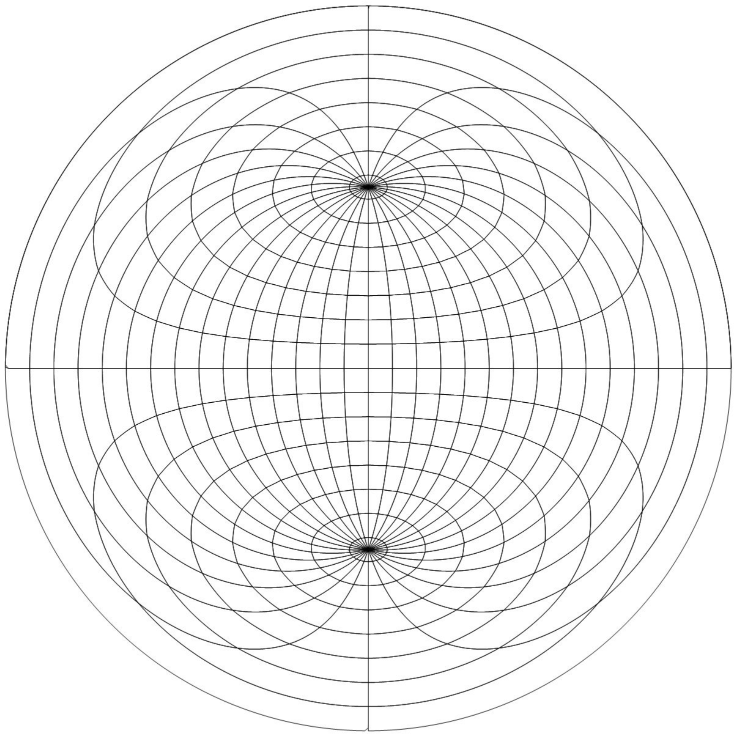
In the anterior hemisphere we approximate parallels by arcs of circles in the manner of [4]: Let h be a parallel of elevation ζ. h intersects the sphere's equator at two points $\mathbf{P_L}$ and $\mathbf{P_R}$ on the left and right side of the sagital plane respectively and intersects the anterior sagital plane at a point $\mathbf{P}$. Then P_L and P_R will be at the disc's equator and $\angle P_R F R = \angle P_L F L = \zeta$, and P will be at the vertical segment $\overline{UD}$, and $|FP| = \angle \mathbf{FOP} = \zeta$. We take the arc of circle $P_R P P_L$ as the approximation to the anterior image of the parallel h. To plot the posterior part of the parallel we make use of the following proposition:

Proposition 4. *Let h be a parallel on the anamorphic sphere. Let $\mathbf{P} \neq \mathbf{F}$ be a point of h. Let $M = \overrightarrow{FP} \cap \varepsilon$ where ε is the equator of the perspective disc. Let Q be the point such that M is the midpoint of $\overline{PQ}$. Then Q is the perspective image of a point of h.*

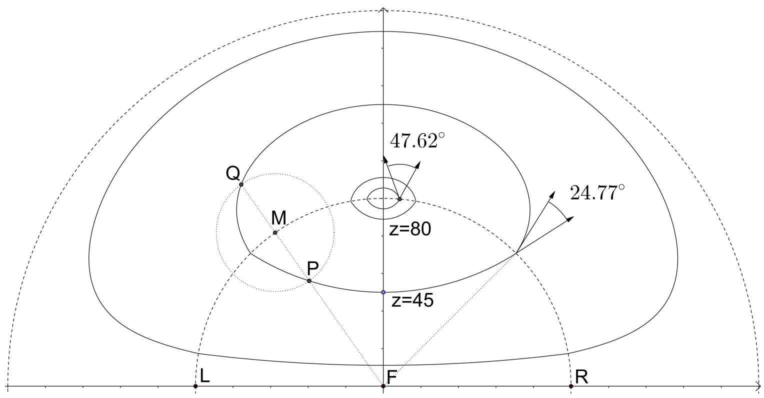
Proof. Parallels and **F**-meridians are invariant by reflection across the observer's plane (because so are their defining planes and the sphere itself and hence their intersection). Then the intersection of a parallel and an **F**-meridian is also invariant for reflection across the observer's plane, and since it is an intersection of circles, it is made up of a whole circle, or of zero, one, or two mirror symmetric points. Let $m \subset \overrightarrow{FP}$ be the radius through P. m is the image of the **F**-meridian C that crosses $\mathbf{P}$. Hence $\mathbf{M} = \tilde{\pi}(M)$ is the point where C crosses the sphere's equator. Since $|PM| = |MQ|$ and m is a measuring line, then $\angle \mathbf{POM} = \angle \mathbf{MOQ}$, and since P and Q lie on the plane of C, orthogonal to the observer's plane, then P and Q are mirror symmetric relative to the observer's plane, hence Q is on h. □

Construction 7. *To plot the posterior half of a parallel h, plot first the anterior half h_a as an arc of circle, then plot a set of measuring lines r_i, intersect them with h_a at points Y_i, find the antipodal points $Y_i^\star$ from Proposition 4, and trace a fat line through the $Y_i^\star$.*

Figure 10.a) shows a computer plot of parallels and verticals calculated directly from Equation 1. Figure 10.b) shows the approximation of the parallels of elevation 10, 45, 80, and 85 degrees plotted by Proposition 4 applied to the inner disc approximation. We see that the curves are not smooth at the equator, this being more noticeable when closer to **U**. This is an artefact of the approximations, as we see from Equation 1 that the perspective images of constant elevation curves are differentiable. The error stems not from Proposition 4, which is exact, but from the initial approximation of the parallel by an arc of circle inside the anterior disc. Near the equator one should favour the method of the previous section instead. The practical draughtsman will however just smooth the edges at the equator and use parallels whenever convenient. In Figure 11 we can see a drawing of the Reading Room of the British Museum seen from a point on the axis of symmetry of the dome. The whole construction is based on verticals and lines of constant elevation hand-plotted using Proposition 4.



(a)



(b)

Figure 10. Lines of constant angular elevation. a) Grid of verticals and parallels, plotted directly from Equation 1. b) Ruler and compass approximation of curves of constant angular elevation of 10,45,80, and 85 degrees. Point Q is such that $|MP| = |MQ|$.

5. Examples

As long as we can draw a grid of squares we can plot any object to any given precision, by caging it inside a fine enough grid and interpolating through well chosen points.

In Figure 12 we solve a central uniform perspective grid. We consider a horizontal grid of squares (a tiled floor) with one axis parallel to $\overrightarrow{\mathbf{OF}}$ and the other parallel to $\overrightarrow{\mathbf{LR}}$. For simplicity assume one of the grid's vertices is directly under the observer. Call ground plane to the plane of the grid and ground line to the intersection of the ground plane with the observer's plane. We make the plane of perspective represent also a top and a back orthogonal view of the scene. We make the back view of $\mathbf{O}$ coincide with F, and scale the sphere to make it tangent to the ground plane at $\mathbf{D}$. We make the top view of $\mathbf{O}$ coincide with D. Hence, a horizontal line through D

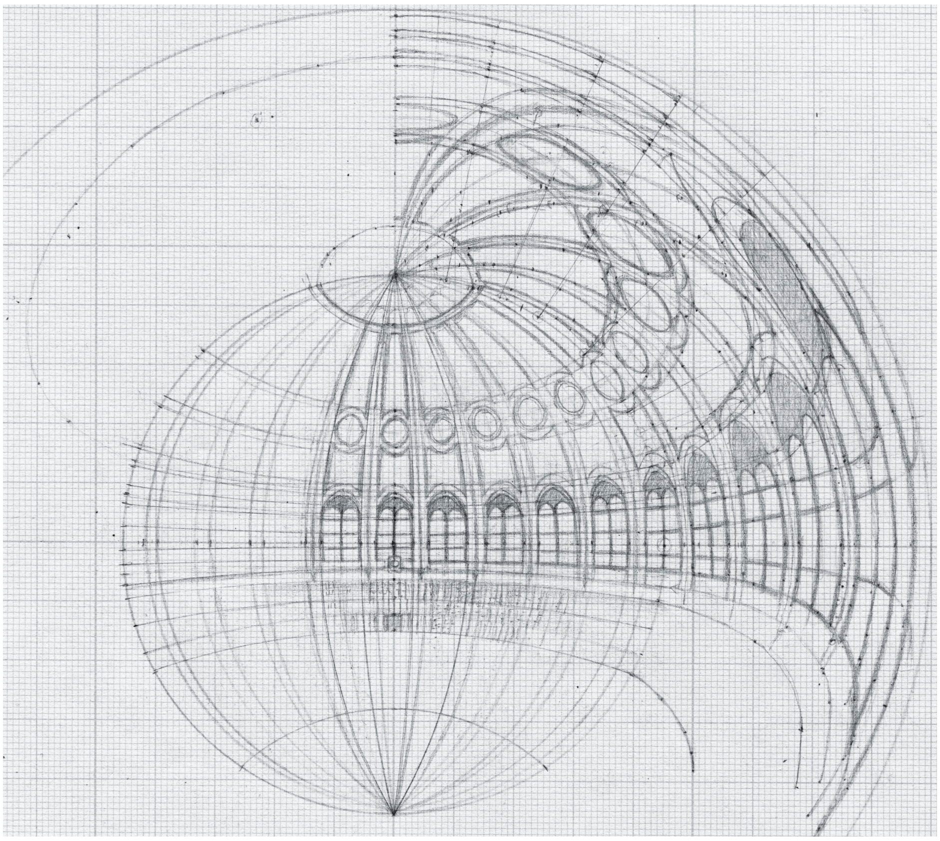


Figure 11. The dome of the British Museum's Reading Room seen from a point on its axis of rotational symmetry. The picture is composed around verticals and lines of constant elevation constructed through Proposition 4. Drawing by the author.

represents both the ground plane on the back view and the observer's plane on top view. There is a grid line coincident with the ground line, and the receding lines of the grid intersect it at points $\mathbf{P_i}$ whose images P_i^b in back view are uniformly spaced. Since $\mathbf{P_i}$ is on the observer's plane, P_i is obtained by intersecting ray $\overrightarrow{FP_i^b}$ with the equator by Construction 1. This ray, extended up to the blowup, is the perspective image of the central receding line of the grid that crosses $\mathbf{P_i}$. Thus the image of the receding lines of the grid is a set of radii l_i going from F to the blowup, through the uniformly spaced P_i^b. Note that this is analogous to the same construction in classical perspective, though with a different interpretation.

To plot the frontal lines of the grid we first trace a line g on the ground plane, such that g makes a 45 degree angle to the right of the observer and crosses D. On top view we see that g will diagonally cross a single square of each row of the grid. Hence it will touch each l_i at a vertex of the grid. We plot the great circle C of the plane defined by O and g. First we plot the anterior half by drawing the arc $C_a = DVU$ where V is the anterior vanishing point of g, that lies on the LR axis, 45 degrees to the right of F. At each intersection of C_a with an l_i we mark a vertex of the grid, G_i, and through it run a frontal line of the grid, drawing the arc of circle LG_iR. For the l_i that intersect C on the posterior ring, intersect the antipodal line of l_i (that is, the radius through $P_i^\star$) with C_a to get a point $G_i^\star$, and take the antipode to find G_i, the vertex in the posterior ring. Draw the auxiliary frontal line $RG_i^\star L$, then construct its antipodal line RG_iL, using the l_i as the natural measuring lines to draw its fat line approximation. This line RG_iL is the frontal posterior grid axis through G_i. In this way we can plot the full 360 degree grid to any given precision and extension. This construction is analogous to that of a 1-point perspective grid in linear perspective, but here we get four vanishing points (counting the blowup a single vanishing point), and we get six if we repeat the construction for the verticals (Figures 12, 13).

In Figure 13 we represent a tiled cubic room drawn from the point of view of an observer at its center, looking straight into the center of one of the walls. The whole setup is drawn very

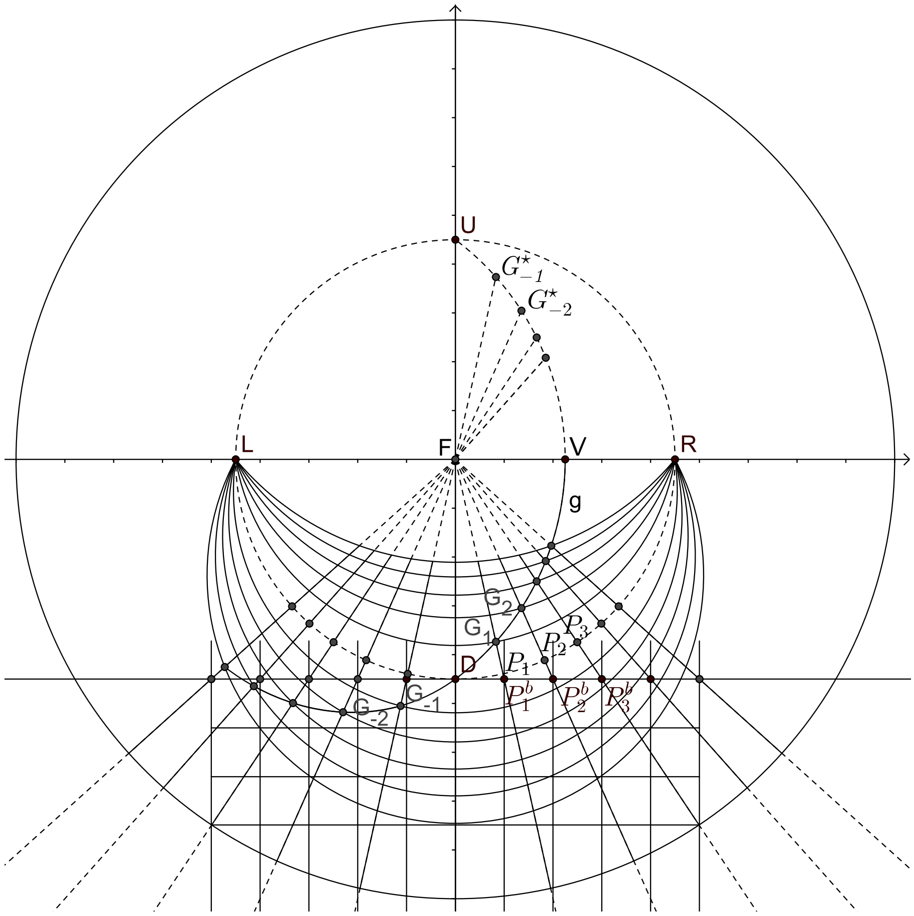


Figure 12. Construction of a uniform central perspective grid. Lines converge to four vanishing points: L, R, F, and B. Recall that the blowup circle identifies with a single vanishing point B through map $\tilde{\pi}$.

simply from a judicious use of vertical and horizontal lines at 45 degrees to the observer; these lines do double duty, as, for instance, the vertical at 45 degrees to the right of the observer has the same great circle as the horizontal that goes under the observer at a 45 degree angle to his right. The same basic grid, with some further refinements, was used to draw Figure 14.

Often we want to draw a grid oriented at some arbitrary angle to the central axis. In Figure 15 we draw a square $ABCE$ on a horizontal plane, below, behind, and to the left of the observer, such that one side of the square makes a 60 degree angle with $\overrightarrow{OF}$. Once again the perspective plane also represents the top and back views of the scene, in the same setup as above. On the top view we draw the square $ABCE$ and project its sides until they intersect the top view of the observer's plane. We draw lines from F to these intersection points and find their projections on the equator. We find the vanishing points, all on the horizontal measuring line, one set of lines converging to the points at 60° and −120° and the other to −30° and 150°. Through these points we find the arcs of circles corresponding to the lines that extend the sides of the square. From the arcs on the anterior perspective we obtain the corresponding fat lines of the posterior perspective. By intersecting these lines we find the perspective images of the points A, B, C, E. Finally, from this square we can plot a grid by an adaptation of the method already described.

Though grids are convenient, we note that our method has no need of them. We can plot all vanishing points and lines as we please. In Figure 1 there are several sets of arbitrary vanishing points and their lines, plotted individually without recourse to any supporting grid.

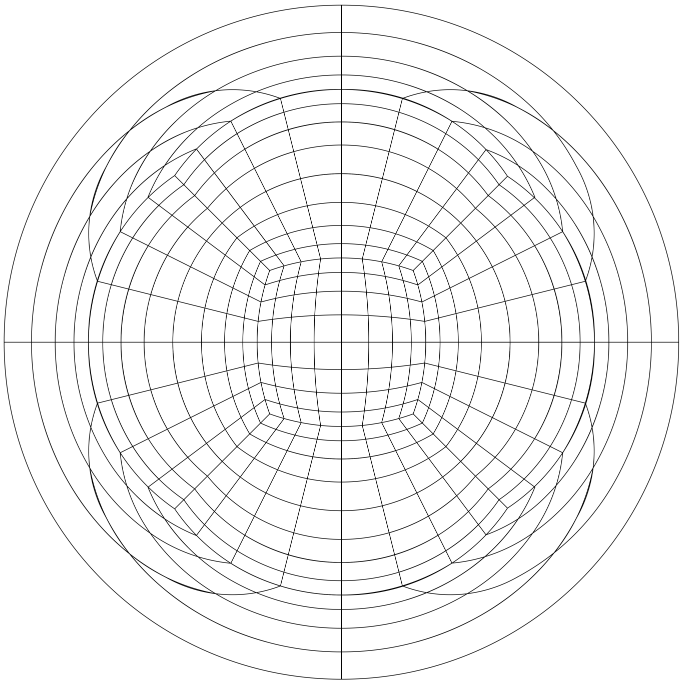


Figure 13. A six-point perspective drawing of a cubical box seen from its center. Lines go up, down, left, right, to the front and to the back of the observer. The blowup circle is seen as the single vanishing point B, behind the observer.

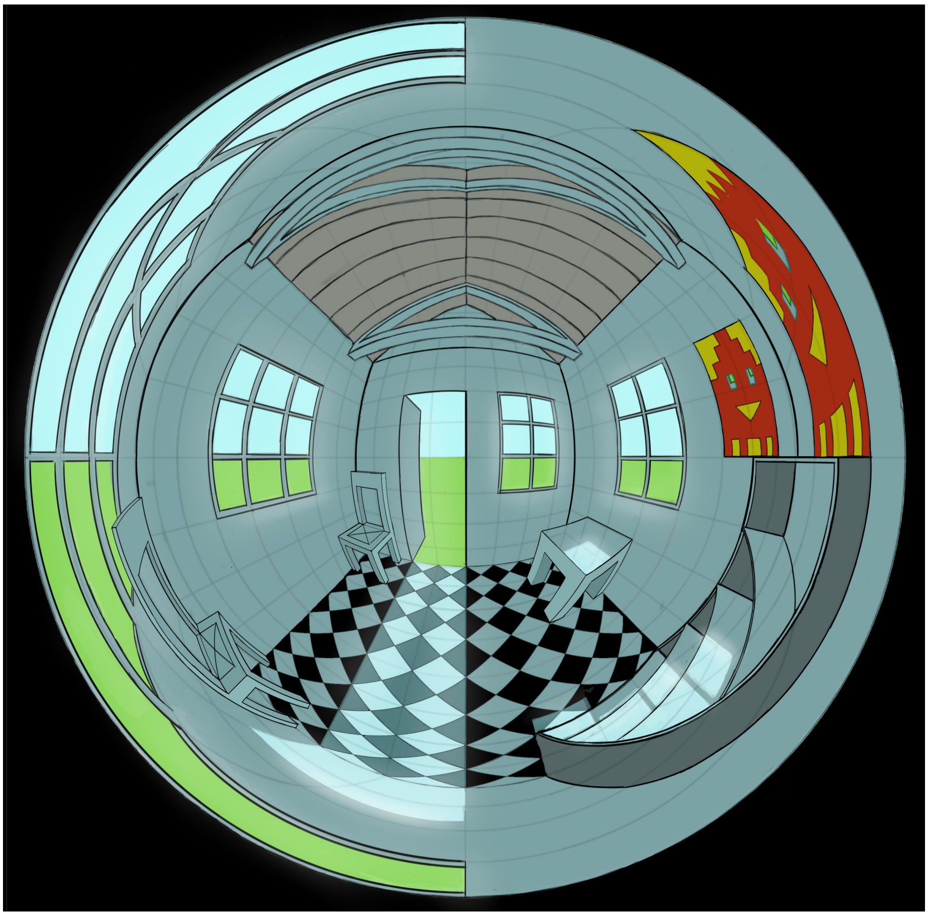


Figure 14. Room 45. Drawing by the author of a cubical room using the construction of Figure 13. The windows on the back and left walls have identical linear measurements, as do the pac-man figures on the right and back walls and the chairs on the front and back walls. This makes apparent the extent and nature of the deformations near the blowup.

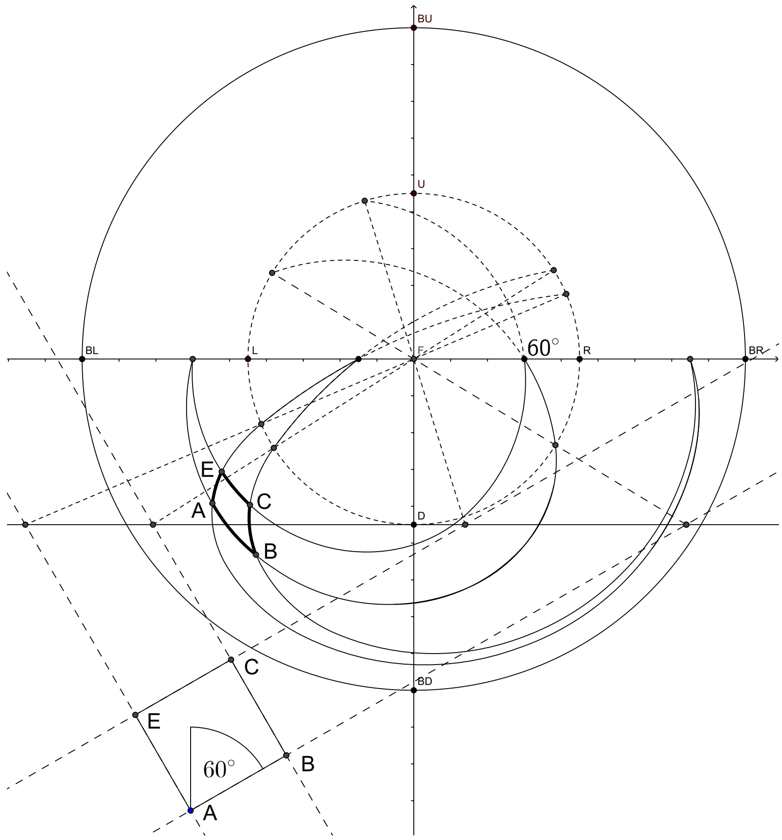


Figure 15. A square, below, behind, and to the left of the observer.

6. Comparison with reflections on a sphere

It is apparent from the plot of the cubical room in Figure 14 that our perspective bears striking resemblances to a reflection on a sphere [7]. It is natural to ask if there is a relation between the two. Recall how reflection works (Figure 16): an observer at E will see a point P reflected at a point R on the sphere such that R is on the intersection of the sphere with the plane EOP and $\angle(\overrightarrow{RP}, \overrightarrow{OR}) = \angle(\overrightarrow{RE}, \overrightarrow{OR})$ (angle of incidence equals angle of reflection).

General reflections are hard to calculate. Given R, it is easy to find the incident and reflected rays, but the inverse problem of obtaining R from P is non-trivial. In general it requires solving an algebraic equation of order four [9].

Also, occlusions are non-trivial. In Figure 16 we can see that points P and Q will have the same reflection R even though they are not in the same ray from either the center O or the observer E. This implies that a general reflection is not a central perspective. Recall that in central perspectives occlusions are always radial since they are determined at the anamorphosis step, whatever the flattening may be.

Finally, total spherical perspective has an angle of view of 360°, while the angle of view captured by a reflection depends on the distance of the observer to the sphere. The points of the sphere define a cone with the observer E at the vertex, the cone of shadow, and every point outside of this cone of shadow will be viewable on the sphere. The field of view will be $360° - \delta$ with $\delta = 2\sin^{-1}(r/d)$, where r is the radius of the sphere and d the distance of the observer from the center of the sphere.

There is however a limiting case where spherical perspective and reflection on a sphere become quite similar. Imagine either moving away from the sphere (preserving its apparent size by looking at it through a telescope) or shrinking it (and seeing it through a microscope). Then r becomes small compared to d and, in the limit $r/d \to 0$, we get a 360° angle of view. ER becomes parallel to EO, the angle of reflexion α becomes equal to $\beta = \angle EOR$, and $\angle ERP \to 2\alpha$ (Figure 17). If

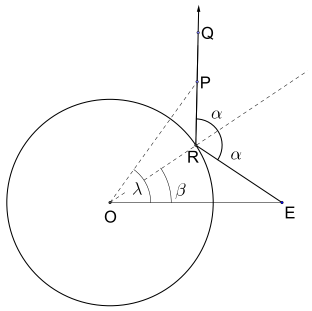


Figure 16. Non-radial occlusion. Points P and Q both project to R although they are not in the same ray from E or O.

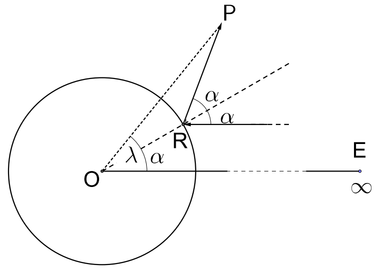


Figure 17. With E at infinity all rays become parallel and angle β becomes equal to α. If P goes to infinity (with fixed λ) then α goes to λ.

furthermore $r \to 0$ (an infinitesimal sphere) or $r/|OP| \to 0$ while $\lambda = \angle EOP$ remains constant (reflection of points on the celestial sphere) then $\lambda \to 2\alpha$. In this limit, the projection becomes radial (therefore making occlusions trivial), and the whole space of directions is mapped onto the hemisphere visible from E. This can be seen as a sphere anamorphosis followed by a uniform contraction onto a hemisphere by halving the angle $\angle EOR$ of each point R of the sphere.

Seen from point E, since all rays ER are parallel to the axis OE, the reflection will look like the orthogonal projection along OE of the image on the sphere. Hence the reflection, seen from E, is anamorphically equivalent to a central perspective (central with respect to O, not E) obtained by anamorphosis onto the sphere followed by a flattening which is the composition of a uniform compression onto a hemisphere followed by an orthogonal projection. In the spherical coordinates of Equation 2 (with the y axis on $\overrightarrow{OE}$ and x, z in the perpendicular plane through O) and rescaling the sphere to $r = 1$, this perspective is the map

$$(\rho, \lambda, \theta) \mapsto (1, \lambda, \theta) \mapsto (1, \lambda/2, \theta) \mapsto \sin(\lambda/2)(cos(\theta), sin(\theta))$$

where the first map is the anamorphosis, the second is the crunching into the anterior hemisphere and the last step is the orthogonal projection onto the disc perpendicular to EO at O.

This is a 360 degree perspective, but different from our spherical perspective. It is not linear along λ, squashing the outer angles more, and cannot be easily used for drawing by hand without the help of pre-computed grids (since we lose the isometry along measuring lines). But we can see why there is a qualitative similarity between the two.

It has been noted in [9] that reflections on a sphere could be used as a form of wide angle perspective. This is well inspired in art history, as reflections drawn from observation of convex mirrors have been the time honoured tool of the artist to represent a wide angle of view; M.C. Escher's *Hand with Reflecting Sphere* (1935) print is probably both the clearest and most well-known example of this device, but recall the picture-within-picture effects in Domenico Remps's *Cabinet of Curiosities* (c. 1690) and Jan van Eyck's *Portrait of John Arnolfini and his Wife* (1434).

But we have seen the difficulties in this approach. First, reflections are hard to calculate. Second, they are not central perspectives, and they have non-trivial occlusions. As noted in [9], this causes difficulties for hidden-face removal algorithms. Even in the limit presented above, where it becomes a central perspective, it is clear that a sphere reflection only makes for a practical perspective for the draughtsman when drawn from observation of an actual sphere.

Spherical perspective is a much more natural proposal for a wide perspective. It allows for up to a 360 degree view, it is easily computed by Equation 1, it is a central perspective and therefore has trivial occlusions, so hidden-face algorithms will work exactly as in the classical case, being calculated at the anamorphosis step. Most important for our purposes, it lends itself to be used by an artist armed only with a ruler, compass, and nail. With some practice even these instruments can be abandoned in favour of reasonably intuitive and accurate freehand drawing from either nature or the imagination.

Supplementary materials

Further notes, computer code and illustrations will be made available at the author's web page at `http://www.univ-ab.pt/~aaraujo/full360.html`

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