DYNAMICS OF A DIFFERENTIAL SYSTEM USING INVARIANT REGIONS

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ABSTRACT. The long-time behaviour of a two dimensional system of ordinary differential equations with singularities is studied using conveniently defined positively invariant sets and auxiliary functions. The approach uses only elementary techniques of phase plane analysis and provides a good geometric insight into the dynamical behaviour of the system. It provides dynamical information analogous to what is usually obtained via centre manifold techniques but does not require the flow to be defined at the limit point.

1. INTRODUCTION

In this paper we present a study of the long-time behaviour of solutions to the following two dimensional ordinary differential system arising from coagulation theory

$$
(1.1) \qquad \begin{cases} \dot{x} &= v - x^2 \\ \dot{v} &= \alpha x - 2xv - \alpha \dfrac{v}{x} + \dfrac{v^2}{x}, \end{cases}
$$

where $\alpha > 0$ is a constant, and $(x, v) \in \mathbf{R}^+ \times \mathbf{R}$. In this system the interesting feature of the dynamical behaviour is the convergence to a singular point of the phase space, namely $(0, 0)$, and the details of this convergence. A study of (1.1) was recently completed in [1] using a centre manifold analysis after a convenient desingularization via a time-scale change. A somewhat different,

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but related, nonautonomous system was also recently considered in [2, Eq. (14)].

In the present paper we study the behaviour of the solutions to (1.1) using the same type of geometric approach, based on the monotonicity properties of auxiliary functions, as well as on the positive invariance under the flow of conveniently (and naturally) defined subsets of $\mathbf{R}^+ \times \mathbf{R}$. This approach, bypassing the need to regularize (1.1) gives a much clearer geometric picture of (1.1) thus allowing for a better insight into its dynamical behaviour and, furthermore, uses only elementary tools, and could, in principle, be used in situations where centre manifold analysis is definitely not applicable.

2. The ODE systems and the main result

In the study of particles undergoing coagulation (see [1] for details), we are led to the following system of differential equations

$$\begin{cases} \dot{y} &= \alpha - xy \\ \dot{x} &= \alpha - xy - x^2, \end{cases} \tag{2.1}$$

where α is a positive constant, and x and y representing physical concentrations must be non negative.

We start by looking at the gross features of the asymptotic behaviour of solutions to (2.1).

Proposition 2.1. *For every nonnegative solution (x,y) of (2.1) the following holds true as $t \to +\infty$: $x(t) \to 0,\ y(t) \to +\infty,$ and $x(t)y(t) \to \alpha$.*

The proof of this result uses only elementary phase plane analysis tools: the tubular flow theorem and the positive invariance under the flow of some subsets of the phase plane. It has already been published in [1] but, since it is very short and resorts to the same type of geometric arguments used later, we shall include it in the next Section, thus also turning the current paper more self-contained.

The behaviour given in Proposition 2.1 is not quite enough for the envisaged application to coagulation systems, and information concerning the rate of approach to the limits is crucial.

Proposition 2.2. *For every nonnegative solution (x,y) of (2.1) we have:*

(i) $\lim_{t\to+\infty}\left(\frac{3}{\alpha}t\right)^{1/3} x(t) = 1$

(ii) $\lim_{t\to+\infty}\left(3\alpha^2 t\right)^{-1/3} y(t) = 1$

(iii) $\lim_{t\to+\infty}\left(\frac{3}{\alpha}t\right)^{2/3} (\alpha - x(t)y(t)) = 1.$

In order to prove this proposition, which deals with the approach to the limit point at infinity, it is convenient to map the limit point at infinity to a point in the phase plane. In this case, and suggested by the result in Proposition 2.1, it is natural to consider the variable $v := \alpha - xy$ and to perform the change of variables $(x, y) \mapsto (x, v)$, which, according to Proposition 2.1, corresponds to mapping the limit point $(0, +\infty)$ to $(0, 0)$. Under this change of variables system (2.1) becomes (1.1) and the region of interest, corresponding to $(x, y) \in \mathbf{R}^+ \times \mathbf{R}^+$, is $(x, v) \in \mathbf{R}^+ \times (-\infty, \alpha)$. Observe that we know, from Proposition 2.1 and the definition of v, that all solutions to (1.1) converge to the origin as $t \to +\infty$. Also, statements (i) and (iii) have direct equivalents for the new variables x (which is actually the same) and v, and (ii) can be easily rephrased by noting that $y = (\alpha - v)/x$. So, the proof of Propostition 2.2 can be done by working directly with system (1.1) in a neighborhood of the singular point $(0, 0)$. This will be done in the next Section using, as main tools, geometric information provided by the positive invariance of certain subsets of the phase plane and analytic information derived from the evolution of some auxiliary functions.

3. PROOFS

Proof of Proposition 2.1 : As stated in the previous section, this proof is based on very basic tools from qualitative theory. First observe that (2.1) does not have equilibria. From the inequalities $\dot{x} = \alpha > 0$ when $x = 0$, and $\dot{y} = \alpha > 0$ when $y = 0$ we immediately conclude the positive invariance of $\mathbf{R}^+ \times \mathbf{R}^+$ of the phase plane. Let Ω be the connected subset of $\mathbf{R}^+ \times \mathbf{R}^+$ whose boundary is $\{y = 0\} \cup \{x = 0\} \cup \{xy = \alpha\}$. Since we have $\dot{y} = 0$ and $\dot{x} = -x^2 < 0$ for points on $\{xy = \alpha\}$, we conclude Ω is positively invariant for the flow of (2.1) (see Figure 1).

Consider first the initial data in the closure of Ω and let Ω_1 be the subset of Ω defined by

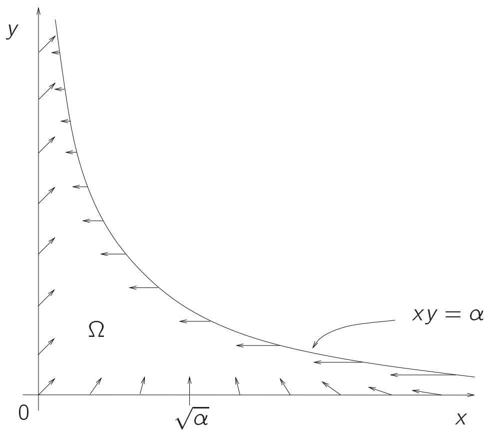

FIGURE 1
Region Ω with a sketch of the flow in $\partial\Omega$.

$$\Omega_1 := \left\{(x,y) \in \Omega : \max\left\{0, \frac{\alpha}{x} - x\right\} \leq y \leq \frac{\alpha}{x}\right\}.$$

Since the flow of (2.1) satisfies $\dot{x} = 0$ and $\dot{y} = x^2 > 0$ on points of $\partial\Omega_1 \setminus \partial\Omega$ we conclude the set Ω_1 is positively invariant. From the absence of equilibria, the inequalities $\dot{x} > 0$ and $\dot{y} > 0$, valid in $\Omega \setminus \Omega_1$, and the tubular flow theorem, we conclude that for any initial condition in Ω the corresponding orbit will eventually enter Ω_1 (see Figure 2).

From this we immediately conclude that, as $t \to +\infty$, we have $x(t) \to 0$ and $y(t) \to +\infty$. Furthermore, for all initial data in Ω, there exists a T (dependent on the initial condition) such that, for all $t > T$, the orbit is in Ω_1, and so

$$t > T \Rightarrow \frac{\alpha}{x} - x \leq y \leq \frac{\alpha}{x} \Leftrightarrow \alpha - x^2 \leq xy \leq \alpha.$$

Letting $t \to +\infty$ and using $x(t) \to 0$ as $t \to +\infty$, we get

$$\lim_{t\to+\infty} x(t)y(t) = \alpha.$$

Consider now initial data $(x_0, y_0) \in \Omega_2 = \mathbf{R}^+ \times \mathbf{R}^+ \setminus \Omega$. Fix $K_1 > x_0$, $K_2 > y_0$ and let $\Omega_2(K_1, K_2) = \Omega_2 \cap ([0, K_1] \times [0, K_2])$.

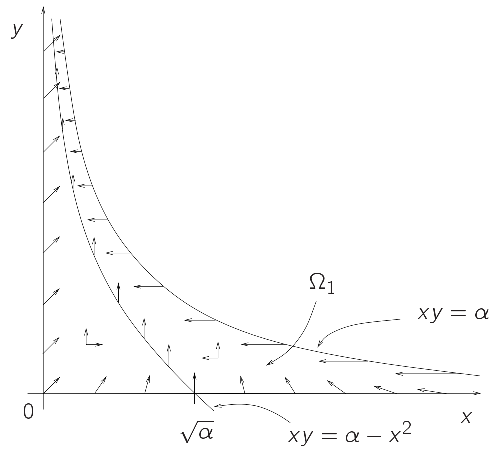

FIGURE 2
Region Ω_1 with a sketch of the flow in $\partial\Omega, \partial\Omega_1$ and in Ω.

By the analysis of the flow in $\partial\Omega_2(K_1, K_2)$ and the tubular flow theorem we conclude the orbit will eventually enter Ω_1 (see Figure 3) and so the previous analysis apply. This concludes the proof. ■

Proof of Proposition 2.2: Considering the change of variables $(x, y) \mapsto (x, v)$ introduced in the previous section, we rephrase part (iii) of Proposition 2.2 as $\lim_{t\to+\infty} (3t\alpha)^{2/3}\, v(t) = 1$. From Proposition 2.1 we know that all orbits will eventually enter Ω_1, i.e., $xy < \alpha$, for sufficiently large times (depending on the orbit). So we need only to consider $v \in (0, \alpha)$. Define the set $A := \mathbf{R}^+ \times (0, \alpha)$. The analysis of the flow of (1.1) on ∂A gives immediately the positive invariance of A.

For the study of the flow of (1.1) in A observe that $\dot{v} = 0 \Leftrightarrow v = v_-(x) := \left(x^2 + \frac{\alpha}{2}\right) - \sqrt{x^4 + \left(\frac{\alpha}{2}\right)^2}$. Let $v_0(x) := x^2$. The analysis of the flow is presented in Figure 4, from which we conclude that the set

$$\widetilde{A} := A \cap \{(x, v) : v_-(x) \leq v \leq v_0(x)\}$$

is positively invariant and every orbit will eventually enter $\widetilde{A}$ for sufficiently

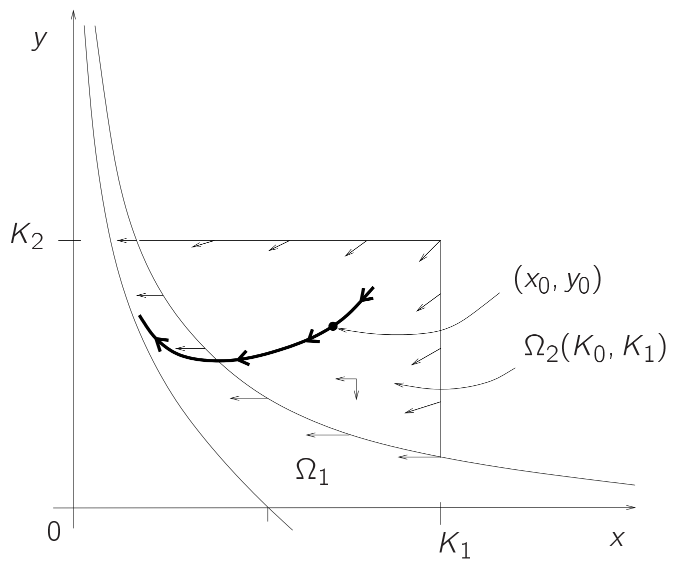

FIGURE 3
Flow in the set $\Omega_2(K_1, K_2)$ surrounding an initial point (x_0, y_0) outside Ω.

large times.

So, for sufficiently large times, solutions satisfy $v \in \big(v_-(x), v_0(x)\big)$.

The upper bound does not give new information about the behaviour of the solutions, since it just implies $x(t)$ is eventually decreasing: $v < v_0(x) \Leftrightarrow v < x^2 \Leftrightarrow v - x^2 < 0 \Leftrightarrow \dot{x} < 0$.

On the other hand, the lower bound $v > v_-(x)$ is a lot more useful: we start by observing that, for all x sufficiently small we can write, using the binomial expansion,

$$v_-(x) = x^2 - \frac{1}{\alpha}x^4 + \frac{1}{\alpha^3}x^8 + O(x^{12}),$$

and denoting by $v_{1/\alpha}(x)$ the second term cut-off, namely $v_{1/\alpha}(x) := x^2 - \frac{1}{\alpha}x^4$, we have, for all sufficiently small x, $v_{1/\alpha}(x) < v_-(x)$. Let us now compute

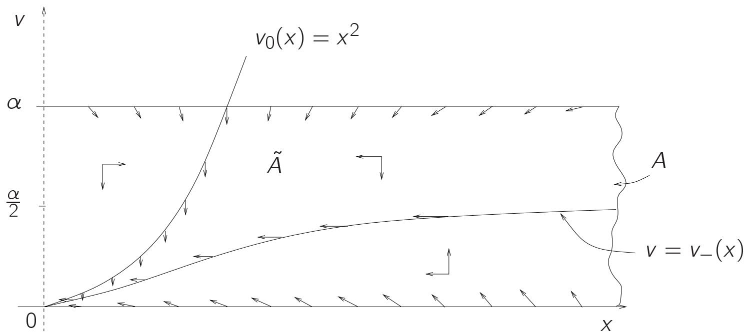

FIGURE 4
Flow of (1.1) in A and in $\widetilde{A}$.

the slope of the orbits at points $(x, v) = \big(x, v_{1/\alpha}(x)\big)$: from (1.1) we have $\dot{x} = -\frac{1}{\alpha}x^4$ and $\dot{v} = \frac{1}{\alpha^2}x^7$ and thus $\dfrac{dv}{dx} = \dfrac{\dot{v}}{\dot{x}} = -\dfrac{1}{\alpha}x^3 < 0$. Since $v'_{1/\alpha}(x) = 2x - \frac{4}{\alpha}x^3 > 0$ for all $x \in \big(0, \sqrt{\frac{\alpha}{2}}\big)$, we conclude that the set

$$A_{1/\alpha} := \left\{(x, v) \in A : x < \sqrt{\frac{\alpha}{2}},\, v_{1/\alpha}(x) < v < v_0(x)\right\}$$

is positively invariant and contains all positive semiorbits for sufficiently large times (see Figure 5).

Take any orbit of (1.1) and let t_0 be a time such that $(x(t_0), v(t_0)) \in A_{1/\alpha}$. Denote $x(t_0)$ by x_0. Using the first equation of (1.1) and the bound $v > v_{1/\alpha}(x)$ we obtain the differential inequality $\dot{x} > -\frac{1}{\alpha}x^4$, which, after integration, yields

$$x(t) \geq \frac{1}{\big(x_0^{-3} + \frac{3}{\alpha}(t - t_0)\big)^{1/3}}\,, \quad \forall t \geq t_0. \tag{3.1}$$

Now, to complete the proof, we need to obtain an upper bound of the same type. In fact, we are going to prove that, for all solutions $(x(t), v(t))$ and all $\beta \in \big(0, \frac{1}{\alpha}\big)$, there exists a $t_\beta \geq t_0$ such that, for all $t \geq t_\beta$ we have

$$x(t) \leq \frac{1}{\Big(x_\beta^{-3} + 3\beta(t - t_\beta)\Big)^{1/3}}\,, \quad \forall t \geq t_\beta \geq t_0, \tag{3.2}$$

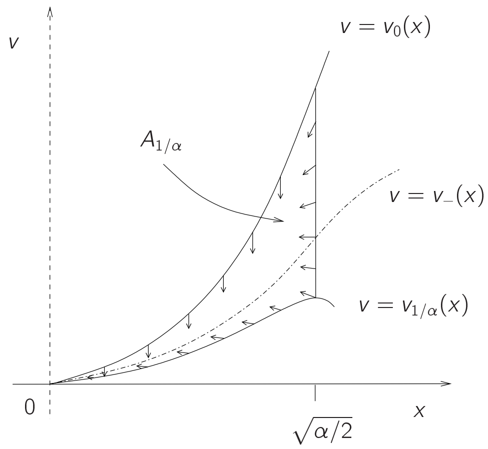

FIGURE 5
Flow of (1.1) on $\partial A_{1/\alpha}$ defined in the proof.

where $x_\beta = x(t_\beta)$. So fix $\beta \in \left(0, \frac{1}{\alpha}\right)$. Consider the curve $v_\beta(x) := x^2 - \beta x^4$. The slopes of the orbits of (1.1) on points $(x, v_\beta(x))$ are obtained from $\dot{x} = v_\beta(x) - x^2 = -\beta x^4$ and $\dot{v} = \alpha x - 2x v_\beta(x) - \alpha \frac{v_\beta(x)}{x} + \frac{v_\beta(x)^2}{x} = -(1-\alpha\beta)x^3 + \beta^2 x^7$, and are given by

$$\frac{dv}{dx} = \frac{\dot{v}}{\dot{x}} = \frac{1-\alpha\beta}{\beta x} - \beta x^3 \longrightarrow +\infty \quad \text{as } x \to 0^+.$$

Since the slope of $v_\beta(x)$ is $v'_\beta(x) = 2x - 4\beta x^3$, and $v'_\beta(x) \to 0$ as $x \to 0^+$, we conclude that, for sufficiently small x, $\frac{dv}{dx} > v'_\beta(x)$. Observe that, since $\{\text{graph}(v_\beta)\}_{\beta\in(0,1/\alpha)}$ is a foliation of $A_{1/\alpha}$ we have, locally, when $\frac{dv}{dx} > v'_\beta(x)$, a situation like the one depicted in Figure 6. Note that the intersections of the orbits with the graphs of the functions v_β are transversal.

We need to determine the points in $A_{1/\alpha}$ for which $\frac{dv}{dx}$ is equal to $v'_\beta(x)$. From the results above, these points are given by

$$\frac{1-\alpha\beta}{\beta x} - \beta x^3 = \frac{dv}{dx} = v'_\beta(x) = 2x - 4\beta x^3$$

and hence

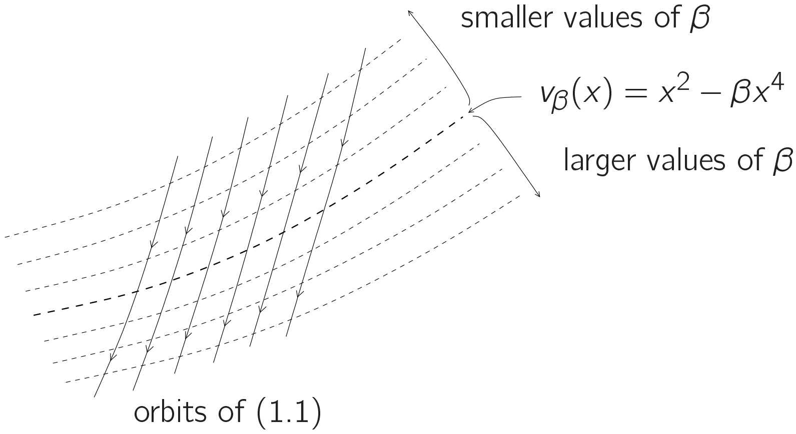

FIGURE 6
Local picture when $\frac{dv}{dx} > v'_\beta(x)$.

$$x^2 = \frac{1 \pm \sqrt{1 - 3(1 - \alpha\beta)}}{3\beta}.$$

Consequently, if $1 - 3(1 - \alpha\beta) < 0 \Leftrightarrow \beta < \frac{2}{3}\frac{1}{\alpha}$, we conclude that $\frac{dv}{dx} > v'_\beta(x)$ always. For $\beta = \frac{2}{3}\frac{1}{\alpha}$ there is a single value of x, namely $x = \sqrt{\frac{\alpha}{2}}$ for which $\frac{dv}{dx} = v'_\beta(x)$, with $\frac{dv}{dx} > v'_\beta(x)$ for all other values of x. Finally, it is not difficult to see that, if $\beta \in \left(\frac{2}{3}\frac{1}{\alpha}, \frac{1}{\alpha}\right)$ we have $\frac{dv}{dx} > v'_\beta(x)$ for $x \in (0, x^*(\beta))$ where

$$x^*(\beta) = \sqrt{\frac{1 - \sqrt{1 - 3(1 - \alpha\beta)}}{3\beta}}.$$

Denote by $v^*(x)$ the curve in $A_{1/\alpha}$ for which $\frac{dv}{dx} = v'_\beta(x)$. From what was done above,

$$v^*(x) = x^2 - \frac{(\alpha + 2x^2) - \sqrt{(\alpha + 2x^2)^2 - 12x^4}}{6}$$

with $x \in \left(0, \sqrt{\frac{\alpha}{2}}\right]$, and the set $A_{1/\alpha}$ can be partitioned into the disjoint union

$$A_{1/\alpha} = A^-_{1/\alpha} \cup \{\text{graph}(v^*)\} \cup A^+_{1/\alpha},$$

where $A^-_{1/\alpha} := A_{1/\alpha} \cap \{v < v^*(x)\}$ and $A^+_{1/\alpha} := A_{1/\alpha} \cap \{v > v^*(x)\}$. For every point $(x(t), v(t))$ of any orbit in $A_{1/\alpha}$, we can consider the value of the function $\beta = \beta(t) := \frac{x^2 - v}{x^4}$. We conclude from the results above that $\beta(t)$ is monotonic decreasing in $A^-_{1/\alpha}$ and monotonic increasing in $A^+_{1/\alpha}$. In Figure 7 we collect the information obtained so far concerning the behaviour of $\beta(t)$ along solutions in $A_{1/\alpha}$.

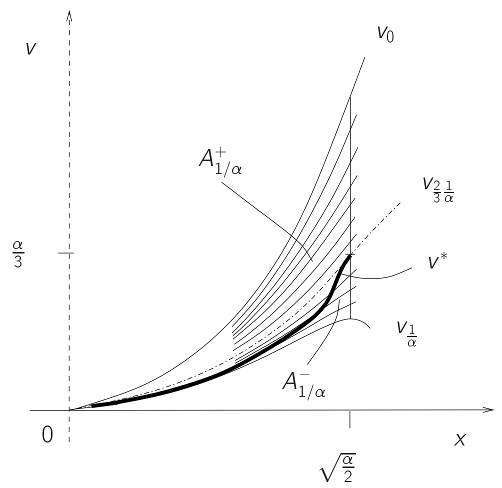

FIGURE 7

Behaviour of $\beta(t)$ along solutions of (1.1) in $A_{1/\alpha}$: $\beta(t)$ is increasing in $A^+_{1/\alpha}$ and decreasing in $A^-_{1/\alpha}$. In the figure are also shown the graphs of functions v_β for several values of β.

From these results we easily conclude that every orbit in $A_{1/\alpha}$ will eventually enter $A^+_{1/\alpha}$ for sufficiently large time, and remain there afterwards. Hence, along solutions, the value of $\beta(t)$ is eventually increasing. Our goal is to prove that $\beta(t) \uparrow 1/\alpha$ as $t \to +\infty$. By the definition of $\beta(t)$ and (1.1) we obtain the following evolution equation for β :

$$\dot{\beta} = \left(2x - 3\frac{v}{x} - \frac{\alpha}{x}\right)\beta + \frac{v}{x^3},$$

and so

$$\dot{\beta} - 2x\beta = -\left(3\frac{v}{x} + \frac{\alpha}{x}\right)\beta + \frac{v}{x^3}. \tag{3.3}$$

In $A^+_{1/\alpha}$ the function $\beta(t)$ is increasing and bounded above by $1/\alpha$; hence $\beta(t)$ converges as $t \to +\infty$ and also $\lim\limits_{t\to+\infty} \dot{\beta}(t) = 0$. Since $x(t) \to 0$ as

$t \to +\infty$, the limit inferior as $t \to +\infty$ of the left-hand side of (3.3) is zero. Because $\beta(t)$ is convergent, v/x is positive, and $\alpha/x \to +\infty$ as $t \to +\infty$, we thus conclude from (3.3) that

$$\lim_{t\to+\infty} \beta(t) = \lim_{t\to+\infty} \frac{1}{3v+\alpha}\frac{v}{x^2} = \frac{1}{\alpha}\lim_{t\to+\infty}\frac{v}{x^2}, \tag{3.4}$$

where in the last equality we used our knowledge that $v(t) \to 0$ as $t \to +\infty$. Suppose now that $\beta(t)$ does not converge to $1/\alpha$ but to some smaller number, $\frac{1}{\alpha} - \varepsilon$, for some $\varepsilon > 0$. Then, from (3.4),

$$\frac{1}{\alpha} - \varepsilon = \frac{1}{\alpha}\lim_{t\to+\infty}\frac{v}{x^2} \Longleftrightarrow \lim_{t\to+\infty}\frac{v}{x^2} = 1 - \varepsilon\alpha < 1.$$

But, from the positive invariance of $A^+_{1/\alpha}$ and the fact that $v^*(x) > v_{1/\alpha}(x)$ we conclude that, in $A^+_{1/\alpha}$,

$$v > x^2 - \frac{1}{\alpha}x^4 \Leftrightarrow \frac{v}{x^2} > 1 - \frac{1}{\alpha}x^2 \Longrightarrow \lim_{t\to+\infty}\frac{v}{x^2} \geq 1.$$

This contradiction implies we must have $\beta(t) \to 1/\alpha$ as $t \to +\infty$.

Now it is straightforward to obtain inequality (3.2): Pick any $\widetilde{\beta}$ arbitrarily close to, and smaller than, $1/\alpha$. Without loss of generality, consider any initial data in $A_{1/\alpha}$. The corresponding positive semi-orbit will enter $A^+_{1/\alpha}$ after a sufficiently long time. Once there, $\beta(t)$ is increasing along the solution and converges to $1/\alpha$. Thus there must exist a $t_{\widetilde{\beta}} \geq t_0$ such that $\beta(t_{\widetilde{\beta}}) = \widetilde{\beta}$ and, for $(x(t), v(t))$ a point of the orbit, we have $v(t) < v_{\widetilde{\beta}}(x(t))$ for all $t > t_{\widetilde{\beta}}$, where $v_{\widetilde{\beta}}(x) = x^2 - \widetilde{\beta}x^4$. Substituting this bound in the equation for $x(t)$ in (1.1), integrating the differential inequality thus obtained, and changing $\widetilde{\beta}$ to β, we obtain the desired result (3.2).

From (3.1) and (3.2) we can write

$$\frac{1}{\left(x_0^{-3} + \frac{3}{\alpha}(t-t_0)\right)^{1/3}} \leq x(t) \leq \frac{1}{\left(x_\beta^{-3} + 3\beta(t-t_\beta)\right)^{1/3}}, \quad \forall t \geq t_\beta \geq t_0. \tag{3.5}$$

Multiplying (3.5) by $\left(\frac{3}{\alpha}t\right)^{1/3}$ and taking $\underline{\lim}_{t\to+\infty}$ and $\overline{\lim}_{t\to+\infty}$ we obtain

$$1 \leq \underline{\lim}_{t\to+\infty}\left(\frac{3}{\alpha}t\right)^{1/3}x(t) \leq \overline{\lim}_{t\to+\infty}\left(\frac{3}{\alpha}t\right)^{1/3}x(t) \leq \frac{1}{\alpha\beta},$$

and since $\beta < 1/\alpha$ is arbitrary, we conclude that

$$\lim_{t\to+\infty}\left(\frac{3}{\alpha}t\right)^{1/3}x(t)=1,$$

which proves (i). Returning to the bound $v_{1/\alpha}<v<v_\beta<v_0$, valid, along any given orbit, for all sufficiently large t, we have

$$x^2(t)-\frac{1}{\alpha}x^4(t)\le v(t)\le x^2(t)-\beta x^4(t)<x^2(t).$$

Using (3.5) we obtain, for all $t\ge t_\beta\ge t_0$,

$$\frac{1}{\left(x_0^{-3}+\frac{3}{\alpha}(t-t_0)\right)^{2/3}}-\frac{1}{\alpha}\frac{1}{\left(x_\beta^{-3}+3\beta(t-t_\beta)\right)^{4/3}}\le$$
$$\le\quad v(t)\quad\le\frac{1}{\left(x_\beta^{-3}+3\beta(t-t_\beta)\right)^{2/3}}.$$

Multiplying by $\left(\frac{3}{\alpha}t\right)^{2/3}$ and taking $\varliminf_{t\to+\infty}$ and $\varlimsup_{t\to+\infty}$ we conclude, as above, by the arbitrariness of $\beta<1/\alpha$, that

$$\lim_{t\to+\infty}\left(\frac{3}{\alpha}t\right)^{2/3}v(t)=1,$$

which establishes (iii). Finally, to prove (ii), observe that for the original variable $y(t)$ we have $y=\frac{\alpha-v}{x}$ and thus, as $t\to+\infty$,

$$(3\alpha^2t)^{-1/3}y(t)=\frac{1}{\left(\frac{3}{\alpha}t\right)^{1/3}x(t)}-\frac{\left(\frac{3}{\alpha}t\right)^{2/3}v(t)}{\left(\frac{3}{\alpha}t\right)^{1/3}x(t)}\,\frac{1}{\alpha\left(\frac{3}{\alpha}t\right)^{2/3}}\to 1,$$

which concludes the proof. ■

REFERENCES

[1] F.P. da Costa, H. van Roessel, J.A.D. Wattis, Long-Time Behaviour and Self-Similarity in a Coagulation Equation With Input of Monomers, Markov Process. Related Fields, **12** (2006) 367-398.

[2] F.P. da Costa, R. Sasportes, Dynamics of a Nonautonomous ODE System Occurring in Coagulation Theory, J. Dyn. Differential Equations, (published online : 3 January 2007, DOI : 10.1007/s10884-006-9067-5).

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