Stability of finite difference schemes for nonlinear complex reaction-diffusion processes*

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Abstract

In this paper we establish the stability condition of a general class of finite difference schemes applied to nonlinear complex reaction-diffusion equations. We consider the numerical solution of both implicit and semi-implicit discretizations. To illustrate the theoretical results we present some numerical examples computed with a semi-implicit scheme applied to a nonlinear equation.

1 Introduction

Complex diffusion is a commonly used denoising procedure in image processing [6]. In particular, nonlinear complex diffusion proved to be a numerically well conditioned technique that has been successfully applied in medical imaging despeckling [3]. The stability condition for finite difference methods applied to the linear diffusion equation has been investigated extensively and it is widely

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documented in literature (see *e.g.* [10, 12]). A stability result for the linear complex case was derived in [5].

The stability properties of a class of finite difference schemes for the nonlinear complex diffusion equation, were studied in [2], where only the explicit and implicit scheme where studied and no reaction term was considered. In this paper we extend those results for nonlinear complex reaction-diffusion equations, considering discretizations also with a semi-implicit finite difference scheme, in addition to the explicit and implicit schemes. Applications of interest include diffusion processes which are commonly used in image processing, as for example in noise removal, inpainting, stereo vision or optical flow (see *e.g.* [3, 4, 6, 7, 8, 9, 13, 14, 15]). Complex difusion with reactive term appears also in the well-known Schrödinger equation, though conservative numerical methods are usually used instead of the finite difference approach [1, 11].

Let Ω be a bounded open set in $\mathbb{R}^d$, $d \geq 1$, with boundary $\Gamma = \partial\Omega$. Typically Ω is the cartesian product of open intervals in $\mathbb{R}$, i.e.,

$$\Omega = \prod_{j=1}^{d} (a_j, b_j), \tag{1}$$

with $a_j, b_j \in \mathbb{R}$. Let $Q = \Omega \times (0, T]$, with $T > 0$, and $v : \bar{Q} = \bar{\Omega} \times [0, T] \longrightarrow \mathbb{C}$. We consider a reaction diffusion process with a non-constant complex coefficient $D(x, t, v) = D_R(x, t, v) + iD_I(x, t, v)$ and non-constant complex reaction term $F(x, t, v) = F_R(x, t, v) + iF_I(x, t, v)$, where $D_R(x, t, v)$, $D_I(x, t, v)$, $F_R(x, t, v)$, $F_I(x, t, v)$ are real functions dependent on v. We need to assume that

$$D_R(x, t, v) \geq 0, \qquad (x, t) \in \bar{Q}, \tag{2}$$

and that there exists a constant $L > 0$ such that

$$|D(x, t, v)| \leq L, \qquad (x, t) \in \bar{Q}. \tag{3}$$

These inequalities (2) and (3) can easily be shown to hold for the diffusion coefficient in [3] and [6].

We define the initial boundary value problem for the unknown complex function u

$$\frac{\partial u}{\partial t}(x, t) = \nabla \cdot (D(x, t, u)\nabla u(x, t)) + F(x, t, u), \quad (x, t) \in Q, \tag{4}$$

under the initial condition

$$u(x, 0) = u^0(x), \quad x \in \overline{\Omega}, \tag{5}$$

and with either the Dirichlet boundary condition

$$u(x, t) = 0, \quad x \in \Gamma, \quad t \in [0, T], \tag{6}$$

or the Neumann boundary condition

$$\frac{\partial u}{\partial \nu}(x, t) = 0, \quad x \in \Gamma, \quad t \in [0, T], \tag{7}$$

where $\dfrac{\partial u}{\partial \nu}$ denotes the derivative in the direction of the exterior normal to Γ.

For the reaction term we will consider the following decomposition

$$F(x,t,v) = F_0(x,t) + F_L(x,t)v + F_{NL}(x,t,v), \tag{8}$$

with $F_0(x,t) = F_{0R}(x,t) + iF_{0I}(x,t)$, $F_L(x,t) = F_{LR}(x,t) + iF_{LI}(x,t)$ and $F_{NL}(x,t,v) = F_{NLR}(x,t,v) + iF_{NLI}(x,t,v)$, where $F_{0R}(x,t)$, $F_{0I}(x,t)$, $F_{LR}(x,t)$, $F_{LI}(x,t)$, $F_{NLR}(x,t,v)$ and $F_{NLI}(x,t,v)$ are real functions. For the nonlinear term, we consider that there exists a complex function χ such that

$$F_{NL}(x,t,v) = F_{NL}(x,t,0) + J(x,t,v)v, \tag{9}$$

with

$$J(x,t,v) = F'_{NL}(x,t,v) + \chi(v), \tag{10}$$

and $|\chi(r)| \longrightarrow 0$ as $|r| \longrightarrow 0$, being F'_{NL} the Fréchet derivative of F_{NL} with respect to the third component.

We assume that the problem is well posed, in the sense that it admits a unique solution and it depends continuously on the data.

Expression (4) involves both Schrödinger type equations and parabolic equations and includes the possibility of having a source term, a linear reaction term, a nonlinear reaction term or none of them (see (8)).

The paper is organized as follows: in Section 2 we describe the implicit and semi-implicit numerical methods simultaneously by embedding them into a two-parameter family of finite difference schemes. In Section 3 we derive a stability result of the numerical methods considered in the previous section. In the last section some numerical experiments are shown to confirm the theoretical analysis.

2 Numerical method

Let us construct a non equidistant rectangular grid on $\overline{Q}$. Let $(h_{k,j_k})_{0\leq j_k\leq N_k-1}$ be a vector of mesh-sizes (i.e. positive numbers) in the kth spatial coordinate direction, $k = 1,...,d$, with $N_k \geq 2$ an integer. We denote by h the maximal mesh-size. We define the space grid by

$$\overline{\Omega}_h = \prod_{k=1}^{d} \overline{\Omega}_{h_1}, \tag{11}$$

where, for $k = 1,...,d$,

$$\overline{\Omega}_{h_k} = \{x_{k,j_k} \in \mathbb{R} : x_{k,0} = a_k,\ x_{k,j_k+1} = x_{k,j_k} + h_{k,j_k},\ j_k = 1,...,N_k-1\}.$$

The set of gird points is denoted by x_j, where $j = (j_1,...,j_d)$, $0 \leq j_1 \leq N_k$. Points in the middle between two adjacent grid points are denoted by

$x_{j+(1/2)e_k} = x_j + h_{k,j_k}/2$ and $x_{j-(1/2)e_k} = x_j - h_{k,j_k-1}/2$, where e_k denotes the kth element of the natural basis in $\mathbb{R}^d$. We will also use the notation $h_{k,j_k-1/2} = (h_{k,j_k-1} + h_{k,j_k})/2$, $j_k = 1, .., N_k - 1$. For the temporal interval we consider the mesh

$$0 = t^0 < t^1 < \cdots < t^{M-1} < t^M = T,$$

where $M \geq 1$ is an integer and $\Delta t^m = t^{m+1} - t^m$, $m = 0, \dots, M-1$. Let $\Delta t = \max \Delta t^m$. We denote by $\overline{Q}_h^{\Delta t}$ the mesh in $\overline{Q}$ defined by the cartesian product of the space grid $\overline{\Omega}_h$ and a grid in the temporal domain. Let $Q_h^{\Delta t} = \overline{Q}_h^{\Delta t} \cap Q$ and $\Gamma_h^{\Delta t} = \overline{Q}_h^{\Delta t} \cap \Gamma \times [0, T]$.

We associate the coordinate $(j, m) = (j_1, ..., j_d, m)$ to the point $(x_j, t^m) \in \overline{Q}_h^{\Delta t}$ and $(j + (1/2)e_k, m)$ and $(j - (1/2)e_k, m)$ to the midpoints $(x_{j+(1/2)e_k}, t^m)$ and $(x_{j-(1/2)e_k}, t^m)$, respectively. We consider the notations $V_j^m = V(x_j, t^m)$, $V_{j+(1/2)e_k}^m = V(x_{j+(1/2)e_k}, t^m)$ and $V_{j-(1/2)e_k}^m = V(x_{j-(1/2)e_k}, t^m)$, for a function V defined on $\overline{\Omega}$. For the formulation of the finite difference approximations, we use the centered finite difference quotients in the kth spatial direction

$$\delta_k V_j^m = \frac{V_{j+(1/2)e_k}^m - V_{j-(1/2)e_k}^m}{h_{k,j_k-1/2}}, \quad \delta_k V_{j-(1/2)e_k}^m = \frac{V_j^m - V_{j-e_k}^m}{h_{k,j_k-1}}, \quad k = 1, 2.$$

On $\overline{Q}_h^{\Delta t}$ we approximate (4)–(5) by the one-parameter family of finite difference schemes

$$\text{(12)} \qquad \frac{U_j^{m+1} - U_j^m}{\Delta t^m} = \sum_{k=1}^{d} \delta_k (D_j^{m,\mu,\theta} \delta_k U_j^{m+\theta}) + F_j^{m,\mu,\theta} \quad \text{in} \quad \tilde{Q}_h^{\Delta t},$$

with

$$\text{(13)} \qquad U_j^0 = u^0(x_j) \quad \text{in} \quad \overline{\Omega}_h,$$

and either

$$\text{(14)} \qquad U_j^m = 0 \quad \text{in} \quad \Gamma_h^{\Delta t},$$

in the case of homogeneous Dirichlet boundary conditions (6), or

$$\text{(15)} \qquad \sum_{k=1}^{d} \left(h_{k,j_k-e_k} \delta_k U_{j+(1/2)e_k}^m + h_{k,j_k+e_k} \delta_k U_{j-(1/2)e_k}^m \right) \nu_k = 0 \quad \text{in} \quad \Gamma_h^{\Delta t},$$

in the case of homogeneous Neumann boundary conditions (7), where U_j^m represents the approximation of $u(x_j, t^m)$. In (12) we consider, for $\mu \in \{0, 1\}$ and $\theta \in [0, 1]$,

$$D_{j^{+k}}^{m,\mu,\theta} = D_{j+(1/2)e_k}^{m,\mu} = \frac{D(x_j, t^{m+\theta}, U_j^{m+\mu\theta}) + D(x_{j+e_k}, t^{m+\theta}, U_{j+e_k}^{m+\mu\theta})}{2},$$

and

$$F_j^{m,\mu,\theta} = F_0(x_j, t^{m+\theta}) + F_L(x_j, t^{m+\theta})U^{m+\theta} + F_{NL}(x_j, t^{m+\theta}, U^{m+\mu\theta}),$$

where

$$U_j^{m+\mu\theta} = \mu U_j^{m+1} + (1-\mu\theta)U_j^m. \tag{16}$$

We use the notation $\tilde{Q}_h^{\Delta t}$ for the set $Q_h^{\Delta t}$ or $\overline{Q}_h^{\Delta t}$, respectively, in the case of Dirichlet or Neumann boundary conditions, and ν_k represents the kth component of the normal vector ν.

Note that, when $\mu = 1$, the cases $\theta = 0$, $\theta = \frac{1}{2}$ and $\theta = 1$ correspond, respectively, to the explicit Euler, Crank-Nicolson an implicit Euler schemes. When $\mu = 0$, we have the semi-implicit case (semi-implicit Euler method when $\theta = 1$), that is, the diffusion coefficient and the non-linear part of the reaction term are treated explicitly.

In this paper we will consider two cases: the case when $\mu = 1$, which corresponds to the usual θ-method, and the case where $\mu = 0$ and $\theta = 1$, i.e. the semi-implicit Euler scheme. For all cases we suppose that

$$F_{LR}(x_j, t^{m+1}) \leq F_{LRmax} \tag{17}$$

and

$$J_R(x_j, t^{m+1}, U_j^{m+\theta}) \leq J_{Rmax}, \tag{18}$$

for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$, where $J_R(x,t,v)$ is the real part of $J(x,t,v)$ given by (10). For $\mu = 1$ and $\theta \in [0, \frac{1}{2})$ or $\mu = 0$ and $\theta = 1$ we also consider

$$J_I(x_j, t^{m+1}, U_j^{m+\theta}) \leq J_{Imax}, \tag{19}$$

for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$, where $J_I(x,t,v)$ is the imaginary part of $J(x,t,v)$ given by (10). In addition, for $\mu = 1$ and $\theta \in [0, \frac{1}{2})$ we also need to assume that

$$F_{LI}(x_j, t^{m+1}) \leq F_{LImax}, \tag{20}$$

for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$. We need the notation

$$|F_{Lmax}|^2 = F_{LRmax}^2 + F_{IRmax}^2, \qquad |J_{max}|^2 = J_{Rmax}^2 + J_{Imax}^2 \tag{21}$$

In what follows, $\|\cdot\|_h$ will denote the discrete L^2 norm, which will be specified in the next section.

3 Stability

In this section we derive the continuous dependence of the numerical solution on the initial data and on the right-hand side.

3.1 Implicit and explicit case

Let us first consider the case where $\mu = 1$. In this case we have the usual θ-method.

Theorem 1 *Let us consider $\mu = 1$ in the numerical method (12)–(13) with (14) or (15) and suppose that (17) and (18) hold, for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$. If $\theta \in [\frac{1}{2}, 1]$ the method is stable under the condition*

$$0 < \zeta \leq 1 - 4\theta^2 \Delta t^m K_\epsilon, \quad \zeta \in \mathbb{R}^+, \tag{22}$$

with, for all $\epsilon \neq 0$,

$$K_\epsilon = F_{LRmax} + J_{Rmax} + \epsilon^2, \tag{23}$$

If $\theta \in [0, \frac{1}{2})$ then the method is stable under the condition (22) with, for all $\epsilon \neq 0$,

$$\begin{aligned} K_\epsilon \;&=\; F_{LRmax} + J_{Rmax} + \epsilon^2 + \Delta t^m \left(\frac{1}{2} - \theta\right)(1 + \epsilon^{-2})(1 + \epsilon^2) \\ &\quad\; \times \left((1 + \epsilon^2)|F_{Lmax}|^2 + (1 + \epsilon^{-2})|J_{max}|^2\right), \end{aligned} \tag{24}$$

and

$$1 - \Delta t^m \left(\frac{1}{2} - \theta\right)(1 + \epsilon^2) \frac{4}{(\min h_{k,j_k})^2} \max_{x_j \in \bar{\Omega}_h} \frac{|D_j^{m+\theta}|^2}{D_{Rj}^{m+\theta}} \geq 0, \tag{25}$$

provided that (19) and (20) hold, for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$, $|D_j^{m,1,\theta}|$ is bounded and

$$0 < \xi \leq D_{Rj}^{m,1,\theta} \qquad \forall j, m. \tag{26}$$

Proof To prove this result we will consider the unidimensional case and Neumann boundary conditions. For higher dimension or Dirichlet boundary conditions, the proof follows the same steps.

We rewrite (12)–(13), (15) as a system by separating the real and imaginary parts, U_R and U_I, respectively, of the main variable $U = (U_0, \ldots, U_N)$. We shall then study the convergence of the family of finite difference schemes: find $U_j^m \approx u(x_j, t^m)$, $j = 0, \ldots, N$, $m = 0, \ldots, M$, such that

(27)
$$\begin{cases} \dfrac{U_{Rj}^{m+1} - U_{Rj}^m}{\Delta t^m} = \delta_x(D_{Rj}^{m+\theta} \delta_x U_{Rj}^{m+\theta}) - \delta_x(D_{Ij}^{m+\theta} \delta_x U_{Ij}^{m+\theta}) + F_{Rj}^{m+\theta}, \\ \qquad\qquad\qquad\qquad j = 0, \ldots, N, \quad m = 0, \ldots, M-1, \\ \dfrac{U_{Ij}^{m+1} - U_{Ij}^m}{\Delta t^m} = \delta_x(D_{Ij}^{m+\theta} \delta_x U_{Rj}^{m+\theta}) + \delta_x(D_{Rj}^{m+\theta} \delta_x U_{Ij}^{m+\theta}) + F_{Ij}^{m+\theta}, \\ \qquad\qquad\qquad\qquad j = 0, \ldots, N, \quad m = 0, \ldots, M-1, \end{cases}$$

with initial condition

$$U_{Rj}^0 = u_R^0(x_j), \quad U_{Ij}^0 = u_I^0(x_j), \quad j = 0, \dots, N, \tag{28}$$

and homogeneous Neumann boundary conditions

$$U_{R-1}^m = U_{R1}^m,\ U_{RN-1}^m = U_{RN+1}^m, U_{I-1}^m = U_{I1}^m,\ U_{IN-1}^m = U_{IN+1}^m,\ m = 0, \dots, M, \tag{29}$$

where

$$D_{j^+}^{m+\theta} = D_{j^+}^{m,1,\theta} = \frac{D(x_{j+1}, t^{m+\theta}, U_{j+1}^{m+\theta}) + D(x_j, t^{m+\theta}, U_j^{m+\theta})}{2}, \tag{30}$$

$j = 1, \dots, N,\ m = 0, \dots, M$, and

$$F_j^{m+\theta} = F_j^{m,1,\theta} = F(x_j, t^{m+\theta}, U_j^{m+\theta}) = F_{Rj}^{m+\theta} + iF_{Ij}^{m+\theta},$$

$j = 0, \dots, N,\ m = 0, \dots, M-1$. In (27) and (29) we need the extra points $x_{-1} = x_0 - h_0$ and $x_{N+1} = x_N + h_{N-1}$ and we define $D_{-1^+}^{m+\theta} = D_{0^+}^{m+\theta}$, $D_{N^+}^{m+\theta} = D_{N-1^+}^{m+\theta}$.

We consider the discrete L^2 inner products

$$(U, V)_h = \sum_{j=0}^{N-1} \frac{h_j}{2} \left(U_j \overline{V}_j + U_{j+1} \overline{V}_{j+1} \right) \tag{31}$$

and

$$(U, V)_{h^*} = \sum_{j=0}^{N-1} h_j U_{j+1/2} \overline{V}_{j+1/2}, \tag{32}$$

and their corresponding norms

$$\|U\|_h = (U, U)_h^{1/2} \quad \text{and} \quad \|U\|_{h^*} = (U, U)_{h^*}^{1/2}. \tag{33}$$

Multiplying both members of the first and second equations of (27) by, respectively, $U_R^{m+\theta}$ and $U_I^{m+\theta}$, according to the discrete inner product $(\cdot, \cdot)_h$ and using summation by parts we obtain

$$\left(\frac{U_R^{m+1} - U_R^m}{\Delta t^m}, U_R^{m+\theta} \right)_h + \left(\frac{U_I^{m+1} - U_I^m}{\Delta t^m}, U_I^{m+\theta} \right)_h + \|(D_{R^+}^{m+\theta})^{1/2} \delta_x U^{m+\theta}\|_{h^*}^2$$
$$= \left(F_R^{m+\theta}, U_R^{m+\theta} \right)_h + \left(F_I^{m+\theta}, U_I^{m+\theta} \right)_h.$$

Since we can write

$$U^{m+\theta} = \Delta t^m \left(\theta - \frac{1}{2} \right) \frac{U^{m+1} - U^m}{\Delta t^m} + \frac{U^{m+1} + U^m}{2}, \tag{34}$$

we get

$$\Delta t^m \left(\theta - \frac{1}{2}\right) \left\| \frac{U^{m+1} - U^m}{\Delta t^m} \right\|_h^2 + \frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \|(D_{R^+}^{m+\theta})^{1/2} \delta_x U^{m+\theta}\|_{h^*}^2 = \left(F_R^{m+\theta}, U_R^{m+\theta}\right)_h + \left(F_I^{m+\theta}, U_I^{m+\theta}\right)_h .$$

If $\theta \in [\frac{1}{2}, 1]$ we immediately obtain that

$$\begin{aligned} \frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} &+ \|(D_{R^+}^{m+\theta})^{1/2} \delta_x U^{m+\theta}\|_{h^*}^2 \\ &\leq \left(F_R^{m+\theta}, U_R^{m+\theta}\right)_h + \left(F_I^{m+\theta}, U_I^{m+\theta}\right)_h . \end{aligned} \tag{35}$$

Let us now look to the right-hand side of (35). Considering the decomposition (8)–(9) we can write

$$\begin{aligned} \left(F_R^{m+\theta}, U_R^{m+\theta}\right)_h &+ \left(F_I^{m+\theta}, U_I^{m+\theta}\right)_h \\ &= (F_R(\cdot, t^{m+\theta}, 0), U_R^{m+\theta})_h + (F_I(\cdot, t^{m+\theta}, 0), U_I^{m+\theta})_h \\ &\quad + (F_{LR}(\cdot, t^{m+\theta}) U_R^{m+\theta}, U_R^{m+\theta})_h + (F_{LR}(\cdot, t^{m+\theta}) U_I^{m+\theta}, U_I^{m+\theta})_h \\ &\quad + (J_R(\cdot, t^{m+\theta}, U^{m+\theta}) U_R^{m+\theta}, U_R^{m+\theta})_h \\ &\quad + (J_R(\cdot, t^{m+\theta}, U^{m+\theta}) U_I^{m+\theta}, U_I^{m+\theta})_h . \end{aligned}$$

Since,

$$(J_R(\cdot, t^{m+\theta}, U^{m+\theta}) U_R^{m+\theta}, U_R^{m+\theta})_h \leq J_{Rmax} \|U_R^{m+\theta}\|_h^2$$

and, with the necessary modifications, we obtain a correspondent inequality for $(J_R(\cdot, t^{m+1}, U^{m+\theta}) U_I^{m+\theta}, U_I^{m+\theta})_h$, using Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left(F_R^{m+\theta}, U_R^{m+\theta}\right)_h &+ \left(F_I^{m+\theta}, U_I^{m+\theta}\right)_h \\ &\leq \|F_R(\cdot, t^{m+\theta}, 0)\|_h \|U_R^{m+\theta}\|_h + \|F_I(\cdot, t^{m+\theta}, 0)\|_h \|U_I^{m+\theta}\|_h \\ &\quad + F_{LRmax} \|U^{m+\theta}\|_h^2 + J_{Rmax} \|U^{m+\theta}\|_h^2 \end{aligned}$$

which leads to

$$\begin{aligned} \left(F_R^{m+\theta}, U_R^{m+\theta}\right)_h &+ \left(F_I^{m+\theta}, U_I^{m+\theta}\right)_h \\ &\leq \frac{1}{4\epsilon^2} \|F(\cdot, t^{m+\theta}, 0)\|_h^2 + \epsilon^2 \|U^{m+\theta}\|_h^2 \\ &\quad + \left(F_{LRmax} + J_{Rmax}\right) \|U^{m+\theta}\|_h^2 , \end{aligned}$$

where $\epsilon \neq 0$. Then, from (35),

$$\begin{aligned} &\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \|(D_{R^+}^{m+\theta})^{1/2} \delta_x U^{m+\theta}\|_{h^*}^2 \\ &\leq \frac{1}{4\epsilon^2} \|F(\cdot, t^{m+\theta}, 0)\|_h^2 + \epsilon^2 \|U^{m+\theta}\|_h^2 + \left(F_{LRmax} + J_{Rmax}\right) \|U^{m+\theta}\|_h^2 \end{aligned} \tag{36}$$

and so

$$\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m}$$
$$\leq \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+\theta}, 0)\|_h^2 + \epsilon^2\|U^{m+\theta}\|_h^2 + (F_{LRmax} + J_{Rmax})\,\|U^{m+\theta}\|_h^2. \quad (37)$$

Using the definition of $U^{m+\theta}$ we get

$$\left(1 - 4\theta^2\Delta t^m K_\epsilon\right)\|U^{m+1}\|_h^2$$
$$\leq \left(1 + 4(1-\theta)^2\Delta t^m K_\epsilon\right)\|U^m\|_h^2 + \frac{\Delta t^m}{2\epsilon^2}\|F(\cdot, t^{m+\theta}, 0)\|_h^2,$$

for $m = 0, \ldots, M-1$, with K_ϵ given by (23). If (22) holds we get

$$\|U^{m+1}\|_h^2$$
$$\leq \frac{1 + 4(1-\theta)^2\Delta t^m K_\epsilon}{1 - 4\theta^2\Delta t^m K_\epsilon}\|U^m\|_h^2 + \frac{\Delta t^m}{2\epsilon^2(1 - 4\theta^2\Delta t^m K_\epsilon)}\|F(\cdot, t^{m+\theta}, 0)\|_h^2$$
$$\leq (1 + 4(\theta^2 + (1-\theta)^2)\zeta^{-1}\Delta t^m K_\epsilon)\|U^m\|_h^2 + \frac{\Delta t^m}{2\epsilon^2\zeta}\|F(\cdot, t^{m+\theta}, 0)\|_h^2.$$

Summing through m and using the Discrete Duhamel's Principle (Lemma 4.1 in Appendix B of [5]) we get

$$\|U^k\|_h^2 \leq e^{4(\theta^2+(1-\theta)^2)\zeta^{-1}K_\epsilon t^k}\left(\|U^0\|_h^2 + \frac{1}{2\epsilon^2\zeta}\sum_{m=0}^{k-1}\|F(\cdot, t^{m+\theta}, 0)\|_h^2\Delta t^m\right),$$

which proves the stability.

We now consider the case where $\theta \in [0, \frac{1}{2})$. In this case we have

$$\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \|(D_{R^+}^{m+\theta})^{1/2}\delta_x^- U^{m+\theta}\|_{h^*}^2$$
$$\leq \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+\theta}, 0)\|_h^2 + \epsilon^2\|U^{m+\theta}\|_h^2 + (F_{LRmax} + J_{Rmax})\,\|U^{m+\theta}\|_h^2$$
$$+\Delta t^m\left(\frac{1}{2} - \theta\right)\left\|\frac{U^{m+1} - U^m}{\Delta t^m}\right\|_h^2. \quad (38)$$

Since

$$\left\|\frac{U^{m+1} - U^m}{\Delta t^m}\right\|_h^2 = \left\|\frac{U_R^{m+1} - U_R^m}{\Delta t^m}\right\|_h^2 + \left\|\frac{U_I^{m+1} - U_I^m}{\Delta t^m}\right\|_h^2 \quad (39)$$

and, following [2], we deduce that

$$\left\|\frac{U^{m+1} - U^m}{\Delta t^m}\right\|_h^2 \leq (1 + \eta_1^2)\frac{4}{(\min h_j)^2}\max_{x_j \in \bar{\Omega}_h}\frac{|D_j^{m+\theta}|^2}{D_{Rj}^{m+\theta}}\|(D_{R^+}^{m+\theta})^{1/2}\delta_x U^{m+\theta}\|_{h^*}^2$$
$$+(1 + \eta_1^{-2})\left(\|F_R^{m+\theta}\|_h^2 + \|F_I^{m+\theta}\|_h^2\right),$$

where $\eta_1 \neq 0$. Using (8)–(9) we get

$$\begin{aligned}\left\|\frac{U^{m+1}-U^m}{\Delta t^m}\right\|_h^2 \leq\ & (1+\eta_1^2)\frac{4}{(\min h_j)^2}\max_{x_j\in\bar{\Omega}_h}\frac{|D_j^{m+\theta}|^2}{D_{Rj}^{m+\theta}}\|(D_{R^+}^{m+\theta})^{1/2}\delta_x U^{m+\theta}\|_{h^*}^2\\ &+(1+\eta_1^{-2})(1+\eta_2^{-2})\|F(\cdot,t^{m+\theta},0)\|_h^2\\ &+(1+\eta_1^{-2})(1+\eta_2^2)(1+\eta_3^2)(F_{LRmax}^2+F_{LImax}^2)\|U^{m+\theta}\|_h^2\\ &+(1+\eta_1^{-2})(1+\eta_2^2)(1+\eta_3^{-2})(J_{Rmax}^2+J_{Imax}^2)\|U^{m+\theta}\|_h^2,\end{aligned}$$

where $\eta_2, \eta_3 \neq 0$. Using the definition of $U^{m+\theta}$ and $\eta_1=\eta_2=\eta_3=\epsilon$ we get

$$\begin{aligned}\left\|\frac{U^{m+1}-U^m}{\Delta t^m}\right\|_h^2 \leq\ & (1+\epsilon^2)\frac{4}{(\min h_j)^2}\max_{x_j\in\bar{\Omega}_h}\frac{|D_j^{m+\theta}|^2}{D_{Rj}^{m+\theta}}\|(D_{R^+}^{m+\theta})^{1/2}\delta_x U^{m+\theta}\|_{h^*}^2\\ &+(1+\epsilon^{-2})^2\|F(\cdot,t^{m+\theta},0)\|_h^2\\ &+2\theta^2(1+\epsilon^{-2})(1+\epsilon^2)\big((1+\epsilon^2)|F_{Lmax}|^2\\ &\qquad+(1+\epsilon^{-2})|J_{max}|^2\big)\|U^{m+1}\|_h^2\\ &+2(1-\theta)^2(1+\epsilon^{-2})(1+\eta^2)\big((1+\epsilon^2)|F_{Rmax}|^2\\ &\qquad+(1+\epsilon^{-2})|J_{max}|^2\big)\|U^m\|_h^2.\end{aligned}$$

Then, considering the previous inequality in (38) and if (25) holds, we get

$$\begin{aligned}\Big(1-4\theta^2\Delta t^m K_\epsilon\Big)\|U^{m+1}\|_h^2 &\leq \Big(1+4(1-\theta)^2\Delta t^m K_\epsilon\Big)\|U^m\|_h^2\\ &+2\Delta t^m\left(\frac{1}{4\epsilon^2}+\Delta t^m\left(\frac{1}{2}-\theta\right)(1+\epsilon^{-2})^2\right)\|F(\cdot,t^{m+1},0)\|_h^2,\end{aligned}$$

for $m=0,\dots,M-1$, with K_ϵ given by (24). If (22) holds, summing through m and using the Discrete Duhamel's Principle we get

$$\begin{aligned}\|U^k\|_h^2 &\leq e^{4(\theta^2+(1-\theta)^2)\zeta^{-1}K_\epsilon t^k}\\ &\times\left(\|U^0\|_h^2+2\left(\frac{1}{4\epsilon^2}+T\left(\frac{1}{2}-\theta\right)(1+\epsilon^{-2})^2\right)\sum_{m=0}^{k-1}\|F(\cdot,t^{m+\theta},0)\|_h^2\Delta t^m\right),\end{aligned}$$

which concludes the proof. ■

Remark 1 *If $F(x,t,0)=0$, we may prove that, for $\theta\in[\frac{1}{2},1]$, if*

$$0<\zeta\leq 1-4\theta^2\Delta t^m K,$$

for some $\zeta\in\mathbb{R}^+$, with

$$K=F_{LRmax}+J_{Rmax},$$

we get

$$\|U^{m+1}\|_h^2\leq(1+4(\theta^2+(1-\theta)^2)\zeta^{-1}\Delta t^m K)\|U^m\|_h^2.$$

Summing through m and using the Discrete Duhamel's Principle we get

$$\|U^k\|_h^2 \leq e^{4(\theta^2+(1-\theta)^2)\zeta^{-1}Kt^k}\|U^0\|_h^2.$$

If, in addition, F_{LRmax} and J_{Rmax} are non-positive, the method is unconditionally stable.

Remark 2 *For $\theta \in [0, \frac{1}{2})$, the following particular cases are easily deduced from the previous theorem.*

1. *If $F(x,t,0) = 0$, the stability conditions are (22) and (25) with*

$$\begin{aligned} K_\epsilon &= F_{LRmax} + J_{Rmax} + \Delta t^m \left(\frac{1}{2} - \theta\right)(1+\epsilon^{-1}) \\ &\qquad \times\Big((1+\epsilon^2)|F_{Lmax}|^2 + (1+\epsilon^{-2})|J_{max}|^2\Big). \end{aligned}$$

2. *If $F_L(x,t) = 0$, the stability conditions are (22) and (25) with*

$$K_\epsilon = J_{Rmax} + \epsilon^2 + \Delta t^m \left(\frac{1}{2} - \theta\right)(1+\epsilon^{-2})(1+\epsilon^2)|J_{max}|^2.$$

3. *If $J(x,t,U) = 0$, the stability conditions are (22) and (25) with*

$$K_\epsilon = F_{LRmax} + \epsilon^2 + \Delta t^m \left(\frac{1}{2} - \theta\right)(1+\epsilon^{-2})(1+\epsilon^2)|F_{Lmax}|^2.$$

Corollary 2 *If Dirichlet boundary conditions and (26) hold, then for $\theta \in [\frac{1}{2}, 1]$ the stability condition is (22) with $K_\epsilon = F_{LRmax} + J_{Rmax}$ (does not depend on ϵ). In addition, if both F_{LRmax} and J_{Rmax} are non-positive, the method is unconditionally stable. For $\theta \in [0, \frac{1}{2})$ the stability conditions are (22) and (25) with*

$$\begin{aligned} K_\epsilon = F_{LRmax} + J_{Rmax} + \Delta t^m \left(\frac{1}{2} - \theta\right)(1+\epsilon^{-2})(1+\epsilon^2) \\ \times\Big((1+\epsilon^2)|F_{Lmax}|^2 + (1+\epsilon^{-2})|J_{max}|^2\Big). \end{aligned}$$

Proof According to the discrete Poincaré-Friedrichs inequality (Lemma 5), there exists a constant $C(\Omega)$, depending on Ω, such that

$$C(\Omega)\|U^m\|_h^2 \leq \|\delta_x U^m\|_{h^*}^2.$$

So, for $\theta \in [\frac{1}{2}, 1]$, inequalities (26) and (36) imply

$$\begin{aligned} &\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \xi C(\Omega)\|U^{m+\theta}\|_h^2 \\ &\quad \leq \frac{1}{4\epsilon^2}\|F(\cdot,t^{m+\theta},0)\|_h^2 + \epsilon^2\|U^{m+\theta}\|_h^2 + (F_{LRmax} + J_{Rmax})\|U^{m+\theta}\|_h^2 \end{aligned} \tag{40}$$

Considering $\epsilon^2 = \frac{1}{2}\xi C(\Omega)$, then $\xi C(\Omega) - \epsilon^2 > 0$ and we obtain

$$\begin{aligned}\left(1 - 4\theta^2 \Delta t^m K\right) \|U^{m+1}\|_h^2 &\leq \left(1 + 4(1-\theta)^2 \Delta t^m K\right) \|U^m\|_h^2 \\ &\quad + \frac{\Delta t^m}{\xi C(\Omega)} \|F(\cdot, t^{m+\theta}, 0)\|_h^2,\end{aligned}$$

for $m = 0, \ldots, M-1$, with

$$K = F_{LRmax} + J_{Rmax}.$$

Then, the stability condition is (22) with $K_\epsilon = K$.

With the same arguments, for $\theta \in [0, \frac{1}{2})$ and Dirichlet boundary conditions, we may prove that, if (26) holds, the stability conditions are (22) and (25) with

$$\begin{aligned}K_\epsilon = F_{LRmax} + J_{Rmax} + \Delta t^m \left(\frac{1}{2} - \theta\right) (1 + \epsilon^{-2})(1 + \epsilon^2) \\ \times \Big((1 + \epsilon^2)|F_{Lmax}|^2 + (1 + \epsilon^{-2})|J_{max}|^2\Big).\end{aligned}$$

■

Corollary 3 *If $F(x,t,v) = F_0(x,t)$ and (26) hold then, for $\theta \in [\frac{1}{2}, 1]$, the method is unconditionally stable and for $\theta \in [0, \frac{1}{2})$ the stability condition is (25).*

Proof If we consider Dirichlet boundary conditions, the result is included in the previous corollary. Let us consider Neumann boundary conditions. According to the discrete Friedrichs inequality (Lemma 6), there exists a constant $C(\Omega)$, depending on Ω, such that

$$C(\Omega)\|U^m - \bar{U}^m\|_h^2 \leq \|\delta_x U^m\|_{h^*}^2,$$

where

$$\bar{U}^m = \frac{1}{|\Omega|}(U^m, \mathbf{1})_h,$$

and $\mathbf{1}$ a vector with all entries equal to one. Then

$$\frac{C(\Omega)}{2}\|U^m\|_h^2 - C(\Omega)\|\bar{U}^m\|_h^2 \leq \|\delta_x U^m\|_{h^*}^2.$$

So, for $\theta \in [\frac{1}{2}, 1]$, inequalities (26) and (36) imply

$$\begin{aligned}\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} + \xi \frac{C(\Omega)}{2}\|U^{m+\theta}\|_h^2 \\ \leq \frac{1}{4\epsilon^2}\|F_0(\cdot, t^{m+\theta})\|_h^2 + \epsilon^2\|U^{m+\theta}\|_h^2 + C(\Omega)\|\bar{U}^{m+\theta}\|_h^2.\end{aligned}$$

Considering $\epsilon^2 = \frac{1}{4}\xi C(\Omega)$, then $\xi \frac{1}{2} C(\Omega) - \epsilon^2 > 0$ and we obtain

$$\|U^{m+1}\|_h^2 \leq \|U^m\|_h^2 + \frac{\Delta t^m}{\xi C(\Omega)}\|F_0(\cdot, t^{m+\theta})\|_h^2 + C(\Omega)\|\bar{U}^{m+\theta}\|_h^2.$$

By Lemma 7 we conclude that

$$\begin{aligned}\|U^{m+1}\|_h^2 &\le \|U^m\|_h^2 + \frac{\Delta t^m}{\xi C(\Omega)}\|F_0(\cdot, t^{m+\theta})\|_h^2 \\ &\quad + \frac{C(\Omega)}{|\Omega|^{1/2}}\left(\|\bar{U}^0\|_h + \sum_{k=0}^{m}\Delta t^k \|F_0(\cdot, t^{k+\theta})\|_h\right)^2.\end{aligned}$$

Then, the method is unconditionally stable. ■

3.2 Semi-Implicit case

Let us now consider the case where $\mu = 0$ and $\theta = 1$, i.e, the semi-implicit Euler method.

Theorem 4 *Let us consider $\mu = 0$, $\theta = 1$ in the numerical method (12)–(13) with (14) or (15) and suppose that (17), (18) and (19) hold, for all $(x_j, t^{m+1}) \in \tilde{Q}_h^{\Delta t}$. The numerical method is stable under the condition*

$$0 < \zeta \le 1 - 2\Delta t^m K_\epsilon, \quad \zeta \in \mathbb{R}^+, \tag{41}$$

with, for all $\epsilon \neq 0$,

$$K_\epsilon = F_{LRmax} + \frac{1}{2}|J_{max}|^2 + \epsilon^2. \tag{42}$$

Proof As for the previous theorem, to prove this result we will consider the unidimensional case and Neumann boundary conditions. For higher dimension or Dirichlet boundary conditions, the proof follows the same steps.

We rewrite (12)–(13), (15) as a system by separating the real and imaginary parts, U_R and U_I, respectively, of the main variable. We shall then study the stability of the family of finite difference schemes: find $U_j^m \approx u(x_j, t^m)$, $j = 0, \ldots, N$, $m = 0, \ldots, M$, such that

$$\begin{cases} \dfrac{U_{Rj}^{m+1} - U_{Rj}^m}{\Delta t^m} = \delta_x(D_{Rj}^{m,0,1}\delta_x U_{Rj}^{m+1}) - \delta_x(D_{Ij}^{m,0,1}\delta_x U_{Ij}^{m+1}) + F_{Rj}^{m,0,1} \\ \qquad\qquad j = 0, \ldots, N, m = 0, \ldots, M-1, \\ \dfrac{U_{Ij}^{m+1} - U_{Ij}^m}{\Delta t^m} = \delta_x(D_{Ij}^{m,0,1}\delta_x U_{Rj}^{m+1}) + \delta_x(D_{Rj}^{m,0,1}\delta_x U_{Ij}^{m+1}) + F_{Ij}^{m,0,1}, \\ \qquad\qquad j = 0, \ldots, N, m = 0, \ldots, M-1, \end{cases} \tag{43}$$

with initial condition and homogeneous Neumann boundary conditions given as in the previous theorem, where

$$D_{j^+}^{m,0,1} = \frac{D(x_{j+1}, t^{m+1}, U_{j+1}^m) + D(x_j, t^{m+1}, U_j^m)}{2}, \tag{44}$$

$j = 1, \dots, N,\ m = 0, \dots, M$, and

$$F^{m,0,1} = F_0(.,t^{m+1}) + F_L(.,t^{m+1})U^{m+1} + F_{NL}(.,t^{m+1},U^m) = F^{m,0,1}_{Rj} + iF^{m,0,1}_{Ij},$$

$j = 0, \dots, N$, $m = 0, \dots, M-1$. In (43) we need the extra points $x_{-1} = x_0 - h_0$ and $x_{N+1} = x_N + h_{N-1}$ and we define $D^{m,0,1}_{-1^+} = D^{m,0,1}_{0^+}$, $D^{m,0,1}_{N^+} = D^{m,0,1}_{(N-1)^+}$.

We consider the discrete L^2 inner products defined by (31)–(32) their corresponding norms.

Multiplying both members of the first and second equations of (43) by, respectively, U^{m+1}_R and U^{m+1}_I, according to the discrete inner product $(\cdot,\cdot)_h$, and using summation by parts we obtain, as for (35),

$$\begin{aligned} \frac{\|U^{m+1}\|^2_h - \|U^m\|^2_h}{2\Delta t^m} &+ \|(D^{m,0,1}_{R^+})^{1/2}\delta_x U^{m+1}\|^2_{h^*} \\ &\leq \left(F^{m,0,1}_R, U^{m+1}_R\right)_h + \left(F^{m,0,1}_I, U^{m+1}_I\right)_h . \end{aligned} \tag{45}$$

Let us now look to the right-hand side of (45). Considering (8)–(9) we obtain

$$\begin{aligned} \left(F^{m,0,1}_R, U^{m+1}_R\right)_h &+ \left(F^{m,0,1}_I, U^{m+1}_I\right)_h \\ &= (F_R(\cdot,t^{m+1},0), U^{m+1}_R)_h + (F_I(\cdot,t^{m+1},0), U^{m+1}_I)_h \\ &\quad + (F_{LR}(\cdot,t^{m+1})U^{m+1}_R, U^{m+1}_R)_h + (F_{LR}(\cdot,t^{m+1})U^{m+1}_I, U^{m+1}_I)_h \\ &\quad + (J_R(\cdot,t^{m+1},U^m)U^m_R, U^{m+1}_R)_h + (J_R(\cdot,t^{m+1},U^m)U^m_I, U^{m+1}_I)_h \\ &\quad - (J_I(\cdot,t^{m+1},U^m)U^m_I, U^{m+1}_R)_h + (J_I(\cdot,t^{m+1},U^m)U^m_R, U^{m+1}_I)_h. \end{aligned}$$

So, using Cauchy-Schwarz inequality, we have

$$(J_R(\cdot,t^{m+1},U^m)U^m_R, U^{m+1}_R)_h \leq J^2_{Rmax}\|U^{m+1}_R\|_h\|U^m_R\|_h$$

and so

$$(J_R(\cdot,t^{m+1},U^m)U^m_R, U^{m+1}_R)_h \leq \frac{1}{2}\left(J^2_{Rmax}\|U^{m+1}_R\|^2_h + \|U^m_R\|^2_h\right)$$

and, with the necessary modifications, we obtain a correspondent inequality for $(J_R(\cdot,t^{m+1},U^m)U^m_I, U^{m+1}_I)_h$. We also have, considering the Cauchy-Schwarz inequality,

$$\begin{aligned} -(J_I(\cdot,t^{m+1},U^m)U^m_I, U^{m+1}_R)_h &+ (J_I(\cdot,t^{m+1},U^m)U^m_R, U^{m+1}_I)_h \\ &\leq \frac{1}{2}\left(J^2_{Imax}\|U^{m+1}\|^2_h + \|U^m\|^2_h\right) \end{aligned}$$

Then, for the right-hand side of (45), we have

$$\begin{aligned} \left(F^{m,0,1}_R, U^{m+1}_R\right)_h &+ \left(F^{m,0,1}_I, U^{m+1}_I\right)_h \\ &\leq \|F_R(\cdot,t^{m+1},0)\|_h\|U^{m+1}_R\|_h + \|F_I(\cdot,t^{m+1},0)\|_h\|U^{m+1}_I\|_h \\ &\quad + F_{LRmax}\|U^{m+1}\|^2_h + \frac{1}{2}\left(J^2_{Rmax} + J^2_{Imax}\right)\|U^{m+1}\|^2_h + \|U^m\|^2_h \end{aligned}$$

which leads to

$$
\begin{aligned}
\left(F_R^{m,0,1}, U_R^{m+1}\right)_h &+ \left(F_I^{m,0,1}, U_I^{m+1}\right)_h \\
&\leq \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+1}, 0)\|_h^2 + \epsilon^2\|U^{m+1}\|_h^2 \\
&\quad + \left(F_{LRmax} + \frac{1}{2}|J_{max}|^2\right)\|U^{m+1}\|_h^2 + \|U^m\|_h^2,
\end{aligned}
$$

where $\epsilon \neq 0$. Then, from (45),

$$
\begin{aligned}
\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} &+ \|(D_{R^+}^{m,0,1})^{1/2}\delta_x U^{m+1}\|_{h^*}^2 \\
&\leq \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+1}, 0)\|_h^2 + \epsilon^2\|U^{m+1}\|_h^2 \\
&\quad + \left(F_{LRmax} + \frac{1}{2}|J_{max}|^2\right)\|U^{m+1}\|_h^2 + \|U^m\|_h^2, \qquad (46)
\end{aligned}
$$

and so

$$
\begin{aligned}
\frac{\|U^{m+1}\|_h^2 - \|U^m\|_h^2}{2\Delta t^m} &\leq \frac{1}{4\epsilon^2}\|F(\cdot, t^{m+1}, 0)\|_h^2 + \epsilon^2\|U^{m+1}\|_h^2 \\
&\quad + \left(F_{LRmax} + \frac{1}{2}|J_{max}|^2\right)\|U^{m+1}\|_h^2 + \|U^m\|_h^2, \qquad (47)
\end{aligned}
$$

Using the definition of $U^{m+\theta}$ we get

$$
(1 - 2\Delta t^m K_\epsilon)\|U^{m+1}\|_h^2 \leq (1 + 2\Delta t^m)\|U^m\|_h^2 + \frac{\Delta t^m}{2\epsilon^2}\|F(\cdot, t^{m+1}, 0)\|_h^2, \qquad (48)
$$

for $m = 0, \ldots, M-1$, with K_ϵ given by (42). If (41) holds, summing through m and using the Discrete Duhamel's Principle we get

$$
\|U^k\|_h^2 \leq e^{2(1+K_\epsilon)\zeta^{-1}t^k}\left(\|U^0\|_h^2 + \frac{1}{2\epsilon^2\zeta}\sum_{m=0}^{k-1}\|F(\cdot, t^{m+1}, 0)\|_h^2\Delta t^m\right),
$$

which concludes the proof. ■

Remark 3 *If* $F(x,t,0) = 0$, *we may prove that if*

$$
0 < \zeta \leq 1 - 2\Delta t^m K, \qquad (49)
$$

for some $\zeta \in \mathbb{R}^+$, *with*

$$
K = F_{LRmax} + \frac{1}{2}|J_{max}|^2, \qquad (50)
$$

we get

$$
\|U^{m+1}\|_h^2 \leq \left(1 + 2\Delta t^m(1+K)\zeta^{-1}\right)\|U^m\|_h^2,
$$

for $m = 0, \ldots, M-1$, and so

$$\|U^k\|_h^2 \leq e^{2(1+K)\zeta^{-1}t^k} \|U^0\|_h^2.$$

If, in addition, $K \leq 0$, then the method is unconditionally stable.

Remark 4 *If we consider the Dirichlet boundary conditions and (26) holds, the stability condition is (49) with K given by (50). In addition, if $F_{NL} \equiv 0$, F_{LRmax} is non-positive, the method is unconditionally stable. We may conclude this result with the same arguments as in Corollary 2.*

Remark 5 *If $F(x,t,v) = F_0(x,t)$, the method is unconditionally stable. We may conclude this result with the same arguments as in Corollary 3.*

4 Numerical examples

In this section we will illustrate the stability results using appropriate numerical examples. We start by noting that the stability condition for the explicit method has already been illustrated in [2], though without a reactive term. Since the numerical results are very similar, we will leave the explicit scheme out of this illustration, referring the reader to [2] for details. We will also leave out of this section the illustration of the stability of the implicit scheme, since we expect that the choice of linearization method may further influence the results. In this way, we will focus the numerical illustrations on the stability of the semi-implicit scheme with Neumann boundary condition, since the stability condition (though similar to the Dirichlet case) is slightly more complex.

Let us consider equation (4) with

$$x_1, x_2 \in (0,\pi) \times (0,\pi), \quad t \in (0,T],$$

with initial and Neumann boundary conditions given, respectively, by

$$u(x_1,x_2,0) = \cos(x_1)\cos(x_2)$$

and

$$\frac{\partial u}{\partial \nu}(0,x_2,t) = \frac{\partial u}{\partial \nu}(\pi,x_2,t) = \frac{\partial u}{\partial \nu}(x_1,0,t) = \frac{\partial u}{\partial \nu}(x_1,\pi,t) = 0.$$

Given a constant $A \in \mathbb{C}$, for

$$F(x_1,x_2,t,v) = (A+2i)v + 2v^2 - \left(\sin^2(x_1)\cos^2(x_2) + \cos^2(x_1)\sin^2(x_2)\right) e^{2At}$$

and

$$D(x_1,x_2,t,v) = i + v,$$

the exact solution is given by

$$u(x_1,x_2,t) = \cos(x_1)\cos(x_2)e^{At}.$$

We also note that with this choice of reactive term F we have

$$\begin{aligned} F_0(x_1,x_2,t) &= -\left(\sin^2(x_1)\cos^2(x_2)+\cos^2(x_1)\sin^2(x_2)\right)e^{2At}, \\ F_L(x,t) &= A+2i, \\ F_{NL}(x,t,v) &= 2v^2 \quad (\text{and } F_{NL}(x,t,0)=0), \\ J(x,t,v) &= 2v^2. \end{aligned}$$

We will now consider two different possibilities for the value of the constant A that will induce different behaviours on the solution and therefore on the stability condition.

4.1 Case 1: $\mathbf{F_{LR} \leq 0}$

For $A=-1+i$, we have that $F_{LR}=-1<0$. We will now consider the upper bound (48) (taking $\epsilon=1$) and compare it with the actual norm $\|U^m\|_h^2$. We also note that if the time step Δt is such that there exists no $\xi>0$ so that (41) is satisfied, then no theoretical upper bound is known and the numerical solution might become unbounded in time (even in cases where the solution is bounded).

The numerical results are shown in figures 1 and 2. It can be seen that for smaller steps in time, the ratio stays bounded by the theoretical upper bound. For higher time steps (namely for time steps that do not satisfy the stability condition), there is no theoretical upper bound and the norm of the approximation increases rapidly.

4.2 Case 2: $\mathbf{F_{LR} > 0}$

For $A=0.1+i$, we have that $F_{LR}=0.1>0$. In this way, the condition (41) is harder to satisfy, since now F_{LRmax} is positive. Again we compare the theoretical the upper bound (48) and the actual norm $\|U^m\|_h^2$.

The numerical results are shown in figures 3 and 4. It can be seen that though in some cases the theoretical bound increases, the numerical results might stay bounded. Similarly to the previous case, for higher steps in time, the approximation's norm increases rapidly.

To better illustrate this phenomenon we also considered no uniform meshes. To this end, we considered 50 points in each spatial direction ramdomly distributed (by a uniform distribution) to define the spatial mesh. Moreover we considered 30 steps in time, corresponding to instants randomly chosen in the interval [0,1]. Evolution in time of numerical norm $\|U^m\|_h^2$ and the theoretical upper bound (48) is given in Figure 5 for four different cases. Again a similar behaviour is observed.

5 Conclusions

In this paper we have established the stability conditions for finite difference schemes in the context of complex diffusion with reactive terms. In this way we

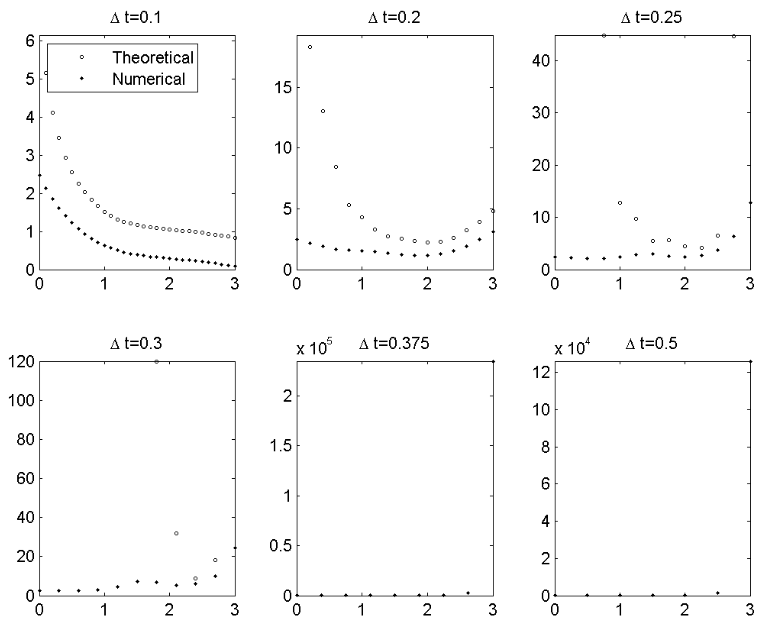

Figure 1: Case 1: Evolution in time of numerical norm $\|U^m\|_h^2$ and the theoretical upper bound (48) for several time steps Δt. No plot on the theoretical upper bound, means there exists no ξ that satifies (41).

have extended a previous stability result [2] to the semi-implicit scheme and to the presence of reactive terms in complex diffusion.

In this way we have shown that both the implicit and semi-implicit schemes are stable under some conditions on the time step. We note that at a fixed time, there is always a small enough time step for which the method is stable, since the stability condition is an upper bound for the time step. As usual, for the explicit scheme, a stability condition that relates the magnitude of the time step and the spatial step size needs to be satisfied.

Finally we have illustrated the theoretical results with numerical examples, to show cases of stability and instability.

Parallel work [**?**] establishes a convergence result for these finite different schemes in the context of complex diffusion with reactive terms.

A Technical results

Lemma 5 (Discrete Poincaré-Friedrichs inequality) *Let U be a discrete function defined on $\bar{\Omega}_h$ given by (11) such that $U = 0$ on $\Gamma \cap \bar{\Omega}_h$. Then there*

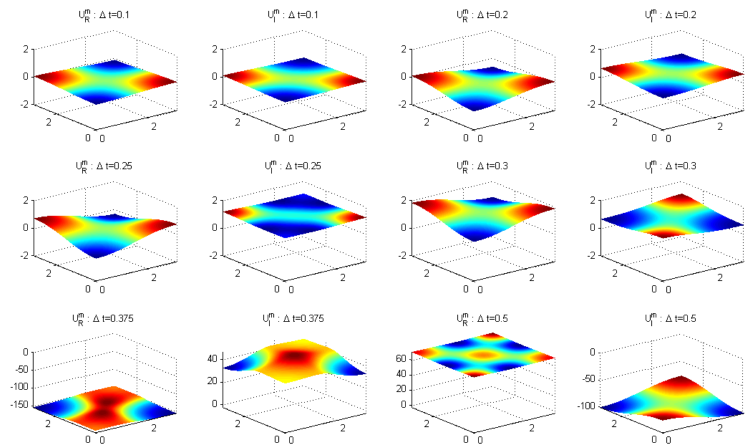

Figure 2: Case 1: Real and imaginary parts of the approximation U^m for the final time $T = 3$ for several time steps Δt.

exists a positive constant $C(\Omega)$ *independent of* U *and* h *such that*

$$C(\Omega)\|U\|_h^2 \leq \|\delta_x U\|_{h^*}^2.$$

Proof Since $\|U\|_h^2 = \|U_R\|_h^2 + \|U_I\|_h^2$ and $\|\delta_x U\|_{h^*}^2 = \|\delta_x U_R\|_{h^*}^2 + \|\delta_x U_I\|_{h^*}^2$ the proof follows from the equivalent result for the real case (see *e.g.* [10]). ■

Lemma 6 (Discrete Friedrich inequality) *Let* U *be a discrete function defined on* $\bar{\Omega}_h$ *given by (11). Then there exists a positive constant* $C(\Omega)$ *independent of* U *and* h *such that*

$$C(\Omega)\|U - \bar{U}\|_h^2 \leq \|\delta_x U\|_{h^*}^2,$$

where

$$\bar{U} = \frac{1}{|\Omega|}(U, \mathbf{1})_h,$$

and $\mathbf{1}$ *a vector with all entries equal to one.*

Proof Let us consider $\bar{U} = \bar{U}_R + i\bar{U}_I$. We just need to prove that

$$C(\Omega)\|U_R - \bar{U}_R\|_h^2 \leq \|\delta_x U_R\|_{h^*}^2.$$

To prove the result we will just consider the unidimensional case. The proof is similar for higher dimensions. Since

$$\bar{U}_R = \frac{1}{|\Omega|}(U_R, \mathbf{1})_h,$$

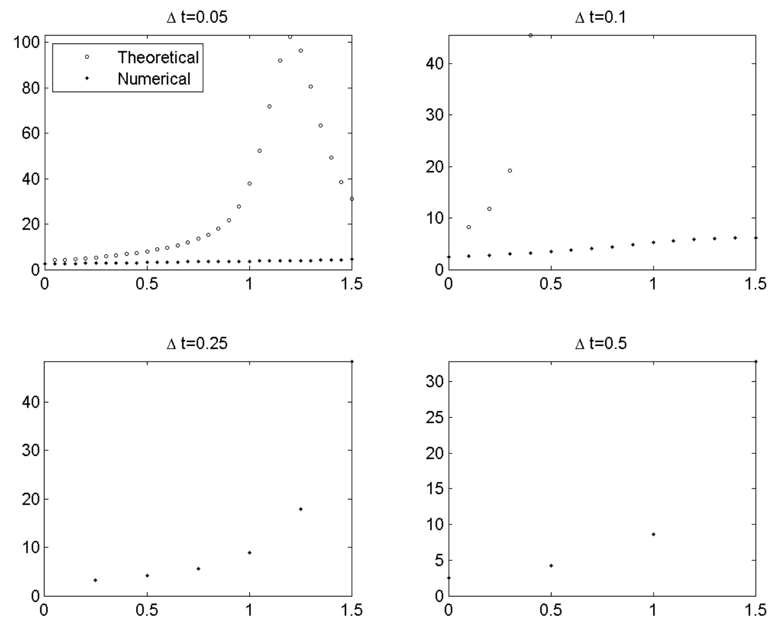

Figure 3: Case 2: Evolution in time of numerical norm $\|U^m\|_h^2$ and the theoretical upper bound (48) for several time steps Δt. No plot on the theoretical upper bound, means there exists no ξ that satisfies (41).

there exists some index j_{max} such that

$$|U_{Rj} - \bar{U}_R| \leq |U_{Rj} - U_{Rj_{max}}|, \qquad j = 0, ..., N.$$

Then, using the Cauchy-Schwarz inequality,

$$(U_{Rj} - \bar{U}_R)^2 \leq \left(\sum_{\ell=\min\{j,j_{max}\}}^{\max\{j,j_{max}\}} h_\ell \delta_x U_{\ell-1/2} \right)^2 \leq |\Omega| \sum_{\ell=1}^{N} h_\ell (\delta_x U_{\ell-1/2})^2, \qquad j = 0, ..., N.$$

Summing through j we get

$$\sum_{j=0}^{N-1} \frac{h_j}{2} \left((U_{Rj} - \bar{U}_R)^2 + (U_{Rj+1} - \bar{U}_R)^2 \right) \leq |\Omega|^2 \sum_{\ell=1}^{N} h_\ell (\delta_x U_{\ell-1/2})^2,$$

which concludes the proof. ■

Lemma 7 (Discrete conservation property) *Let U^m be the solution of (12)–(13) with (14) or (15), respectively. If $F(x,t,v) = F_0(x,t)$ the following discrete*

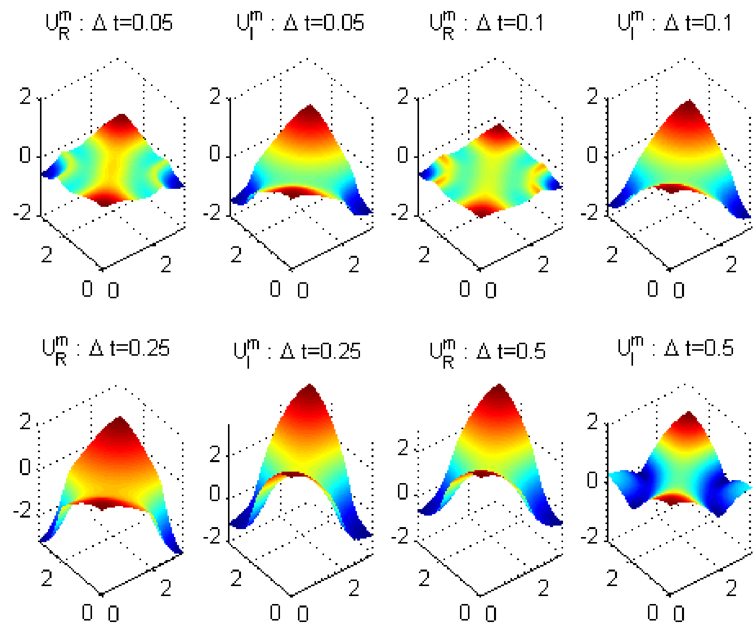

Figure 4: Case 2: Real and imaginary parts of the approximation U^m for final time $T = 1.5$ for several time steps Δt.

conservation property holds

$$(U^m, \mathbf{1})_h = (U^0, \mathbf{1})_h + \sum_{k=0}^{m} \Delta t^k (F_0(\cdot, t^{k+\theta}), \mathbf{1})_h.$$

Proof To prove the result we will just consider the unidimensional case. For higher dimensions, the proof follows the same steps.

Note that we have

$$U_R^{m+1} = U_R^m + \Delta t^m \left(A_1 U_R^{m+\theta} + A_2 U_I^{m+\theta} + F_{0R}^{m+\theta} \right)$$

and

$$U_I^{m+1} = U_I^m + \Delta t^m \left(A_3 U_R^{m+\theta} + A_4 U_I^{m+\theta} + F_{0I}^{m+\theta} \right)$$

where A_ℓ, $\ell = 1, 2, 3, 4$ are matrices that depend on D , U and on the spatial step sizes. Then, summing according to the discrete inner product, and taking into account that $(A_1 U_R^{m+\theta}, \mathbf{1})_h = (A_2 U_I^{m+\theta}, \mathbf{1})_h = (A_3 U_R^{m+\theta}, \mathbf{1})_h = (A_4 U_I^{m+\theta}, \mathbf{1})_h = 0$, we get

$$(U_R^m, \mathbf{1})_h = (U_R^0, \mathbf{1})_h + \sum_{k=0}^{m} \Delta t^k (F_{0R}(\cdot, t^{k+\theta}), \mathbf{1})_h$$

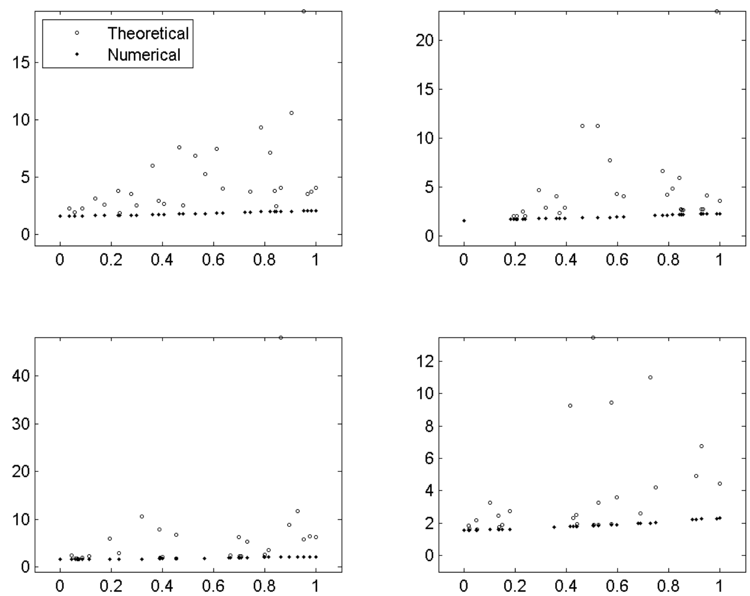

Figure 5: Case 2: Evolution in time of numerical norm $\|U^m\|_h^2$ and the theoretical upper bound (48) for non-uniform time steps Δt. No plot on the theoretical upper bound, means there exists no ξ that satisfies (41).

and

$$(U_I^m, \mathbf{1})_h = (U_I^0, \mathbf{1})_h + \sum_{k=0}^{m} \Delta t^k (F_{0I}(\cdot, t^{k+\theta}), \mathbf{1})_h$$

which concludes the proof. ■

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