LIMITS OF TANGENTS OF A QUASI-ORDINARY HYPERSURFACE

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Abstract. We compute explicitly the limits of tangents of a quasi-ordinary singularity in terms of its special monomials. We show that the set of limits of tangents of Y is essentially a topological invariant of Y.

1. Introduction

The study of the limits of tangents of a complex hypersurface singularity was mainly developped by Le Dung Trang and Bernard Teissier (see [4] and its bibliography). Chunsheng Ban [1] computed the set of limits of tangents Λ of a quasi-ordinary singularity Y when Y has only one very special monomial (see Definition 1.2).

The main achievement of this paper is the explicit computation of the limits of tangents of an arbitrary quasi-ordinary hypersurface singularity (see Theorems 2.8, 2.9 and 2.10). Corollaries 2.11, 2.12 and 2.13 show that the set of limits of tangents of Y comes quite close to being a topological invariant of Y. Corollary 2.12 shows that Λ is a topological invariant of Y when the tangent cone of Y is a hyperplane. Corollary 2.14 shows that the triviality of the set of limits of tangents of Y is a topological invariant of Y.

Let X be a complex analytic manifold. Let $\pi : T^*X \to X$ be the cotangent bundle of X. Let Γ be a germ of a Lagrangean variety of T^*X at a point α. We say that Γ is in *generic position* if $\Gamma \cap \pi^{-1}(\pi(\alpha)) = \mathbb{C}\alpha$. Let Y be a hypersurface singularity of X. Let Γ be the conormal $T^*_Y X$ of Y. The Lagrangean variety Γ is in generic position if and only if Y is the germ of an hypersurface with trivial set of limits of tangents.

Let $\mathcal{M}$ be an holonomic $\mathcal{D}_X$-module. The characteristic variety of $\mathcal{M}$ is a Lagrangean variety of T^*X. The characteristic varieties in generic position have a central role in $\mathcal{D}$-module theory (cf. Corollary 1.6.4 and Theorem 5.11 of [6] and Corollary 3.12 of [5]). It would be quite interesting to have good characterizations of the hypersurface singularities with trivial set of limits of tangents. Corollary 2.14 is a first step in this direction.

After finishing this paper, two questions arose naturally:

Let Y be an hypersurface singularity such that its tangent cone is an hyperplane. Is the set of limits of tangents of Y a topological invariant of Y?

Is the triviality of the set of limits of tangents of an hypersurface a topological invariant of the hypersurface?

Let $p : \mathbb{C}^{n+1} \to \mathbb{C}^n$ be the projection that takes $(x, y) = (x_1, \ldots, x_n, y)$ into x. Let Y be the germ of a hypersurface of $\mathbb{C}^{n+1}$ defined by $f \in \mathbb{C}\{x_1, \ldots, x_n, y\}$. Let W

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be the singular locus of Y. The set Z defined by the equations $f = \partial f/\partial y = 0$ is called the *apparent contour* of f relatively to the projection p. The set $\Delta = p(Z)$ is called the *discriminant* of f relatively to the projection p.
Near $q \in Y\backslash Z$ there is one and only one function $\varphi \in \mathcal{O}_{\mathbb{C}^{n+1},q}$ such that $f(x,\varphi(x)) = 0$. The function f defines implicitly y as a function of x. Moreover,

$$\frac{\partial y}{\partial x_i} = \frac{\partial \varphi}{\partial x_i} = -\frac{\partial f/\partial x_i}{\partial f/\partial y} \text{ on } Y \setminus Z. \tag{1.1}$$

Let $\theta = \xi_1 dx_1 + \ldots \xi_n dx_n + \eta dy$ be the canonical 1-form of the cotangent bundle $T^*\mathbb{C}^{n+1} = \mathbb{C}^{n+1} \times \mathbb{C}_{n+1}$. An element of the projective cotangent bundle $\mathbb{P}^*\mathbb{C}^{n+1} = \mathbb{C}^{n+1} \times \mathbb{P}_n$ i s represented by the coordinates

$$(x_1, \ldots, x_n, y; \xi_1 : \cdots : \xi_n : \eta).$$

We will consider in the open set $\{\eta \neq 0\}$ the chart

$$(x_1, \ldots, x_n, y, p_1, \ldots, p_n),$$

where $p_i = -\xi_i/\eta, 1 \leq i \leq n$. Let Γ_0 be the graph of the map from $Y \setminus W$ into $\mathbb{P}_n$ defined by

$$(x,y) \mapsto \left(\frac{\partial f}{\partial x_1} : \cdots : \frac{\partial f}{\partial x_n} : \frac{\partial f}{\partial y}\right).$$

Let Γ be the smallest closed analytic subset of $\mathbb{P}^*\mathbb{C}^{n+1}$ that contains Γ_0. The analytic set Γ is a Legendrian subvariety of the contact manifold $\mathbb{P}^*\mathbb{C}^{n+1}$. The projective algebraic set $\Lambda = \Gamma \cap \pi^{-1}(0)$ is called the *set of limits of tangents* of Y.

Remark 1.1. It follows from (1.1) that

$$\left(\frac{\partial f}{\partial x_1} : \cdots : \frac{\partial f}{\partial x_n} : \frac{\partial f}{\partial y}\right) = \left(-\frac{\partial y}{\partial x_1} : \cdots : -\frac{\partial y}{\partial x_n} : 1\right) \text{ on } Y \setminus Z.$$

Let $c_1, \ldots, c_n$ be positive integers. We will denote by $\mathbb{C}\{x_1^{1/c_1}, \ldots, x_n^{1/c_n}\}$ the $\mathbb{C}\{x_1, \ldots, x_n\}$ algebra given by the immersion from $\mathbb{C}\{x_1, \ldots, x_n\}$ into $\mathbb{C}\{t_1, \ldots, t_n\}$ that takes x_i into $t_i^{c_i}, 1 \leq i \leq n$. We set $x_i^{1/c_i} = t_i$. Let $a_1, \ldots, a_n$ be positive rationals. Set $a_i = b_i/c_i, 1 \leq i \leq n$, where $(b_i, c_i) = 1$. Given a ramified monomial $M = x_1^{a_1} \cdots x_n^{a_n} = t_1^{b_1} \cdots t_n^{b_n}$ we set $\mathcal{O}(M) = \mathbb{C}\{x_1^{1/c_1}, \ldots, x_n^{1/c_n}\}$.
Let Y be a germ at the origin of a complex hypersurface of $\mathbb{C}^{n+1}$. We say that Y is a quasi-ordinary singularity if Δ is a divisor with normal crossings. We will assume that there is $l \leq m$ such that $\Delta = \{x_1 \cdots x_l = 0\}$.
If Y is an irreducible quasi-ordinary singularity there are ramified monomials $N_0, N_1, \ldots, N_m, g_i \in \mathcal{O}(N_i), 0 \leq i \leq m$, such that $N_0 = 1$, N_{i-1} divides N_i in the ring $\mathcal{O}(N_i)$, g_i is as unity of $\mathcal{O}(N_i), 1 \leq i \leq m$, g_0 vanishes at the origin and the map $x \mapsto (x, \varphi(x))$ is a parametrization of Y near the origin, where

$$\varphi = g_0 + N_1 g_1 + \ldots + N_m g_m. \tag{1.2}$$

Replacing y by $y - g_0$, we can assume that $g_0 = 0$. The monomials $N_i, 1 \leq i \leq m$, are unique and determine the topology of Y (see [3]). They are called the *special monomials* of f. We set $\tilde{\mathcal{O}} = \mathcal{O}(N_m)$.

Definition 1.2. We say that a special monomial N_i, $1 \leq i \leq m$, is *very special* if $\{N_i = 0\} \neq \{N_{i-1} = 0\}$.

Let $M_1, \ldots, M_g$ be the very special monomials of f, where $M_k = N_{n_k}, 1 = n_1 < n_2 < \ldots < n_g, 1 \leq k \leq g$. Set $M_0 = 1, n_{g+1} = n_g + 1$. There are units f_i of $\mathcal{O}(N_{n_{i+1}-1})$, $1 \leq i \leq g$, such that

$$\varphi = M_1 f_1 + \ldots + M_g f_g. \tag{1.3}$$

2. Limits of tangents

After renaming the variables x_i there are integers $m_k, 1 \leq k \leq g+1$, and positive rational numbers $a_{kij}, 1 \leq k \leq g, 1 \leq i \leq k, 1 \leq j \leq m_k$ such that

$$M_k = \prod_{i=1}^{k} \prod_{j=1}^{m_k} x_{ij}^{a_{kij}}, \qquad 1 \leq k \leq g. \tag{2.1}$$

The canonical 1-form of $\mathbb{P}^*\mathbb{C}^{n+1}$ becomes

$$\theta = \sum_{i=1}^{g+1} \sum_{j=1}^{m_i} \xi_{ij} dx_{ij}. \tag{2.2}$$

We set $p_{ij} = -\xi_{ij}/\eta, 1 \leq i \leq g+1, 1 \leq j \leq m_i$. Remark that

$$\frac{\partial y}{\partial x_{ij}} = a_{iij} \frac{M_i}{x_{ij}} \sigma_{ij}, \tag{2.3}$$

where σ_{ij} is a unit of $\tilde{\mathcal{O}}$.

Theorem 2.1. *If $\sum_{i=1}^{m_1} a_{11i} < 1$, $\Lambda \subset \{\eta = 0\}$.*

Proof. Set $m = m_1$, $x_i = x_{1i}$ and $a_i = a_{11i}$, $1 \leq i \leq m$. Given positive integers $c_1, \ldots, c_m$, it follows from (2.3) that

$$\prod_{i=1}^{m} p_i^{c_i} = \prod_{i=1}^{m} x_i^{a_i \sum_{j=1}^{m} c_j - c_i} \phi, \tag{2.4}$$

for some unit ϕ of $\tilde{\mathcal{O}}$. By (1.3) and (2.3),

$$\phi(0) = f_1(0)^{\sum_{j=1}^{m} c_j} \prod_{j=1}^{m} a_j^{c_j}. \tag{2.5}$$

Hence

$$\eta^{\sum_{i=1}^{m} c_i} = \psi \prod_{i=1}^{m} \xi_i^{c_i} x_i^{c_i - a_i \sum_{j=1}^{m} c_j}, \tag{2.6}$$

for some unit ψ. If there are integers $c_1, \ldots, c_m$ such that the inequalities

$$a_k \sum_{j=1}^{m} c_j < c_k, \quad 1 \leq k \leq m, \tag{2.7}$$

hold, the result follows from (2.6). Hence it is enough to show that the set Ω of the m-tuples of rational numbers $(c_1, \ldots, c_m)$ that verify the inequalities (2.7) is non-empty. We will recursively define positive rational numbers l_j, c_j, u_j such that

$$l_j < c_j < u_j, \tag{2.8}$$

j=1,...,m. Let c_1, l_1, u_1 be arbitrary positive rationals verifying $(2.8)_1$. Let $1 < s \leq m$. If l_i, c_i, u_i are defined for $i \leq s-1$, set

$$l_s = \frac{a_s \sum_{j=1}^{s-1} c_j}{1 - \sum_{j=s}^{m} a_j}, \ u_s = (a_s/a_{s-1})c_{s-1}. \tag{2.9}$$

Since $\sum_{j \geq s} a_j < 1$ and

$$\begin{aligned} u_s - l_s &= \frac{a_s}{a_{s-1}(1-\sum_{j=s}^m a_j)} \left((1 - \sum_{j=s-1}^{m} a_j) c_{s-1} - a_{s-1} \sum_{j<s-1} c_j \right) \\ &= \frac{a_s}{a_{s-1}(1-\sum_{j=s}^m a_j)} \left((1 - \sum_{j=s-1}^{m} a_j)(c_{s-1} - l_{s-1}) \right), \end{aligned}$$

it follows from $(2.8)_{s-1}$ that $l_s < u_s$. Let c_s be a rational number such that $l_s < c_s < u_s$. Hence $(2.8)_s$ holds for $s \leq m$.
Let us show that $(c_1, \ldots, c_m) \in \Omega$. Since $c_k < u_k$, then

$$c_k < \frac{a_k}{a_{k-1}} c_{k-1}, \text{ for } k \geq 2.$$

Then, for $j < k$,

$$c_k < \frac{a_k}{a_{k-1}} \frac{a_{k-1}}{a_{k-2}} \cdots \frac{a_{j+1}}{a_j} c_j = \frac{a_k}{a_j} c_j.$$

Hence,

$$a_k c_j < a_j c_k, \text{ for } j > k. \tag{2.10}$$

Since $l_k < c_k$,

$$a_k \sum_{j=1}^{k-1} c_j < c_k - \sum_{j=k}^{m} a_j c_k.$$

Hence, by (2.10),

$$a_k \sum_{j=1}^{k-1} c_j < c_k - \sum_{j=k}^{m} a_k c_j.$$

Therefore $a_k \sum_{j=1}^m c_j < c_k$. □

Theorem 2.2. *Let $1 \leq k \leq g$. Let $I \subset \{1, \ldots, m_k\}$. Assume that one of the following three hypothesis is verified:*

1. $\sum_{j \in I} a_{kkj} > 1$;
2. $k = 1$, $\sum_{j \in I} a_{11j} = 1$ and $\sum_{j=1}^{m_1} a_{11j} > 1$;
3. $k \geq 2$ *and* $\sum_{j \in I} a_{kkj} = 1$.

Then $\Lambda \subset \{\prod_{j \in I} \xi_{kj} = 0\}$.

Proof. Case 1: We can assume that $I = \{1, \ldots, n\}$, where $1 \leq n \leq m_k$. Set $a_i = a_{kki}$. Given positive integers $c_1, \ldots, c_n$, it follows from (2.3) that

$$\prod_{i=1}^{n} \xi_{ki}^{c_i} = \prod_{i=1}^{n} x_{ki}^{a_i \sum_{j=1}^n c_j - c_i} \eta^{\sum_{i=1}^n c_i} \varepsilon, \tag{2.11}$$

where $\varepsilon \in \widetilde{\mathcal{O}}$. Hence it is enough to show that there are positive rational numbers $c_1, \ldots, c_n$ such that

$$a_k(\sum_{j=1}^{n} c_j) - c_k > 0,\ 1 \leq k \leq n. \tag{2.12}$$

We will recursively define $l_j, c_j, u_j \in]0, +\infty]$ such that $c_j, l_j \in \mathbb{Q}$,

$$l_j < c_j < u_j, \tag{2.13}$$

j=1,…,n, and $u_j \in \mathbb{Q}$ if and only if $\sum_{i=j}^{n} a_i < 1$. Choose c_1, l_1, u_1 verifying (2.13). Let $1 < s \leq n-1$. Suppose that l_i, c_i, u_i are defined for $1 \leq i \leq s-1$. If $\sum_{j=s}^{n} a_j < 1$, set

$$l_s = (a_s/a_{s-1})c_{s-1},\ u_s = \frac{a_s \sum_{j=1}^{s-1} c_j}{1 - \sum_{j=s}^{n} a_j}. \tag{2.14}$$

Since

$$\begin{aligned} u_s - l_s &= \frac{a_s}{a_{s-1}(1-\sum_{j=s}^{n} a_j)} \left(a_{s-1} \sum_{j=1}^{s-2} c_j - c_{s-1}(1 - \sum_{j=s-1}^{n} a_j) \right) \\ &\leq \frac{a_s}{a_{s-1}(1-\sum_{j=s}^{n} a_j)} \left((1 - \sum_{j=s-1}^{n} a_j)(u_{s-1} - c_{s-1}) \right), \end{aligned}$$

it follows from $(2.13)_{s-1}$ that $l_s < u_s$.
If $\sum_{j=s}^{n} a_j \geq 1$, set l_s as above and $u_s = +\infty$.
We choose a rational number c_s such that $l_s < c_s < u_s$. Hence $(2.13)_s$ holds for $1 \leq s \leq n$.
Let us show that $c_1, \ldots, c_n$ verify (2.12). We will proceed by induction. First we will show that $c_1, \ldots, c_n$ verify $(2.12)_n$. Suppose that $a_n < 1$. Since $c_n < u_n$, we have that

$$c_n < \frac{a_n \sum_{j=1}^{n-1} c_j}{1 - a_n}.$$

Hence $a_n \sum_{j=1}^{n} c_j > c_n$. If $a_n \geq 1$, then

$$a_n \sum_{j=1}^{n} c_j \geq \sum_{j=1}^{n} c_j > c_n.$$

Hence $(2.12)_n$ is verified. Assume that $c_1, \ldots, c_n$ verify $(2.12)_k$, $2 \leq k \leq n$. Since $c_k > l_k$,

$$a_k \sum_{j=1}^{n} c_j > c_k > \frac{a_k}{a_{k-1}} c_{k-1}.$$

Hence $a_{k-1} \sum_{j=1}^{n} c_j > c_{k-1}$. Therefore $(c_1, \ldots, c_n)$ verify $(2.12)_{k-1}$.
Case 2: Set $a_j = a_{11j}$ and $x_j = x_{1j}$. We can assume that $I = \{1, \ldots, n\}$, where $1 \leq n \leq m_1$. Given positive integers $c_1, \ldots, c_n$, it follows from (1.2) that

$$\prod_{i=1}^{n} \xi_i^{c_i} = \prod_{i=1}^{n} x_i^{a_i \sum_{j=1}^{n} c_j - c_i} \eta^{\sum_{i=1}^{n} c_i} \varepsilon, \tag{2.15}$$

where $\varepsilon \in \widetilde{\mathcal{O}}$ and $\varepsilon(0) = 0$. Hence it is enough to show that there are positive rational numbers $c_1, \ldots, c_n$, such that

$$a_k \sum_{j=1}^{n} c_j = c_k, \ 1 \leq k \leq n. \tag{2.16}$$

We choose an arbitrary positive integer c_1. Let $1 < s \leq n$. If the c_i are defined for $i < s$, set

$$c_s = \frac{a_s}{a_{s-1}} c_{s-1}. \tag{2.17}$$

Let us show that $c_1, \ldots, c_n$ verify (2.16). We will proceed by induction in k. First let us show that $(2.16)_n$ holds.
Let $j < n-1$. By (2.17),

$$c_{n-1} = \frac{a_{n-1}}{a_{n-2}} \frac{a_{n-2}}{a_{n-3}} \cdots \frac{a_{j+1}}{a_j} c_j = \frac{a_{n-1}}{a_j} c_j. \tag{2.18}$$

By (2.17), and since $\sum_{j=1}^{n} a_j = 1$,

$$c_n = \frac{a_n}{a_{n-1}} c_{n-1} = \frac{c_{n-1}}{a_{n-1}} (1 - \sum_{j=1}^{n-1} a_j) = \frac{c_{n-1}}{a_{n-1}} - \sum_{j=1}^{n-1} \frac{a_j}{a_{n-1}} c_{n-1}.$$

Hence, by (2.18)

$$c_n = \frac{c_{n-1}}{a_{n-1}} - \sum_{j=1}^{n-1} c_j.$$

Therefore, $\sum_{j=1}^{n} c_j = c_{n-1}/a_{n-1}$. Hence by (2.17),

$$a_n \sum_{j=1}^{n} c_j = a_n \frac{c_{n-1}}{a_{n-1}} = c_n.$$

Therefore $(2.16)_n$ holds.
Assume $(2.16)_k$ holds, for $2 \leq k \leq n$. Then

$$a_k \sum_{j=1}^{n} c_j = c_k = \frac{a_k}{a_{k-1}} c_{k-1}.$$

Hence, $a_{k-1} \sum_{j=1}^{n} c_j = c_{k-1}$.
Case 3: We can assume that $I = \{1, \ldots, n\}$, where $1 \leq n \leq m_k$. Given positive integers $c_1, \ldots, c_n$, it follows from (2.3) that

$$\prod_{\imath=1}^{n} \xi_{ki}^{c_i} = \left(\prod_{i=1}^{n} x_{ki}^{a_{kki} (\sum_{j=1}^{n} c_j) - c_i} \right) \eta^{\sum_{i=1}^{n} c_i} \varepsilon,$$

where $\varepsilon \in \tilde{\mathcal{O}}$ and $\varepsilon(0) = 0$. We have reduced the problem to the case 2. □

Theorem 2.3. *If* $\sum_{k=1}^{m_1} a_{11j} = 1$, Λ *is contained in a cone.*

Proof. Set $a_i = a_{11i}, i = 1, \ldots m_1$. Given positive integers $c_1, \ldots, c_{m_1}$, there is a unit ϕ of $\tilde{\mathcal{O}}$ such that

$$\prod_{i=1}^{m_1} \xi_i^{c_i} = (-1)^{\sum_{j=1}^{m_1} c_j} \phi \prod_{i=1}^{m_1} x_i^{\sum_{j=1}^{m_1} c_j a_i - c_i} \eta^{\sum_{j=1}^{m_1} c_j}. \tag{2.19}$$

By the proof of case 2 of Theorem 2.2, there is one and only one m_1-tuple of integers $c_1, \dots, c_{m_1}$ such that $(c_1, \dots, c_{m_1}) = (1)$, $a_i \sum_{j=1}^{m_1} c_j = c_i, 1 \leq i \leq m_1$, and Λ is contained in the cone defined by the equation

$$\prod_{i=1}^{m_1} \xi_i^{c_i} - (-1)^{\sum_{j=1}^{m_1} c_j} \phi(0) \eta^{\sum_{j=1}^{m_1} c_j} = 0, \tag{2.20}$$

where $\phi(0)$ is given by (2.5). □

Remark 2.4. Set $D_\varepsilon^* = \{x \in \mathbb{C} : 0 < |x| < \varepsilon\}$, where $0 < \varepsilon << 1$. Set $\mu = \sum_{k=1}^{g+1} m_k$. Let $\sigma : \mathbb{C} \to \mathbb{C}^\mu$ be a weighted homogeneous curve parametrized by

$$\sigma(t) = (\varepsilon_{ki} t^{\alpha_{ki}})_{1 \leq k \leq g+1, 1 \leq i \leq m_k}.$$

Notice that the image of σ is contained in $\mathbb{C}^\mu \setminus \Delta$. Set $\theta_0(t) = 1$ and

$$\theta_{ki}(t) = \frac{\partial \varphi}{\partial x_{ki}}(\sigma(t), \varphi(\sigma(t))),\ 1 \leq k \leq g+1, 1 \leq i \leq m_k,$$

for $t \in D_\varepsilon^*$. The curve σ induces a map from D_ε^* into Γ defined by

$$t \mapsto (\sigma(t), \varphi(\sigma(t)); \theta_{11}(t) : \dots : \theta_{g+1, m_g+1}(t) : \theta_0(t)).$$

Let $\vartheta : D_\varepsilon^* \to \mathbb{P}^\mu$ be the map defined by

$$t \mapsto (\theta_{11}(t) : \dots : \theta_{g+1, m_g+1}(t) : \theta_0(t)). \tag{2.21}$$

The limit when $t \to 0$ of $\vartheta(t)$ belongs to Λ. The functions θ_{ki} are ramified Laurent series of finite type on the variable t. Let h a be ramified Laurent series of finite type. If $h = 0$, we set $v(h) = \infty$. If $h \neq 0$, we set $v(h) = \alpha$, where α is the only rational number such that $\lim\limits_{t \to 0} t^{-\alpha} h(t) \in \mathbb{C} \setminus \{0\}$. We call α the *valuation* of h. Notice that the limit of ϑ only depends on the functions θ_{ki}, θ_0 of minimal valuation. Moreover, the limit of ϑ only depends on the coefficients of the term of minimal valuation of each θ_{ij}, θ_0. Hence the limit of ϑ only depends on the coefficients of the very special monomials of f. We can assume that $m_{g+1} = 0$ and that there are $\lambda_k \in \mathbb{C} \setminus \{0\}, 1 \leq k \leq g$, such that

$$\varphi = \sum_{k=1}^{g} \lambda_k M_k. \tag{2.22}$$

Remark 2.5. Let L be a finite set. Set $\mathbb{C}^L = \{(x_a)_{a \in L} : x_a \in \mathbb{C}\}$. Let $\sum_{a \in L} \xi_a dx_a$ be the canonical 1-form of $T^*\mathbb{C}^L$. Let Λ be the subset of $\mathbb{P}_L$ defined by the equations

$$\prod_{a \in I} \xi_a = 0,\ I \in \mathcal{I}, \tag{2.23}$$

where $\mathcal{I} \subset \mathcal{P}(L)$. Set $\mathcal{I}' = \{J \subset L : J \cap I \neq \emptyset \text{ for all } I \in \mathcal{I}\}$, $\mathcal{I}^* = \{J \in \mathcal{I}'$ such that there is no $K \in \mathcal{I}' : K \subset J, K \neq J\}$. The irreducible components of Λ are the linear projective sets $\Lambda_J, J \in \mathcal{I}^*$, where Λ_J is defined by the equations

$$\xi_a = 0, \qquad a \in J.$$

Let Y be a germ of hypersurface of $(\mathbb{C}^L, 0)$. Let Λ be the set of limits of tangents of Y. For each irreducible component Λ_J of Λ there is a cone V_J contained in the tangent cone of Y such that Λ_J is the dual of the projectivization of V_J. The union of the cones V_J is called the *halo* of Y. The halo of Y is called "la auréole" of Y in [4].

Remark 2.6. If Λ is defined by the equations (2.23), the halo of Y equals the union of the linear subsets $V_J, J \in \mathcal{I}^*$ of $\mathbb{C}^{\,L}$, where V_J is defined by the equations

$$x_a = 0, \qquad a \in L \setminus J.$$

Lemma 2.7. *The determinant of the $n \times n$ matrix $(\lambda_i - \delta_{ij})$ equals*

$$(-1)^n(1 - \sum_{i=1}^{n} \lambda_i).$$

Proof. Notice that $\det(\lambda_i - \delta_{ij}) =$

$$= \left| \begin{array}{ccc|c} & & & 1 \\ & -I_{n-1} & & \vdots \\ & & & 1 \\ \hline \lambda_1 & \cdots & \lambda_{n-1} & \lambda_n - 1 \end{array} \right| = \left| \begin{array}{ccc|c} & & & 1 \\ & -I_{n-1} & & \vdots \\ & & & 1 \\ \hline 0 & \cdots & 0 & \sum_{i=1}^{n} \lambda_i - 1 \end{array} \right|.$$

□

Theorem 2.8. *Assume that $\sum_{i=1}^{m_1} a_{11i} < 1$. Set*

$$L = \cup_{k=2}^{g} \{k\} \times \{1, \dots, m_k\},\ \mathcal{I} = \cup_{k=2}^{g} \{\{k\} \times I : \sum_{j \in I} a_{kkj} \geq 1\}.$$

The set Λ is the union of the irreducible linear projective sets $\Lambda_J, J \in \mathcal{I}^*$, defined by the equations $\eta = 0$ and

$$\xi_{kj} = 0,\ (k,j) \in J. \tag{2.24}$$

The tangent cone of Y equals $\{x_{11} \cdots x_{1m_1} = 0\}$. The halo of Y is the union of the cones V_J, $J \in \mathcal{I}^*$, where V_J is defined by the equations $x_{1j} = 0$, $1 \leq j \leq m_1$, and

$$x_{kj} = 0, (k,j) \in L \setminus J. \tag{2.25}$$

Proof. Let us show that $\Lambda_J \subset \Lambda$. We can assume that there are integers $n_1, \dots, n_g$, $1 \leq n_k \leq m_k$, $1 \leq k \leq g$, such that $J = \cup_{k=1}^{g} \{k\} \times \{n_k + 1, \dots, m_k\}$. We will use the notations of Remark 2.4.
Set $m = \sum_{k=1}^{g} m_k, n = m - \#J$. Assume that there are positive rational numbers $\alpha_k, \beta_k, 1 \leq k \leq g$, such that $\alpha_{ki} = \alpha_k$ if $1 \leq i \leq n_k$, $\alpha_{ki} = \beta_k$ if $n_k + 1 \leq i \leq m_k$, and $\alpha_k > \beta_k$, $1 \leq k \leq g$. Since $v(\theta_{ki}) = v(M_k) - v(x_{ki}) = v(M_k) - \alpha_{ki}$,

$$\lim_{t \to 0} \vartheta(t) \in \Lambda_J.$$

Let $\psi : (\mathbb{C} \setminus \{0\})^n \to \Lambda_J$ be the map defined by

$$\psi(\varepsilon_{ij}) = \lim_{t \to 0} \vartheta(t). \tag{2.26}$$

The map ψ has components ψ_{ki}, $1 \leq i \leq n_k, 1 \leq k \leq g$. In order to prove the Theorem it is enough to show that we can choose the rational numbers α_k, β_k in such a way that the Jacobian of ψ does not vanish identically. We will proceed by induction in k. Let $k = 1$. Since $\sum_{i=1}^{m_1} a_{11i} < 1$, $n_1 = m_1$. Choose positive rationals α_1, β_1, $\alpha_1 > \beta_1$. There is a rational number $v_0 < 0$ such that $v(\theta_{1i}) = v_0$, for all $1 \leq i \leq n_1$.
Assume that there are α_k, β_k such that $v(\theta_{ki}) = v_0$ for $1 \leq i \leq n_k$ and $v(\theta_{ki}) > v_0$ for $n_k + 1 \leq i \leq m_k$, $k = 1, \dots, u$. Set

$$\underline{\alpha}_{u+1} = \frac{\alpha_u + \sum_{k=1}^{u} \sum_{i=1}^{m_k} (a_{u+1,k,i} - a_{uki})\alpha_{ki}}{1 - \sum_{i=1}^{n_{u+1}} a_{u+1,u+1,i}}.$$

Since the special monomials are ordered by valuation and, by construction of Λ_J, $\sum_{i=1}^{n_k} a_{kki} < 1$ for all $1 \leq k \leq g$, $\underline{\alpha}_{u+1}$ is a positive rational number. Choose a rational number β_{u+1} such that $0 < \beta_{u+1} < \underline{\alpha}_{u+1}$. Set

$$\alpha_{u+1} = \underline{\alpha}_{u+1} + \frac{\sum_{i=n_{u+1}+1}^{m_{u+1}} a_{u+1,u+1,i}\beta_{u+1}}{1 - \sum_{i=1}^{n_{u+1}} a_{u+1,u+1,i}}.$$

Then, $v(\theta_{u+1,i}) = v(M_{u+1}) - \alpha_{u+1} = v(M_u) - \alpha_u = v_0$ for $1 \leq i \leq n_{u+1}$.
Set $\widehat{M}_k = \prod_{i=1}^{k} \prod_{j=1}^{m_k} \varepsilon_{ij}^{a_{kij}}, 1 \leq i \leq n_k, 1 \leq k \leq g$. With these choices of α_{ki}, we have that

$$\psi_{ki} = \frac{\widehat{M}_k a_{kki}}{\varepsilon_{ki}},\ 1 \leq i \leq n_k, 1 \leq k \leq g.$$

Let D be the jacobian matrix of ψ. Since $\partial\psi_{ki}/\partial\varepsilon_{uj} = 0$ for all $u > k$, D is upper triangular by blocks. Let D_k be the k-th diagonal block of D, $1 \leq k \leq g$. We have that

$$D_k = \left(\frac{\widehat{M}_k}{\varepsilon_{ki}\varepsilon_{kj}} a_{kki}(a_{kkj} - \delta_{ij}) \right).$$

By Lemma 2.7, $\det(D_k) = \lambda(1 - \sum_{i=1}^{m_k} a_{kki})$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Hence Λ contains an open set of Λ_J. Since Λ is a projective variety and Λ_J is irreducible, Λ contains Λ_J. ☐

Theorem 2.9. *Assume that* $\sum_{i=1}^{m_1} a_{11i} > 1$. *Set*

$$L = \cup_{k=1}^{g} \{k\} \times \{1, \ldots, m_k\},\ \mathcal{I} = \cup_{k=1}^{g} \{\{k\} \times I : \sum_{j \in I} a_{kkj} \geq 1\}.$$

The set Λ *is the union of the irreducible linear projective sets* $\Lambda_J, J \in \mathcal{I}^*$, *defined by the equations (2.24).*
The tangent cone of Y *equals* $\{y = 0\}$. *The halo of* Y *is the union of the cones* V_J, $J \in \mathcal{I}^*$, *where* V_J *is defined by the equations* $y = 0$ *and (2.25).*

Proof. The proof is analogous to the proof of Theorem 2.8. On the first induction step we choose

$$\beta_1 = \left(\frac{1 - \sum_{i=1}^{n_1} a_{11i}}{\sum_{i=n_1+1}^{m_1} a_{11i}} \right) \alpha_1.$$

Hence $\beta_1 < \alpha_1$, $v(\theta_{1i}) = v(\eta) = 0$ for $1 \leq i \leq n_1$ and $v(\theta_{1i}) > 0$ for $n_1 + 1 \leq i \leq m_1$. The rest of the proof proceeds as in the previous case. ☐

Theorem 2.10. *Assume that* $\sum_{i=1}^{m_1} a_{11i} = 1$. *Set*

$$L = \cup_{k=2}^{g} \{k\} \times \{1, \ldots, m_k\},\ \mathcal{I} = \cup_{k=2}^{g} \{\{k\} \times I : \sum_{j \in I} a_{kkj} \geq 1\}.$$

The set Λ *is the union of the irreducible projective algebraic sets* $\Lambda_J, J \in \mathcal{I}^*$, *where* Λ_J *is defined by the equations (2.20) and (2.24).*
There are integers c, d_i *such that* $a_{11i} = d_i/c, 1 \leq i \leq m_1$ *and* c *is the l.c.d. of* $d_1, \ldots, d_{m_1}$. *The tangent cone of* Y *equals*

$$y^c - f(0)^c \prod_{i=1}^{m_1} x_{1i}^{d_i} = 0. \tag{2.27}$$

The halo of Y is the union of the cones V_J, $J \in \mathcal{I}^*$, where V_J is defined by the equations (2.25) and (2.27).

Proof. Following the arguments of Theorem 2.8, it is enough to show that $\Lambda_J \subset \Lambda$ for each $J \in \mathcal{I}^*$. Choose $J \in \mathcal{I}^*$. Let $\tilde{\Lambda}_J$ be the linear projective variety defined by the equations (2.24). We follow an argument analogous to the one used in Theorem 2.8. We have $n_1 = m_1$. We choose positive rational numbers α_1, β_1 such that $\beta_1 < \alpha_1$. Then $v(\theta_{1i}) = 0$ for all $i = 1, \dots, m_1$. The remaining steps of the proof proceed as before. Hence

$$\lim_{t \to 0} \vartheta(t) \in \tilde{\Lambda}_J.$$

Let $\psi : (\mathbb{C} \setminus \{0\})^n \to \tilde{\Lambda}_J$ be the map defined by (2.26). By Theorem 2.3 the image of ψ is contained in Λ_J. By Lemma 2.7, $\det(D_1) = 0$. Let D'_1 be the matrix obtained from D_1 by eliminating the m_1-th line and column. Then $det(D'_1) = \lambda'(1 - \sum_{i=1}^{m_1-1} a_{kki})$ for some $\lambda' \in \mathbb{C} \setminus \{0\}$. Hence, $\Lambda_J \subset \Lambda$. □

Let Y be a quasi-ordinary hypersurface singularity.

Corollary 2.11. *The set of limits of tangents of Y only depends on the tangent cone of Y and the topology of Y.*

Corollary 2.12. *If the tangent cone of Y is a hyperplane, the set of limits of tangents of Y only depends on the topology of Y.*

Corollary 2.13. *Let $x_1^{\alpha_1} \cdots x_k^{\alpha_k}$ be the first special monomial of Y. If $\alpha_1 + \cdots + \alpha_k \neq 1$, the set of limits of tangents of Y only depends on the topology of Y.*

Corollary 2.14. *The triviality of the set of limits of tangents of Y is a topological invariant of Y.*

Proof. The set of limits of tangents of Y is trivial if and only if all the exponents of all the special monomials of Y are greater or equal than 1. □

References

1. C. Ban, "Auréole of a quasi-ordinary singularity" *Proc. Amer. Math. Soc.* , 120 (1994), 393-404.
2. J. Lipman, "Quasi-ordinary singularities of embedded surfaces, *Ph.D. thesis, Harvard University*, 1965.
3. J. Lipman, "Topological invariants of quasi-ordinary singularities", *Mem. Amer. Math. Soc.*, No. 74, Amer. Math. Soc., Providence, RI, 1988, pp. 1-107.
4. D.T. Lé and B. Teissier, "Limites d'espaces tangentes en géometrie analitique", *Comment. Math. Helv.*, 63, (1988), 540-578.
5. O. Neto, "A microlocal Riemann-Hilbert correspondence.", *Compositio. Math.*, 127, (2001), No. 3, 229-241.
6. M. Kashiwara and T. Kawai, "On holonomic systems of microdifferential equations. III. Systems with regular singularities", *Publ. Res. Inst. Math. Sci.*, Vol. 17, No. 3, (1981), 813-979.

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