INTEGRATION OF POSITIVE CONSTRUCTIBLE FUNCTIONS AGAINST EULER CHARACTERISTIC AND DIMENSION

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Abstract. Following recent work of R. Cluckers and F. Loeser [*Fonctions constructible et intégration motivic I*, Comptes rendus de l'Académie des Sciences, **339** (2004) 411 - 416] on motivic integration, we develop a direct image formalism for positive constructible functions in the globally subanalytic context. This formalism is generalized to arbitrary first-order logic models and is illustrated by several examples on the p-adics, on the Presburger structure and on o-minimal expansions of groups. Furthermore, within this formalism, we define the Radon transform and prove the corresponding inversion formula.

1. Introduction

1.1. By a subanalytic set we will always mean a globally subanalytic subset $X \subset \mathbf{R}^n$, meaning that X is subanalytic in the classical sense inside $\mathbf{P}^n(\mathbf{R})$ under the embedding $\mathbf{R}^n = \mathbf{A}^n(\mathbf{R}) \subseteq \mathbf{P}^n(\mathbf{R})$. By a subanalytic function a function whose graph is a (globally) subanalytic set.

By Sub we denote the category of subanalytic subsets $X \subset \mathbf{R}^n$ for all $n > 0$, with subanalytic maps as morphisms. We work with the Euler characteristic $\chi : \mathrm{Sub} \to \mathbf{Z}$ and the dimension $\dim : \mathrm{Sub} \to \mathbf{N}$ of subanalytic sets as defined for o-minimal structures in [8].

Note that if $X \in \mathrm{Sub}$, then by the o-minimal triangulation theorem in [8], the o-minimal Euler characteristic $\chi(X)$ coincides with the Euler characteristic $\chi_{BM}(X)$ of X with respect to the Borel-Moore homology. If $X \in \mathrm{Sub}$ is locally compact, the o-minimal Euler characteristic $\chi(X)$ coincides with the Euler characteristic $\chi_c(X)$ of X with respect to sheaf cohomology of X with compact supports and constant coefficient sheaf.

1.2. By [8], the Euler characteristic $\chi : \mathrm{Sub} \to \mathbf{Z}$ satisfies the following

$$\chi(\emptyset) = 0,$$

$$\chi(X) = \chi(Y) \text{ if } X \text{ and } Y \text{ are isomorphic in Sub}$$

and

$$\chi(X \cup Y) = \chi(X) + \chi(Y)$$

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whenever $X, Y \in$ Sub are disjoint. The last equality for χ_{BM} and χ_c follows from the long exact (co)homology sequence. If we take X to be the unit circle in the plane $\mathbf{R}^2$ and Y a point in X, we see that this equality does not hold for the Euler characteristic associated to the topological singular (co)homology.

Thus we can think of $\chi : \text{Sub} \to \mathbf{Z}$ as a measure with values in the Grothendieck ring $K_0(\text{Sub})$ of the category Sub and, for any $X \in$ Sub and any function $f : X \to \mathbf{Z}$ with finite range and the property that $f^{-1}(a) \in$ Sub for all $a \in \mathbf{Z}$ (constructible functions) one has an obvious definition for

$$\int_X f\,\chi$$

such that $\chi(X) = \int_X 1_X\,\chi$ (cf. [17]).

This measure and integration against Euler characteristic is what is considered by Viro [17], Shapira [15], [16] and Brocker [1]. However, for the measure $\chi : \text{Sub} \to \mathbf{Z}$ it is not true that $\chi(X) = \chi(Y)$ if and only if X and Y are isomorphic in Sub. Following the recent work of the first author and François Loeser [5], [6], [7] on motivic integration, we construct the universal measure μ for the category Sub with values in the Grothendieck semi-ring $SK_0(\text{Sub})$ of Sub such that $\mu(X) = \mu(Y)$ if and only if X and Y are isomorphic in Sub. Furthemore, we develop a direct image formalism for positive constructible functions, i.e., functions $f : X \to SK_0(\text{Sub})$ with finite range and the property that $f^{-1}(a) \in$ Sub for all $a \in SK_0(\text{Sub})$. This formalism is generalized to arbitrary first-order logic models and is illustrated by several examples on the p-adics, on the Presburger structure and on o-minimal expansions of groups. Moreover, within this formalism, we define the Radon transform and prove the corresponding inversion formula.

2. Positive constructible functions

We start by pointing out that instead of Sub we can work in this section with any o-minimal expansion of a field R using the category Def whose objects are definable sets and whose morphisms are definable maps.

2.1. By a semigroup we mean a commutative monoid with a unit element. Likewise a semi-ring is a set equipped with two semigroup structures: addition and multiplication such that 0 is a unit element for the addition, 1 is the unit element for multiplication and the two operations are connected by $x(y + z) = xy + xz$ and $0x = 0$. A morphism of semirings is a mapping compatible with the unit elements and the operations.

2.2. Let $A := \mathbf{Z} \times \mathbf{N}$ be the semi-ring where addition is given by $(a, b) + (a', b') = (a + a', \max(b, b'))$, the additive unit element is $(0, 0)$, multiplication is given by $(a, b)(a', b') = (aa', b + b')$ and the multiplicative unit is $(1, 0)$. Note that the ring generated by A by inverting additively any element of A is $\mathbf{Z}$ with the usual ring structure.

For $Z \in \mathrm{Sub}$ we define $\mathcal{C}_+(Z)$, as the semi-ring of functions $Z \to A$ with finite image and whose fibers are subanalytic sets. We call $\mathcal{C}_+(Z)$ the semi-ring of positive constructible functions on Z. In particular, $\mathcal{C}_+(\{0\}) = A$.

2.3. If $Z \in \mathrm{Sub}$ then we denote by Sub_Z the category of subanalytic maps $X \to Z$ for $X \in \mathrm{Sub}$ with morphisms subanalytic maps which make the obvious diagrams commute. We define the Grothendieck semigroup $SK_0(\mathrm{Sub}_Z)$ as the quotient of the free abelian semigroup over symbols $[Y \to Z]$ with $Y \to Z$ in Sub_Z by relations

$$[\emptyset \to Z] = 0, \tag{2.3.1}$$

$$[Y \to Z] = [Y' \to Z] \tag{2.3.2}$$

if $Y \to Z$ is isomorphic to $Y' \to Z$ in Sub_Z and

$$[(Y \cup Y') \to Z] + [(Y \cap Y') \to Z] = [Y \to Z] + [Y' \to Z] \tag{2.3.3}$$

for Y and Y' subsets of some $X \to Z$. There is a natural semi-ring structure on $SK_0(\mathrm{Sub}_Z)$ where the multiplication is induced by taking fiber products over Z.

We write $SK_0(\mathrm{Sub})$ for $SK_0(\mathrm{Sub}_{\{0\}})$ and $[X]$ for $[X \to \{0\}]$. Note that any element of $SK_0(\mathrm{Sub}_Z)$ can be written as $[X \to Z]$ for some $X \in \mathrm{Sub}_Z$ because we can take disjoint unions in Sub corresponding to finite sums in $SK_0(\mathrm{Sub}_Z)$.

2.3.1. **Proposition.** *For $Z \in \mathrm{Sub}$ there is a natural isomorphism of semi-rings*

$$T : SK_0(\mathrm{Sub}_Z) \to \mathcal{C}_+(Z)$$

induced by sending $[X \to Z]$ in Sub_Z to $Z \to A : z \mapsto (\chi(X_z), \dim(X_z))$, where X_z is the fiber above z. By consequence, $SK_0(\mathrm{Sub}) = A$.

Proof. This follows immediately from the trivialisation property for definable maps in any o-minimal expansion of a field. See [8]. □

By means of this result we may identify $SK_0(\mathrm{Sub}_Z)$ and $\mathcal{C}_+(Z)$.

2.4. **Positive measures.** A general notion of positive measures on a Boolean algebra $\mathcal{S}$ of sets is a map $\mu : \mathcal{S} \to G$ with G a semigroup satisfying

$$\mu(X \cup Y) = \mu(X) + \mu(Y)$$

and

$$\mu(\emptyset) = 0$$

whenever $X, Y \in \mathcal{S}$ are disjoint. Often one has a notion of isomorphisms between sets in $\mathcal{S}$ under which the measure should be invariant and which allows one to take disjoint unions of given sets in $\mathcal{S}$ (by taking disjoint isomorphic copies of the sets).

We let $\mu : \mathrm{Sub} \to A$ be the positive measure which sends X to $(\chi(X), \dim(X))$. This measure is a universal measure on Sub with the property that $\mu(X) = \mu(Y)$ whenever there exists a subanalytic bijection between X and Y and where universal means that any other positive measure with this property factorises through μ.

Note that μ measures is in some sense the topological size since, by the cell decomposition theorem from [8], $\mu(A) = \mu(B)$ will hold for two subanalytic sets

A, B if and only if for any fixed $n \geq 0$ there exists a finite partition of A, resp. B into subanalytic C^n-manifolds $\{A_i\}_{i=1}^m$, resp. $\{B_i\}_{i=1}^m$ and subanalytic maps $A_i \to B_i$ which are isomorphisms of C^n-manifolds.

Now we can define the integral of any positive function $f \in \mathcal{C}_+(Z)$ as

$$\int_Z f\,\mu := \sum_i f_i \mu(Z_i)$$

where $\{Z_i\}$ is any finite partition of Z into subanalytic sets such that f is constant on each part Z_i with value f_i.

To show that this is independent of the partition $\{Z_i\}$ we just note that there is a unique $[X \to Z]$ in $SK_0(Z)$ which corresponds to f under T and that $\sum_i f_i\mu(Z_i)$ corresponds to $[X] = (\chi(X), \dim(X))$ in $A = SK_0(\mathrm{Sub})$. This independence follows also from the cell decomposition theorem ([8]).

2.5. **Pushforward.** For $f : X \to Y$ there is an immediate notion of push-forward $f_! : \mathcal{C}_+(X) \to \mathcal{C}_+(Y)$, resp. $f_! : SK_0(\mathrm{Sub}_X) \to SK_0(\mathrm{Sub}_Y)$, which is given by

$$f_!(g)(y) = \int_{f^{-1}(y)} g_{|f^{-1}(y)}\,\mu$$

for $g \in \mathcal{C}_+(X)$, resp. by

$$f_!([Z \to X]) = [Z \to Y],$$

for $Z \to X$ in Sub_X and where $Z \to Y$ is given by composition with $X \to Y$. Note that these pushforwards are compatible with T.

If $Y = \{0\}$, then $SK_0(\mathrm{Sub}_Y) = A$ and we write $\mu([Z \to X])$ for $f_!([Z \to X])$ which is the integral of $[Z \to X]$. Thus the functoriality condition $(h \circ f)_! = h_! \circ f_!$ can be interpreted as Fubini's Theorem, since

$$\int_X g\,\mu = \int_Y \Big(\int_{f^{-1}(y)} g_{|f^{-1}(y)}\,\mu\Big)\,\mu$$

for $g \in \mathcal{C}_+(X)$ and $h : Y \to \{0\}$.

2.6. **Pullback.** For $f : X \to Y$ a morphism in Sub there is an immediate notion of pullback $f^* : \mathcal{C}_+(Y) \to \mathcal{C}_+(X)$, resp. $f^* : SK_0(\mathrm{Sub}_Y) \to SK_0(\mathrm{Sub}_X)$, which is given by

$$f^*(g) = g \circ f$$

for $g \in \mathcal{C}_+(Y)$, resp. by

$$f^*([Z \to Y]) = [Z \otimes_Y X \to X],$$

for $Z \to Y$ in Sub_Y and where $Z \otimes_Y X \to X$ is the projection and $Z \otimes_Y X$ the set-theoretical fiber product. Note that these pullbacks are also compatible with T and satisfy the functoriality property $(f \circ h)^* = h^* \circ f^*$.

2.6.1. **Proposition** (Projection formula)**.** *Let $f : X \to Y$ be a morphism in* Sub *and let g be in $\mathcal{C}_+(X)$ and h in $\mathcal{C}_+(Y)$. Then*

$$f_!(g f^*(h)) = f_!(g) h.$$

Proof. This is immediate at the level of SK_0 since both the multiplication in SK_0 and the pullback are defined by fiber product. □

2.7. **Radon Transform.** Let $S \subset X \times Y$, X, Y be subanalytic sets and write $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ for the projections and $q_X = \pi_{X|S}$ and $q_Y = \pi_{Y|S}$. For $g \in \mathcal{C}_+(X)$, we define the Radon transform $\mathcal{R}_S(g) \in \mathcal{C}_+(Y)$ by

$$\mathcal{R}_S(g) = q_{Y!} \circ q_X^*(g) = \pi_{Y!} \circ (\pi_X^*(g) 1_S)$$

where 1_S is the characteristic function on S.

2.8. **Example.** Consider the case $X = \mathbf{R}^n$, $Y = \mathrm{Gr}(n)$ with $S = \{(p, \Pi) : p \in \Pi\}$. Let $Z \subseteq \mathbf{R}^n$ be a subanalytic subset and $\sigma_Z : \mathrm{Gr}(n) \to A : \Pi \mapsto (\chi(\Pi \cap Z), \dim(\Pi \cap Z))$. Then $\sigma_Z = \mathcal{R}_S(1_Z)$.

Let $S' \subset Y \times X$ be another subanalytic set and put $q'_X = \pi_{X|S'}$ and $q'_Y = \pi_{Y|S'}$. The following proposition is proved just like in [16].

2.8.1. **Proposition** (Inversion formula)**.** *Let $r : S \otimes_Y S' \to X \times X$ be the projection and suppose that the following hypothesis hold*
(*) there exists $\lambda \in A$ such that $[r^{-1}(x, x')] = \lambda$ for all $x \neq x'$, $x, x' \in X$;
(**) there exists $0 \neq \theta \in A$ such that $[r^{-1}(x, x)] = \theta + \lambda$ for all $x \in X$.
If g is in $\mathcal{C}_+(X)$, then

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(g) = \theta g + \lambda \int_X g\, \mu \tag{2.8.1}$$

and this is independent of the choice of θ.

Proof. Let h and h' be the projections from $S \otimes_Y S'$ to S and S' respectively. Then by definition of fiber product, $q_Y \circ h = q'_Y \circ h'$, and so by functoriality of pullback and pushforward we have $h'_! \circ h^* = q'^*_Y \circ q_{Y!}$. Thus $\mathcal{R}_{S'} \circ \mathcal{R}_S(g) = q'_{X!} \circ (q'_Y)^* \circ q_{Y!} \circ q_X^*(g) = q'_{X!} \circ h'_! \circ h^* \circ q_X^*(g)$.

The last formula is also equal to $p_{2!} \circ r_! \circ r^* \circ p_1^*(g)$ where $p_1, p_2 : X \times X \to X$ are the projections onto the first and second coordinates respectively, since $q_X \circ h = p_1 \circ r$ and $q'_X \circ h' = p_2 \circ r$. The hypothesis shows that $r_!(1_{S \otimes_Y S'}) = \theta 1_{\Delta_X} + \lambda 1_{X \times X}$, moreover, this expression is independent of the choice of θ. By the projection formula, $r_!(r^*(p_1^*(g))) = r_!(1_{S \otimes_Y S'} r^*(p_1^*(g))) = r_!(1_{S \otimes_Y S'}) p_1^*(g) = (\theta 1_{\Delta_X} + \lambda 1_{X \times X}) p_1^*(g)$ holds, hence we obtain $p_{2!}((\theta 1_{\Delta_X} + \lambda 1_{X \times X}) p_1^*(g)) = \theta p_{2!}(1_{\Delta_X} p_1^*(g)) + \lambda p_{2!}(p_1^*(g)) = \theta g + \lambda \int_X g\, \mu$ as required.

We now show that the inversion formula is independent of the choice of θ. If $\theta + \lambda = \theta' + \lambda$ and $\theta \neq \theta'$, then necessarily $\lambda_2 > \theta_2$, $\lambda_2 > \theta'_2$ and $\theta_1 = \theta'_1$ with $\lambda = (\lambda_1, \lambda_2)$, $\theta = (\theta_1, \theta_2)$ and $\theta' = (\theta'_1, \theta'_2)$. Hence, $\theta g + \lambda \int_X g\, \mu = \theta' g + \lambda \int_X g\, \mu$ for all $x \in X$. □

2.9. **Example.** Consider the case $X = \mathbf{R}^n$, $Y = \mathrm{Gr}(n)$ with $S = \{(p, \Pi) : p \in \Pi\}$ and $S' = \{(\Pi, p) : p \in \Pi\}$. Then $[r^{-1}(x, x)] = [\mathbf{P}^{n-1}]$ and $[r^{-1}(x, x')] = [\mathbf{P}^{n-2}]$ for all $x, x' \in \mathbf{R}^n$ with $x \neq x'$. Since $[\mathbf{P}^n] = (\frac{1+(-1)^n}{2}, n)$, we have

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(g) = ((-1)^{n+1}, n-1)g + (\frac{1+(-1)^n}{2}, n-2) \int_X g\, \mu.$$

In particular, we have

$$\mathcal{R}_{S'} \circ \mathcal{R}_S(1_Z) = ((-1)^{n+1}, n-1)1_Z + (\frac{1+(-1)^n}{2}, n-2)[Z]$$

for every subanalytic subset Z of $\mathbf{R}^n$.

3. Direct image formalism in model theory

Let $\mathcal{M}$ be a model of a theory in a language $\mathcal{L}$ with at least two constant symbols c_1, c_2 satisfying $c_1 \neq c_2$. For Z a definable set we define the category $\mathrm{Def}_Z(\mathcal{M})$, also written Def_Z for short, whose objects are definable sets X with a definable map $X \to Z$ and whose morphisms are definable maps which make the obvious diagram commute. We write $\mathrm{Def}(\mathcal{M})$ or Def for $\mathrm{Def}_{\{c_1\}}(\mathcal{M})$. In $\mathcal{M}$ one can pursue the usual operations of set theory like finite unions, intersections, Cartesian products, disjoint unions and fiber products.

We define the Grothendieck semigroup $SK_0(\mathrm{Def}_Z)$ as the quotient of the free abelian semigroup over symbols $[Y \to Z]$ with $Y \to Z$ in Def_Z by relations

$$[\emptyset \to Z] = 0, \tag{3.0.1}$$

$$[Y \to Z] = [Y' \to Z] \tag{3.0.2}$$

if $Y \to Z$ is isomorphic to $Y' \to Z$ in Def_Z and

$$[(Y \cup Y') \to Z] + [(Y \cap Y') \to Z] = [Y \to Z] + [Y' \to Z] \tag{3.0.3}$$

for Y and Y' subsets of some $X \to Z$. There is a natural semi-ring structure on $SK_0(\mathrm{Def}_Z)$ where the multiplication is induced by taking fiber products over Z. Note that any element of $SK_0(\mathrm{Def}_Z)$ can be written as $[X \to Z]$ for some $X \to Z \in \mathrm{Def}_Z$ because we can take disjoint unions in $\mathcal{M}$ corresponding to finite sums in $SK_0(\mathrm{Def}_Z)$.

The map $\mathrm{Def} \to SK_0(\mathrm{Def})$ sending X to its class $[X]$ is a universal positive measure with the property that two sets have the same measure if there exists a definable bijection between them. For $f : X \to Y$ there is an immediate notion of push-forward $f_! : SK_0(\mathrm{Def}_X) \to SK_0(\mathrm{Def}_Y)$ given by

$$f_!([Z \to X]) = [Z \to Y],$$

for $Z \to X$ in Def_X and where $Z \to Y$ is given by composition with $X \to Y$.

If $Y = \{c_1\}$, then we write $\mu([Z \to X])$ for $f_!([Z \to X])$ which we call the integral of $[Z \to X]$, note that $\mu([Z \to X])$ is just $[Z]$ in $SK_0(\mathrm{Def})$. Thus the functoriality condition $(f \circ h)_! = f_! \circ h_!$ can be interpreted as Fubini's Theorem.

There is also an immediate notion of pullback $f^* : SK_0(\mathrm{Def}_Y) \to SK_0(\mathrm{Def}_X)$ given by

$$f^*([Z \to Y]) = [Z \otimes_Y X \to X],$$

for $Z \to Y$ in Def_Y and where $Z \otimes_Y X \to X$ is the projection and $Z \otimes_Y X$ the set-theoretical fiber product. The pullback is functorial, i.e., $(f \circ h)^* = h^* \circ f^*$.

3.0.1. **Proposition** (Projection formula). *Let $f : X \to Y$ be a morphism in* Def *and let g be in $SK_0(\mathrm{Def}_X)$ and h in $SK_0(\mathrm{Def}_Y)$. Then*

$$f_!(gf^*(h)) = f_!(g)h.$$

Proof. Exactly the same proof as for the subanalytic sets above works. □

3.1. **Radon transform and inversion formula.** One can also define the Radon transform in this context in exactly the same way as in the subanalytic case. Furthermore, the same argument as in the subanalytic case gives the corresponding inversion formula. However, since in general there is no trivialisation theorem, the conditions (*) and (**) in Proposition (2.8.1) have to be replaced by global conditions.Using the embedding $SK_0(\mathrm{Def}) \to SK_0(\mathrm{Def}_U)$ sending $[W]$ to $[W \times U \to U]$ where $W \times U \to U$ is the projection, the statement becames:

Let $r : S \otimes_Y S' \to X \times X$ be the projection and suppose that the following hypothesis hold

(*) there exists Z_1 in Def such that in $SK_0(\mathrm{Def}_{X_1})$ we have

$$[B_1 \to X_1] = [Z_1],$$

(**) there exists Z_2 in Def such that in $SK_0(\mathrm{Def}_{\Delta_X})$ we have

$$[B_2 \to \Delta_X] = [Z_1] + [Z_2]$$

where $X_1 = X \times X \setminus \Delta_X$, $B_1 = S \otimes_Y S' \setminus r^{-1}(\Delta_X)$, $B_2 = S \otimes_Y S' \cap r^{-1}(\Delta_X)$ and $B_1 \to X_1$ and $B_2 \to \Delta_X$ are the restrictions of the projection $r : S \otimes_Y S' \to X \times X'$. If $Z \to X$ is in Def_X, then

$$\mathcal{R}_{S'} \circ \mathcal{R}_S([Z \to X]) = [Z_2][Z \to X] + [Z_1][Z] \tag{3.1.1}$$

and this is independent of the choice of Z_2.

4. Examples

4.1. **Semialgebraic and subanalytic sets in $\mathbf{Q}_p$.** For K any finite field extension of the field $\mathbf{Q}_p$ of p-adic numbers, one can calculate explicitely the semi-ring of semialgebraic sets $SK_0(K, \mathrm{Sem})$, resp. of globally subanalytic sets $SK_0(K, \mathrm{Sub})$, using work of [2] for semialgebraic sets, resp. using work of [4] for the subanalytic sets. In both cases it is a subset of $\mathbf{N} \times \mathbf{N}$ and the class of a semialgebraic set X, resp. a subanalytic set X, is $(\sharp X, 0)$ if X is finite and $(0, \dim X)$ if X is infinite. This is because there exists a semialgebraic bijection between two infinite semialgebraic sets if and only if they have the same dimension, and similarly for subanalytic sets. However, no trivialisation theorem is known hence the relative semi-Grothendieck rings $SK_0(K, \mathrm{Sem}_Z)$, resp. $SK_0(K, \mathrm{Sub}_Z)$ for Z semialgebraic, resp. subanalytic, are expected to be much more complicated than maps $Z \to \mathbf{N} \times \mathbf{N}$ with finite image.

4.2. **Presburger sets.** Consider the Presburger structure on $\mathbf{Z}$ by using the Presburger language

$$\mathcal{L}_{\mathrm{PR}} = \{+, -, 0, 1, \leq\} \cup \{\equiv_n \mid n \in \mathbf{N},\ n > 1\},$$

with $\equiv_n$ the equivalence relation modulo n. Again one can can calculate explicitely the semi-ring $SK_0(\mathbf{Z}, \mathcal{L}_{\mathrm{PR}})$, using work of [3]. It is a subset of $\mathbf{N} \times \mathbf{N}$ and the class of a Presburger set X is $(\sharp X, 0)$ if X is finite and $(0, \dim X)$ if X is infinite, where the dimension of [3] is used. Again this is because there exists a Presburger bijection between two infinite Presburger sets if and only if they have the same dimension. Again, no trivialisation theorem is known hence the relative semi-Grothendieck rings are expected to be more complicated.

4.3. **Semilinear sets.** Let $K = (K, 0, 1, +, \cdot, <)$ be an ordered field and consider the structure $\mathcal{M} = (K, 0, 1, +, (\lambda_c)_{c \in K}, <)$, where λ_c is the scalar multiplication by $c \in K$. The category Def in this case is the category of K-semilinear sets with K-semilinear maps.

By [12], the Grothendieck ring $K_0(\mathrm{Def})$ is isomorphic to $E = \mathbf{Z}[x]/(x(x+1))$ and there is a universal Euler characteristic $\epsilon : \mathrm{Def} \to E$ (see also [10]).

Let D be the set whose elements are of the form $\sum_{i=1}^{n} y^{k_i} z^{l_i} \in \mathbf{N}[y, z]$ with $k_i \leq l_i$, and for $i \neq j$, $\neg(y^{k_i} z^{l_i} = y^{k_j} z^{l_j}) \wedge \neg(y^{k_i} z^{l_i} \prec y^{k_j} z^{l_j}) \wedge \neg(y^{k_j} z^{l_j} \prec y^{k_i} z^{l_i})$. Here, $y^{k_i} z^{l_i} \prec y^{k_j} z^{l_j}$ if and only if $k_i < k_j$ and $l_i < l_j$.

The set D can be equipped with a semi-ring structure in the following way: the zero element 0_D is $\sum_{i=1}^{0} y^{k_i} z^{l_i}$, the identity element 1_D is $y^0 z^0$, the addition is given by

$$\sum_{i=1}^{n} y^{k_i} z^{l_i} +_D \sum_{i=1}^{m} y^{k'_i} z^{l'_i} = \sum \max\nolimits_{\prec} \{y^k z^l : y^k z^l \text{ a monomial in } \sum_{i=1}^{n} y^{k_i} z^{l_i} + \sum_{i=1}^{m} y^{k'_i} z^{l'_i}\}$$

and multiplication by

$$\sum_{i=1}^{n} y^{k_i} z^{l_i} \cdot_D \sum_{i=1}^{m} y^{k'_i} z^{l'_i} = \sum \max\nolimits_{\prec} \{y^k z^l : y^k z^l \text{ a monomial in } \sum_{i=1}^{n} y^{k_i} z^{l_i} \cdot \sum_{i=1}^{m} y^{k'_i} z^{l'_i}\}$$

where the symbol $\sum \max_{\prec} S$ mean that we sum up the $\prec$-maximal elements of the finite set S.

By [12], there is a universal abstract dimension $\delta : \mathrm{Def} \to D$ and two sets in Def are isomorphic in Def if and only if they have the same universal Euler characteristic and the same universal abstract dimension. Thus, if A is the semi-ring $E \times D$, then the Grothendieck semi-ring $SK_0(\mathrm{Def})$ is isomorphic to A and the map $\mu : \mathrm{Def} \to A$ given by $\mu(X) = (\epsilon(X), \delta(X))$ is the positive universal measure on Def.

Note that the results we used above from [12] were proved in the field of real numbers but the same arguments hold in any arbitrary ordered field K.

4.4. **Semi-bounded sets.** Let $K = (K, 0, 1, +, \cdot, <)$ be a real closed field and consider the structure $\mathcal{M} = (K, 0, 1, +, (\lambda_c)_{c \in K}, B, <)$, where λ_c is the scalar multiplication by $c \in K$ and B is the graph of multiplication on a bounded interval. The category Def in this case is the category of K-semibounded sets with K-semibounded maps.

By [11] all bounded semialgebraic subsets are in Def and, by [14], $\mathcal{M}$ is, up to definability, the only o-minimal structure properly between $(K, 0, 1, +, (\lambda_c)_{c \in K}, <)$ and $(K, 0, 1, +, \cdot, <)$.

By [12], the Grothendieck ring $K_0(\text{Def})$ is isomorphic to $E = \mathbf{Z}[x]/(x(x+1))$ and there is a universal Euler characteristic $\epsilon : \text{Def} \to E$ (see also [10]). Furthermore, if D is the semi-ring of Example 4.3, then there is a universal abstract dimension $\delta : \text{Def} \to D$ and two sets in Def are isomorphic in Def if and only if they have the same universal Euler characteristic and the same universal abstract dimension. Thus, if A is the semi-ring $E \times D$, then the Grothendieck semi-ring $SK_0(\text{Def})$ is isomorphic to A and the map $\mu : \text{Def} \to A$ given by $\mu(X) = (\epsilon(X), \delta(X))$ is the positive universal measure on Def.

The results we used above from [12] were proved in the field of real numbers and are based on Peterzil's [13] structure theorem for semibounded sets in the real numbers. However, the same arguments hold in any arbitrary real closed field K using the structure theorem from [9].

References

1. L. Bröcker, *Euler integration and Euler multiplication*, Adv. Geom., **5** (2005), 145-169.
2. R. Cluckers, *Classification of semialgebraic p-adic sets up to semialgebraic bijection*, J. Reine Angew. Math., **540** (2001), 105-114.
3. R. Cluckers, *Presburger sets and p-minimal fields*, J. Symbolic Logic, **68** (2003), 153–162.
4. R. Cluckers, *Analytic p-adic cell decomposition and integrals*, Trans. Amer. Math. Soc., **356** (2004), 1489 - 1499. Available at arXiv:math.NT/0206161.
5. R. Cluckers, F. Loeser, *Fonctions constructibles et intégration motivique I*, C. R. Math. Acad. Sci. Paris **339** (2004), 411–416.
6. R. Cluckers, F. Loeser, *Fonctions constructibles et intégration motivique II*, C. R. Math. Acad. Sci. Paris **339** (2004), 487–492.
7. R. Cluckers, F. Loeser, *Constructible motivic functions and motivic integrals*, Available at arXiv:math.AG/0410203.
8. L. van den Dries, *Tame topology and o-minimal structures*, Cambridge University Press, Lecture note series, **248**, (1998).
9. M. Edmundo, *Structure theorems for o-minimal expansions of groups*, Ann. Pure Appl. Logic, **102** (2000), 159–181.
10. M. Kageyama, M. Fujita, *Grothendieck rings of o-minimal expansions of ordered abelian groups*, to appear in J. Algebra. Available at arXiv:math.LO/0505341.
11. D. Marker, Y. Peterzil, A. Pillay, *Additive reducts of real closed fields*, J. Symbolic Logic, **57** (1992), 109–117.
12. J. Maříková, *Geometric properties of definable sets*, Diplomová Práce. Univerzita Karlova v Praze, Praha, 2003.
13. Y. Peterzil *A structure theorem for semibounded sets in the reals*, J. Symbolic Logic **57** (1992), 779–794.

14. A. Pillay, P. Scowcroft, C.Steinhorn, *Between groups and rings*, Rocky Mountain J. Math. **9** (1989), 871–885.
15. P. Schapira, *Operations on constructible functions*, J. Pure Appl. Algebra, **72** (1991), 83–93.
16. P. Schapira, *Tomography of constructible functions* In: AAECC, G.Cohen, M.Giusti and T.Mora (eds.) Lecture Notes in Computer Sciences, Springer Verlag, **948** (1995), 427–435.
17. O. Viro, *Some integral calculus based on Euler characteristic*, Lecture Notes in Math, Springer-Verlag, **1346** (1988), 127–138.

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