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Orientation matters for NIMreps

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Abstract

The problem of finding boundary states in CFT, often rephrased in terms of "NIM-reps" of the fusion algebra, has a natural extension to CFT on non-orientable surfaces. This provides extra information that turns out to be quite useful to give the proper interpretation to a NIMrep. We illustrate this with several examples. This includes a rather detailed discussion of the interesting case of the simple current extension of A_2 level 9, which is already known to have a rich structure. This structure can be disentangled completely using orientation information. In particular we find here and in other cases examples of *diagonal* modular invariants that do not admit a NIMrep, suggesting that there does not exist a corresponding CFT. We obtain the complete set of NIMreps (plus Moebius and Klein bottle coefficients) for many exceptional modular invariants of WZW models, and find an explanation for the occurrence of more than one NIMrep in certain cases. We also (re)consider the underlying formalism, emphasizing the distinction between oriented and unoriented string annulus amplitudes, and the origin of orientation-dependent degeneracy matrices in the latter.

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1. Introduction

Boundary states in conformal field theory (CFT) have been studied for at least two rather different reasons. First of all one may need these states in a particular physical problem in which boundaries of the Riemann surface are important, such as open string theory or applications of statistical mechanics. But on the other hand it has also been suggested that the mere (non)-existence of (a complete set [1] of) boundaries may tell us something about the existence of the CFT itself.

Completeness of boundaries in usually rephrased [1], [2] in terms of an equation involving the annulus coefficients ${A_{ia}}^b$, which are non-negative integers appearing in the decomposition of the open string spectrum (more precise definitions will be given below). These integer matrices must form a non-negative integer matrix representation (NIMrep) of the fusion algebra:

$$ {A_{ia}}^b {A_{jb}}^c = {N_{ij}}^k {A_{ka}}^c \tag{1.1} $$

A general solution found by Cardy [3] is to choose A equal to the fusion coefficients. The issue of finding non-trivial NIMreps was considered shortly thereafter in [4] in the context of RSOS models. These authors also gave some non-trivial solutions, in particular for $SU(3)$ WZW models. The relation to completeness of boundaries was understood in [1], after which many papers appeared giving solutions in various cases with various degrees of explicitness, see *e.g.* [2] and [5-15].

The link of a given NIMrep to closed CFT is made by simultaneously diagonalizing the integer matrices A in terms of irreducible representations of fusion algebra and matching the representations that appear with the C-diagonal terms in the torus partition function matrix Z_{ij} (by "C-diagonal" we mean the coefficients Z_{ii^c}).

The issue of existence of a CFT usually arises in situations where the existence of a modular invariant partition function (MIPF) is the only hint of a possible CFT. This occurs frequently in the case of rational CFT's, where such partition functions are determined by a non-negative integer matrix that commutes with the modular transformation matrices, a simple algebraic problem that can be solved by a variety of methods, including brute force. It is well known that the existence of a MIPF is necessary, but not sufficient for the existence of a CFT. Unfortunately the necessary additional checks are considerably harder to carry out, since they involve fusing and braiding matrices that are available in only a few cases (for recent progress and an extensive list of references see [16]). It has been suggested that the existence of a complete set of boundaries might be used as an additional consistency check. This point was in particular emphasized recently by Gannon [13].

The implication of this point of view is that a CFT that cannot accommodate a complete set of boundaries is inconsistent even if one is only interested in closed Riemann surfaces. By the same logic, a symmetric† CFT that cannot exist on unorientable

† We mean here a CFT with a symmetric modular invariant partition function.

surfaces, i.e. Riemann surfaces with crosscaps, should then be inconsistent, even if one is only interested in orientable surfaces.

The advantage of using crosscap consistency is that there is a very simple consistency check, namely

$$\sum_m S_{im} K_m = 0 \text{ , if } i \text{ is } \underline{\text{not}} \text{ an Ishibashi label} \tag{1.2}$$

where K_m are the Klein bottle coefficients, which must satisfy the conditions $K_m = Z_{mm} \bmod 2$ and $|K_m| \leq Z_{mm}$, where Z_{mn} is the integer matrix defining the torus partition function.

There are, however, two caveats. First of all, non-symmetric modular invariants manifestly do not allow non-orientable surfaces, since an orientation reversal interchanges the left and right Hilbert spaces. Nevertheless such partition functions may well correspond to sensible CFT's. It is not obvious that asymmetry of the modular invariant is the *only* possible obstruction to orientation reversal. In other words, there might exist examples of symmetric CFT's allowing boundaries but no crosscaps. Secondly, the Klein bottle sum rule (1.2) can only be used successfully with some additional restrictions on the allowable Klein bottle coefficients, since the integrality constraints usually allow too many solutions. These additional restrictions are unfortunately only conjectures, and hence on less firm ground than the integrality constraints. One of them, the "Klein bottle constraint", that requires the signs of the non-vanishing coefficients K_i to be preserved in fusion:

$$K_i K_j K_k > 0 \quad \text{if} \quad N_{ij}{}^k \neq 0 \ , \tag{1.3}$$

is known to be violated in some simple current examples [17]. Therefore either there is something wrong with those examples, or a more precise formulation of the Klein bottle constraint is needed (we consider the latter possibility to be the most likely one). The other restriction is the trace formula [18]

$$\sum_a A^i_{aa} = \frac{S^{i\ell}}{S_{0\ell}} Y^{\ell}_{\ 00} K_\ell \ , \tag{1.4}$$

to which no counter examples are known, but which is still only a conjecture. In this paper we will consider the extra information from non-orientable surfaces always in conjunction with boundary information, and we will see that violations of (1.2) usually give the right hint regarding the existence of boundaries and presumably the existence of the CFT. We investigated the validity of the Klein bottle constraint as well as the trace formula (1.4) in many examples, and found additional counter examples to the former, but additional support for the conjecture (1.4).

Another way in which orientation information can be used is to search for Moebius and Klein bottle amplitudes belonging to an already given NIMrep. This can be done by solving a set of "polynomial equations" given in [18]. In this case only the first caveat applies: if the polynomial equations have no solution, then either the modular invariant is not sensible or it does not admit an orientifold. Without further information this cannot be decided unambiguously, but the examples strongly suggest a definite answer.

The orientation information is encoded in terms of a set of integer data that extends the notion of a NIMrep. This is most naturally done in two stages, first to an "S-NIMrep" (symmetrized NIM-rep) and then to a "U-NIMrep" (unoriented NIMrep), by adding Moebius and Klein bottle coefficients. A given NIMrep can admit several S-NIMreps (or none at all), which in their turn may have any number of U-NIMreps. For example, the charge conjugation invariant has a well-known NIMrep (the Cardy solution), which extends to at least one U-NIMrep, but often to more than one [19] [20], obtained using simple currents. For all simple current modifications⋆ of charge conjugation modular invariant a class of U-NIMreps has been determined in [12] (this includes all U-NIMreps that exist for generic simple current invariants, but there might exist additional exceptional solutions). In [6] the U-NIMreps were presented for all the A_1 automorphism invariants, including the exceptional "E_7" invariant at level 16. Another interesting class are the diagonal invariants. A general prescription for finding their NIMreps was given in [10]. This involves the construction of a "charge conjugation orbifold", which does not necessarily exist. If the orbifold CFT *does* exist, the NIMrep is constructed from it using simple currents. Then, using the results of [12], one may also construct U-NIMreps for the diagonal invariant. The charge conjugation orbifold theory was constructed explicitly for WZW-models (recently, the NIMreps for SU(N) diagonal invariants were obtained by different methods in [15], while the case $N = 3$ was already considered in [4]; these authors did not consider U-NIMreps). Finally, it is straightforward to extend the known NIMreps for the exceptional invariants of A_1 [2] to U-NIMreps. This summarizes what is presently known about U-NIMreps.

As remarked above, the existence of (S,U)-NIMreps may be used as a guiding principle to the (non)-existence of a CFT. In [13] several examples were given of MIPF without a corresponding NIMrep. In these examples, the MIPF is an extension of the chiral algebra. It was already known on other grounds that this extension was inconsistent, and hence the non-existence of the complete set of boundaries just gives an additional confirmation. Here we will present some examples that are automorphism invariants, for which no simple consistency checks are available, other than modular invariance itself. We will find examples that do not admit a complete set of boundaries, and examples that do not admit crosscap coefficients, even though a complete set of boundaries does exist. Perhaps surprisingly, among the examples for which a complete set of boundaries does not exist are diagonal invariants (*i.e.* the Cardy case modified by charge conjugation). This presumably implies that the charge conjugation orbifold theory does not exist.

⋆ Note that simple currents are used in two distinct ways here: to change the Klein bottle projection for a given modular invariant, and to change the modular invariant itself.

Apart from existence of a NIMrep, one may also worry about uniqueness of a NIM-rep. At least one example is known [4][13] where a given modular invariant appears to have more than one NIMrep. We find many more, and show how the ambiguity is resolved by computing in addition to the annulus coefficients also the Moebius and Klein bottle coefficients. Multiple NIMreps can be expected if there are degeneracies in the spectrum, *i.e.* if $Z_{ii^c} > 1$ for some i. There are exceptions: if the degeneracies are due to simple currents, or if the degeneracy is absorbed into the extended characters (rather than leading to several Virasoro-degenerate extended characters) we see no reason to expect multiple NIMreps.

This paper is consists of two parts: a discussion of the formalism for general modular invariants, and a set of examples illustrating several special features. In the next chapter we discuss the formalism, with particular emphasis on the distinction between annuli for oriented and unoriented strings. We believe this clarifies some points that have at best remained implicit in the existing literature. This discussion is presented for the general case, allowing for Ishibashi state multiplicities larger than 1, which usually occur for modular invariants of extension type. Chapters 3-7 contain various classes of examples, illustrating in different ways what can be learned about NIM-reps by paying attention to orientation issues. The examples in section 3 concern automorphisms of simple current extended WZW models. Here we discover an example of a diagonal invariant for which no NIMrep exists. In chapter 4 we discuss this in more generality, and find some additional examples of this kind. In section 5 we consider automorphisms of $c = 1$ orbifolds. Since our results suggest that not all the allowed automorphisms of extended WZW-models yield sensible CFT's, it is natural to inspect also pure WZW automorphisms. These were completely enumerated in [21]. We examine some of these MIPF's in section 6, and we find no inconsistencies. Finally in section 7 we consider exceptional extensions of WZW models, including cases with multiplicities larger than 1 and extensions by higher spin (*i.e.* higher than 1) currents. The results for the spin 1 extensions can be understood remarkably well from the point of view of the extended theory, whereas the results for higher spin extensions confirm the consistency of the new CFT's obtained after the extension.

We will not present the explicit NIMreps here in order to save space, but all data are available from the authors on request. The examples where obtained by solving (1.1) on a computer, using the fact that all matrix elements are bounded from above by quantum dimension [13]. Apart from (1.1) we also imposed the condition that the matrix representation should be diagonalizable in terms of the correct one-dimensional fusion representations, with multiplicity $\mathbf{Z}_{ii^c}$. Although this is a finite search in principle, in practice a straightforward search is impossible in essentially all cases. However, using a variety of methods – which will not be explained here – we were able to do an exhaustive search in most cases of interest.

2. Orientation issues for NIM-reps

Here want to discuss two issues that have remained unsatisfactory in the literature so far:

— Two different expressions for "annulus coefficients" are in use, one of the form "$S\mathcal{B}\mathcal{B}^*$" and the other form "$S\mathcal{B}\mathcal{B}$"

— For a given CFT there are sometimes several choices for the complete set of boundary coefficients.

Here "S" stands for the modular transformation matrix and "$\mathcal{B}$" for the boundary coefficients (for reasons explained later we use the notation $\mathcal{B}$ instead of B here). The indices of these quantities are as follows. A complete basis for the boundaries in a CFT are the Ishibashi states, labelled by a pair of chiral representations (i, i^c). The MIPF determines how often each Ishibashi state actually appears in a CFT. The completeness hypothesis (recently proved in [16]) states that the number of boundaries is equal to the number of Ishibashi states. Given a MIPF, parametrized by a non-negative integral matrix Z_{ij}, the number of boundaries should thus be given by $\sum_i Z_{ii^c}$. Since an Ishibashi state may appear more than once we need an additional degeneracy label α, so that the relevant Ishibashi labels are $(i, \alpha), \alpha = 1, \ldots, Z_{ii^c}$. We denote the Ishibashi states as $|i, \alpha\rangle\rangle$. The coupling of such a state to boundary a is given by a boundary coefficient $\mathcal{B}_{(i,\alpha)a}$, and the completeness condition states that this should be a square, invertible matrix. The boundary coefficients appear in the expansion of boundary states in terms of Ishibashi states, $|\mathcal{B}_a\rangle\rangle = \sum_{i,\alpha} \mathcal{B}_{(i,\alpha)a}|i, \alpha\rangle\rangle$

Regarding the first point above, some authors do seem to understand that the expression $S\mathcal{B}\mathcal{B}^*$ is to be used for oriented strings, and $S\mathcal{B}\mathcal{B}$ for unoriented ones, but this is rarely stated, and these expressions are often presented without justification. The former expression is the one obtained most straightforwardly. The usual derivation of the open string partition function is to transform to the closed string channel, where the amplitude can be written as a closed string exchange between two branes. This amplitude can be evaluated easily in the transverse (closed string tree) channel by taking matrix elements of the closed string propagator U between boundary states: $\langle\langle B_b|U|B_a\rangle\rangle$. This describes closed strings propagating from boundary a to boundary b. Transforming this expression back to the open string loop channel results in the following expression for the annulus coefficients

$$A_{ia}{}^b = \sum_j \sum_\alpha S_{ij}\mathcal{B}_{(j,\alpha)a}\mathcal{B}^*_{(j,\alpha)b} \tag{2.1}$$

In general the resulting expression is not symmetric in the two indices a and b, indicating that this amplitude describes oriented strings. Obviously a symmetric expression is obtained by dropping the complex conjugation, but that step would require some justification.

Regarding the second point, the paradox is that on general grounds one would expect there to exist a unique set of boundary coefficients B, given the CFT and the MIPF.

This is because they are the complete set of irreducible (and hence one-dimensional) representations of a "classifying algebra" [7][1] [22]. This is an abelian algebra of the form[⋆]

$$B_{(i,\alpha)a}B_{(j,\beta)a} = \sum_{k,\gamma} X_{(i,\alpha)(j,\beta)}{}^{(k,\gamma)} S_{0a} B_{(k,\gamma)a} \tag{2.2}$$

whose coefficients X can be expressed in terms of CFT data like OPE coefficients and fusing matrices. Although these coefficients are rarely available in explicit form, they are completely fixed by the algebra. Nevertheless it was found that in many cases – where (2.2) is not available and only integrality conditions were solved – a given bulk CFT may have more than one set of boundary coefficients $\mathcal{B}$, giving rise to different annulus coefficients when the form $S\mathcal{B}\mathcal{B}$ is used. In all these cases, the different annuli are related to each other by changing the coefficients $\mathcal{B}_{(i,\alpha)a}$ by a (i,α)-dependent phase. This changes the annulus coefficients in the $S\mathcal{B}\mathcal{B}$ form, but it does not change the $S\mathcal{B}\mathcal{B}^*$ annuli. Obviously such phases do not respect the classifying algebra,[†] and hence at most one of the different sets $\mathcal{B}$ can be equal to the unique solution B to (2.2). We use the symbol B to denote the solution to (2.2), and $\mathcal{B}$ to denote any complete set of boundary coefficients that satisfy the annulus integrality conditions. In principle, one can compute orientable annuli (NIMreps) $A_{ia}{}^b$ for each choice of $\mathcal{B}$, and yet another one by replacing $\mathcal{B}$ by the solution B of (2.2). If indeed the coefficients $\mathcal{B}$ and B differ only by phases (or unitary matrices) on the degeneracy spaces, all these NIMreps are identical, and we might as well write $A_{ia}{}^b$ in terms of B:

$$A_{ia}{}^b = \sum_j \sum_\alpha S_{ij} B_{(j,\alpha)a} B^*_{(j,\alpha)b} \tag{2.3}$$

Recent work on orientifold planes in group manifolds provides some useful geometrical insight into the solution of the two problems stated above. The different boundaries $|B_a\rangle\rangle$ correspond to D-branes at different positions. The coefficients $B_{(j,\alpha)a}$ specify the couplings of these branes to closed strings. If one adds orientifold planes, there are no changes to the positions of the D-branes, but they are identified by the orientifold reflection. Hence one would expect the couplings $B_{(j,\alpha)a}$ to remain unchanged, in agreement with the classifying algebra argument given above. Nevertheless there is an important change in the system: without O-planes, a stack of N_a D-branes gives rise to a $U(N_a)$ Chan-Paton gauge group. In the presence of an O-plane, self-identified branes give rise to SO or Sp groups, whereas pairwise identified branes give U groups.

In string theory, oriented open strings occur when one considers branes that are not space-time filling in an oriented closed string theory, for example the type-II superstring.

⋆ Our normalization differs from [7], in order to obtain a simpler form for the annuli.

† Alternatively, one may define orientation-dependent generalizations of the classifying algebra that incorporate the sign changes, as was done in [18]. However, there will always be one special choice with structure coefficients derived from the CFT data in the canonical way.

They give rise to $U(N)$ Chan-Paton groups. The $U(N)$ gauge bosons come from the spectrum of open strings with both endpoints attached to the same brane. When the branes are space-time filling tadpole cancellation (or RR-charge cancellation) requires the the addition of O-planes. In this case, SO, Sp and U can occur, and the $U(N)$ gauge bosons come from open string with their endpoints attached to *different* branes, identified by the orientifold map. This suggests that in order to describe such a system one should consider a different partition function, which in the closed string channel correspond to propagation of strings from a brane to the orientifold reflection of a brane.

An orientifold projection involves an operator Ω that reverses the orientation of the worldsheet. In a geometric description Ω does not only invert the worldsheet orientation, but it may also act non-trivially on the target space. This operator has some realization in CFT (denoted by the same symbol Ω) which in any case interchanges the left and right Hilbert space, and which may have a non-trivial action on the representations, which must square to 1. On Ishibashi states this yields[‡]

$$\Omega|i,\alpha\rangle\rangle = \sum_\beta \Omega^i_{\alpha,\beta}|i^c,\beta\rangle\rangle$$

Of course the action of Ω must be defined on all closed string states, but we only need it on the Ishibashi states here. The condition that Ω is a reflection implies

$$\sum_\beta \Omega^i_{\alpha\beta}\Omega^{i^c}_{\beta\gamma} = \delta_{\alpha\gamma} \tag{2.4}$$

Furthermore Ω must be unitary, i.e

$$\sum_\beta \Omega^i_{\alpha\beta}(\Omega^i_{\gamma\beta})^* = \delta_{\alpha\gamma}\ . \tag{2.5}$$

This implies

$$\Omega^i = (\Omega^{i^c})^\dagger \tag{2.6}$$

The reasoning in the foregoing paragraph suggests that the quantity of interest is $\langle\langle B_b|U\Omega|B_a\rangle\rangle$, transformed to the open string channel. The computation results in

$$A^\Omega_{iab} = \sum_j \sum_{\alpha,\beta} S_{ij} B_{(j,\alpha)a}\Omega^j_{\alpha\beta}B^*_{(j^c,\beta)b}\ . \tag{2.7}$$

If this is the right expression, it should be symmetric. To show that it is, one may replace the left hand side by its complex conjugate (since the right hand side should obviously be real), use (2.6), change the summation from j to j^c, and finally use $S_{ij} = (S_{ij^c})^*$.

‡ The matrix Ω introduced here is a generalization of the signs ϵ defined in [6].

In order to write (2.7) in manifestly symmetric form we need a relation of the form

$$(B_{(j^c,\alpha)a})^* = \sum_\beta C^j_{\alpha\beta} B_{(j,\beta),a} \ , \tag{2.8}$$

where C^j must be unitary, and is uniquely determined if we know B. Such a relation should always hold as a consequence of CPT invariance, since it relates the emission of a closed string state (j, j^c) to the absorption of its charge conjugate. Conjugating twice we find $\sum_\beta (C^j_{\alpha\beta})^* C^{j^c}_{\beta\gamma} = \delta_{\alpha\gamma}$. This relation may be used to show that

$$A^\Omega_{iab} = \sum_j \sum_{\alpha,\beta} S_{ij} B_{(j,\alpha)a} [\Omega^j C^j]_{\alpha\beta} B_{(j,\beta)b} \ . \tag{2.9}$$

is indeed symmetric, if A^Ω_{iab} is real. The latter relation may be written as

$$A^\Omega_{iab} = \sum_j \sum_{\alpha,\beta} S_{ij} B_{(j,\alpha)a} g^j_{\alpha\beta} B_{(j,\beta)b} \ . \tag{2.10}$$

Conversely, if A^Ω_{iab} is symmetric and B is invertible (as it is in the case of a complete set of boundaries), then $g^j_{\alpha\beta}$ must be symmetric. Such a "degeneracy matrix" was first introduced in [12] for a rather different purpose, and further discussed in [18]. Here we even encounter it in the absence of degeneracies. Note that one can of course take a square root of g^j and absorb it into the definition of $B_{(j,\alpha)a}$. This the yields the quantity previously denoted as $\mathcal{B}$. This was implicitly or explicitly done in several previous papers, *e.g.* [12] and [18], and leads to orientation-dependent boundary coefficients. Both formalisms are of course completely equivalent if one is only interested in partition functions.

Note that in most cases we do not know g^j (or Ω^j) and $B_{(j,\alpha)}$ separately, but we only know the integer data $A^i{}_a{}^b$ or A^Ω_{iab}. Obviously, from the former we cannot determine Ω, and we can only determine $B_{(j,\alpha)}$ up to Ishibashi-dependent phases (or unitary matrices in case of degeneracies). From A^Ω_{iab} we can determine $B_{(j,\alpha)}$ up to signs (or orthogonal matrices in case of degeneracies), provided we know Ω. Furthermore, since the coefficients $B_{(j,\alpha)}$ as well as the matrices C^j are assumed to be independent of the orientifold choice, we can determine relative signs of two choices of Ω. In the examples discussed so far two choices of Ω (or g^j) always had *relative* eigenvalues ± 1, but that need not be the case in general. If one knows the classifying algebra coefficients one can obviously determine B completely, and then also Ω and C.

The allowed choices of Ω are determined by the requirement that it must be a symmetry of the CFT. A necessary condition is that the coefficients Ω^j must respect the OPE's of the bulk CFT, and hence presumably the fusion rules. For automorphism invariants (*i.e.* no degeneracies) an obvious solution to that condition is $\Omega^j = 1$, but there might be further constraints in the full CFT. Obviously these matrices will have to satisfy the appropriate generalization (to allow for degeneracies) of the crosscap constraint of [23] and [6], but we will not pursue that here, because we are only considering constraints that do not require knowledge of fusing matrices and OPE coefficients.

A related issue is that of boundary conjugation. The boundary conjugation matrix is defined as

$$C^B_{ab} = A^{\Omega}_{0ab} \ .$$

The matrix A^{Ω}_{0ab} must be an involution in order to get meaningful Chan-Paton groups in string theory. Since it is an involution, we can define a^c as the boundary conjugate to a, satisfying $C^B_{aa^c} = 1$. The boundary conjugation matrix may be used to raise and lower indices. Then we may define

$$A^{\Omega}_{ia}{}^{b} = \sum_c A^{\Omega}_{iac} A^{\Omega}_{0cb} \tag{2.11}$$

For a complete set of boundaries this quantity satisfies eqn (1.1). If the latter have a unique solution (for a given set of Ishibashi states), it follows that all quantities $A^{\Omega}_{ia}{}^{b}$ are identical. Conversely, one can say that all quantities A^{Ω}_{iab} are different symmetrizations of ${A_{ia}}^b$. We will refer to such a symmetrization as an S-NIMrep.

Given a NIMrep, one can always write the oriented annulus coefficient ${A_{ia}}^b$ in the form (2.3), since this just amounts to diagonalizing a set of commuting normal[⋆] matrices [14]. We will now show that given an S-NIMrep, one can write A^{Ω}_{iac} in the form (2.10). Note that[†]

$$A^{\Omega}_{iab} = {A_{ia}}^{b^c} = A^{\Omega}_{iba} = {A_{ib}}^{a^c}$$

so that

$${A_{ia^c}}^{b^c} = A^{\Omega}_{iba^c} = {A_{ib}}^{a} \ .$$

Contracting this with a matrix S_{im} we get

$$\sum_{\alpha} B_{(m,\alpha)a^c} (B_{(m,\alpha)b^c})^* = \sum_{\alpha} B_{(m,\alpha)b} (B_{(m,\alpha)a})^* \ ,$$

where $B_{(m,\alpha)a}$ are the set of matrices in terms of which (2.3) holds. Completeness implies

⋆ A normal matrix is a square matrix that commutes with its adjoint. This is automatically true if ${A_{ia}}^b = {A_{i^cb}}^a$ and ${A_{ia}}^b$ is real, properties that should be treated as part of the definition of a NIMrep. It is not hard to show that the matrices $A^{\Omega}_{ia}{}^{b}$ satisfy this.

† We are assuming here for simplicity that the NIMrep $A^{\Omega}_{ia}{}^{b^c}$ is unique, so we may drop the superscript Ω. If it is not unique, not only ${A_{ia}}^{b^c}$ but also B get an additional label Ω, but the rest of the discussion is unaffected.

that the matrix $B_{(m,\alpha)a^c}$ has a right inverse X,

$$\sum_a B_{(m,\alpha)a^c} X_{a^c(n,\gamma)} = \delta_{mn}\delta_{\alpha\gamma} \ .$$

This leads to the following expression

$$\sum_\alpha \delta_{mn}\delta_{\alpha\gamma}(B_{(m,\alpha)b^c})^* = \sum_\alpha B_{(m,\alpha)b} \sum_a (B_{(m,\alpha)a})^* X_{a^c(n,\gamma)} \ .$$

which implies

$$\begin{aligned}(B_{(m,\gamma)b^c})^* &= \sum_\alpha B_{(m,\alpha)b} \sum_a (B_{(m,\alpha)a})^* X_{a^c(m,\gamma)} \\ &= \sum_\alpha B_{(m,\alpha)b} V^m_{\alpha\gamma} \ .\end{aligned}$$

Iterating this we see that V^m must be a unitary matrix. Hence we get the desired answer,

$$A^\Omega_{iab} = \sum S_{im} B_{(m,\alpha)a)} V^m_{\alpha\gamma} B_{(m,\gamma)b}$$

Since the left hand side is symmetric, V^m must be a symmetric matrix. It can be written as $\Omega^m C^m$ to extract Ω^m, but nothing guarantees that the resulting Ω is indeed a symmetry. Indeed there may well exist symmetrizations that have no interpretation in terms of orientifold maps.

We now have two ways of arriving at an S-NIMrep, a.) by computing $\langle\langle B_b|U\Omega|B_a\rangle\rangle$ and transforming to the open string loop channel, and b.) by starting from the oriented amplitude $\langle\langle B_b|U|B_a\rangle\rangle$, and contracting it with C^B_{ab}. Comparing the two expressions and using completeness of boundaries, we arrive at the following expression for the action of Ω on a boundary state

$$\Omega|B_a\rangle\rangle = C^B_{ab}|B_b\rangle\rangle \ . \tag{2.12}$$

An S-NIMrep is only interesting as a first ingredient in a complete set of annuli, Moebius and Klein amplitudes. We will refer to such a set of data as an U-NIMrep (with "U" for unoriented). This problem can also be phrased (almost) entirely in terms of integers [18]. The full set of conditions is then that the following quantities must exist

— NIMrep: a set of non-negative integer matrices ${A_{ia}}^b = {A_{i^c b}}^a$ satisfying (1.1)

— S-NIMrep: In addition, an involution C^B_{ab} such that $A_{iab} \equiv {A_{ia}}^c C^B_{cb}$ is symmetric

— U-NIMrep: In addition to the two foregoing requirements, a set of Moebius coefficients M_{ia} and Klein bottle coefficients K_i such that $\frac{1}{2}(A_{iaa}+M_{ia})$ and $\frac{1}{2}(Z_{ii}+K_i)$ are non-negative integers, and the following polynomial equations are satisfied

$$\begin{aligned}\sum_b A_{ia}{}^b M_{jb} &= \sum_l Y_{ij}{}^l M_{la} \\ \sum_{a,b} C^B_{ab} M_{ia} M_{jb} &= \sum_l Y^l{}_{ij} K_l \ .\end{aligned} \tag{2.13}$$

The derivation of (2.13) in [18] uses the following formulas for $M_{\ell a}$ and K_ℓ.

$$K_\ell = \sum_{m,\alpha} S_{\ell m} \Gamma_{m\alpha} g^m_{\alpha\beta} \Gamma_{m\beta} \tag{2.14}$$

$$M_{\ell a} = \sum_{m,\alpha} P_{\ell m} \Gamma_{m\alpha} g^m_{\alpha\beta} B_{(m\beta)a} \tag{2.15}$$

In the computation one uses the completeness relation

$$\sum_a S_{0m} B_{(m,\alpha)a} B^*_{(\ell,\beta)a} = \delta_{m\ell}\delta_{\alpha\beta} \ , \tag{2.16}$$

which follows from $A_{0a}{}^b = \delta^b_a$ plus the completeness assumption that B is an invertible matrix. In an analogous way one derives from $A_{0ab} = C^B_{ab}$:

$$\sum_{a,b} S_{0m} B_{(m,\alpha)a} C^B_{ab} B_{(\ell,\beta)b} = \delta_{m\ell} (g^m)^{-1}_{\alpha\beta} \ , \tag{2.17}$$

which implies

$$B^*_{(m,\alpha)a} = \sum_{b,\beta} C^B_{ab} g^m_{\alpha\beta} B_{(m,\beta)b} \tag{2.18}$$

The formulas for $M_{\ell a}$ and K_ℓ follow from the closed string amplitudes $\langle\langle B_a|U\Omega|\Gamma\rangle\rangle$ and $\langle\langle\Gamma|U\Omega|\Gamma\rangle\rangle$, rather than the corresponding expressions without Ω. Actually, this should not matter, since we expect $|\Gamma\rangle\rangle$ to be an eigenstate of Ω: a crosscap corresponds to an orientifold plane, formed by the fixed points of Ω. This would still seem to allow two possibilities:

$$\Omega|\Gamma\rangle\rangle = \pm|\Gamma\rangle\rangle \ ,$$

but there are several reasons to believe that the + sign is the correct one. First of all Ω acts on self-conjugate boundaries only with a + sign, which might suggest that the same should be true for O-planes; secondly, if this relation holds with a $-$ sign, one finds $M_{ia}\hat{\mathcal{X}}_i = -M_{ia^c}\hat{\mathcal{X}}_i$ (where $\hat{\mathcal{X}}$ are the usual Moebius characters). This is obviously inconsistent for self-conjugate boundaries, which must have a non-vanishing M_{0a}. Hence in theories with at least one self-conjugate boundary, $|\Gamma\rangle\rangle$ *must* be an Ω eigenstate with eigenvalue $+1$. This includes most CFT's but leaves open the possibility of a rare exception. Finally, if different expressions were used for the annulus, Moebius and Klein bottle amplitudes, tadpole factorization for the genus 1 open string amplitudes would not work. While none of these arguments are totally convincing, it seems nevertheless reasonable to assume that $\Omega|\Gamma\rangle\rangle = |\Gamma\rangle\rangle$, which implies $M_{ia}\hat{\mathcal{X}}_i = M_{ia^c}\hat{\mathcal{X}}_i$. In CFT's with degenerate Virasoro characters the implication for the Moebius coefficients is ambiguous, but we can be more precise.

The relation $\Omega|\Gamma\rangle\rangle = |\Gamma\rangle\rangle$ implies a relation for the basis coefficients $\Gamma_{m,\alpha}$ appearing in the expansion

$$|\Gamma\rangle\rangle = \sum_{m,\alpha} \Gamma_{(m,\alpha)}|m,\alpha\rangle\rangle \ ,$$

namely

$$\Gamma_{m^c,\beta} = \sum_{\alpha} \Gamma_{m,\alpha}\Omega^m_{\alpha\beta} \tag{2.19}$$

Furthermore we will assume that the coefficients Γ satisfy the same CPT relation (2.8) as the coefficients B, *i.e.*

$$(\Gamma_{(j^c,\alpha)})^* = \sum_{\beta} C^j_{\alpha\beta}\Gamma_{(j,\beta)} \ , \tag{2.20}$$

which is plausible since the action is entirely on the space of Ishibashi states. Then we can derive

$$\begin{aligned} M_{\ell^c a^c} &= \sum_{m,\alpha} P_{\ell^c m}\Gamma_{m,\alpha} g^m_{\alpha\beta} B_{(m\beta)a^c} \\ &= \sum_{m,\alpha,\beta,\gamma} P_{\ell m} C^{m^c}_{\alpha\beta}\Gamma_{m,\gamma}\Omega^m_{\gamma\beta}B_{(m\alpha)a} \\ &= \sum_{m,\alpha,\beta,\gamma} P_{\ell m}\Gamma_{m,\gamma}\Omega^m_{\gamma\beta}C^m_{\beta\alpha}B_{(m\alpha)a} = M_{\ell a} \end{aligned} \tag{2.21}$$

where we used (2.18), reality of $M_{\ell,a}$, (2.20), (2.19) and finally the relation $C^{m^c} = (C^m)^T$ derived just after (2.8). This relation is an additional constraint that must hold if Ω is actually a symmetry. In the next chapter we will encounter solutions to (2.13) that violates this constraint, and must therefore be rejected.

On might think that the polynomial equations (2.13) are not sufficient, just as the NIMrep condition (1.1) is not sufficient to relate a NIMrep to a modular invariant. We must also require that in the loop channel only valid Ishibashi states propagate. In other words, the proper channel transformation applied to the Moebius and Klein bottle coefficients that solve these equations should give zero on non-Ishibashi labels. In fact, more is required: a set of boundary coefficients $B_{(i,\alpha)a}$ and crosscap coefficients $\Gamma_{(i,\alpha)}$ that reproduce the integers. We will now show that this already follows from (2.13), *i.e.* that (2.15) and (2.14) not only imply (2.13), but that the converse is also true.

Suppose the set of integers M_{ia} satisfies the first equation. We can always write them as $M_{ia} = \sum_m P_{im} X_{ma}$ since P is invertible. Plugging this into the first equation and using (2.3), plus several inversions of P and S we get

$$X_{\ell a} = \sum_{\alpha} B_{(\ell,\alpha)a} \left[\sum_b B_{(\ell,\alpha)b} X_{\ell b} \right] \equiv \sum_{\alpha} B_{(\ell,\alpha)a} C_{(\ell,\alpha)} \ . \tag{2.22}$$

Substituting the resulting expression for M_{ia} into the second equation we find that any set of Klein bottle coefficients satisfying it must be of the form

$$K_i = \sum_{m,\alpha,\beta} S_{im} C_{m,\alpha} (g^m)^{-1}_{\alpha\beta} C_{m,\beta} \ , \tag{2.23}$$

where we used (2.17). Defining $C_{m,\alpha} = \sum_\beta g_{\alpha\beta} \Gamma_{m,\beta}$ then gives us both K_i and M_{ia} in the required form. Note that (2.22) does not determine the crosscap coefficients outside the space spanned by the Ishibashi states. However, (2.23) shows that there is no room for additional components, so that $C_{m,\alpha}$ must vanish on non-Ishibashi labels. Hence no further conditions are required.

Note that in both cases there are far more equations than variables. In practice the first set usually reduces to a number of independent equations slightly smaller than the number of Moebius coefficients (or equal, in which case the only solution is $M_{ia} = K_i = 0$). Since the equations are homogeneous, the open string integrality conditions are needed to cut the space of solutions down to a finite set. The second set of equations produces a unique set of coefficients K_ℓ for any solution of the first. These coefficients are then subject to the closed sector integrality conditions.

One may expect the symmetry Ω to be closely related to the choice of Klein bottle projection K_i, which determines the projection in the closed string sector. The link between the two is provided by a relation postulated in [18]:

$$\sum_{a,\alpha\beta} S_{0j} B_{(j,\alpha)a} g^j_{\alpha\beta} B_{(j,\beta)a} = Y^j_{\ 00} K_j \ , \tag{2.24}$$

conjectured to hold *without* summation on j. Here we restored the dependence on g, that was absorbed into the definition of B in [18]. Even though we do not know g^j itself, it is clear that if this relation holds, and if we limit ourselves to automorphism invariants and to real CFT's,[⋆] any sign change of Ω^j implies a sign change of K_j for the same value of j. However, this does not necessarily imply that Ω^i is equal to K^i.

⋆ Note that $Y^j_{\ 00}$ is equal to the Frobenius-Schur indicator [24][20], which vanishes for complex fields.

A closely related issue is the so-called "Klein bottle constraint". This condition would imply that the signs of the Klein bottle are preserved by the fusion rules. This is obviously true if $\Omega^i = K^i$, since Ω is a symmetry. However, there are cases where the Klein bottle constraint is violated in otherwise consistent simple current U-NIMreps [17]. In those examples either $\Omega^i \neq K^i$, or Ω is not a symmetry. Since we cannot determine Ω without further information, as discussed earlier, this cannot be decided at present. One important check can be made, however. The examples admit (at least) two choices for K^i, both violating the Klein bottle constraint. The sign flips relating these choices imply, via (2.24), corresponding sign flips in Ω^i, which must themselves respect the fusion rules if each choice of Ω^i does. We have checked that this is indeed true.

3. Examples I: Automorphisms of Extended WZW models

In the following chapters we consider a variety of modular invariant partition functions of CFT's, for which we compute all the NIMreps that satisfy ${A^i}_a{}^b = {A^{i^c}}_b{}^a$, all the symmetrizations of each NIMrep, and all the Moebius and Klein bottle coefficients that satisfy the polynomial equations (2.13), together with the usual mod-2 conditions.

Additional constraints that may be imposed are

1. The Klein bottle constraint (1.3).
2. The orientifold condition (2.21).
3. The trace formula (1.4).
4. Positivity of the Klein bottle coefficient of the vacuum.
5. Reality of the crosscap coefficients.†

We have seen examples violating any of these conditions, but Nr. 3 was only violated if Nr. 4 (which is clearly necessary in applications to string theory) was violated as well. In those cases it turns out that (1.4) holds with a $-$ sign, and furthermore there was a second solution that satisfies (1.4) and has the signs of all Klein bottle coefficients reversed (note that the polynomial equations allow a sign flip of all Moebius coefficients, but not in general a sign flip of all Klein bottle coefficients). We consider any violation of conditions Nr 1,2,3 and 4 as signs of an inconsistency, but we will mention such solutions whenever they occur.

The number of solutions we present is after removing all equivalences (*i.e.* the overall Moebius sign choice, boundary permutations that respect the NIMrep and relate different S-NIMreps, and boundary permutations that respect an S-NIMrep and relate different sets of Moebius coefficients).

† This is a conjecture due to A. Sagnotti.

The examples we consider in this chapter occur in CFT's obtained by extending the chiral algebra of a WZW-model by simple currents. We emphasize that we work in the extended CFT and hence we only consider boundaries that respect the extended symmetries. Nevertheless we will get information about broken boundaries from other modular invariants of the extended theory. The cases we consider have one fixed point representation, and we apply the method of fixed point resolution of [25] to obtain the modular transformation matrix of the extended theory.

We will be interested in automorphism modular invariants of the extended theory. Usually the fixed point resolution leads to a CFT that has one or more non-trivial fusion rule automorphisms in which the resolved fixed points are interchanged. Here we consider three cases in which there are additional automorphisms interchanging resolved fixed points with other representations. This phenomenon was first observed in the so-called "E_7"-type modular invariant [26] of A_1 level 16 [27], which is the first in a short series including also A_2 level 9 [28] and A_4 level 5 [29].

3.1. A_1 level 16

Consider first as a warm-up example A_1 level 16. The extended theory has six primaries, which we label as (0), (1), (2), (3), (4^+) and (4^-), where the integers indicate the smallest $SU(2)$-spin in the ground state representation. The spin 4 representation is a fixed point of the simple current, and hence is split into two degenerate representations. In addition the representation (1) has the same conformal weight as (4^+) and (4^-), up to integers. All representations are self-conjugate.

This extended algebra has six modular invariants, corresponding to the six permutations of (1), (4^+) and (4^-)

$$\begin{aligned}
a:&\quad (0)^2+(1)^2+(2)^2+(3)^2+(4^+)^2+(4^-)^2\\
c:&\quad (0)^2+(1)^2+(2)^2+(3)^2+(4^+)\times(4^-)+(4^-)\times(4^+)\\
b^+:&\quad (0)^2+(1)\times(4^+)+(2)^2+(3)^2+(4^+)\times(1)+(4^-)^2\\
b^-:&\quad (0)^2+(1)\times(4^-)+(2)^2+(3)^2+(4^-)\times(1)+(4^+)^2\\
d^+:&\quad (0)^2+(1)\times(4^+)+(2)^2+(3)^2+(4^-)\times(1)+(4^+)\times(4^-)\\
d^-:&\quad (0)^2+(1)\times(4^-)+(2)^2+(3)^2+(4^+)\times(1)+(4^-)\times(4^+)
\end{aligned}$$

When we write these partition functions in terms of $SU(2)$ characters (which do not distinguish (4^+) from (4^-)) on obtains respectively the invariants of types (D,D,E,E,E,E) of A_1.

The number of Ishibashi labels is 6,4,4,4,3,3 respectively, and we found precisely one complete set of boundaries in all six cases (for the unextended theory, the NIMreps were presented in the second paper of [2]; the U-NIMrep of the invariants "b" in the extended theory were obtained in [6].) All these sets are symmetric and have no other symmetrizations. Furthermore in all cases except d^+ and d^- a U-NIMrep exists. In this case we know that all these boundaries are indeed physically meaningful. The cases (a, c), (b^+, d^+) and (b^-, d^-) are pairwise related to each other by a left conjugation (*i.e.* a (4^+), (4^-) interchange), whereas (a, c), (b^+, d^-) and (b^-, d^+) are pairwise related by a right conjugation. These conjugations are like T-dualities, which from the point of view of the unextended A_1 theory (the orbifold theory of this conjugation) interchange the automorphism type of the boundary. Hence the 6 boundaries of the a-invariant plus the 4 boundaries of the c-invariant together form the 10 boundaries expected for the D-type invariant of A_1 level 16; the 4 boundaries of the d-invariant plus the 3 boundaries of the e-invariant form the 7 boundaries of the E-type invariant of A_1 level 16.

In the extended theory, orientation does not add much information, since all boundary coefficients were expected to exist, and since furthermore the cases d^+ and d^- are asymmetric invariants that do not admit an orientifold projection.

3.2. A_2 level 9

Now consider another case with similar features, namely A_2 level 9. The extended theory now has 8 primaries, which we will label as follows

$$\begin{array}{rlrlrl}
(0): & (0,0) & (1): & (0,3) & (2): & (3,0) \\
(3): & (1,1) & (4): & (4,4) & (5): & (2,2) \\
(6^i): & (3,3), & \multicolumn{4}{l}{i = 1,2,3}
\end{array}$$

Here (a, b) are A_2 Dynkin labels of one of the ground state representations. The representations (1) and (2) are conjugate to each other, and all others are self-conjugate.

The extended theory has a total of 48 distinct modular invariants, obtained by combining charge conjugation with the 24 permutations of (3), (6^1), (6^2) and (6^3). These modular invariants are related to the following four distinct ones of A_2 level $9^\star$

$$\begin{aligned}
D: &\quad [0]^2 + [1][2] + [2][1] + [3]^2 + [4]^2 + [5]^2 + 3[6]^2 \\
DC: &\quad [0]^2 + [1]^2 + [2]^2 + [3]^2 + [4]^2 + [5]^2 + 3[6]^2 \\
E: &\quad [0]^2 + [1][2] + [2][1] + [3][6] + [6][3] + [4]^2 + [5]^2 + 2[6]^2 \\
EC: &\quad [0]^2 + [1]^2 + [2]^2 + [3][6] + [6][3] + [4]^2 + [5]^2 + 2[6]^2,
\end{aligned}$$

where $[i]$, $i \neq 6$ stands for the full simple current orbit of the orbit representatives listed above, and [6] is the fixed point. These modular invariants admit respectively 21, 15, 17 and 11 Ishibashi states.

⋆ Here "D" stands for "type D" as in ADE, and not for "diagonal". Note that we have included a charge conjugation in the definitions, so that D corresponds to the C-diagonal or Cardy case, and DC to the diagonal invariant.

In the following table we show the number of NIMreps (of various kinds) that exist for these 48 invariants. The results of this table are based on an exhaustive search, *i.e.* the set of solutions is complete.

Nr.	Permutation	Proj.	Ish.	NIMreps	S-NIMreps	U-NIMreps
1	$<>$	D	9	1	1	1
2	$< a, b >$	D	7	0	0	0
3	$< 3, b >$	E	7	0	0	0
4	$< 1, 2 >$	DC	7	0	0	0
5	$< 1, 2 >< a, b >$	DC	5	1	2	$1 + 1(^*)$
6	$< 1, 2 >< 3, b >$	EC	5	1	2	$1 + 1(^*)$
7	$< 3, a >< b, c >$	E	5	1	2	1
8	$< 3, a, b, c >$	E	5	1	2	1
9	$< 1, 2 >< 3, a >< b, c >$	EC	3	1	1	0
10	$< 1, 2 >< 3, a, b, c >$	EC	3	1	1	0
11	$< a, b, c >$	D	6	2(**)	0	0
12	$< 3, a, b >$	E	6	2(**)	0	0
13	$< 1, 2 >< a, b, c >$	DC	4	0	0	0
14	$< 1, 2 >< 3, a, b >$	EC	4	0	0	0

(*) Inconsistent solution to the polynomial equations.
(**) Asymmetric NIMrep, one solution is the transpose of the other.

In this table, a, b and c denote the three resolved fixed points. The solutions are essentially the same if a, b or c are replaced by 3, but we represent these cases separately because they have a different interpretation. Furthermore cases 7,8 and 9,10 have pairwise the same solutions, because boundary CFT is only sensitive to the diagonal and C-diagonal partition function, which is identical in these cases (note that when solving (2.13) we do not take into account the left-right symmetry of the modular invariant). In cases 5 and 6 the notation is as follows: the unique NIMrep allows two symmetrizations. For each of these symmetrizations there is one set of Moebius coefficients that satisfy the polynomial equations (2.13) (there are 40 independent equations for the 45 Moebius coefficients). The Moebius coefficients for the second S-NIMrep (indicated with a (*) in the table) should be rejected, however. They give rise to complex crosscap coefficients and to Klein bottle coefficients that violate the Klein bottle constraint. More importantly, however, they violate the condition $M_{ia} = M_{i^c a^c}$, indicating that there is no orientifold symmetry that underlies these coefficients.

As was the case for A_1 we may expect these NIMreps to correspond to those of the unextended theory. Furthermore the symmetry breaking boundaries may be expected to correspond to two different modular invariants of the extended theory, obtained by applying the cyclic permutations $< a,b,c >$ and $< a,c,b >$ to the "symmetric" partition function. This leads to the following identifications of partition functions of the unextended theory with triplets of extended theory partition functions

$$\begin{aligned} D: &\quad <>;\ < a,b,c >;\ < a,c,b > \\ DC: &\quad < 1,2 >< a,b >;\ < 1,2 >< b,c >;\ < 1,2 >< a,c > \\ E: &\quad < 3,a >< b,c >;\ < 3,a,b >;\ < 3,a,c > \\ EC: &\quad < 1,2 >< 3,a >;\ < 1,2 >< 3,a,b,c >;\ < 1,2 >< 3,a,c,b > , \end{aligned} \tag{3.1}$$

The first of these identifications can actually be verified, because we can compute the NIMrep matrices for the simple current extension of A_2 level 9 using the formalism developed in [12]. Consider the matrices $A^{i\ \ b}_{a}$, with i a zero charge primary. Since the boundaries are in one-to-one correspondence with simple current orbits, we can assign the corresponding charge to each boundary. As is well known [8], zero charge boundaries are symmetry preserving, the others symmetry breaking. Now consider the matrix elements $A^{i_0\ \ b_0}_{a_0}$, $A^{i_0\ \ b_1}_{a_1}$, $A^{i_0\ \ b_2}_{a_2}$, where the subscript on the labels denotes three times the charge. As expected, these matrix elements are precisely given, respectively, by the NIMreps for the modular invariants $<>$, $< a,b,c >$ and $< a,c,b >$ (we assume that the two mutually transpose solutions in row 11 of the table are associated with the latter two modular invariants). All other matrix elements of A^{i_0} vanish due to charge conjugation. With considerably more work one should be able to compute also the NIMreps for charged primaries. This requires extracting the boundary coefficients from the NIMreps and in particular resolving the phase ambiguities.

The same comparison made above for the D invariant, can in principle also be made for the other three invariants, DC, E and EC of A_2 level 9. This requires an explicit expression for the corresponding NIMreps of the unextended theory. These NIMreps can in principle be extracted with a considerable amount of work from [4][14] (the four cases D, DC, E and EC correspond to $\mathcal{D}^{(12)*}$, $\mathcal{D}^{(12)}$, ${\mathcal{E}_4}^{(12)*}$ and ${\mathcal{E}_5}^{(12)}$ in the notation of these authors; the remaining cases are discussed at the end of this section).

Remarkably the following four cases do not work (we list here symmetric invariants and their cyclic permutations)

$$D: \quad < 1,2 >;\ < 1,2 >< a,b,c >;\ < 1,2 >< a,c,b >$$

This is the diagonal invariant of the extension. None of the three permutations admits any NIMreps.

$$DC: \quad < a, b >; \; < b, c >; \; < a, c >$$

Again, none of the permutations has a NIMrep.

$$E: \quad < 3, a >; \; < 3, a, b, c >; \; < 3, a, c, b >$$

Here it seems that the two permutations of $< 3, a >$ do admit NIMreps, but that solution just happens to coincide with the one of $< 3, a >< b, c >$. Apparently this solution should be assigned to the latter modular invariant, which is needed in (3.1). This also explains why it exists on non-orientable surfaces, not admitted by the modular invariant $< 3, a, b, c >$.

$$EC: \quad < 1, 2 >< 3, a >< b, c >; \; < 1, 2 >< 3, a, b >; \; < 1, 2 >< 3, a, c >$$

This is precisely the opposite of the previous case: there appears to be a solution for $< 1, 2 >< 3, a >< b, c >$, but it does not admit a crosscap coefficient, and should be assigned to $< 1, 2 >< 3, a, b, c >$, needed in (3.1). We see thus that there is precisely one way of relating the four modular invariants of the unextended theory to groups of three modular invariants of the extended theory, related by Z_3 permutations. The solutions 8 and 9 are fake ones. Removing them we get also a consistent result for the absence of solutions for Z_3-related modular invariants.

The extension to E_6

There is an additional modular invariant of A_2 level 9 corresponding to the conformal embedding in E_6. It can be obtained by means of an exceptional extension on top of the simple current extension. The case has the interesting feature of allowing two (or three) distinct NIMreps (see refs. [4][14]).

$$|[1] + [4]|^2 + 2 \times |[5]|^2 \tag{3.2}$$

We have analyzed this modular invariant from the point of view of the extended algebra, and find a result that seems in agreement with [4][14]: in a complete search we did indeed find precisely three distinct NIMreps. Each of these has 4 S-NIMreps, but only two of the three NIMreps allow a U-NIMrep. These two distinct U-NIMreps differ in a very interesting way: one of them has a Klein bottle coefficient equal to 0 on the degenerate field (*i.e.* the field [5]), the other has this coefficient equal to 2. The interpretation is now immediately obvious. The extended theory, E_6 level 1, is complex, and has two modular invariants, charge conjugation and the diagonal invariant. Both invariants have a U-NIMrep, according to refs. [3] and [10] respectively. The Klein bottle coefficients are respectively $(1, 0, 0)$ and $(1, 1, 1)$. In terms of the exceptional invariant (3.2), which cannot distinguish the complex field from its conjugate, these two cases reveal themselves through a different value of the Klein bottle coefficient on the degenerate field, just as discussed in [30]. The main novelty in this example is the fact that the two distinct Klein bottle choices (which we knew *a priori*) leads to two distinct NIMreps, whereas in all cases studied so far (*i.e.* the simple current extensions) different Klein bottle choices gave rise to different unoriented annuli, but the same NIMrep.

Note that the existence of two distinct NIMreps seems to clash with the discussion on uniqueness of the classifying algebra (and hence its representations) in chapter 2. This example suggests a very obvious way out. Clearly the modular invariant (3.2) belongs to two distinct CFT's. Hence it seems plausible that after a proper resolution of the degeneracy of the field [5] one will in fact obtain two distinct sets of classifying algebra coefficients, although the details will have to be worked out. This phenomenon can be expected to occur in general for modular invariants with multiplicities larger than 1, except for simple current invariants.

The two NIMreps that admit a U-NIMrep probably correspond to the cases $\mathcal{E}_1^{(12)}$ and $\mathcal{E}_2^{(12)}$ in [14], whereas the third NIMrep is likely to correspond to $\mathcal{E}_3^{(12)}$, which is discarded in [14] for not very transparent reasons.

Since the proper interpretation of these NIMreps appears to be a longstanding problem in the literature [4][14][13] we present the NIMreps here explicitly. We present them here only in terms of the simple current extension of A_2 level 9.*

The one with Klein bottle coefficients $K^0 = K^4 = 1$, all others zero is:

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A^4 = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 2 & 2 & 2 & 13 \end{pmatrix} \qquad A^5 = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \\ 2 & 2 & 2 & 14 \end{pmatrix}$$

$$A^i = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 6 \end{pmatrix} , \qquad i = 1, 2, 3, 6^1, 6^2, 6^3$$

The one with Klein bottle coefficients $K^0 = K^4 = 1$, $K^5 = 2$, all others zero is:

⋆ We have also obtained the three solutions in the unextended theory, where for this modular invariant an exhaustive search was possible. However in the unextended theory one gets 55 12×12 matrices, which we will not present here.

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A^4 = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 5 & 4 & 4 \\ 2 & 4 & 5 & 4 \\ 2 & 5 & 5 & 4 \end{pmatrix} \qquad A^5 = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 4 & 5 & 5 \\ 2 & 5 & 4 & 5 \\ 2 & 5 & 5 & 4 \end{pmatrix}$$

$$A^i = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 \end{pmatrix} \quad , \quad i = 1, 2, 3, 6^1, 6^2, 6^3$$

and finally the one not admitting any Moebius coefficients is

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A^4 = \begin{pmatrix} 3 & 2 & 2 & 4 \\ 2 & 3 & 2 & 4 \\ 2 & 2 & 3 & 4 \\ 4 & 4 & 4 & 7 \end{pmatrix} \qquad A^5 = \begin{pmatrix} 2 & 3 & 3 & 4 \\ 3 & 2 & 3 & 4 \\ 3 & 3 & 2 & 4 \\ 4 & 4 & 4 & 8 \end{pmatrix}$$

$$A^i = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix} \quad , \quad i = 1, 2, 3, 6^1, 6^2, 6^3$$

Looking at these matrices it is immediately clear that our interpretation is indeed the correct one. Namely, the annulus partition function in the first case, $\sum_i A^i{}_a{}^b \mathcal{X}_i$ restricted to the first three boundaries turns into

$$(\mathcal{X}_0 + \mathcal{X}_4)\delta_a^b + A_a^5{}^{\ b} \mathcal{X}_5$$

where $(\mathcal{X}_0 + \mathcal{X}_4)$ is the identity character of E_6 and $\mathcal{X}_5$ is the Virasoro character of the (27) and $(\overline{27})$ representations (this follows directly from the conformal embedding). By inspection, $A^5{}_a{}^b$ restricted to the first three boundaries is the sum of the E_6 annulus coefficients for the (27) and the $(\overline{27})$. The fourth boundary must then be a symmetry breaking one, that respects the simple current extension of A_2 level 9, but not the exceptional extension on top of it.

The interpretation of the second NIMrep is completely analogous, but less instructive, because it only concerns the first boundary. The other three are symmetry breaking. The last NIMrep does not admit a rewriting in terms of E_6 characters for any choice of boundaries.

The power of orientation is of course very well demonstrated in the latter example. We consider some other examples of this type in chapter 7.

3.3. A_4 LEVEL 5

Consider now the simple current extension of A_4 level 5. This extension has 10 representations,namely

$$(0) : (0,0,0,0);\ (1) : (1,0,0,1);\ (2) : (2,0,0,2)$$

$$(3) : (0,2,2,0);\ (4) : (0,1,1,0);\ (5^i) : (1,1,1,1), i = 1,\ldots,5$$

All 720 permutations of the representations (1) and (5^i) yield modular invariant partition functions. In this case the table of solutions (which is complete, as in the previous case) is as follows

Ish.	NIMreps	S-NIMreps	U-NIMreps
10	1	1	1
8	0	0	0
7	0	0	0
6	5	$(1+1+1+1+2)$	$(0+0+0+0+(0+2))$
5	2	$(1+1)$	0
4	1	1	0

Since the six primaries (1) and (5^i) are completely on equal footing as far as the modular data (*i.e.* S and T) are concerned, the solution depends only on the number of Ishibashi states, i.e. the number of primaries not involved in a permutation. The 720 modular invariants are related to one of the following two modular invariants of A_4 level 5

$$\begin{aligned} D: &\quad [0]^2+[1]^2+[2]^2+[3]^2+[4]^2+5[5]^2 \\ E: &\quad [0]^2+[1][5]+[5][1]+[2]^2+[3]^2+[4]^2+4[5]^2\ , \end{aligned}$$

where $[0]\ldots[4]$ denotes a complete simple current orbit. If [1] is non-trivially involved in the permutation, the invariant of the extended theory is related to E, and otherwise to D. The notation is as in the previous table: for example, the permutation $<1,a><b,c>$ gives rise to six Ishibashi states. We read from the table that there exist 5 NIMreps, four of which yield one S-NIMrep, while the fifth one yields two. Of those two, one allows no U-NIMreps, and one allows two (which are only very marginally different: the Klein bottle coefficients are the same, but a few Moebius coefficients have opposite signs). The five NIMreps are all completely different, and clearly not related by some unidentified symmetry.

The most striking feature is again the complete absence of NIMreps for the symmetric automorphism modular invariants $< 1, a >$ and $< a, b >$, which presumably are unphysical. For the automorphism $< 1, a >< b, c >< d, e >$ a NIMrep and even an S-NIMrep is available, but it does not allow a U-NIMrep.

As in the previous case, we can compare the NIMreps for the simple current extension of A_4 level 5 with the results from the table. This time we consider ${A^{i_0}}_{a_0}{}^{b_0}$, ${A^{i_0}}_{a_1}{}^{b_1}$, ${A^{i_0}}_{a_2}{}^{b_2}$, ${A^{i_0}}_{a_3}{}^{b_3}$ and ${A^{i_0}}_{a_4}{}^{b_4}$, and we find that the result coincides with the NIMreps of $<>$ and $< a, b, c, d, e >$ in the table. In the latter case (five Ishibashi states) we found two NIMreps, which coincide respectively with the entries of the charge $\pm 1/5$ and $\pm 2/5$ boundaries.

Identifying other solutions is harder than in the previous case, partly because there is an ordering problem in determining cyclic symmetries. From the point of view of the fusion algebra, the five resolved fixed point representations are completely on equal footing, so that the fusion rules are invariant under all their permutations. If we make the assumption that boundaries with broken symmetries in the unextended case correspond to boundaries for cyclically permuted modular invariants in the extended case, (as is the case thus far), then an apparent contradiction arises. For example, the two modular invariants $< 1, 5^1 >< 5^2, 5^3 >$ and $< 1, 5^1 >< 5^2, 5^5 >$, which appear to be totally equivalent, yield a different structure of permutations if one cyclically permutes the fixed point labels. The cyclic permutations of the symmetric modular invariants are as follows

Nr.	invariant	type	permutations
1	$< ab >$	D	$2 < abcd > +2 < abc >< de >$
2	$< 1a >$	E	$4 < 1abcde >$
3	$< ab >< cd >$	D	$4 < ab >< cd >$
4	$< 1a >< cd >$	E	$2 < 1abcd > +2 < 1abc >< de >$
5	$< 1a >< cd >$	E	$2 < 1abcd > +2 < 1ab >< cde >$
6	$< 1a >< cd >< ef >$	E	$4 < 1ab >< de >$
7	$< 1a >< cd >< ef >$	E	$2 < 1abcde > +2 < 1abc >$

Here only the permutation group element is indicated. where a, b, c, d, e stand for the fixed point fields in an unspecified order. For example $< abc >< de >$ indicates a product of and order 3 and an order 2 cyclic permutation, but is not meant to imply an actual ordering of the labels. In the last four rows, the actual choice of labels in column 1 determines which of the two options is valid. Note that the number of Ishibashi states in columns 1 and 3 correctly adds up to the number of Ishibashi states of the type D and E modular invariants (30 and 24, respectively).

Clearly, there is no NIMrep for cases 1 and 2, since there is no NIMrep for the symmetric invariant. Presumably the same is true for cases 6 and 7, since there is no U-NIMreps for the symmetric invariant, which has 4 Ishibashi states. There does exist a NIMrep in this case, but this is precisely needed for the a-symmetric invariants $< 1abc >< de >$ and $< 1ab >< cde >$ appearing in case 4 and 5, which is the only remaining chance to realize the E-invariant in terms of the extended theory.

4. Examples II: Diagonal invariants

The problems that we encountered above with the diagonal invariant of the extension of A_2 level 9 can be expected to occur quite generally for similar invariants. Consider a CFT with a simple current of odd (and, for simplicity, prime) order N of integer spin. Suppose the unextended CFT has a complete U-NIMrep for the diagonal invariant. This assumption holds in particular for (tensor products of) WZW models, for which the boundary states and NIMreps were constructed by means of an charge conjugation orbifold theory [10]. Although orientation issues were not discussed in that paper, the results were obtained using a simple current extension of the orbifold theory, to which the results of [12] apply. Those results can then be used to derive also the crosscap states.

If a full U-NIMrep is available for the diagonal invariant, this implies that there exists a set of Klein bottle coefficients that satisfies the following sum rule

$$\sum_j S_{ij} K^j = 0 \quad \text{if } i \text{ is complex}$$

In the case of WZW models such a sum rule must in fact hold for $K^i = 1$ (and also for $K^i = \nu^i$, the Frobenius-Schur indicator), for all i. It should be possible to derive this directly from the Kac-Peterson formula for S, but follows in any case from the argument in the previous paragraph.

Now consider a complex representation i with vanishing simple current charge (so that it survives in the extended theory), and whose conjugate lies on a different orbit. Then all other representations on its orbit are necessarily complex as well, and hence

$$0 = \sum_j S_{Ji,j} K^j = \sum_j S_{ij} e^{2\pi i Q_J(j)} K^j$$

This implies that the sum must vanish for each value of the simple current charge separately, and in particular

$$\sum_{j, Q_J(j)=0} S_{i,j} K^j = 0 \tag{4.1}$$

This appears to be the sum rule needed for the simple current extended theory, and indeed it is if there are no fixed points. But if there are fixed points, their total contribution to the sum is enhanced by a factor N, so that the correct sum rule for the extended theory does not hold.⋆

There are two ways to remedy this. One is to flip the signs of some of the Klein bottle coefficients on the fixed points. This would violate the Klein bottle constraint, but even if one accepts that, in the examples we studied no corresponding complete set of boundaries could be found. The second solution is to modify the modular invariant in such a way that in addition to the charge conjugation, $N-1$ resolved fixed points are off-diagonal. This removes $N-1$ contributions K^j for each fixed point, so that the sum rule holds again. Note that this also introduces new sum rules for the new off-diagonally paired fields themselves, which are no longer Ishibashi states. The latter sum rules automatically follow from (4.1) and the general structure of the matrix S on resolved fixed points [29]. Note, however, that for $N>3$ the $N-1$ off-diagonal elements need not be paired: there are other (non-symmetric) modular invariants in which they are off-diagonal.

The foregoing argument only shows that a plausible set of Klein bottle and crosscap coefficients can be found if in addition to the charge conjugation one also pairs $N-1$ resolved fixed points off-diagonally, simultaneously for all fixed points. It also shows that without the latter, the existence of a U-NIMrep is unlikely. A few examples of this kind can be studied.

In the series of simple current extension of A_2 level $3k$ the first complex theory is A_2 level 9, discussed in the previous chapter. In the next case, A_2 level 12, we found a U-NIMrep for the diagonal invariant with interchange of two resolved fixed points, but no NIMrep for the diagonal invariant or interchange invariant separately (this statement is based on an exhaustive search). Extrapolating these results to levels 3 and 6 (where the extended theory is real) one might anticipate the existence of NIMreps for the fixed point interchange invariants. This is indeed correct. For A_2 level 3, the extension is D_4 level 1, and the fixed point interchange is a simple current invariant, hence the NIMrep is the same as the one given in [12]. Another example is the extension of E_6 level 3 with the spin-2 simple current. This has a novel feature in having two resolved fixed points, f^i and g^i, $i=1,2,3$. Charge conjugation is trivial in the extended theory, and the invariant $<f^1,f^2><g^1,g^2>$ has one NIMrep (with one S- and U-NIMrep). Note that $<f^1,f^2><g^2,g^3>$ is not modular invariant.

Other cases with similar features occur for tensor products and coset CFT's. Examples are $A_{2,3}\times A_{2,3}$ and $A_{2,3}\times E_{6,3}$, extended with the current (J,J) and the coset CFT $A_{2,3}\times A_{2,3}/A_{2,6}$, where the "extension" is by the identification current. In the first two cases, there is no NIMrep for the diagonal invariant and one NIMrep exists for the diagonal invariant plus fixed point interchange (yielding 4 S-NIMreps and 2 U-NIMreps). This was demonstrated using an exhaustive search. For the coset CFT (and many analogous ones) we expect similar results, but it was unfortunately not possible to verify this.

⋆ It turns out the the fixed point contribution is in general non-vanishing. For simplicity we assume that the fixed point fields themselves are real.

5. Examples III: c=1 orbifolds

Here we analyse some results from [11] on modular invariants of $c = 1$ orbifolds. In [11] we considered orbifolds at radius $R^2 = 2pq$ (p, q odd primes) (with $pq+7$ primaries), and considered the automorphism that occurs [31],[21] if the radius factorizes into two primes.

Two cases where considered: the automorphism acting on the charge conjugation invariant (denoted "$C+A$" in [11]) and acting on the diagonal invariant (denoted "$D+A$"). In both cases, two U-NIMreps where found, but this was not claimed to be the complete set of solutions. Meanwhile we have verified that it is indeed complete in the simplest case, $pq = 15$. The question whether the corresponding NIMreps are also distinct was not addressed in [11], but from the explicit boundary coefficients one may verify that they are not. The solutions present some features that are worth mentioning in the present context.

In the language of the present paper the results can be described as follows

— $D + A$: one NIMrep, yielding 4 S-NIMreps. Of these four, three do not allow any U-NIMreps and should presumably be viewed as unphysical. The fourth one allows two distinct U-NIMreps, each with a different Klein bottle as described in [11].

— $C + A$: one NIMrep, yielding 2 S-NIMreps. Each of these yields one U-NIMrep, with the two Klein bottles described in [11].

Both cases contain a novelty with respect to the previous examples. In the D+A case the two Klein bottle choices correspond to the same S-NIMrep. Hence there is no change in the unprojected open string partition function, but some of the Moebius signs are flipped. This means that the only difference between the two orientation choices is the symmetrization of the representations. With both choices, the same branes are identified, but with different signs.

In the C+A case the novelty is that the boundary coefficients of the two choices differ by 8th roots of unity rather than fourth roots, as in all simple current cases. This implies that at least one of the projections Ω has coefficients Ω^j that are phases, rather than signs. This possibility is allowed by (2.4) provided that $\Omega^j = (\Omega^{j^c})^*$. We cannot check this directly because we cannot determine Ω^j, but we can at least check it for the ratio of the two different orientifold projections.

From the annuli $A^{\Omega_x}_{iab}$ we can compute $B_{ja}\Omega^j_x C^j B_{ja}$, where $x = 1, 2$, labelling the two cases. Since neither B nor C depended on the orientifold projection, the ratio of these quantities for $x = 1, 2$ determines the ratio Ω^j_2/Ω^j_1. We find that this ratio is $\pm i$ on the twist fields, $j = \sigma_i$ or τ_i (and ± 1 on all other fields), and that it does indeed respect (2.4).

6. Examples IV: Pure WZW automorphisms

The automorphism modular invariants of WZW models were completely classified in [21]. The foregoing results on automorphisms of extended WZW models may raise some doubts about their consistency as CFT's. We will not answer that question completely here, but we will consider the "most exceptional" cases. The results are summarized (together with those of the next section) in a table in the appendix.

Four basic types (which may be combined) may be distinguished: Dynkin diagram automorphisms (*i.e.* charge or spinor conjugation and triality), simple current automorphisms, infinite series for SO(N) level 2 if N contains two or more odd prime factors, and the three cases G_2 level 4, F_4 level 3 and E_8 level 4. The first of these was dealt with in [10], the second in [12], but combinations still have to be considered. For even N, some cases follow from the results for $c = 1$ orbifolds, using the relation derived in [32], but further work is needed for completing this and extending it to odd N.

This leaves the three "doubly exceptional" cases. The first two are closely related since the fusing rings are isomorphic. The NIMrep of G_2 level 4 is already known, and the one of F_4 level 3 turns out to be identical. In both cases we find four S-NIMreps, but three of them do not allow a U-NIMrep. The fourth one gives rise to one U-NIMrep.

For E_8 level 4 we find one NIMrep, yielding one S-NIMrep, which in its turn gives one U-NIMrep. The matrix elements of $R_{ma} = B_{ma}\sqrt{S_{m0}}$ are all of the form $\pm\frac{2}{\sqrt{17}}\sin(\frac{2\pi\ell}{17})$, $\ell = 1, \ldots 8$, and are up to signs equal to those of the modular transformation matrix of the twisted affine Lie algebra (denoted as in [33]) $\tilde{B}_1^{(2)}$, level 14 or, equivalently, $\tilde{B}_7^{(2)}$, level 2, although the significance of this observation is not clear to us.

7. Examples V: WZW extensions

In the table in the appendix we list the number of (S-,U-)NIMreps some extensions of WZW-models. In all cases the results are based on a complete search. The extensions are either conformal embeddings (denoted as $H \subset G$, listed in [34]), simple current extensions (denoted as "SC"), or higher spin extensions (HSE). If the CFT is complex, there are in principle two invariants, obtained by extending the charge conjugation invariant or the diagonal invariant. The latter possibility is indicated by a $(*)$ in the first column. Most of the extensions of complex CFT's in the table are themselves real (although some become complex after fixed point resolution). In a few cases only the extension of the diagonal invariant was tractable, since it has fewer Ishibashi states than the extension of the charge conjugation invariant.

We only list those simple current extensions where we found more solutions than those obtained in [12], although we considered all accessible low-level cases in order to check whether the set of solutions of [12] is complete. The higher spin extensions appeared in the list of $c = 24$ meromorphic CFT's [35], although some were obtained before, or could be inferred from rank-level duality. Since the existence of these CFT's is on less firm ground than the existence of the other kinds of extensions, it is especially interesting to find out if NIMreps exist.

In the U-NIMreps column those cases denoted with a single asterisk have Klein bottle coefficients violating the Klein bottle constraint (but no other conditions); those with a double asterisk violate the orientifold condition $M_a^i = M_{a^c}^{i^c}$ (and often also the Klein bottle constraint). The ones with a triple asterisk have a value for K_0 equal to -1 (and hence violate the Klein bottle constraint), and they also violate the trace identity (1.4). Often these solutions also violate the orientifold condition, and in some case they give rise to imaginary crosscap coefficients. We will assume that the latter violations are unacceptable, but on the other hand we will be forced to conclude that the Klein bottle constraint violations must be accepted.

All the U-NIMreps that are listed have different Klein bottle coefficients. Those of the spin-1 extensions can all be understood in terms of simple-current related Klein bottle choices in the extended algebra (which is a level-1 WZW model), as described in [12].

Some of the NIMreps in the table (in particular those for the algebras of type A) have been discussed elsewhere (see *e.g.* [4] [36]), most others are new, as far as we know. Although it is impossible to present all matrices here explicitly, they are available on request.

The following features are noteworthy (the characters refer to the last column)

A. In this case there are two NIMreps, one corresponding the interpretation of the MIPF as a C-diagonal invariant of the extended algebra, and one corresponding to the diagonal interpretation.

B. In this case there are two U-NIMreps, related to the fact that the extended algebra admits two Klein bottle choices, generated by simple currents as discussed in [20].

C. This is as case B, except that the second Klein bottle choice violates the Klein bottle constraint in the unextended theory (although it does satisfy it in the extended theory).

D. These are combinations of cases A, B and C: the modular invariant corresponds to both diagonal and C-diagonal interpretations, each of which allows two Klein bottle choices. Since the diagonal invariant of D_{2n+1} is a simple current invariant, all four Klein bottle choices follow in fact from [12].

E. In this case the extension is by a simple current, so that [12] applies. However, that paper contains only a single NIMrep for this case, and we find two. A plausible explication is that the $A_{2,3m}$ series bifurcates for $m \geq 3$ into a series of simple current modifications of the C-diagonal and of the diagonal invariant of $A_{2,3m}$. For $m < 3$ these invariants coincide. The formula of [12] only applies to the first series, and the second solution must be interpreted as part of the other series. This interpretation agrees with the values of the Klein bottle coefficients on the fixed point field (resp. 3 and 1), and also with the discussion at the end of section 4.

F. Here the same remarks apply as in case E, but in addition the resolved fixed points become simple currents (of $SO(8)$ level 1), allowing an additional Klein bottle choice (or rather three, related by triality). The three cases in the table (not including the double asterisk ones) have Klein bottle values $3, 1$ and -1 on the fixed point field. This corresponds precisely to the three values that follow from [12] for $SO(8)$ level 1 (for respectively the diagonal invariant, the simple current automorphism and the Klein bottle simple current).

G. Here "SC 2" means that the invariant is obtained using the simple current with Dynkin labels $J^2 = (0, 2, 0)$. This case does not really belong in the table, because both from the point of view of the unextended theory, A_3 level 2, as in the extended theory, A_5 level 1, the entire result can be obtained using [12]. In A_3 level 2 they follow from by combining both choices for $\alpha(J)$ in formula (11) of [12], applied to the extension by J^2, with the two allowed Klein bottle choices, $K = 1$ and $K = J$. In A_5 level 1 one obtains the same four Klein bottles from two distinct modular invariants which cannot be distinguished prior to fixed point resolution. The invariants are the diagonal one, and the one generated by the simple current J^2 of A_5. Both can be modified by the Klein bottle simple current J.

H. This is the unextended analog of the example that was explicitly presented in section (3.2.1). It is in complete agreement with the results presented there.

One can make sense of the higher spin extension results in a very similar way, because also in those cases we know the fusion rules of the extended algebras. In some cases this knowledge is available via rank-level duality of the unextended algebras. Some of these dualities are quite clear in the table, for example $C_{7,2}$ with $C_{2,7} \equiv B_{2,7}$. In other cases one can make use of the fact that the exceptionally extended CFT's combine with known CFT's to form a meromorphic CFT, and hence must have the complex conjugate matrix S of that CFT. For example, the E_7 level 3 extension has the same fusion rules as A_5 level 1. The results for the latter CFT match those of the exceptional extension: we find the same number of distinct U-NIMreps as in case G. The same is true, with appropriate modifications, in all other cases.

These examples demonstrate once more the importance of orientation, as well as the completeness of the results of [12]. They also give the clearest evidence so far that the Klein bottle constraint may well be correct in some form, but that the precise formulation requires more work.

8. Conclusions

Orientation effects add useful additional structure to the a NIMreps. Furthermore, given a NIMrep, this information is relatively easy to get, and it is therefore a pity that many authors decide to ignore it.

Taken into account orientation leads naturally to a different formula for the annulus than the one commonly used in the NIMrep literature. The distinctions between the expression for oriented annuli, (2.3) and unoriented ones (2.9) are:

— The former does not depend on the choice of orientifold projection, the latter does;

— The former does not imply any boundary conjugation, the latter does.

We have given examples of modular invariants without NIMreps (including, rather surprisingly, diagonal invariants), modular invariants with several NIMreps, NIMreps without S-NIMreps (and hence no boundary conjugation), NIMreps with several S-NIMreps, S-NIMreps without U-NIMreps and S-NIMreps with more than one U-NIMrep, and U-NIMreps with identical Klein bottle coefficients but different Moebius coefficients.

Two distinct modular invariants may have the same Ishibashi states and hence the same NIMreps. This happens in particular when asymmetric invariants are considered. In that case the existence of a U-NIMrep is useful information to determine which NIMrep belongs to which modular invariant.

Although there appear to exist NIMreps that are accidental solutions to the equations, and that have no physical interpretation, we have not found any U-NIMreps that are clearly not sensible.

In all cases we know, the occurrence of multiple NIMreps for symmetric modular invariants can be interpreted in a sensible way if orientation information is taken into account, and if those NIMreps that lack a corresponding U-NIMrep are rejected. Furthermore, the occurrence of multiple U-NIMreps is in complete agreement with the results of [12], wherever a comparison is possible. It seems plausible that only if the partition function has multiplicities larger than 1, there can be more than one "sensible" NIMrep.

The results in section 7 (including a large set of cases not appearing in the table because they agree with expectations) suggest that the results of [12] are in fact generically complete, *i.e.* no other boundary and crosscap states can be written down that are valid for any simple current invariant. Sporadically there may be additional solutions, but the absence of these solutions in other cases with the same simple current orbit structure proves that they are not generic (unfortunately there is a infinity of distinct orbits structures, and hence "generic completeness" cannot be proved rigorously in this way).

Although our results shed some light on the infamous "Klein bottle constraint", its precise formulation clearly requires further thought. The examples in the last section show that it does not hold in its naive form, at least not for modular invariants of extension type.

The purpose of the examples in this paper is only to demonstrate various features. They should be useful to avoid incorrect conjectures, but no general conclusion can be drawn from them with confidence. Indeed, even the combined information from the existence of a modular invariant, NIMreps, S-NIMreps and U-NIMreps is probably ultimately insufficient to decide whether a given candidate CFT really exists. Fortunately, in applications to string theory we rarely need to consider such exotic cases. Simple current invariants is all one usually needs, and for that class the solution is known.

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APPENDIX

Number of NIMreps for WZW modular invariants

MIPF	NIM	S-NIM	U-NIM	remark
$A_{1,10} \subset SO(5)$	1	2	1+1	B
$A_{1,16}$(“E_7” inv.)	1	1	1	
$A_{1,28} \subset G_2$	1	1	1	
$A_{2,3}$ (SC/$\subset SO(8)$)	2	2+3	$(0+1)+(0+1+(1^{*}+1^{***}))$	F
$A_{2,5} \subset SU(6)$	1	2	$1+1^{*}$	C
$A_{2,6}$ (SC)	2	2+1	$(0+1)+(1)$	E
$A_{2,9} \subset E_6$	3	2+2+2	$(1+0)+(1+0)+(0+0)$	H
$A_{3,2}$ (SC 2/$\subset SU(6)$)	1	4	1+1+1+1	G
$A_{3,4} \subset SO(15)$	1	4	$0+0+(1+2^{**})+(1+2^{**})$	B
$A_{9,2}$ (HSE*)	1	2	1 +1	
$B_{2,3} \subset SO(10)$	2	4+4	$(0+0+1+(1^{*}+1^{***}))+(0+0+1+1)$	D
$B_{2,7} \subset SO(14)$	2	4+4	$(0+0+1+(1^{*}+1^{***}))+(0+0+1+1)$	D
$B_{2,12} \subset E_8$	1	4	1	
$B_{12,2}$ (HSE)	1	2	1	
$C_{3,2} \subset SO(14)$	2	4+4	$(0+0+1+(1^{*}+1^{***}))+(0+0+1+1)$	D
$C_{3,4} \subset SO(21)$	1	16	$1+1+14\times 0$	B
$C_{4,3} \subset SO(27)$	1	16	$1+1+14\times 0$	B
$C_{7,2}$ (HSE)	2	4+4	$(0+0+1+(1^{*}+1^{***}))+(0+0+1+1)$	
$D_{7,3}$ (HSE*)	2	2+2	$(0+0)+(0+2)$	
$D_{7,3}$ (HSE)	2	2+2	$(0+0)+(1+1)$	
$C_{10,1}$ (HSE)	1	2	1 +1	
$D_{9,2}$ (HSE)	2	2+5	$(1+1)+(5\times 0)$	
$D_{9,2}$ (HSE*)	3	4+4+4	$(1+1+0+0)+(4\times 0)+(4\times 0)$	
$G_{2,3} \subset E_6$	2	2+2	$(0+1)+(0+1)$	A
$G_{2,4}$ (automorphism)	1	4	1	
$G_{2,4} \subset SO(14)$	2	4+4	$(0+0+1+(1^{*}+1^{***}))+(0+0+1+1^{*})$	D
$F_{4,3}$ (automorphism)	1	4	1	
$F_{4,3} \subset SO(26)$	2	4+4	$(0+0+1+(1^{*}+1^{***}))+(0+0+1+1^{*})$	D
$E_{6,4}$ (HSE*)	2	2+2	$(0+0)+(1+1^{*})$	
$E_{7,3}$ (HSE)	2	4+4	$2\times(0+0+1+1)$	
$E_{8,4}$ (automorphism)	1	1	1	