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An Uniqueness Result for a Class of Wiener-Space Valued Stochastic Differential Equations

M. J. OLIVEIRA
Universidade Aberta and Grupo de Física-Matemática da Universidade de Lisboa, Av. Prof. Gama Pinto, 2, P-1699 Lisboa Codex, Portugal. (e-mail: oliveira@alf1.cii.fc.ul.pt)

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Abstract. We prove a generalization of Bismut-Itô-Kunita formula to infinite dimensions and derive an uniqueness result for Wiener space valued processes which holds for a special class of Bernstein processes.

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1. Introduction

Let $(\mu, \mathbf{H}, \mathbf{X})$ be the classical Wiener space, namely $\mathbf{X}$ is the space of continuous functions $x: [0,1] \to \mathbb{R}^n$ such that $x(0) = 0$, μ the Wiener measure and $\mathbf{H}$ the corresponding Cameron–Martin space. Considering on a probability space $(\Omega, \mathcal{F}, P)$ a $\mathbf{X}$-valued semimartingale with respect to a $\mathbf{X}$-valued Brownian motion W_t, X, such that the image measure $X_t(P)$ has a density in $L^2(\mu)$, we prove in paragraph 3 a generalization of Bismut–Itô–Kunita formula [1, 2] for $F_t(X_t)$ where F is a Wiener functional

$$F: [0,T] \times \mathbf{X} \times \Omega \to \mathbf{X}$$

and, for each $x \in \mathbf{X}$, $F_{\cdot}(x)$ is another continuous semimartingale. This generalization will be essential to establish in section 4 the uniqueness result for Wiener space valued processes (Theorem 4.2). This theorem allows to establish the pathwise uniqueness for a particular class of Wiener space valued Bernstein processes (paragraph 5).

We mention that in infinite dimensions not many general results about existence and uniqueness for stochastic differential equations are available, particularly in the cases where the drift and diffusion coefficients are not smooth or are time-dependent. In the framework of Dirichlet forms, the situation where the coefficients are time-independent has been intensively studied (for example, [3]). To the class of Bernstein processes treated in the end of this paper these methods do not apply.

2. Sobolev Spaces for Banach Space Valued Wiener Functionals

In this paper we will work with the Sobolev spaces for Banach space valued Wiener functionals introduced by [4], which are constructed using the operator norm for the derivatives instead of the Hilbert–Schmidt norm. The corresponding Sobolev norms are strictly weaker than the usual Malliavin Sobolev norms. For the construction of these weaker Sobolev spaces we refer to [4].

Let **B** be a Banach space and $L_n(\mathbf{H};\mathbf{B})$ the Banach space of bounded n-linear **B**-valued operators defined on **H** with the norm

$$\|T\|_{Op} = \sup_{h_i\in\mathbf{H},\|h_i\|=1} \|T(h_1,\ldots,h_n)\|_{\mathbf{B}}.$$

Let $L_1^0(\mathbf{H};\mathbf{B})$ be the subspace of finite dimensional linear mappings defined on **H** with values in **B**, and let $\overline{L}_1^0(\mathbf{H};\mathbf{B})$ be its closure in $L_1(\mathbf{H};\mathbf{B})$. The spaces $\overline{L}_n^0(\mathbf{H};\mathbf{B})$ are defined recursively using the canonical identification,

$$L_n(\mathbf{H};\mathbf{B}) \longleftrightarrow L_1(\mathbf{H};L_{n-1}(\mathbf{H};\mathbf{B})).$$

Namely, $\overline{L}_n^0(\mathbf{H};\mathbf{B}) = \overline{L}_1^0(\mathbf{H};\overline{L}_{n-1}^0(\mathbf{H};\mathbf{B}))$.

In order to define the **H**-derivative of a **B**-valued functional, we consider the set $W^1(\mathbf{X};\mathbf{B})$ of the **B**-valued Wiener functionals f such that, for each $h\in\mathbf{H}$ there is a functional f_h verifying

(i) $f_h(x) = f(x)$ μ a.e. in x;

(ii) The functional $f_h(x+th)$ is strongly $\mathcal{B}_{\mathbf{X}}/\mathcal{B}_{\mathbf{B}}$-measurable and there exists a $L_1(\mathbf{H};\mathbf{B})$-valued strongly measurable functional Df for which the equality

$$f_h(x+th) = f_h(x) + \int_0^t Df(x+sh)h\,\mathrm{d}s,$$

is satisfied for every $t\in\mathbb{R}$ and μ almost everywhere in x.

The Wiener functional Df will be called the **H**-derivative of f and is uniquely defined up to a set of measure zero (cf. [5], Prop. 3.2). This defines the first **H**-derivative of elements in $W^1(\mathbf{X};\mathbf{B})$.

The construction of the higher order **H**-derivatives D^l is made by recurrence. For that, the following spaces of k-jets are introduced

$$B_0 := \mathbf{B},$$

$$B_k := \mathbf{B}\times L_1(\mathbf{H};\mathbf{B})\times\cdots\times L_k(\mathbf{H};\mathbf{B}),\quad k>0,$$

with the norms,

$$\|(S_0,S_1,\ldots,S_k)\|_{B_{k,p}} = \left(\|S_0\|_{\mathbf{B}}^p + \sum_{i=1}^k \|S_i\|_{Op}^p\right)^{1/p},\quad 1\leqslant p<+\infty.$$

Given $f \in W^1(\mathbf{X};\mathbf{B})$, the 1-jet of f is defined as the B_1-valued functional

$$j_1(f) := (f, Df).$$

Hence, for any $f \in W^1(\mathbf{X};\mathbf{B})$ such that $j_1(f) \in W^1(\mathbf{X};B_1)$ one can define the second derivative of f and, in a recursive way, the higher order derivatives and the sets

$$W^k(\mathbf{X};\mathbf{B}) := \{f \in W^{k-1}(\mathbf{X};\mathbf{B})\colon j_{k-1}(f) \in W^1(\mathbf{X};B_{k-1})\}.$$

By definition, given $f \in W^k(\mathbf{X};\mathbf{B})$, $D^k f$ is the $L_k(\mathbf{H};\mathbf{B})$-valued functional obtained by taking the $L_k(\mathbf{H};\mathbf{B})$ component of $j_1(j_{k-1}(f))$. For a $f \in W^k(\mathbf{X};\mathbf{B})$, we will denote $j_k(f) = (f, Df, \ldots, D^k f)$.

The weaker Sobolev spaces $\widetilde{W}_k^p(\mathbf{X};\mathbf{B})$ are defined, for each k, $p \geqslant 1$, by

$$\widetilde{W}_k^p(\mathbf{X};\mathbf{B}) := \left\{ f \in W^k(\mathbf{X};\mathbf{B})\colon \int_{\mathbf{X}} \|j_k(f)\|_{B_{k,p}}^p \,\mathrm{d}\mu < \infty \right\},$$

which is a Banach space for the norm

$$\|f\|_{k,p} := \left(\int_{\mathbf{X}} \|j_k(f)\|_{B_{k,p}}^p \,\mathrm{d}\mu \right)^{1/p}.$$

We observe that for each k, $p > 1$,

$$W_k^p(\mathbf{X};\mathbf{B}) \subset \widetilde{W}_k^p(\mathbf{X};\mathbf{B}),$$

where $W_k^p(\mathbf{X};\mathbf{B})$ are the Sobolev spaces introduced in [6], provided with the norm defined in [6] also denote here by $\|\cdot\|_{k,p}$.

3. A Bismut–Itô–Kunita Formula in Infinite Dimensions

In this paragraph we generalize the Bismut–Itô–Kunita formula [1, 2] for our Wiener space framework. This formula gives the rule for the composition of Wiener functionals and semimartingales with values on the Wiener space.

On a probability space $(\Omega, \mathcal{F}, P)$ provided with a complete right continuous filtration, consider a semimartingale on the Wiener space $\mathbf{X}$ of the form

$$X_t = X_0 + \int_0^t A_s \,\mathrm{d}s + \int_0^t (I + B_s) \,\mathrm{d}W_s,$$

where W is a $\mathbf{X}$-valued Brownian motion and $A \in L^4([0,T] \times \Omega; \mathbf{X})$, $B \in L^8([0,T] \times \Omega; \mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H}))$ are progressively measurable processes.

We shall assume the existence, for each $t \in [0,T]$, of a function $K_t\colon \mathbf{X} \to \mathbb{R}$ such that K as function of t and $x \in \mathbf{X}$ is square integrable and

$$Ef(X_t) = \int_{\mathbf{X}} f(x) K_t(x)\, \mathrm{d}\mu(x), \tag{3.1}$$

for all functionals f for which both sides of the last equality make sense. Assume that K_0 belongs to the $L^2(\mathbf{X})$ space.

Under these conditions we derive the following result.

THEOREM 3.1. *Let $F\colon [0,T] \times \mathbf{X} \times \Omega \to \mathbf{X}$ be a Wiener functional such that for each $x \in \mathbf{X}$ $F.(x)$ is a continuous semimartingale of the form*

$$F_t(x) = F_0(x) + \int_0^t a_s(x)\, \mathrm{d}s + \int_0^t (I + b_s(x))\, \mathrm{d}\widetilde{W}_s, \tag{3.2}$$

where

(i) $\widetilde{W}$ *is a* **X***-valued Brownian motion.*

(ii) *For* $\mu -$ a.e.x *in* $\mathbf{X}$, $a(x)\colon [0,T] \times \Omega \to \mathbf{X}$, $b(x)\colon [0,T] \times \Omega \to \mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H})$ *are progressively measurable processes.*

(iii) *For* $P -$ a.e.ω, $t \in [0,T]$, $a_t(w) \in \widetilde{W}_3^p(\mathbf{X};\mathbf{X})$, $b_t(w) \in W_3^p(\mathbf{X};\mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H}))$ *and the integrals*

$$E\int_0^T \|a_t\|_{3,p}^p\, \mathrm{d}t, \qquad E\int_0^T \|b_t\|_{3,p}^p\, \mathrm{d}t$$

are finite for each $2 \leqslant p < \infty$. *If*

(iv) *for each* t *and* x, $a_t(x) - x \in \mathbf{H}$;

(v) *for each* $0 \leqslant t \leqslant T$, $l = 1,2,3$, $D^l a_t(x) \in \overline{L}_l^0(\mathbf{H};\mathbf{X})$, μ-a.e. *in* x.

(vi) *Almost surely* $F_0 \in \cap_p \widetilde{W}_3^p(\mathbf{X};\mathbf{X})$ *and* $F_0 \in L^2(\Omega;\widetilde{W}_3^p)$ *for every* $p \geqslant 2$.

(vii) ∇F_0 *is the form* $I + H_0$ *with* H_0 *a* $\mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H})$*-valued functional,* $D^n F_0(x) \in \overline{L}_n^0(\mathbf{H};\mathbf{X})$ *for* $n = 1,2,3$, μ-a.e. *in* x,

then $F_t(X_t)$ *is a stochastic process with an almost surely continuous version given by the following equation*

$$\begin{aligned} F_t(X_t) \;=\; & F_0(X_0) + \int_0^t (I + b_s(X_s))\, \mathrm{d}\widetilde{W}_s \\ & + \int_0^t a_s(X_s)\, \mathrm{d}s + \int_0^t (\nabla F_s(X_s), A_s)\, \mathrm{d}s \\ & + \int_0^t (\nabla F_s(X_s), (I + B_s)\, \mathrm{d}W_s) \\ & + \frac{1}{2}\int_0^t \mathrm{Tr}((I + B_s)^* \nabla^2 F_s(X_s)(I + B_s))\, \mathrm{d}s \end{aligned}$$

$$+\int_0^t \nabla b_s(X_s)(I+B_s)\,\mathrm{d}\langle W,\widetilde{W}\rangle_s. \tag{3.3}$$

REMARK 3.2. This theorem generalizes the Itô formula shown in [7] for non random Wiener functionals F.

Proof. Let $\{e_i\}_{i=1}^{\infty}$ be an Hilbertian basis for **H**. For each n we denote by $E^{V_n}\cdot$ the conditional expectation on V_n, where V_n is the finite dimensional subspace of **H** generated by $\{e_1,\dots,e_n\}$, by $\Pi^n\colon \mathbf{X}\to\mathbb{R}^n$ the extension to **X** of the orthogonal projection of $\mathbf{H}\to\mathbb{R}^n$, and by $\Pi_n\cdot$ the orthogonal projection of the separable Hilbert space $\mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H})$ to $\mathbb{R}^n\times\mathbb{R}^n$.

For simplicity, denote by F_{nt}, a_{nt}, b_{nt}, respectively, the approximations $\Pi^n(E^{V_n}F_t)$, $\Pi^n(E^{V_n}a_t)$, $\Pi_n(E^{V_n}b_t)$, $t\in[0,T]$. It follows from (3.2) that

$$F_{nt}(x)=F_{n0}(x)+\int_0^t a_{ns}(x)\,\mathrm{d}s+\int_0^t (I+b_{ns}(x))\,\mathrm{d}\widetilde{W}_s. \tag{3.4}$$

Assuming the assumptions (i)–(vi), it follows that, for each t, almost surely F_t belongs to all spaces $\widetilde{W}_3^p(\mathbf{X};\mathbf{X})$, $p\geqslant 2$. Thus, by [8] and [4], we may conclude that

$$F_{nt},\ a_{nt}\in\bigcap_p W_3^p(V_n;V_n),\qquad b_{nt}\in\bigcap_p W_3^p(V_n;\mathbb{R}^n\times\mathbb{R}^n). \tag{3.5}$$

Furthermore, denoting generically F_t or a_t by f_t, $t\in[0,T]$, it holds by [8] and [4],

$$\begin{aligned}&\|\Pi^n(E^{V_n}f_t)\|_{3,p}\leqslant\|f_t\|_{3,p},\\&\|\Pi_n(E^{V_n}b_t)\|_{3,p}\leqslant\|b_t\|_{3,p},\quad\forall n\in\mathbb{N},\quad t\in[0,T],\end{aligned} \tag{3.6}$$

and the sequence $\{\Pi^n(E^{V_n}f_t)\}_n$ (resp., $\{\Pi_n(E^{V_n}b_t)\}_n$) converges to f_t (resp., b_t) in $\widetilde{W}_3^p$ (resp., W_3^p) for all integer $p\geqslant 2$.

Conditions (3.5) and the Sobolev imersion theorem enables us to rewrite each function F_{nt}, a_{nt}, b_{nt}, $t\in[0,T]$, on the form, respectively

$$\begin{aligned}&F_{nt}(x)=f_{nt}(x_1,\dots,x_n),\\&a_{nt}(x)=\alpha_{nt}(x_1,\dots,x_n),\\&b_{nt}(x)=\beta_{nt}(x_1,\dots,x_n),\quad x_i=(x,e_i)_{\mathbf{H}},\ x\in\mathbf{X},\ t\in[0,T],\end{aligned} \tag{3.7}$$

for P-almost everywhere C^2 functions on $\mathbb{R}^n$, f_{nt}, $\alpha_{nt}\colon\mathbb{R}^n\to\mathbb{R}^n$, $\beta_{nt}\colon\mathbb{R}^n\to\mathbb{R}^n\times\mathbb{R}^n$. Consequently, we are in conditions to apply the Bismut–Itô–Kunita formula ([1, 2]) to the process (3.4). We obtain

$$\begin{aligned}F_{nt}(X_t)\;=\;&F_{n0}(X_0)+\int_0^t (I+b_{ns}(X_s))\,\mathrm{d}\widetilde{W}_s+\int_0^t a_{ns}(X_s)\,\mathrm{d}s\\&+\int_0^t(\nabla F_{ns}(X_s),A_s)\,\mathrm{d}s+\int_0^t(\nabla F_{ns}(X_s),(I+B_s)\,\mathrm{d}W_s)\end{aligned}$$

$$+\frac{1}{2}\int_0^t \mathrm{Tr}((I+B_s)^*\nabla^2 F_{ns}(X_s)(I+B_s))\,\mathrm{d}s$$

$$+\int_0^t \nabla b_{ns}(X_s)(I+B_s)\,\mathrm{d}\langle W,\widetilde{W}\rangle_s. \tag{3.8}$$

Hence, from (3.8), we can deduce,

$$\begin{aligned}
&\left(E\left[\sup_{0\leqslant t\leqslant T}\|F_{nt}(X_t)-F_{mt}(X_t)\|^2_{\mathbf{X}}\right]\right)^4\\
&\leqslant k\Bigg\{(E\|F_{n0}(X_0)-F_{m0}(X_0)\|^2_{\mathbf{X}})^4\\
&\quad+\left(E\left[\sup_{0\leqslant t\leqslant T}\left\|\int_0^t(b_{ns}(X_s)-b_{ms}(X_s))\,\mathrm{d}\widetilde{W}_s\right\|^2_{\mathbf{H}}\right]\right)^4\\
&\quad+\left(E\left[\sup_{0\leqslant t\leqslant T}\left\|\int_0^t(a_{ns}(X_s)-a_{ms}(X_s))\,\mathrm{d}s\right\|^2_{\mathbf{X}}\right]\right)^4\\
&\quad+\left(E\left[\sup_{0\leqslant t\leqslant T}\left\|\int_0^t(\nabla F_{ns}(X_s)-\nabla F_{ms}(X_s),A_s)\,\mathrm{d}s\right\|^2\right]\right)^4\\
&\quad+\left(E\left[\sup_{0\leqslant t\leqslant T}\left\|\int_0^t(\nabla F_{ns}(X_s)-\nabla F_{ms}(X_s),\mathrm{d}W_s)\right\|^2\right]\right)^4\\
&\quad+\left(E\left[\sup_{0\leqslant t\leqslant T}\left\|\int_0^t(\nabla F_{ns}(X_s)-\nabla F_{ms}(X_s),B_s\,\mathrm{d}W_s)\right\|^2\right]\right)^4\\
&\quad+\frac{1}{2^4}\left(E\left[\sup_{0\leqslant t\leqslant T}\left\|\int_0^t\mathrm{Tr}[(I+B_s)^*(\nabla^2F_{ns}(X_s)\right.\right.\right.\\
&\qquad\left.\left.\left.-\nabla^2F_{ms}(X_s))(I+B_s)]\,\mathrm{d}s\right\|^2\right]\right)^4\\
&\quad+\left(E\left[\sup_{0\leqslant t\leqslant T}\left\|\int_0^t(\nabla b_{ns}(X_s)\right.\right.\right.\\
&\qquad\left.\left.\left.-\nabla b_{ms}(X_s))(I+B_s)\,\mathrm{d}\langle W,\widetilde{W}\rangle_s\right\|^2\right]\right)^4\Bigg\},
\end{aligned}\tag{3.9}$$

where k is a positive constant.

Next, we will estimate the norms envolved in the right-hand side of the inequality (3.9). For example, considering the first, the second and the penultimate terms of the right-hand side of (3.9) we have, respectively,

$$\begin{aligned}
&E\|F_{n0}(X_0)-F_{m0}(X_0)\|_{\mathbf{X}}^2\\
&\quad\leqslant \left(\int_{\mathbf{X}}(K_0(x))^2\,\mathrm{d}\mu(x)\right)^{1/2}\cdot\left(\int_{\mathbf{X}}E\|F_{n0}(x)-F_{m0}(x)\|_{\mathbf{X}}^4\,\mathrm{d}\mu(x)\right)^{1/2}\\
&\quad\leqslant \|K_0\|_{L^2(\mathbf{X})}(E\|F_{n0}-F_{m0}\|_{L^8(\mathbf{X};\mathbf{X})}^8)^{1/4}\\
&\quad\leqslant c^I(E\|F_{n0}-F_{m0}\|_{\widetilde{W}_3^8(\mathbf{X};\mathbf{X})}^8)^{1/4}\quad (c^I \text{ independent of } n \text{ and } m);
\end{aligned}$$

$$\begin{aligned}
&E\left[\sup_{0\leqslant t\leqslant T}\left\|\int_0^t(b_{ns}(X_s)-b_{ms}(X_s))\,\mathrm{d}\widetilde{W}_s\right\|_{\mathbf{H}}^2\right]\\
&\quad\leqslant 4E\int_0^T\|b_{nt}(X_t)-b_{mt}(X_t)\|_{\mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H})}^2\,\mathrm{d}t\\
&\quad= 4\int_0^T\int_{\mathbf{X}}E\|b_{nt}(x)-b_{mt}(x)\|_{\mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H})}^2K_t(x)\,\mathrm{d}\mu(x)\,\mathrm{d}t\\
&\quad\leqslant 4\left(\int_0^T\int_{\mathbf{X}}E\|b_{nt}(x)-b_{mt}(x)\|_{\mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H})}^4\,\mathrm{d}\mu(x)\,\mathrm{d}t\right)^{1/2}\cdot\\
&\qquad\cdot\left(\int_0^T\int_{\mathbf{X}}(K_t(x))^2\,\mathrm{d}\mu(x)\,\mathrm{d}t\right)^{1/2}\\
&\quad\leqslant c'\left(\int_0^T E\|b_{nt}-b_{mt}\|_{L^8(\mathbf{X};\mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H}))}^8\,\mathrm{d}t\right)^{1/4}\\
&\quad\leqslant c^{II}\left(\int_0^T E\|b_{nt}-b_{mt}\|_{W_3^8(\mathbf{X};\mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H}))}^8\,\mathrm{d}t\right)^{1/4}\\
&\qquad (c^{II} \text{ independent of } n \text{ and } m),
\end{aligned}$$

by the Doob and Hölder inequalities; and

$$\begin{aligned}
&E\left[\sup_{0\leqslant t\leqslant T}\left\|\int_0^t\mathrm{Tr}[(I+B_s)^*(\nabla^2F_{ns}(X_s)-\nabla^2F_{ms}(X_s))(I+B_s)]\,\mathrm{d}s\right\|^2\right]\\
&\quad\leqslant 4TE\int_0^T(\|B_t\|^4+4\|B_t\|^2+1)\|\nabla^2F_{nt}(X_t)-\nabla^2F_{mt}(X_t)\|^2\,\mathrm{d}t
\end{aligned}$$

$$\leqslant 4T\int_0^T E\|\nabla^2 F_{nt}(X_t)-\nabla^2 F_{mt}(X_t)\|^2\,\mathrm{d}t$$

$$+4T\left(\int_0^T E\|\nabla^2 F_{nt}(X_t)-\nabla^2 F_{mt}(X_t)\|^4\,\mathrm{d}t\right)^{1/2}\cdot$$

$$\cdot\left\{\left(E\int_0^T \|B_t\|^8\,\mathrm{d}t\right)^{1/2}+4\left(E\int_0^T \|B_t\|^4\,\mathrm{d}t\right)^{1/2}\right\}$$

$$\leqslant 4T\int_0^T\int_{\mathbf{X}} E\|\nabla^2 F_{nt}(x)-\nabla^2 F_{mt}(x)\|^2 K_t(x)\,\mathrm{d}\mu(x)\,\mathrm{d}t$$

$$+c\left(\int_0^T\int_{\mathbf{X}} E\|\nabla^2 F_{nt}(x)-\nabla^2 F_{mt}(x)\|^4 K_t(x)\,\mathrm{d}\mu(x)\,\mathrm{d}t\right)^{1/2}$$

$$\leqslant c'\left\{\left(\int_0^T\int_{\mathbf{X}} E\|\nabla^2 F_{nt}(x)-\nabla^2 F_{mt}(x)\|^4\,\mathrm{d}\mu(x)\,\mathrm{d}t\right)^{1/2}\right.$$

$$\left.+\left(\int_0^T\int_{\mathbf{X}} E\|\nabla^2 F_{nt}(x)-\nabla^2 F_{mt}(x)\|^8\,\mathrm{d}\mu(x)\,\mathrm{d}t\right)^{1/4}\right\}$$

$$\leqslant c^{III}\left(\int_0^T E\|F_{nt}-F_{mt}\|^8_{\widetilde{W}_3^8(\mathbf{X};\mathbf{X})}\,\mathrm{d}t\right)^{1/4}$$

$$(c^{III}\text{ independent of } n \text{ and } m),$$

applying the Hölder inequality.

The remaining terms of (3.9) can be estimated analogously. Thus we conclude from (3.9) that

$$\left(E\left[\sup_{0\leqslant t\leqslant T}\|F_{nt}(X_t)-F_{mt}(X_t)\|^2_{\mathbf{X}}\right]\right)^4$$

$$\leqslant c\left\{E\int_0^T \|a_{nt}-a_{mt}\|^8_{\widetilde{W}_3^8(\mathbf{X};\mathbf{X})}\,\mathrm{d}t\right.$$

$$+E\int_0^T \|b_{nt}-b_{mt}\|^8_{W_3^8(\mathbf{X};\mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H}))}\,\mathrm{d}t$$

$$+E\int_0^T \|F_{nt}-F_{mt}\|^8_{\widetilde{W}_3^8(\mathbf{X};\mathbf{X})}\,\mathrm{d}t$$

$$+E\|F_{n0}-F_{m0}\|^8_{\widetilde{W}^8_3(\mathbf{X};\mathbf{X})}\Bigg\}, \tag{3.10}$$

for a positive real number c independent of n and m.

Since F_{nt} $(=\Pi^n(E^{V_n}F_t))$ converges to F_t in $\widetilde{W}^8_3(\mathbf{X};\mathbf{X})$, $t\in[0,T]$, and by (3.6) it holds

$$\begin{aligned}\|\Pi^n(E^{V_n}F_t)-F_t\|^8_{3,8} &\leqslant c\|\Pi^n(E^{V_n}F_t)\|^8_{3,8}+c\|F_t\|^8_{3,8}\\ &\leqslant 2c\|F_t\|^8_{3,8},\ \ \forall n\in\mathbb{N},\end{aligned}$$

for a positive constant c, with

$$\begin{aligned}&E\int_0^T\|F_t\|^8_{3,8}\,\mathrm{d}t\\ &\quad\leqslant k\left\{E\|F_0\|^8_{3,8}+E\int_0^T\|a_t\|^8_{3,8}\,\mathrm{d}t+E\int_0^T\|I+b_t\|^8_{3,8}\,\mathrm{d}t\right\},\end{aligned}$$

we may conclude by the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty}E\int_0^T\|F_{nt}-F_t\|^8_{\widetilde{W}^8_3(\mathbf{X};\mathbf{X})}\,\mathrm{d}t=0.$$

Therefore, F_n, $n\in\mathbb{N}$, is a Cauchy sequence in $L^8(\Omega\times[0,T];\widetilde{W}^8_3(\mathbf{X};\mathbf{X}))$, *i.e.*,

$$\lim_{n,m\to\infty}E\int_0^T\|F_{nt}-F_{mt}\|^8_{\widetilde{W}^8_3(\mathbf{X};\mathbf{X})}\,\mathrm{d}t=0. \tag{3.11}$$

Analogously, we derive the following equalities

$$\begin{aligned}&\lim_{n,m\to\infty}E\int_0^T\|a_{nt}-a_{mt}\|^8_{\widetilde{W}^8_3(\mathbf{X};\mathbf{X})}\,\mathrm{d}t=0,\\ &\lim_{n,m\to\infty}E\int_0^T\|b_{nt}-b_{mt}\|^8_{W^8_3(\mathbf{X};\mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H}))}\,\mathrm{d}t=0,\\ &\lim_{n,m\to\infty}E\|F_{n0}-F_{m0}\|^8_{\widetilde{W}^8_3(\mathbf{X};\mathbf{X})}=0.\end{aligned} \tag{3.12}$$

Thus, by (3.10), (3.11) and (3.12) we obtain

$$\lim_{n,m\to\infty}E\left[\sup_{0\leqslant t\leqslant T}\|F_{nt}(X_t)-F_{mt}(X_t)\|^2_{\mathbf{X}}\right]=0,$$

which implies the existence of a subsequence of F_n (still denoted by F_n), such that

$$\lim_{n,m\to\infty}\sup_{0\leqslant t\leqslant T}\|F_{nt}(X_t)-F_{mt}(X_t)\|_{\mathbf{X}}=0\ \text{ q.s. }\ P;$$

these means, by Cauchy criteria, almost everywhere the sequence $F_{nt}(X_t)$, $n \in \mathbb{N}$, converges uniformly in the interval $[0,T]$.

Finally, since X is a stochastic process with continuous sample paths and F_{nt} is a continuous semimartingale – (3.4) – which admits a representation as a C^2 function (as we observed on (3.7)), it follows that the application $t \mapsto F_{nt}(X_t)$ is continuous, which enables us to conclude (by the uniform convergence) the continuity almost everywhere of the application $t \mapsto F_t(X_t)$. □

REMARK 3.3. The estimates we obtained for each term of the right-hand side of (3.9) also enable us to conclude that each term of (3.9) converges uniformly on $[0,T]$ to the correspondent term of the right-hand side of (3.3).

4. An Uniqueness Result

We shall use the theorem obtained in last paragraph to derive an uniqueness result for solutions of Wiener space valued stochastic differential equations. This will be useful to study the Bernstein processes in next section.

Given $a\colon [0,T] \times \mathbf{X} \to \mathbf{X}$, $b\colon [0,T] \times \mathbf{X} \to \mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H})$ such that, for each $t \in [0,T]$,

$$a_t \in \bigcap_p \widetilde{W}_3^p(\mathbf{X};\mathbf{X}), \qquad b_t \in \bigcap_p W_4^p(\mathbf{X};\mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H})), \tag{4.1}$$

and, for every $2 \leqslant p < \infty$, the integrals

$$\int_0^T \|a_t\|_{3,p}^p \,\mathrm{d}t, \qquad \int_0^T \|b_t\|_{4,p}^p \,\mathrm{d}t \tag{4.2}$$

are finite, consider the following stochastic differential equation

$$\mathrm{d}X_t = a(t,X_t)\,\mathrm{d}t + (I+b)(t,X_t)\,\mathrm{d}W_t. \tag{4.3}$$

We assume that, μ-a.e. $x \in \mathbf{X}$, exists a weak solution,

$$((X_{1t}(x), W), (\Omega, \mathcal{F}, P), \{\mathcal{F}_t^1\}), \tag{4.4}$$

of (4.3) for the initial condition $X(0) = x$. We also assume the existence, for each $t \in [0,T]$, of a function $K_{1t}\colon \mathbf{X} \to \mathbb{R}$ such that

$$K_{1\cdot}(\cdot) \in L^2([0,T] \times \mathbf{X};\mathbb{R}), \tag{4.5}$$

and

$$\int_{\mathbf{X}} Ef(X_{1t}(x))\,\mathrm{d}\mu(x) = \int_{\mathbf{X}} f(x)K_{1t}(x)\,\mathrm{d}\mu(x), \tag{4.6}$$

for all functionals f for which both sides of (4.6) make sense. With respect to (4.4) we also shall assume the existence of two subsets of $\mathbf{X}$, C and D, of measure one, such that

$$P \text{ q.s. } \omega, C \ni x \mapsto X_{1t}(x) \in D \text{ is a bijection, for each } t \in [0,T]. \tag{4.7}$$

We remark that this assumption enables us to think about the inverse stochastic process $X_{1t}^{-1}(x)$. About this one assumes

$$\begin{aligned}&X_{1t}^{-1}(x) \text{ is a diffusion process, say } \mathrm{d}X_{1t}^{-1}(x)\\ &\quad = A(t,x)\,\mathrm{d}t + B(t,x)\,\mathrm{d}W_t;\end{aligned} \tag{4.8}$$

with,

$$\begin{aligned}&A_t \in \cap_p \widetilde{W}_3^p(\mathbf{X};\mathbf{X})\, P \text{ a.e.,} \quad \text{and}\\ &E\int_0^T \|A_t\|_{3,p}^p\,\mathrm{d}t < \infty \text{ for every } 2 \leqslant p < \infty;\end{aligned} \tag{4.9}$$

$$\forall t \in [0,T], \quad n = 1,2,3, \quad D^n A_t \in \overline{L}_n^0(\mathbf{H};\mathbf{X}); \tag{4.10}$$

$$\begin{aligned}&B \text{ is the form } B = I + B' \text{ with}\\ &B_t' \in \cap_p W_3^p(\mathbf{X};\mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H}))P \text{ a.e.,} \quad \text{and}\\ &E\int_0^T \|B_t'\|_{3,p}^p\,\mathrm{d}t < \infty \text{ for every } 2 \leqslant p < \infty;\end{aligned} \tag{4.11}$$

$$\begin{aligned}&\text{For each } t, \nabla X_{1t}^{-1} \text{ is the form } I + L_t, \text{ with}\\ &L_t \text{ a } \mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H})\text{–valued functional;}\end{aligned} \tag{4.12}$$

For each t, exists a real function K_t square integrable on $[0,T] \times \mathbf{X}$ such that

$$\int_{\mathbf{X}} Ef(X_{1t}^{-1}(x))\,\mathrm{d}\mu(x) = \int_{\mathbf{X}} f(x)K_t(x)\,\mathrm{d}\mu(x), \tag{4.13}$$

for every functional f for which this equality make sense.

PROPOSITION 4.1. *Under the conditions* (4.1)–(4.2), (4.5)–(4.13), μ-a.e. $x \in \mathbf{X}$,

$$((X_{1t}^{-1}(x), W), (\Omega, \mathcal{F}, P), \{\mathcal{F}_t^1\})$$

is the unique weak solution of the stochastic differential equation

$$\begin{aligned}\mathrm{d}Y_t(x) \;=\; & -(\nabla Y_t(x), a(t,x))\,\mathrm{d}t + (\nabla Y_t(x), \nabla b(t,x)(I + b(t,x)))\,\mathrm{d}t\\ & +\tfrac{1}{2}\mathrm{Tr}((I+b)^*(t,x)\nabla^2 Y_t(x)(I+b)(t,x))\,\mathrm{d}t\\ & -(\nabla Y_t(x), (I+b)(t,x)\,\mathrm{d}W_t),\end{aligned} \tag{4.14}$$

for the initial condition $Y(0)(x) = x$.

Proof. μ-almost every $x \in \mathbf{X}$, let $((Z_t(x), W), (\Omega, \mathcal{F}, P), \{\mathcal{F}_t^1\})$, be a weak solution of (4.14) for the initial condition $Y(0)(x) = x$. As usually, we denote this solution briefly by $Z_t(x)$.

Suppose $Z_t(x)$ satisfy all the hypothesis of the Theorem 3.1 with the condition (3.1) replaced by (4.6). Thus, by application of the Theorem 3.1 we get

$$\begin{aligned}
&\mathrm{d}(Z_t(X_{1t}(x))) \\
&\quad = -(\nabla Z_t(X_{1t}(x)), a(t, X_{1t}(x)))\,\mathrm{d}t \\
&\qquad +(\nabla Z_t(X_{1t}(x)), \nabla b(t, X_{1t}(x))(I+b)(t, X_{1t}(x)))\,\mathrm{d}t \\
&\qquad +\tfrac{1}{2}\mathrm{Tr}((I+b)^*(t, X_{1t}(x))\nabla^2 Z_t(X_{1t}(x))(I+b)(t, X_{1t}(x)))\,\mathrm{d}t \\
&\qquad -(\nabla Z_t(X_{1t}(x)), (I+b)(t, X_{1t}(x))\,\mathrm{d}W_t) \\
&\qquad +(\nabla Z_t(X_{1t}(x)), a(t, X_{1t}(x)))\,\mathrm{d}t \\
&\qquad +(\nabla Z_t(X_{1t}(x)), (I+b)(t, X_{1t}(x))\,\mathrm{d}W_t) \\
&\qquad +\tfrac{1}{2}\mathrm{Tr}((I+b)^*(t, X_{1t}(x))\nabla^2 Z_t(X_{1t}(x))(I+b)(t, X_{1t}(x)))\,\mathrm{d}t \\
&\qquad -\mathrm{Tr}((I+b)^*(t, X_{1t}(x))\nabla^2 Z_t(X_{1t}(x))(I+b)(t, X_{1t}(x)))\,\mathrm{d}t \\
&\qquad -(\nabla Z_t(X_{1t}(x)), \nabla b(t, X_{1t}(x))(I+b)(t, X_{1t}(x)))\,\mathrm{d}t \\
&\quad = 0, \quad \mu\,\text{a.e.}\,x;
\end{aligned}$$

which enables us to conclude

$$\forall t \in [0, T], \quad \mu\,\text{a.e.}\,x, Z_t \circ X_{1t}(x) = x, \text{ a.e. } P. \tag{4.15}$$

since $Z_0(X_{10}(x)) = x$, μ a.e. $x \in \mathbf{X}$. Hence, given the existence of the inverse process $X_{1t}^{-1}(x)$, it follows from (4.15)

$$\forall t \in [0, T], \quad \mu\,\text{a.e.}\,x, Z_t(x) = X_{1t}^{-1}(x), \text{ a.e. } P.$$

Summarizing what we have been proved we conclude μ a.e. x in $\mathbf{X}$ there exists at most one weak solution of (4.14) for the initial condition $Y(0)(x) = x$. Furthermore, if such solution exists, then necessarilly it will be $((X_{1t}^{-1}(x), W), (\Omega, \mathcal{F}, P), \{\mathcal{F}_t^1\})$. Thus, it remains to prove that $X_{1t}^{-1}(x)$ satisfy the Equation (4.14) to complete this proof.

Applying once more the Theorem 3.1 we will get

$$\begin{aligned}
0 &= \mathrm{d}(X_{1t}^{-1}(X_{1t}(x))) \\
&= A(t, X_{1t}(x))\,\mathrm{d}t + B(t, X_{1t}(x))\,\mathrm{d}W_t \\
&\quad +(\nabla X_{1t}^{-1}(X_{1t}(x)), (I+b)(t, X_{1t}(x))\,\mathrm{d}W_t) \\
&\quad +(\nabla X_{1t}^{-1}(X_{1t}(x)), a(t, X_{1t}(x)))\,\mathrm{d}t \\
&\quad +\tfrac{1}{2}\mathrm{Tr}((I+b)^*(t, X_{1t}(x))\nabla^2 X_{1t}^{-1}(X_{1t}(x))(I+b)(t, X_{1t}(x)))\,\mathrm{d}t \\
&\quad +(\nabla B(t, X_{1t}(x))(I+b)(t, X_{1t}(x)))\,\mathrm{d}t,
\end{aligned}$$

which enables us to conclude, by the nullity of the coefficients drift and dispersion of the stochastic differential equation on the right-hand side of the above sequence of equalities,

$$A(t,x) = -(\nabla X_{1t}^{-1}(x), a(t,x)) + (\nabla X_{1t}^{-1}(x), \nabla b(t,x)(I+b(t,x))) + \tfrac{1}{2}\mathrm{Tr}((I+b)^*(t,x)\nabla^2 X_{1t}^{-1}(x)(I+b)(t,x)),$$

$$B(t,x)(h) = -(\nabla X_{1t}^{-1}(x), (I+b)(t,x)(h)), \quad h \in \mathbf{H},$$

i.e., $X_{1t}^{-1}(x)$ is a solution of the Equation (4.14). □

With this result we can now prove the above mentioned uniqueness result.

THEOREM 4.2. *Given* a, b *satisfying* (4.1)–(4.2)*, consider the stochastic differential equation*

$$\mathrm{d}X_t = a(t,X_t)\,\mathrm{d}t + (I+b)(t,X_t)\,\mathrm{d}W_t. \tag{4.16}$$

For almost every $x \in \mathbf{X}$*, let* $((X_{1t}(x), W), (\Omega, \mathcal{F}, P), \{\mathcal{F}_t^1\})$ *be a weak solution of the equation* (4.16) *for the initial condition* $X(0) = x$ *on the conditions* (4.5)–(4.13)*. If* μ a.e. $x \in \mathbf{X}$*,* $((X_{2t}(x), W), (\Omega, \mathcal{F}, P), \{\mathcal{F}_t^2\})$ *is another solution of the Equation* (4.16) *for the same initial condition such that for each* $t \in [0,T]$ *there exists a function* $K_{2t}\colon \mathbf{X} \to \mathbb{R}$ *square integrable on* $[0,T] \times \mathbf{X}$ *in such a way that*

$$\int_{\mathbf{X}} Ef(X_{2t}(x))\,\mathrm{d}\mu(x) = \int_{\mathbf{X}} f(x) K_{2t}(x)\,\mathrm{d}\mu(x),$$

for all functionals f *for which both sides of this equality make sense, then*

$$P[X_{1t}(x) = X_{2t}(x), \ \forall t \in [0,T]] = 1,$$

i.e., the pathwise uniqueness holds for the Equation (4.16).

Proof. By the proposition above and by application of the Theorem 3.1 we have

$$\begin{aligned}
&\mathrm{d}(X_{1t}^{-1}(X_{2t}(x)))\\
&\quad = -(\nabla X_{1t}^{-1}(X_{2t}(x)), a(t, X_{2t}(x)))\,\mathrm{d}t\\
&\qquad +(\nabla X_{1t}^{-1}(X_{2t}(x)), \nabla b(t, X_{2t}(x))(I+b)(t, X_{2t}(x)))\,\mathrm{d}t\\
&\qquad +\tfrac{1}{2}\mathrm{Tr}((I+b)^*(t, X_{2t}(x))\nabla^2 X_{1t}^{-1}(X_{2t}(x))(I+b)(t, X_{2t}(x)))\,\mathrm{d}t\\
&\qquad -(\nabla X_{1t}^{-1}(X_{2t}(x)), (I+b)(t, X_{2t}(x))\,\mathrm{d}W_t)\\
&\qquad +(\nabla X_{1t}^{-1}(X_{2t}(x)), a(t, X_{2t}(x)))\,\mathrm{d}t\\
&\qquad +(\nabla X_{1t}^{-1}(X_{2t}(x)), (I+b)(t, X_{2t}(x))\,\mathrm{d}W_t)
\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1}{2}\mathrm{Tr}((I+b)^*(t,X_{2t}(x))\nabla^2 X_{1t}^{-1}(X_{2t}(x))(I+b)(t,X_{2t}(x)))\,\mathrm{d}t\\
&-\mathrm{Tr}((I+b)^*(t,X_{2t}(x))\nabla^2 X_{1t}^{-1}(X_{2t}(x))(I+b)(t,X_{2t}(x)))\,\mathrm{d}t\\
&-(\nabla X_{1t}^{-1}(X_{2t}(x)),\nabla b(t,X_{2t}(x))(I+b)(t,X_{2t}(x)))\,\mathrm{d}t\\
&= 0,\ \ \mu\,\text{a.e.}\,x,
\end{aligned}
$$

which enables us to conclude, by $X_{10}^{-1}(X_{20}(x)) = x$ μ almost everywhere in x,

$$\forall t\in[0,T],\ \ \mu\,\text{a.e.}\,x,\ \ X_{1t}(x) = X_{2t}(x)\,\text{a.e.}\,P.$$

Thus, for all $t\in[0,T]$,

$$\int_{\mathbf{X}} E\|X_{1t}(x) - X_{2t}(x)\|_{\mathbf{X}}\,\mathrm{d}\mu(x) = 0,$$

which implies,

$$
\begin{aligned}
&\forall t\in[0,T]\exists A_t\subseteq\mathbf{X}, B_t\subseteq\Omega\colon \mu(\mathbf{X}\setminus A_t) = P(\Omega\setminus B_t) = 0 \quad\text{and}\\
&X_{1t}(x)(\omega) = X_{2t}(x)(\omega),\ \ \forall x\in A_t,\ \ \omega\in B_t.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
&\forall t\in[0,T]\cap\mathbb{Q},\ \ X_{1t}(x)(\omega) = X_{2t}(x)(\omega),\\
&\quad\forall x\in \bigcap_{t\in[0,T]\cap\mathbb{Q}} A_t,\quad \omega\in\bigcap_{t\in[0,T]\cap\mathbb{Q}} B_t;
\end{aligned}
$$

following by the continuity of the processes $X_{1t}(x)$, $X_{2t}(x)$,

$$\mu\,\text{a.e.}\,x\in\mathbf{X},\ \ P[X_{1t}(x) = X_{2t}(x),\ \ \forall t\in[0,T]] = 1,$$

as we wanted to prove. □

5. Bernstein Processes

In this section we shall apply an adaptation of Theorem 4.2 to prove the uniqueness of some Bernstein processes taking values on the Wiener space. These are processes Z_t, $t\in[-1,1]$, which are weak solutions of the forward Itô stochastic differential equation,

$$\mathrm{d}Z_t = \mathrm{d}W_t - (Z_t - \nabla\log\eta_t(Z_t))\,\mathrm{d}t,\ \ t\in[0,1], \tag{5.1}$$

for the initial probability distribution

$$P(Z_0\in\Gamma) = \int_\Gamma \eta_0\eta_0^*\,\mathrm{d}\mu,\ \ \Gamma\subset\mathbf{X},$$

where W is a **X**-valued Brownian motion adapted to the increasing filtration of the past events $\mathcal{P}_t$, and $\eta_t = e^{-(1-t)H}\theta$, with $t \leqslant 1$, is a solution of the infinite dimensional forward heat equation $\partial\eta_t/\partial t = H\eta_t$ associated to the Hamiltonian $H = -\mathcal{L} + V$, $\mathcal{L}$ being the Ornstein–Uhlenbeck operator on the Wiener space (cf. [6], [9]) and V a functional potential. The probability density of Z at a fixed time $t \in [0, 1]$ is given by $\eta_t\eta_t^* \,\mathrm{d}\mu$. In [10] the existence of such processes, *i.e.*, of weak solutions of (5.1), was proved. The conditions assumed in [10] were

(A.1) V is a positive Wiener functional belonging to the space $W_2^{4p}(\mathbf{X}; \mathbb{R})$, for some $2 \leqslant p < +\infty$;
(A.2) θ is a bounded positive functional belonging to $W_2^{2p}(\mathbf{X}; \mathbb{R})$ and such that $\log\theta \in L^2(\mathbf{X})$.

Applying to the time-reversible property of the Bernstein processes (cf. [10], [11]), these are also solutions of the backward Itô stochastic differential equation

$$\mathrm{d}_*Z_t = \mathrm{d}_*W_t^* + (Z_t - \nabla\log\eta_t^*(Z_t))\,\mathrm{d}t, \quad t \in [-1, 0], \tag{5.2}$$

where W^* is a Brownian motion adapted to the decreasing filtration of the future events $\mathcal{F}_t$, d_* denotes the backward differentiation, and $\eta_t^* = e^{-(t+1)H}\theta^*$, with $t \geqslant -1$, is a solution of the infinite dimensional backward heat equation $-\partial\eta_t^*/\partial t = H\eta_t^*$. An additional condition is assumed in [10]

(A.3) θ^* is a bounded positive functional belonging to $W_2^{2p}(\mathbf{X}; \mathbb{R})$ and such that $\log\theta^* \in L^2(\mathbf{X})$.

In this case, we are dealing with a process defined as a weak solution of a stochastic differential equation for a initial distribution which is not concentrated on a given point of **X**. So, in order to apply Theorem 4.2, we must consider a more general notion of inverse process. Let us denote by $X_t(\rho)$ the probability distribution of a process X at time t with initial probability density ρ. A process X_t is defined as inverse of the Bernstein process Z_t if for each fixed time t, the probability distribution of X at time t is $\eta_0\eta_0^*\,\mathrm{d}\mu$ having as initial probability distribution $\eta_t\eta_t^*\,\mathrm{d}\mu$. Using the notation introduced above, this means,

$$X_t(Z_t(\eta_0\eta_0^*\,\mathrm{d}\mu)) = X_t(\eta_t\eta_t^*\,\mathrm{d}\mu) = \eta_0\eta_0^*\,\mathrm{d}\mu$$

or, in an equivalent way and using the same notation, $X_t(Z_t) = I$ in law.

With this definition, we have the following result.

LEMMA 5.1. *The inverse of the Bernstein process* Z_t, $t \in [-1, 1]$ *exists and is a solution of the backward stochastic differential equation*

$$\mathrm{d}_*X_t = \mathrm{d}_*W_t^* + (X_t - \nabla\log\varphi_t^*(X_t))\,\mathrm{d}t, \quad t \in [-1, 0],$$

and the forward stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}W_t - (X_t - \nabla\log\varphi_t(X_t))\,\mathrm{d}t, \quad t \in [0, 1],$$

for the initial probability density $\varphi_0\varphi_0^*\,\mathrm{d}\mu$, *where* $\varphi_t^* = e^{-(1+t)H}\theta$, *with* $t \geqslant -1$, *and* $\varphi_t = e^{-(1-t)H}\theta^*$, *with* $t \leqslant 1$. *In particular,* $Z_t^{-1} = Z_{-t}$.

Proof. The result follows from the time-reversible property of the Bernstein processes. □

We observe that in the case of Bernstein processes, if conditions (4.1), (4.2), (4.9) and (4.11) are verified, the proof of Theorem 4.2 (adapted to this case) enables us to conclude that given Z'_t, $t \in [0,1]$, another solution of (5.1) for the initial density $\eta_0\eta_0^*\,\mathrm{d}\mu$, then

$$\mu[Z_t = Z'_t,\ \forall t \in [0,1]] = 1,$$

that is, the pathwise uniqueness for the Equation (5.1) holds. Hence, to be able to apply this result it remains to prove that the assumptions (4.1)–(4.2), (4.9) and (4.11) are satisfied. We devote the rest of this section to verify (4.1)–(4.2), (4.9) and (4.11), namely to prove that the integral

$$\int_{-1}^{1} \|\nabla \log \eta_t\|^p_{W_3^p(\mathbf{X};\mathbf{H})}\,\mathrm{d}t$$

is finite, for each $2 \leqslant p < \infty$, which is enough for our purpose. In order to show this, we recall the Feynman–Kac formula on the Wiener space proved in [10]. Let X_t be the Ornstein–Uhlenbeck process having $\mathcal{L}$ as its generator (cf. [9]). One possible way to describe this process is by the expression

$$X_t(s) = e^{-t}x(s) + \int_0^t e^{-(t-\xi)}\,\mathrm{d}W_\xi(s), \tag{5.3}$$

for $s \in [0,1]$. Under the conditions (A.1) and (A.2) the following Feynman–Kac representation in terms of the Ornstein–Uhlenbeck process X_t holds

$$\eta_t(x) = E_{x,t}\left(\theta(X_1)\exp - \int_t^1 V(X_s)\,\mathrm{d}s\right), \tag{5.4}$$

for all $t < 1$, μ-almost everywhere in x; where $E_{x,t}$ denotes a $\mathcal{P}_t$ conditional expectation given that $X_t(\cdot)$ is the path $x \in \mathbf{X}$. Using this representation and the Jensen inequality one gets the following estimation

$$\eta_t(x) \geqslant \exp E_{x,t}\left(\log\theta(X_1) - \int_t^1 V(X_s)\,\mathrm{d}s\right). \tag{5.5}$$

LEMMA 5.2. *Let* V *and* θ *be two Wiener functionals under the conditions, respectively,* (A.1) *and* (A.2). *If, in addition,*

(i) V *and* θ *belong to every* $W_4^p(\mathbf{X};\mathbb{R})$;

(ii) θ^{-1} *and* $\exp(V)$ *are* L^p *integrables, for all* $1 \leqslant p < \infty$*, then*

$$\int_{-1}^{1} \|\nabla \log \eta_t\|^p_{W_3^p(\mathbf{X};\mathbf{H})}\, \mathrm{d}t < \infty,$$

for each $2 \leqslant p < \infty$.

REMARK 5.3. The assumption $E_\mu \exp(pV) < \infty$ for all $1 \leqslant p < \infty$ could be relaxed so that potentials with quadratic growth could be considered.

Proof. For the framework of this proof we refer [10].

Step 1. We start by proving

$$\int_{-1}^{1} E_\mu \|\nabla \log \eta_t\|^p_{\mathbf{H}}\, \mathrm{d}t < \infty,$$

for each $2 \leqslant p < \infty$.

Observing that

$$E_\mu \|\nabla \log \eta_t\|^p = E_\mu \left\| \frac{\nabla \eta_t}{\eta_t} \right\|^p \leqslant (E_\mu \|\nabla \eta_t\|^{2p})^{1/2} (E_\mu |\eta_t|^{-2p})^{1/2}, \tag{5.6}$$

it follows by the estimation (5.5) that

$$E_\mu |\eta_t|^{-2p} \leqslant E_\mu \exp E_{x,t} \left(-2p \log \theta(X_1) + 2p \int_t^1 V(X_s)\, \mathrm{d}s \right);$$

where, by the Jensen inequality (J.I.), the properties of the conditional expectation (C.E.P.), the positivity of V (P.) and the invariance of the measure μ for the Ornstein–Uhlenbeck process X_t (I.M.), we get the following estimation for the right-hand side of the above inequality

$$\begin{aligned}
&E_\mu \exp E_{x,t} \left(-2p \log \theta(X_1) + 2p \int_t^1 V(X_s)\, \mathrm{d}s \right) \quad \text{(by J.I.)} \\
&\quad \leqslant E_\mu E_{x,t} \exp\left(-2p \log \theta(X_1) + 2p \int_t^1 V(X_s)\, \mathrm{d}s \right) \quad \text{(by C.E.P.)} \\
&\quad = E_\mu \left[(\theta(X_1))^{-2p} \exp 2p \int_t^1 V(X_s)\, \mathrm{d}s \right] \quad \text{(by P.)} \\
&\quad \leqslant (E_\mu (\theta(X_1))^{-4p})^{1/2} \left(E_\mu \exp 8p \int_{-1}^1 V(X_s) \frac{\mathrm{d}s}{2} \right)^{1/2} \quad \text{(by J.I.)} \\
&\quad \leqslant (E_\mu (\theta(X_1))^{-4p})^{1/2} \left(E_\mu \frac{1}{2} \int_{-1}^1 \exp(8pV(X_s))\, \mathrm{d}s \right)^{1/2} \\
&\qquad\qquad \text{(by I.M.)} \\
&\quad \leqslant (E_\mu \theta^{-4p})^{1/2} (E_\mu \exp(8pV))^{1/2}.
\end{aligned}$$

Consequently, taking the expectations on (5.6) we obtain

$$\begin{aligned}\int_{-1}^{1} E_\mu\|\nabla \log \eta_t\|^p \,\mathrm{d}t &\\ &\leqslant \int_{-1}^{1} (E_\mu\|\nabla\eta_t\|^{2p})^{1/2}(E_\mu|\eta_t|^{-2p})^{1/2}\,\mathrm{d}t\\ &\leqslant \int_{-1}^{1} (E_\mu\|\nabla\eta_t\|^{2p})^{1/2}(E_\mu\theta^{-4p})^{1/4}(E_\mu \exp(8pV))^{1/4}\,\mathrm{d}t\\ &\leqslant c(E_\mu\theta^{-4p})^{1/4}\left(E_\mu \exp(8pV)\right)^{1/4}\left(\int_{-1}^{1} E_\mu\|\nabla\eta_t\|^{2p}\,\mathrm{d}t\right)^{1/2}.\end{aligned}$$

It remains to prove the implication

$$\theta, V \in W_1^{2p}(\mathbf{X};\mathbb{R}) \Longrightarrow\ a_t \in W_1^p(\mathbf{X};\mathbb{R}),\quad t\in[-1,1], \tag{5.7}$$

analogous to another one of the same kind obtained on [10]. Indeed, using the representations (5.3) and (5.4) we have

$$\begin{aligned}\nabla\eta_t(x) \ =\ & e^{-1}E_{x,t}\left(\nabla\theta(X_1)\exp - \int_t^1 V(X_s)\,\mathrm{d}s\right)\\ & -E_{x,t}\left(\theta(X_1)\int_t^1 \nabla V(X_s)e^{-s}\,\mathrm{d}s \exp - \int_t^1 V(X_s)\,\mathrm{d}s\right),\end{aligned} \tag{5.8}$$

which implies, by the positivity of V and by the invariance of the measure μ for the process X_t, an estimation of the type

$$\begin{aligned}E_\mu\|\nabla\eta_t\|^p \ \leqslant\ & E_\mu\left\|\left\{e^{-1}\nabla\theta(X_1) - \theta(X_1)\int_t^1 \nabla V(X_s)e^{-s}\,\mathrm{d}s\right\}\right.\\ &\qquad \left.\times \exp - \int_t^1 V(X_s)\,\mathrm{d}s\right\|^p\\ \leqslant\ & c(p)\left\{e^{-p}E_\mu\|\nabla\theta\|^p + E_\mu\left\|\theta(X_1)\int_t^1 \nabla V(X_s)e^{-s}\,\mathrm{d}s\right\|^p\right\},\end{aligned}$$

for a positive constant $c(p)$ (depending only on p); where

$$\begin{aligned}E_\mu &\left\|\theta(X_1)\int_t^1 \nabla V(X_s)e^{-s}\,\mathrm{d}s\right\|^p\\ &\leqslant (E_\mu\theta^{2p})^{1/2}\left(E_\mu\left(\int_t^1 \|\nabla V(X_s)\|e^{-s}\,\mathrm{d}s\right)^{2p}\right)^{1/2}\end{aligned}$$

$$\leqslant (\sqrt{2})^{2p-1} e^p (E_\mu \theta^{2p})^{1/2} \left(E_\mu \int_t^1 \|\nabla V(X_s)\|^{2p} \,\mathrm{d}s \right)^{1/2}$$

$$= (2e)^p (E_\mu \theta^{2p})^{1/2} (E_\mu \|\nabla V\|^{2p})^{1/2}.$$

Hence, taking the expectations we get

$$\int_{-1}^1 E_\mu \|\nabla \eta_t\|^p \,\mathrm{d}t \leqslant c(p)\{E_\mu \|\nabla\theta\|^p + (E_\mu \theta^{2p})^{1/2} (E_\mu \|\nabla V\|^{2p})^{1/2}\},$$

which valid the implication (5.7).

Step 2. Next, we prove that

$$\int_{-1}^1 E_\mu \|\nabla^2 \log \eta_t\|^p_{\mathcal{L}_{H.S.}(\mathbf{H};\mathbf{H})} \,\mathrm{d}t < \infty,$$

for each $2 \leqslant p < \infty$.

We need the L^p norms estimations of $\mathcal{L}F$, with $F \in W_2^p(\mathbf{X};\mathbb{R})$, in terms of Sobolev norms introduced in [6] by means of Poincaré-type inequalities due to Krée and Meyer,

$$c_1 \|\mathcal{L}F\|_{L^p} \leqslant \|\nabla^2 F\|_{L^p} \leqslant c_2 \|\mathcal{L}F\|_{L^p}, \tag{5.9}$$

where c_1 and c_2 are positive constants, $1 < p < +\infty$.

Since we have

$$\mathcal{L} \log \eta_t = \frac{\mathcal{L}\eta_t}{\eta_t} - \tfrac{1}{2}\|\nabla \log \eta_t\|^2, \tag{5.10}$$

then, twice applying (5.9) we get

$$\int_{-1}^1 E_\mu \|\nabla^2 \log \eta_t\|^p \,\mathrm{d}t$$

$$\leqslant c_2^p \int_{-1}^1 E_\mu |\mathcal{L} \log \eta_t|^p \,\mathrm{d}t$$

$$\leqslant c(p) \left\{ \int_{-1}^1 E_\mu \left| \frac{\mathcal{L}\eta_t}{\eta_t} \right|^p \mathrm{d}t + \int_{-1}^1 E_\mu \|\nabla \log \eta_t\|^{2p} \,\mathrm{d}t \right\}$$

$$\leqslant c(p) \left\{ \left(\int_{-1}^1 E_\mu \|\nabla^2 \eta_t\|^{2p} \,\mathrm{d}t \right)^{1/2} \left(\int_{-1}^1 E_\mu |\eta_t|^{-2p} \,\mathrm{d}t \right)^{1/2} \right.$$

$$\left. + \int_{-1}^1 E_\mu \|\nabla \log \eta_t\|^{2p} \,\mathrm{d}t \right\}.$$

Therefore, by the first step, we just need to estimated the integral

$$\int_{-1}^{1} E_\mu \|\nabla^2 \eta_t\|^p \, \mathrm{d}t,$$

with $1 < p < +\infty$.

Using the representations (5.3) and (5.4) we obtain

$$\begin{aligned}\nabla^2 \eta_t \;=\; & e^{-2} E_{x,t} \left(\nabla^2 \theta(X_1) \exp - \int_t^1 V(X_s)\, \mathrm{d}s \right) \\ & -2e^{-1} E_{x,t} \left(\nabla \theta(X_1) \int_t^1 \nabla V(X_s) e^{-s} \, \mathrm{d}s \exp - \int_t^1 V(X_s)\, \mathrm{d}s \right) \\ & -E_{x,t} \left(\theta(X_1) \int_t^1 \nabla^2 V(X_s) e^{-2s} \, \mathrm{d}s \exp - \int_t^1 V(X_s)\, \mathrm{d}s \right) \\ & +E_{x,t} \left(\theta(X_1) \left(\int_t^1 \nabla V(X_s) e^{-s} \, \mathrm{d}s \right)^2 \exp - \int_t^1 V(X_s)\, \mathrm{d}s \right). \end{aligned} \tag{5.11}$$

Hence, taking the expectations on both sides of the above equality we get an estimation of the type

$$\begin{aligned} & \int_{-1}^{1} E_\mu \|\nabla^2 \eta_t\|^p \, \mathrm{d}t \\ & \leqslant c(p) \{ E_\mu \|\nabla^2 \theta\|^p + (E_\mu \|\nabla \theta\|^{2p})^{1/2} (E_\mu \|\nabla V\|^{2p})^{1/2} \\ & +(E_\mu \theta^{2p})^{1/2} ((E_\mu \|\nabla^2 V\|^{2p})^{1/2} + (E_\mu \|\nabla V\|^{4p})^{1/2}) \}, \end{aligned}$$

which proves the implication

$$\theta \in W_2^{2p}(\mathbf{X}; \mathbb{R}), \;\; V \in W_2^{4p}(\mathbf{X}; \mathbb{R}) \Longrightarrow \eta_t \in W_2^{p}(\mathbf{X}; \mathbb{R}), \;\; t \in [-1, 1],$$

similar to another one proved in [10].

Step 3. Now, we will prove that

$$\int_{-1}^{1} E_\mu \|\nabla^3 \log \eta_t\|^p_{\mathcal{L}^2_{H.S.}(\mathbf{H};\mathbf{H})} \, \mathrm{d}t < \infty, \tag{5.12}$$

for each $2 \leqslant p < \infty$, where $\mathcal{L}^i_{H.S.}(\mathbf{H};\mathbf{H})$ means the class of i-linear Hilbert–Schmidt operators.

From the third derivative of $\log \eta_t$, and using the same arguments above, we can deduce

$$\int_{-1}^{1} E_\mu \|\nabla^3 \log \eta_t\|^p \, \mathrm{d}t$$
$$\leqslant c(p) \left\{ \left(\int_{-1}^{1} E_\mu \|\nabla^3 \eta_t\|^{2p} \, \mathrm{d}t \right)^{1/2} \left(\int_{-1}^{1} E_\mu |\eta_t|^{-2p} \, \mathrm{d}t \right)^{1/2} \right.$$
$$+ \left(\int_{-1}^{1} E_\mu \left\| \frac{\nabla^2 \eta_t}{\eta_t} \right\|^{2p} \mathrm{d}t \right)^{1/2} \left(\int_{-1}^{1} E_\mu \|\nabla \log \eta_t\|^{2p} \, \mathrm{d}t \right)^{1/2}$$
$$\left. + \left(\int_{-1}^{1} E_\mu \|\nabla^2 \log \eta_t\|^{2p} \, \mathrm{d}t \right)^{1/2} \left(\int_{-1}^{1} E_\mu \|\nabla \log \eta_t\|^{2p} \, \mathrm{d}t \right)^{1/2} \right\}.$$

Therefore, by the first and second steps it remains to estimate the first integral of the right-hand side of the above inequality to conclude (5.12). Using once more the representations (5.3) and (5.4) we have

$$\begin{aligned}
\nabla^3 \eta_t \;=\; & e^{-3} E_{x,t} \left(\nabla^3 \theta(X_1) \exp - \int_t^1 V(X_s) \, \mathrm{d}s \right) \\
& -(1 + 2e^{-2}) E_{x,t} \left(\nabla^2 \theta(X_1) \int_t^1 \nabla V(X_s) e^{-s} \, \mathrm{d}s \right. \\
& \qquad \left. \times \exp - \int_t^1 V(X_s) \, \mathrm{d}s \right) \\
& -3e^{-1} E_{x,t} \left(\nabla \theta(X_1) \int_t^1 \nabla^2 V(X_s) e^{-2s} \, \mathrm{d}s \exp - \int_t^1 V(X_s) \, \mathrm{d}s \right) \\
& +3e^{-1} E_{x,t} \left(\nabla \theta(X_1) \left(\int_t^1 \nabla V(X_s) e^{-s} \, \mathrm{d}s \right)^2 \exp - \int_t^1 V(X_s) \, \mathrm{d}s \right) \\
& -E_{x,t} \left(\theta(X_1) \int_t^1 \nabla^3 V(X_s) e^{-3s} \, \mathrm{d}s \exp - \int_t^1 V(X_s) \, \mathrm{d}s \right) \\
& +3E_{x,t} \left(\theta(X_1) \int_t^1 \nabla^2 V(X_s) e^{-2s} \, \mathrm{d}s \int_t^1 \nabla V(X_s) e^{-s} \, \mathrm{d}s \right. \\
& \qquad \left. \times \exp - \int_t^1 V(X_s) \, \mathrm{d}s \right)
\end{aligned}$$

$$-E_{x,t}\left(\theta(X_1)\left(\int_t^1 \nabla V(X_s)e^{-s}\,\mathrm{d}s\right)^3 \exp - \int_t^1 V(X_s)\,\mathrm{d}s\right).$$

Hence, taking the expectations and estimating the L^p norms of the third derivative of η_t, one can prove, in an analogous way as we have done on the above steps, that it holds the implication

$$\theta \in W_3^{2p}(\mathbf{X};\mathbb{R}),\quad V \in W_3^{6p}(\mathbf{X};\mathbb{R}) \Longrightarrow \eta_t \in W_3^p(\mathbf{X};\mathbb{R}),\quad t\in[-1,1].$$

Step 4. To complete the proof, we will prove that

$$\int_{-1}^1 E_\mu \|\nabla^4 \log \eta_t\|^p_{\mathcal{L}^3_{H.S.}(\mathbf{H};\mathbf{H})}\,\mathrm{d}t < \infty,$$

for each $2 \leqslant p < \infty$.

By computation of the fourth derivative of $\log \eta_t$ and applying once more the above arguments, we obtain

$$\begin{aligned}
&\int_{-1}^1 E_\mu \|\nabla^4 \log \eta_t\|^p\,\mathrm{d}t \\
&\quad\leqslant c(p)\Bigg\{\left(\int_{-1}^1 E_\mu\|\nabla^4\eta_t\|^{2p}\,\mathrm{d}t\right)^{1/2}\left(\int_{-1}^1 E_\mu|\eta_t|^{-2p}\,\mathrm{d}t\right)^{1/2} \\
&\qquad + \left(\int_{-1}^1 E_\mu\left\|\frac{\nabla^3\eta_t}{\eta_t}\right\|^{2p}\mathrm{d}t\right)^{1/2}\left(\int_{-1}^1 E_\mu\|\nabla\log\eta_t\|^{2p}\,\mathrm{d}t\right)^{1/2} \\
&\qquad + \left(\int_{-1}^1 E_\mu\left\|\frac{\nabla^2\eta_t}{\eta_t}\right\|^{2p}\mathrm{d}t\right)^{1/2}\left(\int_{-1}^1 E_\mu\|\nabla\log\eta_t\|^{4p}\,\mathrm{d}t\right)^{1/2} \\
&\qquad + \left(\int_{-1}^1 E_\mu\left\|\frac{\nabla^2\eta_t}{\eta_t}\right\|^{2p}\mathrm{d}t\right)^{1/2}\left(\int_{-1}^1 E_\mu\|\nabla^2\log\eta_t\|^{2p}\,\mathrm{d}t\right)^{1/2} \\
&\qquad + \left(\int_{-1}^1 E_\mu\|\nabla^3\log\eta_t\|^{2p}\,\mathrm{d}t\right)^{1/2}\left(\int_{-1}^1 E_\mu\|\nabla\log\eta_t\|^{2p}\,\mathrm{d}t\right)^{1/2} \\
&\qquad + \int_{-1}^1 E_\mu\|\nabla^2\log\eta_t\|^{2p}\,\mathrm{d}t\Bigg\}.
\end{aligned}$$

Hence, from the above steps, it remains to study the L^p-integrability of the fourth derivative of η_t. To do this, we apply the same scheme used before: from the representations (5.3) and (5.4) we get the expression of the derivative $\nabla^4\eta_t$ and, taking the expectation and estimating the L^p norms of $\nabla^4\eta_t$ we prove that the implication holds

$$\theta \in W_4^{2p}(\mathbf{X};\mathbb{R}),\ \ V \in W_4^{8p}(\mathbf{X};\mathbb{R}) \Longrightarrow \eta_t \in W_4^p(\mathbf{X};\mathbb{R}),\ \ t \in [-1,1],$$

which complete the proof. □

REMARK 5.4. Given θ^* on the conditions (A.3), belonging to every spaces $W_4^p(\mathbf{X};\mathbb{R})$ in such way that θ^{*-1} is L^p integrable, for all $1 \leqslant p < \infty$, we may conclude, by the same steps of this proof, that $\nabla \log \eta_t^*$ is $L^p([-1,1]; W_3^p(\mathbf{X};\mathbf{H}))$ integrable, with $2 \leqslant p < \infty$.

The results of this paragraph and the above mentioned adaptation of Theorem 4.2 allow to derive the following theorem of uniqueness to Bernstein processes.

THEOREM 5.5. *Under the conditions of Lemma* 5.2*, the pathwise uniqueness for the Equation* (5.1) *holds*.

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