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Homoclinic tangencies *versus* uniform hyperbolicity for conservative 3-flows

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ABSTRACT

We prove that a volume-preserving three-dimensional flow can be C^1-approximated by a volume-preserving Anosov flow or else by another volume-preserving flow exhibiting a homoclinic tangency. This proves the conjecture of Palis for conservative 3-flows and with respect to the C^1-topology.

1. Introduction and statement of the results

Let M^3 be a compact, connected and boundaryless smooth Riemannian manifold and let μ denote the Lebesgue measure induced by a fixed volume form on M^3. Let $\mathfrak{X}^r(M^3)$ be the space of C^r vector fields, for any $r \geqslant 1$, and let $\mathfrak{X}^r_\mu(M^3)$ be its subset of divergence-free vector fields, that is, $X \in \mathfrak{X}^r_\mu(M^3)$ is such that $\nabla \cdot X = 0$. By Liouville's formula these vector fields define volume-preserving (or conservative, incompressible) flows.

It was conjectured by Palis (see [13, Conjecture 1]) that any dynamical system can be C^r approximated, $r \geqslant 1$, by a hyperbolic one without cycles or by one exhibiting either a homoclinic tangency or a heterodimensional cycle.

In a remarkable work [15], Pujals and Sambarino proved the conjecture in the context of two-dimensional dissipative and discrete-time case with the C^1 topology. Notice that, with this low-dimensional assumption the existence of heterodimensional cycles is discarded. Using new flow-type ingredients Arroyo and Hertz, in [3], proved the continuous-time version of [15], namely:

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Theorem 1.1. *Any vector field $X \in \mathfrak{X}^1(M^3)$ can be C^1-approximated by another one $Y \in \mathfrak{X}^1(M^3)$ satisfying one of the following phenomena*:

(1) *Uniform hyperbolicity with the no-cycles condition*;
(2) *A homoclinic tangency*;
(3) *A singular cycle.*

It is conjectured [13, Conjecture 4] that the third item can be replaced by a singular-hyperbolic set (see [10] for the definition). Since there are no singular-hyperbolic sets for three-dimensional volume-preserving flows (see [2, Corollary 4.1]) one has good reasons to believe that, in the three-dimensional conservative flows setting, will prevail (densely) the dichotomy hyperbolicity or homoclinic tangencies. Actually, our result is the following.

Main Theorem. *Any vector field $X \in \mathfrak{X}^1_\mu(M^3)$ can be C^1-approximated by another one $Y \in \mathfrak{X}^1_\mu(M^3)$ satisfying one of the following properties*:

(1) *Y is Anosov or else*
(2) *Y^t has a homoclinic tangency associated to a hyperbolic closed orbit.*

Notice that homoclinic tangency phenomena is strongly related with elliptic points (see [12] and [7]) in the sense that near homoclinic tangencies there exist many elliptic points. Therefore, the existence of homoclinic tangencies is a sufficient condition to have elliptic points. We will see that it is also a necessary condition for, at least, a sufficiently C^1-close system. For that we recall the following result proved in [4, Theorem 1.3] for divergence-free vector fields without singularities and then extended in [1, Corollary 1.4] for the all class of divergence-free vector fields.

Theorem 1.2. *There exists a C^1-residual subset $\mathcal{R} \subset \mathfrak{X}^1_\mu(M^3)$ such that, if $X \in \mathcal{R}$ then X is Anosov or else the elliptical periodic points of X^t are dense in M^3.*

For surface area-preserving diffeomorphisms the existence of smooth invariant curves is associated to the existence of elliptic points. Actually, Mora and Romero [9] developed a mechanism to create open sets containing a dense set of maps exhibiting homoclinic tangencies once one has a smooth invariant curve. A key step to prove this result is [9, Proposition 7], which also plays a crucial role in the proof of the Main Theorem. To state this proposition let us define

$$\mathbb{A} = \{(\theta, r)\colon \theta \in \mathbb{S}^1,\ r \in \mathbb{R}\} \quad \text{and} \quad \mathbb{A}_\delta = \{(\theta, r)\colon \theta \in \mathbb{S}^1,\ r \in\,]-\delta, \delta[\}.$$

Theorem 1.3. *Let $f : \mathbb{A}_\delta \to \mathbb{A}$ be a C^∞ area-preserving map of the annulus leaving invariant some C^∞ curve*

$$\Lambda = \{(\theta, \Phi(\theta)),\ \theta \in \mathbb{S}^1\},$$

where $\Phi\colon \mathbb{S}^1 \to \mathbb{R}$, and such that $f|_\Lambda$ has an irrational rotation number. Then, for $s \geqslant 1$ and $\epsilon > 0$, f can be ϵ-C^s-approximated by an area-preserving g exhibiting homoclinic tangencies such that for some $\delta' < \delta$ we have

$$g|_{\mathbb{A}_\delta \setminus \mathbb{A}_{\delta'}} = f|_{\mathbb{A}_\delta \setminus \mathbb{A}_{\delta'}}.$$

We end this introduction giving the guidelines of the proof of the Main Theorem. Assume that $X \in \mathfrak{X}^1_\mu(M^3)$ cannot be C^1 approximated by an Anosov vector field. Then the proof follows in four steps:

(1) The first step consists in using directly Theorem 1.2, a theorem of Zuppa [16] and Franks' lemma for conservative vector fields [5] to obtain $\tilde{X} \in \mathfrak{X}^\infty_\mu(M^3)$ exhibiting an elliptic point with period arbitrarily large such that the associated first return map, f, is a linear rotation. In Section 2.3 we present the statements of the referred results.
(2) Next we apply Theorem 1.3 to the return map f in a local normal section of the orbit to obtain a local diffeomorphism g having homoclinic tangencies.
(3) The third step consists in using a suitable change of coordinates, in a small tubular neighborhood of the elliptic orbit, that allows us to perform local perturbations.
(4) Finally we realize g as return map associated to a new divergence-free vector field C^1-close to the initial one.

In the next section we explain how we obtain the adequate coordinates mentioned in (3). Finally, in Section 3, we show how to realize g by constructing explicitly the divergence-free vector field. Given a closed orbit for a flow, its return map f and a small perturbation g of f, this last mentioned construction gives us a way to explicit a conservative vector field having g has its first return map. Since we could not find a proof in the literature we present here a detailed proof for future use. We observe that in [14, Section 7A] Pugh and Robinson do a similar type of construction but in the dissipative flow setting.

2. Preliminaries

2.1. Basic notions

Given a vector field X defined in a smooth Riemannian manifold M^3 we denote by $Sing(X)$ the set of all the *singularities* of X, that is the points $p \in M^3$ such that $X(p) = \vec{0}$. Let $R := M^3 \setminus Sing(X)$ be the set of *regular* points. Given $p \in R$ we consider its normal bundle $N_p = X(p)^\perp \subset T_pM$ and define the associated *linear Poincaré flow* by $P^t_X(p) := \Pi_{X^t(p)} \circ DX^t(p)$ where $\Pi_{X^t(p)} : T_{X^t(p)}M \to N_{X^t(p)}$ is the projection along the direction of $X(X^t(p))$.

Let $\Lambda \subseteq M$ be an invariant set. A P^t_X-invariant splitting $N = N^1 \oplus N^2$ over Λ is said to be a *hyperbolic* set for the linear Poincaré flow if there exists $k \in \mathbb{N}$ such that $\|(P^k_X(p) \cdot u)^{-1}\| \leqslant 1/2$ (expanding), for all $p \in \Lambda$ and any unit vector $u \in N^1(p)$, and $\|P^k_X(p) \cdot u\| \leqslant 1/2$ (contracting), for all $p \in \Lambda$ and any unit vector $u \in N^2(p)$.

If M is hyperbolic for the linear Poincaré flow of X we say that X is an *Anosov vector field*. Due to Proposition 1.1 of [6] it follows that this definition is equivalent to the usual one, that is, when the manifold is hyperbolic for the tangent map DX^t. The Anosov systems form an open subset of $\mathfrak{X}^1_\mu(M)$.

Given a closed hyperbolic orbit $\mathcal{O}$ and $p \in \mathcal{O}$ let W^s_p (respectively W^u_p) denotes the stable (respectively unstable) manifold of p for the associated return map defined in a local normal section at p. We say that $\mathcal{O}$ has a *homoclinic tangency* at $q \neq p$ if:

- $T_qW^s_p \cap T_qW^u_p$ contains a nonzero vector and
- $T_qW^s_p \oplus T_qW^u_p \oplus X(q) \neq T_qM^3$.

Given $X \in \mathfrak{X}^1_\mu(M^3)$ and a regular point p we consider a *linear differential system* over the orbit of p in the following way:

$$S^t : \mathbb{R}^2_p \to \mathbb{R}^2_{X^t(p)}$$

is such that

- $S^t \in SL(2, \mathbb{R})$, for every t;
- $S^{t+r} = S^t \circ S^r$, for every r, t;
- $S^0 = Id$ and
- S^t is differentiable in t.

It follows that

$$\big(S^t\big(X^\tau(p)\big)\big)'_{|t=s} = A\big(X^{\tau+s}(p)\big) \circ S^\tau,$$

for some $A(X^{\tau+s}(p)) \in \mathfrak{sl}(2,\mathbb{R})$, that is, its trace is equal to zero. For simplicity we write this linear variational equation as

$$\big(S^t(\tau)\big)'_{|t=s} = A(\tau+s) \circ S^\tau.$$

2.2. Good coordinates

We consider a vector field $X \in \mathfrak{X}^4_\mu(M^3)$, $\tau > 0$, and a point $p \in M^3$ such that $X^t(p) \neq p$, for all $t \in [0,\tau]$. We define $\Gamma(p,\tau) = \{X^t(p)\colon t \in [0,\tau]\}$. In what follows, up to a smooth conservative change of coordinates Ψ_0 defined on a neighborhood $\mathcal{U}$ of $\Gamma(p,\tau)$, we can assume that we are working on the Euclidean space $\mathbb{R}^3$, that $p = \vec{0}$ and that $\frac{1}{\|X(p)\|}X(p) = \frac{\partial}{\partial x_1} = v$ (see [11]).

Let $W \subset \mathbb{R}^3$ be the two-dimensional vector subspace orthogonal to the unitary vector v. Given $r > 0$ let $B_r(p)$ denote the ball of radius r, centered at p and contained in $N_p = X(p)^\perp = W$. For $r > 0$ and $\delta > 0$ define the tubular neighborhood

$$\mathcal{T} = \mathcal{T}(p,\tau,r,\delta) = \bigcup_{t\in]-\delta,\tau+\delta[} X^t\big(B_r(p)\big).$$

If $r > 0$ and $\delta > 0$ are small enough the set $\mathcal{T}$ is an open neighborhood of $\Gamma(p,\tau)$; by definition this set is foliated by orbits of the flow and for this reason we call it a *flowbox*.

We fix a linear isometry $\iota_p : N_p \to W$ and choose a family $\{\iota_t\}_{t\in]-\delta,\tau+\delta[}$, such that, for each $t \in\,]-\delta,\tau+\delta[$, ι_t is a linear isometry from $N_{X^t(p)}$ onto W, $\iota_0 = \iota_p$, and this family is C^1 on the parameter t. Such an isometry can be obtained by considering M^3 embedded in $\mathbb{R}^N$, for some N, and then choosing τ_t, a one-parameter family of isometries of $\mathbb{R}^N$ which are C^1 on the parameter, such that $\tau_t(N_{X^t(p)}) = N_p$ and τ_0 is the identity; finally we define $\iota_t = \iota_0 \circ \tau_t|_{N_{X^t(p)}}$.

In the local coordinates $(\Psi_0,\mathcal{U})$ fixed previously, for any $q \in \mathcal{T}$, we can write $q = \lambda_q v + w_q$, where $w_q \in W$ and $\lambda_q \in \mathbb{R}$. Define $\ell(t) = \int_0^t \|X(X^s(p))\|\,ds$ and notice that there exists $t_q \in\,]-\delta,\tau+\delta[$ such that $\ell(t_q) = \lambda_q$. We observe that $t_p = 0$.

Let us now define the *Poincaré flow* $\hat{X}^t$ associated to X on $\mathcal{T}$.

For t such that $t_q + t \in\,]-\delta,\tau+\delta[$ define

$$\hat{X}^t(q) = \ell(t_q+t)v + \iota_{(t_q+t)} \circ P^t_X\big(X^{t_q}(p)\big) \circ \iota^{-1}_{t_q}(w_q).$$

Clearly $\hat{X}^0 \equiv Id$ and $\hat{X}^{t+t'}(q) = \hat{X}^t(\hat{X}^{t'}(q))$, when defined.

Let $\hat{X}$ be the vector field associated to the flow $\hat{X}^t$. $\hat{X}$ is of class C^2 and it is divergence-free. In fact a direct computation gives that the matrix of $D\hat{X}^t(q)$ relatively to the decomposition $\mathbb{R}^n = W \oplus \langle v\rangle$ is

$$\begin{pmatrix} \frac{\|X(X^{t_q+t}(p))\|}{\|X(X^{t_q}(p))\|} & 0 \\ \star & \iota_{t_q+t} \circ P^t_X(X^{t_q}(p)) \circ \iota^{-1}_{t_q} \end{pmatrix}.$$

As X^t is volume-preserving and the maps ι_s are linear isometries we get that for all t

$$\det D\hat{X}^t(q) = \frac{\|X(X^{t_q+t}(p))\| \det P^t_X(X^{t_q}(p))}{\|X(X^{t_q}(p))\|} = 1. \tag{1}$$

Thus, according to Liouville's formula

$$e^{\int_0^t \nabla\cdot\hat{X}(\hat{X}^s(q))\,ds} = \det D\hat{X}^t(q),$$

hence it follows that $\nabla\cdot\hat{X} = 0$.

We also observe that $P^t_{\hat{X}}(q) = \iota_{t_q+t}\circ P^t_X(X^{t_q}(p))\circ\iota^{-1}_{t_q}$; in particular $P^t_{\hat{X}}(0) = \iota_t\circ P^t_X(p)\circ\iota^{-1}_0$.

Next lemma gives the adequate coordinates to perform perturbations and its formulation in a general setting and its proof can be found in [5, Lemma 3.1].

Lemma 2.1. *Let $X\in\mathfrak{X}^4_\mu(M^3)$, $\tau>0$, and $p\in M^3$ such that $X^t(p)\neq p$, $\forall t\in[0,\tau]$. There exists a C^2-conservative change of coordinates Ψ, defined on a neighborhood of $\Gamma(p,\tau)$, such that*

$$\hat{X} = \Psi_* X \quad\text{and}\quad \Psi\big(X^t(p)\big) = \hat{X}^t(0), \quad \forall t\in[0,\tau].$$

We point out that, from the proof of this lemma, it follows that if the initial vector field is of class C^∞ then the change of coordinates Ψ is also of class C^∞.

We also observe that, using a new C^∞-change of coordinates $\tilde{\Psi}$ also obtained using Lemma 2.1, which essentially consists in parameterizing the curve $\hat{X}^t(p)$ by arc length, for points $q\in W$ we can write

$$\hat{X}^t(q) = tv + S^t(q),$$

where

$$S^t = \left(\sqrt{\frac{\|X(X^t(p))\|}{\|X(p)\|}}\right)\iota_t\circ P^t_X(p)\circ\iota^{-1}_0. \tag{2}$$

It is clear that $\tilde{\Psi}\circ\Psi(\mathcal{T}) = \bigcup_{q\in W}\hat{X}^{[0,\tau]}(q)$.

Our perturbation will be carried out in the linearizing coordinates provided by $\Phi = \tilde{\Psi}\circ\Psi$, after which Arbieto–Matheus' Pasting Lemma (see [2]) is used to extend (in a volume-preserving way) the linear vector field into a zero divergence vector field that coincides with the original vector field outside a small neighborhood of the periodic orbit.

2.3. Fundamental results

In this subsection we recall four theorems that will be used in the proof of the main result.

The first result allows us to approximate any C^1 divergence-free vector field by a C^∞ divergence-free one.

Theorem 2.2. *(See Zuppa [16].) The set of C^∞ divergence-free vector fields on M^3 is C^1 dense in $\mathfrak{X}^1_\mu(M^3)$.*

The second result allows us to realize C^1 local perturbations in the conservative class.

Theorem 2.3 *(Arbieto–Matheus Pasting Lemma). (See [2].) Given $\epsilon>0$ there exists $\delta>0$ such that if $X\in\mathfrak{X}^\infty_\mu(M)$, $K\subset M$ is a compact set and $Y\in\mathfrak{X}^\infty_\mu(M)$ is δ-C^1-close to X in a small neighborhood $U\supset K$, then there exist $Z\in\mathfrak{X}^\infty_\mu(M)$, V and W with $K\subset V\subset U\subset W$ such that:*

(1) $Z|_V = Y$;
(2) $Z|_{int(W^c)} = X$;
(3) *Z is ϵ-C^1-close to X.*

Finally, the next result is a version of Franks' lemma for divergence-free vector fields.

Theorem 2.4. *(See [5].) Given $\epsilon > 0$ and a vector field $X \in \mathfrak{X}^4_\mu(M)$ there exists $\xi_0 = \xi_0(\epsilon, X)$ such that $\forall \tau \in [1,2]$, for any periodic point p of period greater than 2, for any sufficient small flowbox $\mathcal{T}$ of $\Gamma(p,\tau)$ and for any one-parameter linear family $\{A_t\}_{t\in[0,\tau]}$ such that $\|A'_t A_t^{-1}\| < \xi_0$, $\forall t \in [0,\tau]$, there exists $Y \in \mathfrak{X}^1_\mu(M)$ satisfying the following properties*:

(1) Y *is* ϵ-C^1-*close to* X;
(2) $Y^t(p) = X^t(p)$, *for all* $t \in \mathbb{R}$;
(3) $P^\tau_Y(p) = P^\tau_X(p) \circ A_\tau$ *and*
(4) $Y|_{\mathcal{T}^c} \equiv X|_{\mathcal{T}^c}$.

3. Proof the Main Theorem

Let $X_0 \in \mathfrak{X}^1_\mu(M^3)$ be a vector field that cannot be C^1-approximated by an Anosov vector field. As $\mathfrak{X}^1_\mu(M^3)$ endowed with the C^1 topology is a Baire space we make use of Theorem 1.2 to approximate, in the C^1-topology, X_0 by $X_1 \in \mathfrak{X}^1_\mu(M^3)$ such that the elliptic points of X_1 are dense on M^3. In particular X_2 as an elliptic point p with period, say, greater than two. Now, using Theorem 2.2 and the stability of elliptic orbits, we approximate, in the C^1-topology, X_1 by $X_2 \in \mathfrak{X}^\infty_\mu(M^3)$ such that p is an elliptic point, of period $\pi > 2$, for X_2.

Next we consider the linear action DX_2 in a small neighborhood U of the orbit of p. Now, taking $Y = DX_2$ in the fixed neighborhood U, Theorem 2.3 allows to C^1-approximate X_2 by $X_3 \in \mathfrak{X}^\infty_\mu(M^3)$, such that

- $X_3|_V = DX_2|_V$, where V is a neighborhood of the orbit of p and is contained in U,
- p still is an elliptic point of period π for X_3,
- $X_3 = X_2$ outside W, where W is an open set containing V, and
- there exists an X^t_3-invariant tubular neighborhood $\mathfrak{T}$ where the first return map at a normal section of p, $\mathcal{N}'_p$, is a rotation of angle θ.

We assume that θ is irrational. In fact assume that θ is rational, say $P^\pi_{X_3}(p) = R_\theta$, where R_θ denotes a rotation of rational angle θ. Then we can perturb X_3 to get $X_4 \in \mathfrak{X}^\infty_\mu(M^3)$ with $P^\pi_{X_4}(p) = R_\alpha$, α irrational, by using Theorem 2.4. For that we take $A'_t \circ A_t^{-1}$ equal to the infinitesimal generator of a small rotation chosen in such a way that

$$R_\alpha = P^\pi_{X_4}(p) = P^\tau_{X_4}\big(X_3^{\pi-\tau}(p)\big) \circ P^{\pi-\tau}_{X_3}(p) = P^\tau_{X_3}\big(X_3^{\pi-\tau}(p)\big) \circ A_\tau \circ P^{\pi-\tau}_{X_3}(p).$$

Let $X = X_3$ (or $X = X_4$). Define $\mathcal{N}_p = \mathcal{N}'_p \cap \mathfrak{T}$; let f be the first return map at $\mathcal{N}_p$. By construction f is an area-preserving diffeomorphism, in fact it is an irrational rotation. For δ small we fix the annulus $\mathbb{A}_\delta \subset \mathcal{N}_p$, and, for every small $\epsilon > 0$, we can apply Theorem 1.3 to get an area-preserving diffeomorphism g, ϵ-C^1-close to f, exhibiting homoclinic tangencies, and such that, for some $\delta' < \delta$, one has $g|_{\mathbb{A}_\delta \setminus \mathbb{A}_{\delta'}} = f|_{\mathbb{A}_\delta \setminus \mathbb{A}_{\delta'}}$. Moreover, for a fixed $s \in \mathbb{N}$, one can obtain the diffeomorphism g of class C^s.

Next step consists in the realization of the map g has a first return map associated to a periodic orbit of a C^1 divergence-free vector field, Y, arbitrarily C^1-close to X, and equal to X outside a small tubular neighborhood of the X^t-orbit of p, which ends the proof of the theorem.

Let us fix a normal section at $X^1(p)$, $\mathcal{N}'_{X^1(p)}$, and define $\mathcal{N}_1 = \mathcal{N}'_{X^1(p)} \cap \mathfrak{T}$, and

$$\mathcal{N}^\star_{\pi-1} = X^{\pi-2}(\mathcal{N}'_{X^1(p)}) \cap \mathfrak{T}.$$

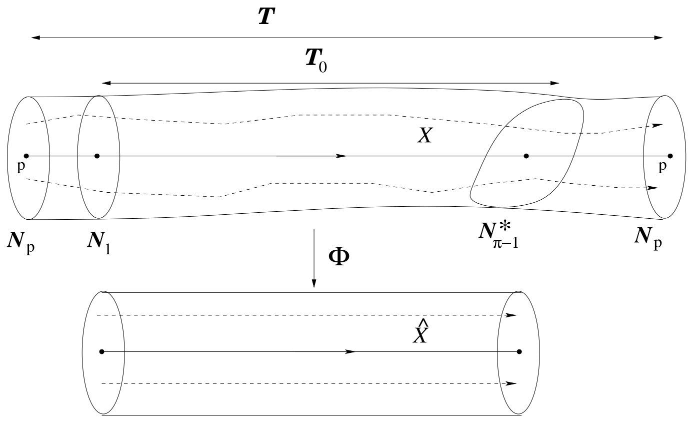

Fig. 1.

Consider the tubular neighborhood

$$\mathfrak{T}_0 = \bigcup_{q \in \mathcal{N}_1} X^{[0,\pi-2]}(q),$$

of the segment $X^{[1,\pi-1]}(p)$, see Fig. 1. For each $q \in \mathcal{N}_p$ let $\tau_0(q)$ be the first arrival time to $\mathcal{N}_1$, and for each $r \in \mathcal{N}^\star_{\pi-1}$ let $\tau_1(r)$ be the first arrival time to $\mathcal{N}_p$. Let $F : \mathcal{N}_1 \to \mathcal{N}^\star_{\pi-1}$ be defined by

$$F(q) = X^{-\tau_1(r)} \circ f \circ X^{-\tau_0(q)}(q),$$

where $r = f \circ X^{-\tau_0(q)}(q)$; in the same way let $G : \mathcal{N}_1 \to \mathcal{N}^\star_{\pi-1}$ be defined by

$$G(q) = X^{-\tau_1(r)} \circ g \circ X^{-\tau_0(q)}(q),$$

where $r = g \circ X^{-\tau_0(q)}(q)$.

It is clear that, if we choose g arbitrarily C^s-close to f, then F and G are also arbitrarily C^s-close. Therefore to obtain the desired divergence-free vector field Y it is enough to perturb X in $\mathfrak{T}_0$ in such a way that the transition from $\mathcal{N}_1$ to $\mathcal{N}^\star_{\pi-1}$ is given by G.

Now we apply Lemma 2.1 to get good coordinates in $\mathfrak{T}_0$ using the C^∞ volume-preserving change of coordinates Φ. Let $\hat{F} = \Phi \circ F \circ \Phi^{-1}$ and $\hat{G} = \Phi \circ G \circ \Phi^{-1}$; as the change of coordinates is of class C^2 it is clear that if F and G are C^2-arbitrarily close then $\hat{F}$ and $\hat{G}$ are C^2-arbitrarily close.

Let $\mathcal{N}^\Phi_1 = \Phi(\mathcal{N}_1)$ and $\mathcal{N}^\Phi_{\pi-1} = \Phi(\mathcal{N}^\star_{\pi-1})$. We will construct a di vergence-free vector field $\hat{Y}$ (close to $\hat{X}$) such that its $(\pi-2)$-time Poincaré map $\hat{G} : \mathcal{N}^\Phi_1 \to \mathcal{N}^\Phi_{\pi-1}$ is $\hat{G} = \Phi \circ G \circ \Phi^{-1}$.

Let $(0, x, y) \in \mathcal{N}^\Phi_1$. In the adequate coordinates introduced in Section 2.2 we have:

$$\hat{X}^t\big((0, x, y)\big) = \big(t, S^t(x, y)\big),$$

where $S^t : N_p \to N_{X^t(p)}$ is such that $S^t \in SL(2, \mathbb{R})$ for every $t \in \mathbb{R}$. Actually, S^t has the same dynamics of the linear Poincaré flow modulo the distortion factor given by a ratio involving the norm of the vector field (see (1) and (2)).

The family of smooth curves $(t, S^t(x, y))$ is a foliation of $\Phi(\mathfrak{T}_0)$ and has well-defined "product" coordinates given by $(t, \bar{x}, \bar{y}) := (t, S^t(x, y))$. Clearly, for each point $(0, x, y) \in \mathcal{N}^\Phi_1$ there exists a unique curve passing throughout it.

Taking the time derivative at $t=\tau$ we obtain:

$$\frac{d}{dt}\hat{X}^t\big((0,x,y)\big)\big|_{t=\tau}=\hat{X}\big(\big(\tau,S^\tau(x,y)\big)\big)=\big(1,\big(S^t\big)'(x,y)\big|_{t=\tau}\big),$$

where $'$ denotes the derivative in order of t.

Using the linear variational equation associated to S^t,

$$\big(S^t\big)'=A(t)\cdot S^t,$$

we get

$$\hat{X}(\tau,\bar{x},\bar{y})=\hat{X}\big(\big(\tau,S^\tau(x,y)\big)\big)=\big(1,A(\tau)\cdot S^\tau(x,y)\big)=\big(1,A(\tau)\cdot(\bar{x},\bar{y})\big). \tag{3}$$

Since the trace of $A(\tau)$ is zero, the equality (3) defines locally, in the coordinates $(t,\bar{x},\bar{y})$, a divergence-free vector field.

In order to obtain the desired perturbation we define a family of one-parameter local, C^2 on the parameter, area-preserving C^∞ diffeomorphism ψ_t on $\mathcal{N}_1^\Phi$ in the following way:

(I) $\psi_t=Id$ for $t\leqslant 0$,
(II) $\psi_t=\hat{F}^{-1}\circ\hat{G}$ for $t\geqslant\pi-2$,
(III) ψ_t is a smooth arc in the arc-connected space of local area-preserving diffeomorphisms, joining Id to $\hat{F}^{-1}\circ\hat{G}$ and such that ψ_t is C^1-close to Id for every $t\in\mathbb{R}$,
(IV) $\max\{\|{\psi_t}'\|,\|\psi_t''\|\}\approx 0$ and
(V) $\|D{\psi_t}'\|\approx 0$.

These conditions are achieved if we choose g arbitrarily C^1-close to f. Moreover some care is needed in the choice on the thickness of the initial tubular neighborhood, $\mathfrak{T}$, of the elliptic closed orbit. In fact, for example, to construct the desired vector field Y, C^1-close to X, we need to use two coordinate systems and assure that the associated change of coordinates is C^1 sufficiently close to the identity. This is achieved taking the thickness of $\mathfrak{T}$ very small (see Remark 3.1).

Given $(x,y)\in\mathcal{N}_1^\Phi$ and $t\geqslant 0$ we define the following family of smooth curves

$$\big(t,S^t\circ\psi_t(x,y)\big). \tag{4}$$

This family is a foliation of $\Phi(\mathfrak{T}_0)$ and has coordinates given by

$$(t,\mathbf{x},\mathbf{y})=\big(t,S^t\circ\psi_t(x,y)\big).$$

Of course that, for each $(0,x,y)\in\mathcal{N}_1^\Phi$, there exists a unique curve passing throughout $(0,x,y)$.

We consider the vector field $\hat{Y}$ which is induced by taking time derivatives in (4)

$$\begin{aligned}\hat{Y}\big(t,S^t\big(\psi_t(x,y)\big)\big)&=\big(1,\big(S^t\circ\psi_t\big)'(x,y)\big)\\&=\big(1,\big[A(t)+S^t\psi_t'\psi_t^{-1}\big(S^t\big)^{-1}\big]\cdot S^t\big(\psi_t(x,y)\big)\big),\end{aligned}$$

that is

$$\hat{Y}(t,\mathbf{x},\mathbf{y})=\big(1,\big[A(t)+S^t\psi_t'\psi_t^{-1}\big(S^t\big)^{-1}\big]\cdot(\mathbf{x},\mathbf{y})\big) \tag{5}$$

$$=\big(1,\big[A(t)+\mathfrak{R}\big]\cdot(\mathbf{x},\mathbf{y})\big). \tag{6}$$

We claim that $\hat{Y}$ is arbitrarily C^1-close to $\hat{X}$, and it is a divergence-free vector field which, by construction, generates a conservative flow $\hat{Y}^t$ such that the Poincaré map $\mathcal{P}_{\hat{Y}}^{\pi-2}(\Phi(X^1(p))) : \mathcal{N}_1^{\Phi} \to \mathcal{N}_{\pi-1}^{\Phi}$ realizes the map $\hat{G}$.

Remark 3.1. If $\|(x, y)\|$ is sufficiently small, that is if the initial tubular neighborhood $\mathfrak{T}$ is (arbitrarily) thin, as ψ_t is C^1-close to the identity and $\|\psi_t'\|$ is close to zero, it follows that the change of coordinates from $(t, \bar{x}, \bar{y})$ to $(t, \mathbf{x}, \mathbf{y})$ is (arbitrarily) C^1-close to the identity. We also observe that $(0, \bar{x}, \bar{y}) = (0, \mathbf{x}, \mathbf{y}) = (0, x, y)$.

The C^1-closeness. Using Remark 3.1 we obtain that,

$$\hat{X}(t, \mathbf{x}, \mathbf{y}) \overset{C^0}{\approx} \hat{X}\big((t, \bar{x}, \bar{y})\big) = \big(1, A(t) \cdot (\bar{x}, \bar{y})\big). \tag{7}$$

As $\|\psi_t'\| \approx 0$, from Eqs. (5) and (7) and Remark 3.1, it follows that $\|S^t \psi_t' \psi_t^{-1} (S^t)^{-1}\| \approx 0$ and therefore $\hat{Y}$ is C^0-close to $\hat{X}$.

Let $\mathbf{q} = (t, \mathbf{x}, \mathbf{y})$ and $q = (t, \bar{x}, \bar{y})$. Notice that $D_{(t,\bar{x},\bar{y})} \hat{X}(\mathbf{q}) \approx D_{(t,\bar{x},\bar{y})} \hat{X}(q)$.

Taking derivatives in (3) we obtain

$$D_{(t,\bar{x},\bar{y})} \hat{X}(q) = \begin{pmatrix} 0 & 0 \quad 0 \\ \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix} & A(t) \end{pmatrix},$$

where ϑ_i are time derivatives of $A(t)$ calculated at the point q, in the coordinates $(t, \bar{x}, \bar{y})$, that is $(\vartheta_1, \vartheta_2) = (A_t)' \cdot (\bar{x}, \bar{y})$.

The derivative of $\hat{Y}$ at the point $\mathbf{q}$ can also be represented in a matricial form by:

$$D_{(t,\mathbf{x},\mathbf{y})} \hat{Y}(\mathbf{q}) = \begin{pmatrix} 0 & 0 \quad 0 \\ \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} & [A(t) + D_{(\mathbf{x},\mathbf{y})} \mathfrak{R}] \end{pmatrix}$$

where, recall, $\mathfrak{R} = S^t \psi_t' \psi_t^{-1} (S^t)^{-1}$, and σ_i are time derivatives of $A(t) + S^t \psi_t' \psi_t^{-1} (S^t)^{-1}$ calculated at the point $\mathbf{q}$ in the coordinates $(t, \mathbf{x}, \mathbf{y})$, that is $(\sigma_1, \sigma_2) = (A(t) + \mathfrak{R})'(\mathbf{x}, \mathbf{y})$.

Now, using condition (V), we get

$$\begin{aligned} D_{(\mathbf{x},\mathbf{y})} \mathfrak{R} &= S^t \circ D_{(\mathbf{x},\mathbf{y})} \big(\psi_t' \psi_t^{-1}\big) \big(S^t\big)^{-1} \\ &= S^t \circ D_{(x,y)} \psi_t' \circ D_{(\mathbf{x},\mathbf{y})} \big(\psi_t^{-1}\big) \big(S^t\big)^{-1} \approx 0. \end{aligned}$$

We are left to see that the time derivatives of $A(t)$ are close to the time derivatives of $A(t) + \mathfrak{R}$. To see this we observe that

$$\begin{aligned} \big(A(t) + \mathfrak{R}\big)' = \big(A(t)\big)' &+ \big(S^t\big)' \psi_t' \psi_t^{-1} \big(S^t\big)^{-1} + S^t \psi_t'' \psi_t^{-1} \big(S^t\big)^{-1} \\ &+ S^t \psi_t' \big(\psi_t^{-1}\big)' \big(S^t\big)^{-1} + S^t \psi_t' \psi_t^{-1} \big(\big(S^t\big)^{-1}\big)'. \end{aligned}$$

Finally, as the change of coordinates from $(\bar{x}, \bar{y})$ to $(\mathbf{x}, \mathbf{y})$ is C^1-close to the identity (see Remark 3.1), condition (IV) on ψ_t implies that $(A(t))'$ and $(A(t) + \mathfrak{R})'$ are C^0-close in the (x, y) coordinates.

$\hat{Y}$ is divergence-free. Clearly,

$$\mathrm{tr}\big(A(t,x,y) + S^t \circ D\big(\psi_t' \circ \psi_t^{-1}\big) \circ \big(S^t\big)^{-1}\big) = \mathrm{tr}\big(D\big(\psi_t' \circ \psi_t^{-1}\big)\big).$$

For simplicity we drop the subscribe t from ψ_t and also write $\psi = (\psi_1, \psi_2)$.

$$\begin{aligned} tr\big(D\big(\psi' \circ \psi^{-1}\big)\big) &= \frac{\partial \psi_1'}{\partial x}\frac{\partial \psi_2}{\partial y} - \frac{\partial \psi_1'}{\partial y}\frac{\partial \psi_2}{\partial x} + \frac{\partial \psi_2'}{\partial y}\frac{\partial \psi_1}{\partial x} - \frac{\partial \psi_2'}{\partial x}\frac{\partial \psi_1}{\partial y} \\ &= \frac{\partial}{\partial t}\left(\frac{\partial \psi_1}{\partial x}\frac{\partial \psi_2}{\partial y} - \frac{\partial \psi_1}{\partial y}\frac{\partial \psi_2}{\partial x}\right) \\ &= 0, \end{aligned}$$

where the last equality follows from the fact that $\det D\psi_t = 1$.

Finally, the vector field Y is defined by the pullback of $\hat{Y}$, that is

$$Y(q) = D\big(\Phi^{-1}\big)\big(\hat{Y}\big(\Phi(q)\big)\big).$$

From the construction it follows that if g is arbitrarily C^1-close to f then Y is arbitrarily C^1-close to X. As the map g is a return map of the Y^t-orbit of p it follows that Y has homoclinic tangencies.

4. Final remark

In the higher-dimensional case ($d \geqslant 4$) we do not have a version of the Mora–Romero theorem that allows us to obtain, using our approach, tangencies for vector fields $X \in \mathfrak{X}^1_\mu(M^d)$. Also, due to the non-stability of elliptic closed orbits, and to the existence of open sets of partially hyperbolic and non Anosov conservative flows, there is no hope to obtain a generalization of Theorem 1.2.

To construct an open set of non Anosov partially hyperbolic flows in dimension greater or equal to 4 we proceed as follows. Consider Mañé's example of a partially hyperbolic diffeomorphism on $\mathbb{T}^3$ [8]. Mañé's construction can be changed in order to get a similar example in the class of volume-preserving diffeomorphisms. This map has two hyperbolic saddles of different index. Now we consider the suspension of this map. In this way we obtain a partially hyperbolic volume-preserving flow, defined on a four-dimensional manifold, M^4, with two hyperbolic periodic closed orbits of different index; it follows that this flow is not Anosov. As partial hyperbolicity is an open property and hyperbolic closed orbits persist with the same index under perturbations, a small open neighborhood, in $\mathfrak{X}^1_\mu(M^4)$, of this flow gives the required open set.

We finish this paper with the following question, in the conservative flow setting, which goes in the spirit of Palis' conjecture.

Question. Can any $X \in \mathfrak{X}^1_\mu(M^d)$, $d \geqslant 4$, be C^1-approximated by a conservative flow exhibiting some form of hyperbolicity in M^d, or by one exhibiting homoclinic tangencies or else by one having a heterodimensional cycle?

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