Analytic Aspects of Poissonian White Noise Analysis*

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Abstract

General structures of Poissonian white noise analysis are presented. Simultaneously, the theory is developed on Poisson and Lebesgue-Poisson space. Both spaces have an own S-transform, well known in the Gaussian case. They give an extra connection between these two spaces via the Bargmann-Segal space. Test and generalized functions, different types of convolutions, and representations of creation and annihilation operators in the aforementioned spaces are considered.

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1 Introduction

Poissonian measures (PM) appear in several areas of mathematics and their applications to problems of physics, biology, chemistry, economics and other parts of modern science. In particular, PM are related with the study of point processes in probability theory, representation theory for diffeomorphism groups and current algebras, models of non-relativistic quantum fields, classical, quantum statistical mechanics, etc.. Any reasonable list of references here would be noncomplete. Let us mention only the paper [AKR99] for an overview concerning some applications in mathematical physics and annotated references. From another point of view, PM are subject of internal interest inside of infinite dimensional analysis. This is due to the fact that Gaussian and Poissonian measures are classes of measures on infinite dimensional spaces for which the corresponding analysis has a constructive character and a very reach structure, see e.g. books [BK88], [Hid80], [HKPS93] for detailed explanations concerning Gaussian analysis.

Poissonian analysis (PA) (i.e., an analysis on a space equipped with a Poisson measure π) has many specific aspects and directions from which we suppose discuss only particular ones. We start with a description of the general structure of Poissonian white noise analysis. This analysis is essentially based on the so-called chaos decomposition of Poissonian L^2-spaces. From the standpoint of probability theory this decomposition is related to the construction of multiple stochastic integrals with respect to compensated PM, see e.g. [Itô56], [NV95]. From the standpoint of representation theory it is related to the fact that the regular representation in the sense of Mackey of the diffeomorphism group on the configuration space equipped with the Poisson measure π can be also realized in the symmetric Fock space cf. [VGG75]. Alternatively, the chaos decomposition can be introduced using the orthogonal system of Charlier polynomials on Poisson space as it was done in [HI67], [Ito88], [IK88]. The latter can be considered as a particular case of the general concept of generalized Appell systems in non-Gaussian infinite dimensional analysis elaborated in [KSS97], [KSWY98]. (Note that useful additional aspects of Appell systems theory were developed in a generalization of this theory to the analysis on hypergroups [Ber98].) The chaos decomposition gives a unitary isomorphism between the space $L^2(\pi)$ and the symmetric Fock space. The latter space has a natural realization as an L^2-space with respect to the so-called Lebesgue-Poisson measure λ. This isomorphism uses the concept of symmetric measure spaces well known in infinite dimensional analysis, see e.g. [CP72], [Gui72]. In probability theory such isomorphism is known as a description of the symmetric Fock space in terms of counting measures and finite point processes, see [FF91], [FW93]. Let us mention that this is only one example illustrating the present situation: several constructions and concepts of PA are known in multiple versions. They were discovered independently by specialists from stochastics, mathematical physics and infinite dimensional analysis and form essentially disconnected areas. Establishing some links between different approaches to PA is one of the aims of this paper.

Poisson and Lebesgue-Poisson measures are discussed in Chapter 2. The aforementioned unitary isomorphism $I_{\pi\lambda}$ between $L^2(\pi)$ and $L^2(\lambda)$ is an essential technical tool in our considerations, see Chapter 3. This isomorphism can be considered as a version of the Fourier transform in PA. Such interpretation of the operator $I_{\pi\lambda}$ is motivated by the so-called spectral approach to PA developed in [Ber98], [Ber00], [BLL95]. Chapter 4 is devoted to the explanation of some aspects of generalized functions theory in PA. Because of the chaos decomposition this theory can be developed analogously to the Gaussian case as it was done in fact in [Ito88], [IK88]. But the approach used in this paper has two essential new moments. First of all, one Gaussian L^2-space splits into two spaces, $L^2(\pi)$ and $L^2(\lambda)$, and all constructions of test and generalized functions are presented now in two versions. And secondly, instead of the well-known S-transform in Gaussian analysis we have in our case two versions of this operator: S_π and S_λ which maps the spaces $L^2(\pi)$ and $L^2(\lambda)$, respectively, into the Bargmann-Segal space of analytic functions. Note that the chaos decomposition on the Poisson space was used the first time in [Ito88], [IK88] for the construction of a Hida type distribution theory in PA. The spaces of generalized functions over the Poisson space considered in our paper are of a similar but different type and corresponds to the spaces of test and generalized functions of the Gaussian analysis introduced in [Kon78], see also [HØUZ96] for a detailed explanation and applications in the Gaussian case.

Some bilinear operations are considered in Chapter 5. We discuss here the Wick product on Poisson space and the $*$-convolution on Lebesgue-Poisson space. The latter convolution plays an important role in applications to statistical physics, see [Rue69]. These operations are related by the unitary operator $I_{\pi\lambda}$. Other products on the Lebesgue-Poisson space are the Wiener- and Poisson-convolutions, see e.g. [Maa85], [LM90], [Lin90], [LP89] and [LP90], respectively.

The isomorphism between $L^2(\pi)$ and the Fock space, on the one hand, and the isomorphism between the Fock space and $L^2(\lambda)$, on the other hand, give us the possibility to transport some operators from the Fock space to $L^2(\pi)$ and $L^2(\lambda)$. In particular, the creation and annihilation operators well known in both spaces, see e.g. [KSSU98], [NV95] for the Poissonian case and [FW93] for the Lebesgue-Poisson one. We discuss more operators of PA in Chapter 6, see e.g. [Rue69]. Again, a specific point of our considerations is the use of the Fourier transform between the spaces $L^2(\pi)$ and $L^2(\lambda)$ in the study of these operators. In contrast to the theory of generalized functions on Poisson spaces we not only use that the Poisson measure can be realized as a measure on a space of distributions. In fact, we need that the support of this measure is smaller and is even contained in the set of all locally finite configurations on the underlying space. (We embed such configuration space into the distribution space by taking the sum of the Dirac's delta over the corresponding configuration.) This localization of the support becomes important also in several other situations in the study of PA.

Note that several important directions of PA are not at all reflected in this paper. Among them we would like to mention recent developments in the geometrical and differentiable structure of PA, see [AKR96], [AKR98a]. Moreover, PA should be considered as a part of a general configuration space analysis. This more general point of view gives us the possibility to exploit in PA such powerful techniques as the combinatorial harmonic analysis on configuration spaces which was elaborated in [KK99b], [KK02], [Kun99]. This harmonic analysis together with the Poissonian white noise analysis described in this paper produce new technical possibilities in applications of PA, in particular, to problems of mathematical physics. We will discuss in detail some related problems in a forthcoming paper [KKO02].

2 Poisson and Lebesgue-Poisson spaces

In order to describe the framework to be used along this work, first we recall the definition of configuration spaces over a manifold and their measurable structures.

We consider a measure space $(X, \mathcal{B}(X), \sigma)$ where X is a geodesically complete connected oriented (non-compact) C^∞-Riemannian manifold (describing the *position space* of the particles), $\mathcal{B}(X)$ is the Borel σ-algebra on X and σ is a Radon measure on $(X, \mathcal{B}(X))$. We assume that σ is non-degenerate (*i.e.*, $\sigma(O) > 0$ for all non-empty open sets $O \subset X$) and σ is non-atomic (*i.e.*, $\sigma(\{x\}) = 0$ for every $x \in X$). Note that $\sigma(\Lambda)$ is finite for each Λ belonging to the class $\mathcal{B}_c(X)$ of all $\mathcal{B}(X)$-measurable sets with compact closure. (Elements from $\mathcal{B}_c(X)$ are called *finite volumes*.) Having in mind the most interesting applications, additionally we assume that $\sigma(X) = \infty$.

2.1 Configuration spaces over a manifold

Definition 2.1 *The configuration space $\Gamma := \Gamma_X$ over the manifold X is defined as the set of all locally finite subsets of X:*

$$\Gamma := \{\gamma \subset X : |\gamma \cap K| < \infty \text{ for every compact } K \subset X\},$$

where $|A|$ denotes the cardinality of a set A. The elements of the space Γ are called configurations.

Given a subset $Y \subset X$ we denote by Γ_Y the space of all configurations contained in Y, *i.e.*,

$$\Gamma_Y := \{\gamma \in \Gamma : |\gamma \cap (X \backslash Y)| = 0\}.$$

Let us describe the measurable structure on the configuration space Γ. For each $\Lambda \in \mathcal{B}_c(X)$ consider the mapping $N_\Lambda : \Gamma \to \mathbb{N}_0$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$,

$$N_\Lambda(\gamma) := |\Lambda \cap \gamma|, \quad \gamma \in \Gamma.$$

The σ-algebra $\mathcal{B}(\Gamma)$ is defined as the smallest σ-algebra on Γ such that all mappings N_Λ, $\Lambda \in \mathcal{B}_c(X)$, are measurable, *i.e.*,

$$\mathcal{B}(\Gamma) := \sigma\left(\{N_\Lambda : \Lambda \in \mathcal{B}_c(X)\}\right).$$

Analogously, for $Y \in \mathcal{B}(X)$ the σ-algebra $\mathcal{B}(\Gamma_Y)$ is defined. This σ-algebra is σ-isomorphic to the σ-algebra $\mathcal{B}_Y(\Gamma)$ defined on Γ by

$$\mathcal{B}_Y(\Gamma) := \sigma\left(\{N_\Lambda : \Lambda \in \mathcal{B}_c(X), \Lambda \subset Y\}\right).$$

There exists a natural measurable mapping $p_\Lambda : \Gamma \to \Gamma_\Lambda$ for $\Lambda \in \mathcal{B}_c(X)$. Similarly we can consider measurable mappings $p_{\Lambda',\Lambda} : \Gamma_{\Lambda'} \to \Gamma_\Lambda$ for Λ, $\Lambda' \in \mathcal{B}_c(X)$, $\Lambda \subset \Lambda'$. They are defined, respectively, by

$$\begin{aligned} p_\Lambda(\gamma) &:= \gamma_\Lambda, \quad \gamma \in \Gamma, \\ p_{\Lambda',\Lambda}(\gamma) &:= \gamma_\Lambda, \quad \gamma \in \Gamma_{\Lambda'}, \end{aligned}$$

where $\gamma_\Lambda := \gamma \cap \Lambda$. It can be shown (see, *e.g.*, [Shi94]) that the measurable space $(\Gamma, \mathcal{B}(\Gamma))$ coincides (up to an isomorphism) with the projective limit of the measurable spaces $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$, $\Lambda \in \mathcal{B}_c(X)$, with respect to the above projections p_Λ.

Another description of $\mathcal{B}(\Gamma)$ is also possible. We can identify each $\gamma \in \Gamma$ with the positive integer valued Radon measure

$$\sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}_{\mathbb{N}_0}(X) \subset \mathcal{M}^+(X),$$

where ε_x is the Dirac measure at x, $\sum_{x\in\emptyset} \varepsilon_x :=$ zero measure and $\mathcal{M}^+(X)$ (resp., $\mathcal{M}_{\mathbb{N}_0}(X)$) denotes the set of all positive (resp., $\mathbb{N}_0$-valued) Radon measures on $\mathcal{B}(X)$. In this way, Γ can be endowed with the topology induced by the *vague topology* on $\mathcal{M}^+(X)$, *i.e.*, the weakest topology on Γ with respect to which all mappings

$$\Gamma \ni \gamma \longmapsto \langle \gamma, f\rangle := \int_X f(x)\, d\gamma(x) = \sum_{x\in\gamma} f(x)$$

are continuous for each $f \in C_0(X)$ (:= the space of all real continuous functions on X with compact support). The corresponding Borel σ-algebra coincides with the above defined σ-algebra $\mathcal{B}(\Gamma)$ (see, *e.g.*, [Kal83]).

Definition 2.2 *Given $Y \in \mathcal{B}(X)$, $n \in \mathbb{N}_0$, the space of n-point configurations $\Gamma_Y^{(n)}$ is the subset of Γ_Y defined by*

$$\Gamma_Y^{(n)} := \{\gamma \in \Gamma_Y : |\gamma| = n\}, n \in \mathbb{N}, \quad \Gamma_Y^{(0)} := \{\emptyset\}.$$

Using the mapping

$$\begin{array}{ll} \mathrm{sym}_Y^n : \tilde{Y}^n & \to \Gamma_Y^{(n)}, \quad Y \in \mathcal{B}(X), n \in \mathbb{N}, \\ (x_1, ..., x_n) & \longmapsto \{x_1, ..., x_n\}, \end{array}$$

where

$$\tilde{Y}^n := \{(x_1, ..., x_n) : x_i \in Y, x_i \neq x_j \text{ if } i \neq j\},$$

one may construct a bijective mapping between $\Gamma_Y^{(n)}$ and the symmetrization $\tilde{Y}^n \diagup S_n$ of $\tilde{Y}^n$, where S_n is the permutation group over $\{1, ..., n\}$. This mapping induces a σ-algebra on $\Gamma_Y^{(n)}$ denoted by $\mathcal{B}(\Gamma_Y^{(n)})$.

Note that for $\Lambda \in \mathcal{B}_c(X)$ we have $\Gamma_\Lambda = \bigsqcup_{n=0}^\infty \Gamma_\Lambda^{(n)}$ which provides a representation for the space Γ_Λ and gives an equivalent description of the σ-algebra $\mathcal{B}(\Gamma_\Lambda)$ as the σ-algebra of a disjoint union.

Definition 2.3 *The space of finite configurations* Γ_0 *is the subset of* Γ *defined by*

$$\Gamma_0 := \{\gamma \in \Gamma : |\gamma| < \infty\} = \bigsqcup_{n=0}^{\infty} \Gamma_X^{(n)}. \tag{1}$$

The space Γ_0 is naturally equipped with the σ-algebra $\mathcal{B}(\Gamma_0)$ given by the disjoint union of the measurable spaces $(\Gamma_X^{(n)}, \mathcal{B}(\Gamma_X^{(n)}))$.

2.2 Poisson and Lebesgue-Poisson measures

There are several ways to define the Poisson measure on the space Γ, see, *e.g.*, [GV68], [IK88], [Ito88], [AKR98b] and the references therein. Following the above scheme, first we define it in a constructive way. An alternative definition, through the Laplace transform, will also be presented.

Given a $n \in \mathbb{N}$ let us consider the product measure $\sigma^{\otimes n}$ on the measurable space $(X^n, \mathcal{B}(X^n))$. Of course, $\sigma^{\otimes n}(X^n \backslash \tilde{X}^n) = 0$. For $Y \in \mathcal{B}(X)$ we consider the measure $\sigma^{\otimes n}$ restricted to $(\tilde{Y}^n, \mathcal{B}(\tilde{Y}^n))$ and the mapping sym_Y^n. We define a measure $\sigma_Y^{(n)}$ on $(\Gamma_Y^{(n)}, \mathcal{B}(\Gamma_Y^{(n)}))$ by $\sigma_Y^{(n)} := \sigma^{\otimes n} \circ (\mathrm{sym}_Y^n)^{-1}$. For $n = 0$ we put $\sigma_Y^{(0)}(\{\emptyset\}) := 1$.

Definition 2.4 *The* σ*-finite measure defined on* $(\Gamma_0, \mathcal{B}(\Gamma_0))$ *by*

$$\lambda_\sigma := \sum_{n=0}^{\infty} \frac{1}{n!} \sigma_X^{(n)}$$

is called the Lebesgue-Poisson measure with intensity measure σ*. The measure space* $(\Gamma_0, \mathcal{B}(\Gamma_0), \lambda_\sigma)$ *is called the Lebesgue-Poisson space.*

To simplify the notation, we denote the measure $\sigma_Y^{(n)}$ by $\sigma^{(n)}$ if $Y = X$. Throughout this work, L^p-spaces consist of complex valued functions. A subscript "*Re*" will be added if we consider the corresponding real space. The L^p-spaces with respect to a measure μ are denoted by $L^p(\mu)$ if the underlying measurable space is clear from the context.

Remark 2.5 If the measure σ is finite, then the Lebesgue-Poisson measure is also a finite measure and $\lambda_\sigma(\Gamma_0) = \exp(\sigma(X))$.

Let us consider the restriction of the measure λ_σ to the measurable space $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ $(\Lambda \in \mathcal{B}_c(X))$ which is a finite measure with $\lambda_{\sigma|\Gamma_\Lambda}(\Gamma_\Lambda) = \exp(\sigma(\Lambda))$. Thus we can define a probability measure on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ by

$$\pi_\sigma^\Lambda := \exp(-\sigma(\Lambda))\lambda_{\sigma|\Gamma_\Lambda}.$$

This family of probability measures allows us to introduce a probability measure π_σ on $(\Gamma, \mathcal{B}(\Gamma))$ through a version of the Kolmogorov theorem for the projective limit space $(\Gamma, \mathcal{B}(\Gamma))$ (cf. [Par67, Chap. V Theorem 5.1]).

Definition 2.6 *The probability measure π_σ is called the Poisson measure with intensity measure σ. The measure space $(\Gamma, \mathcal{B}(\Gamma), \pi_\sigma)$ is called the Poisson space.*

Remark 2.7 If the measure σ is finite, then $\pi_\sigma = \exp(-\sigma(X))\lambda_\sigma$.

Let $f \in C_0(X)$ be given. A simple calculation yields

$$l_\sigma(f) := \int_\Gamma \exp(\langle\gamma, f\rangle) d\pi_\sigma(\gamma) = \exp\left(\int_X (e^{f(x)} - 1) d\sigma(x)\right).$$

The above equality may be used for another description of the Poisson measure. Let us consider the Schwartz space $\mathcal{D} = C_0^\infty(X)$ of all real C^∞-functions on X with compact support and its dual space $\mathcal{D}'$ (with respect to $L^2_{Re}(\sigma)$). On $\mathcal{D}'$ we fix the σ-algebra $\mathcal{C}_\sigma(\mathcal{D}')$ generated by the *cylinder sets*

$$\{\omega \in \mathcal{D}' : (\langle\omega, \varphi_1\rangle, ..., \langle\omega, \varphi_n\rangle) \in B\}, \quad \varphi_i \in \mathcal{D}, B \in \mathcal{B}(\mathbb{R}^n), n \in \mathbb{N}.$$

Through the Minlos theorem, l_σ uniquely determines a probability measure π_σ on the measurable space $(\mathcal{D}', \mathcal{C}_\sigma(\mathcal{D}'))$, see, *e.g.*, [GV68, Vol. IV]. In order to obtain an independent construction, additionally, one needs to show that the support of the measure π_σ consists of generalized functions of the form $\sum_{x\in\gamma} \varepsilon_x$, $\gamma \in \Gamma$.

Remark 2.8 The Poisson measure π_σ can either be consider on $(\Gamma, \mathcal{B}(\Gamma))$ or on $(\mathcal{D}', \mathcal{C}_\sigma(\mathcal{D}'))$ where, in contrast to Γ, $\mathcal{D}' \supset \Gamma$ is a linear space. Since $\pi_\sigma(\Gamma) = 1$, the measure space $(\mathcal{D}', \mathcal{C}_\sigma(\mathcal{D}'), \pi_\sigma)$ can, in this way, be regarded as a linear extension of the Poisson space $(\Gamma, \mathcal{B}(\Gamma), \pi_\sigma)$.

3 Fourier transform in Poissonian analysis

A description of the elements of the space $L^2(\pi_\sigma)$ can be given by the corresponding chaos decomposition. For this purpose, first we recall the notion of Fock space over a Hilbert space, see, *e.g.*, [Gui72, Chapter 2], [RS75], [Seg56].

Given a complex separable Hilbert space $\mathcal{H}$, for each $n \in \mathbb{N}$ let $\mathcal{H}^{\otimes n}$ be the n-th tensor power of $\mathcal{H}$ and let $\mathcal{H}^{\hat{\otimes} n}$ be the n-th symmetric tensor power of $\mathcal{H}$. In the case $\mathcal{H} = L^2(X, \mathcal{B}, \tau)$ for a given measure space $(X, \mathcal{B}, \tau)$, the space $\mathcal{H}^{\hat{\otimes} n}$ is unitarily isomorphic to the subspace $\hat{L}^2(X^n, \tau^{\otimes n}) \subset L^2(X^n, \tau^{\otimes n})$ of all symmetric square integrable functions. Therefore, we may identify the space $\mathcal{H}^{\hat{\otimes} n}$ with $\hat{L}^2(X^n, \tau^{\otimes n})$.

Definition 3.1 *The Fock space (Bose or symmetric)* $\mathrm{Exp}\mathcal{H}$ *over* $\mathcal{H}$ *is defined by the Hilbertian direct sum*

$$\mathrm{Exp}\mathcal{H} := \bigoplus_{n=0}^{\infty} \mathrm{Exp}_n\mathcal{H}, \quad \mathrm{Exp}_0\mathcal{H} := \mathbb{C},$$

where $\mathrm{Exp}_n\mathcal{H}$ *(called n-particle subspace) is the space* $\mathcal{H}^{\hat{\otimes} n}$ *provided with the inner product* $n!(\cdot,\cdot)_{\mathcal{H}^{\otimes n}}$, $n \in \mathbb{N}$. *The subspace* $\mathrm{Exp}_0\mathcal{H}$ *is called the vacuum subspace.*

Thus, a generic element F of the Fock space $\mathrm{Exp}\mathcal{H}$ is a sequence $F = (f^{(n)})_{n=0}^{\infty}$ with $f^{(n)} \in \mathrm{Exp}_n\mathcal{H}$ for each $n \in \mathbb{N}$, and

$$\|F\|^2_{\mathrm{Exp}\mathcal{H}} = \sum_{n=0}^{\infty} n! \left|f^{(n)}\right|^2_{\mathcal{H}^{\otimes n}} < \infty.$$

A special class of elements of the Fock space consists of the so-called *coherent states* (or *exponential vectors*) $e(f)$ corresponding to the one-particle state $f \in \mathcal{H}$:

$$e(f) := \left(1, f, \frac{1}{2!} f^{\otimes 2}, ..., \frac{1}{n!} f^{\otimes n}, ...\right), \qquad f^{\otimes n} := f \otimes ... \otimes f (n \text{ times}).$$

Given a dense subspace $\mathcal{L} \subset \mathcal{H}$, the family of coherent states $\{e(f) : f \in \mathcal{L}\} \subset \mathrm{Exp}\mathcal{H}$ is total in $\mathrm{Exp}\mathcal{H}$ (see, *e.g.*, [BK88]).

For what follows it is also useful to recall the notion of holomorphic function defined on a general locally convex topological vector space $\mathcal{E}$ (over the complex field $\mathbb{C}$) see, *e.g.*, [Bar85], [Col82] and [Din81]. A function $F : \mathcal{U} \to \mathbb{C}$ defined on an open set $\mathcal{U} \subset \mathcal{E}$ is called *holomorphic* if for all $\zeta \in \mathcal{U}$ there exists an open neighborhood $\mathcal{V}$ of zero such that the series $\sum_{n=0}^{\infty} \frac{1}{n!} \widehat{d^n F(\zeta)}(\theta)$, $\widehat{d^n F(\zeta)}(\theta) := d^n F(\zeta)(\theta, ..., \theta)$ (the n-th directional derivative of F at the point ζ) converges on $\theta \in \mathcal{V}$ to a continuous function. In particular, a holomorphic function $F : \mathcal{E} \to \mathbb{C}$ is called *entire*. F is called *holomorphic at a point* $\theta_0 \in \mathcal{E}$ if there exists an open neighborhood $\mathcal{U} \subset \mathcal{E}$ of θ_0 such that $F : \mathcal{U} \to \mathbb{C}$ is holomorphic.

Keeping in mind the embeddings $\Gamma \subset \mathcal{M}^+(X) \subset \mathcal{D}'$, let us consider the triple

$$\mathcal{D} \subset L^2_{Re}(\sigma) \subset \mathcal{D}'.$$

In the sequel $\mathcal{D}_{\mathbb{C}}$ denotes the complexified space of $\mathcal{D}$.

Introducing the function α,

$$\alpha(\varphi)(x) := \log(1 + \varphi(x)), \quad \varphi \in \mathcal{D}_{\mathbb{C}}, x \in X,$$

which is invertible and holomorphic on a neighborhood $\mathcal{U} \subset \mathcal{D}_{\mathbb{C}}$ of zero, we can define the *normalized exponential* $e_\pi(\cdot, \cdot)$ by

$$e_\pi(\varphi, \omega) := \frac{\exp(\langle \omega, \alpha(\varphi) \rangle)}{l_\sigma(\alpha(\varphi))} = \exp\left(\langle \omega, \log(1 + \varphi) \rangle - \langle \varphi \rangle_\sigma\right),$$

for $\omega \in \mathcal{D}'$, $\varphi \in \mathcal{U}$, and

$$\langle \varphi \rangle_\sigma := \int_X \varphi(x) d\sigma(x).$$

Using the holomorphy of $e_\pi(\cdot, \omega)$ on $\mathcal{U}$ we consider its Taylor expansion which, by the Cauchy formula, the polarization identity, and the kernel theorem (see, *e.g.*, [BK88], [KSS97], [KSWY98]) provides the decomposition

$$e_\pi(\varphi, \omega) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle C_n^\sigma(\omega), \varphi^{\otimes n} \rangle, \quad \omega \in \mathcal{D}',$$

where $C_n^\sigma : \mathcal{D}' \to \mathcal{D}'^{\hat{\otimes} n}$, $n \in \mathbb{N}$, are the so-called *generalized Charlier kernels*. Hence, for any $\varphi^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}$, $n \in \mathbb{N}$, we can define the *smooth Charlier monomial of order* n as $\langle C_n^\sigma(\omega), \varphi^{(n)} \rangle$, $\omega \in \mathcal{D}'$.

The following orthogonality relation holds:

$$\left(\langle C_n^\sigma, \varphi^{(n)} \rangle, \langle C_m^\sigma, \phi^{(m)} \rangle\right)_{L^2(\pi_\sigma)} = \delta_{n,m} n! (\varphi^{(n)}, \phi^{(n)})_{L^2(\sigma^{\otimes n})}, \tag{2}$$

which allows the use of an approximation procedure to extend the class of smooth Charlier monomials to measurable monomials $\langle C_n^\sigma, f^{(n)}\rangle$ with kernels $f^{(n)} \in \hat{L}^2(X^n, \sigma^{\otimes n})$ in such a way that the above orthogonality property still holds for this extension.

Consider the space $\mathcal{P}(\mathcal{D}')$ of *smooth continuous polynomials on* $\mathcal{D}'$:

$$\mathcal{P}(\mathcal{D}') := \left\{ \Phi : \Phi(\omega) = \sum_{n=0}^{N} \langle C_n^\sigma(\omega), \varphi^{(n)}\rangle, \varphi^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}, \omega \in \mathcal{D}', N \in \mathbb{N}_0 \right\}.$$

We endow $\mathcal{P}(\mathcal{D}')$ with the natural topology such that $\mathcal{P}(\mathcal{D}')$ becomes a nuclear space (see, *e.g.*, [BK88]). We observe that $\mathcal{P}(\mathcal{D}')$ is independent of the Poisson measure, see, *e.g.*, [KSS97] for details.

Since the space $\mathcal{P}(\mathcal{D}')$ is densely embedded into $L^2(\pi_\sigma)$ [Sko74, Section 10, Theorem 1], it follows that for any $F \in L^2(\pi_\sigma)$ there exists a sequence $\left(f^{(n)}\right)_{n=0}^\infty \in \mathrm{Exp}L^2(\sigma)$ such that

$$F = \sum_{n=0}^{\infty} \langle C_n^\sigma, f^{(n)}\rangle \tag{3}$$

and, moreover,

$$\|F\|^2_{L^2(\pi_\sigma)} = \sum_{n=0}^{\infty} n! \left|f^{(n)}\right|^2_{L^2(\sigma^{\otimes n})} = \left\| \left(f^{(n)}\right)_{n=0}^\infty \right\|^2_{\mathrm{Exp}L^2(\sigma)}.$$

Vice versa, any series of the form (3) with $\left(f^{(n)}\right)_{n=0}^\infty \in \mathrm{Exp}L^2(\sigma)$ defines a function from $L^2(\pi_\sigma)$. As a result, we have the so-called *chaos decomposition* (3) which yields a unitary isomorphism I_π between $L^2(\pi_\sigma)$ and the Fock space $\mathrm{Exp}L^2(\sigma)$ defined by

$$I_\pi \left(\sum_{n=0}^{\infty} \langle C_n^\sigma, f^{(n)}\rangle \right) := \left(f^{(n)}\right)_{n=0}^\infty.$$

Remark 3.2 The unitary operator $I_\pi^{-1} : \mathrm{Exp}L^2(\sigma) \longrightarrow L^2(\pi_\sigma)$ can be considered as the Fourier transform corresponding to a commuting family of "generalized field operators" in $\mathrm{Exp}L^2(\sigma)$, see, *e.g.*, [Ber98], [Ber00], [BLL95], [Lyt95].

The space $L^2(\lambda_\sigma)$ can be regarded as another realization of the Fock space. Indeed, by the decomposition (1), each function G in $L^2(\lambda_\sigma)$ is well defined if we know $G(\{x_1,...,x_n\})$ for $\{x_1,...,x_n\} \in \Gamma_X^{(n)}$, $n \in \mathbb{N}$, and $G(\emptyset)$. Thus, there exists a natural (linear) correspondence between the space $L^2(\lambda_\sigma)$ and the Fock space $\mathrm{Exp}L^2(\sigma)$:

$$\begin{array}{rcl} I_\lambda : L^2(\lambda_\sigma) & \rightarrow & \mathrm{Exp}L^2(\sigma) \\ G & \longmapsto & \left(f^{(n)}\right)_{n=0}^{\infty} \end{array}$$

where $f^{(n)}(x_1,...,x_n) := \frac{1}{n!}G(\{x_1,...,x_n\})$, $n \in \mathbb{N}$, $f^{(0)} := G(\emptyset)$. Since the measure σ is non-atomic, the kernels $f^{(n)}$ are well defined as elements of the space $\hat{L}^2(X^n, \sigma^{\otimes n})$. Moreover, we have the following equalities:

$$\begin{aligned} \|I_\lambda G\|^2_{\mathrm{Exp}L^2(\sigma)} &= \sum_{n=0}^{\infty} n! \int_{X^n} \left|f^{(n)}(x_1,...,x_n)\right|^2 d\sigma^{\otimes n}(x_1,...,x_n) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{X^n} \left|G\left(\mathrm{sym}_X^n(x_1,...,x_n)\right)\right|^2 d\sigma^{\otimes n}(x_1,...,x_n) \\ &= \|G\|^2_{L^2(\lambda_\sigma)}. \end{aligned}$$

This means that I_λ defines a unitary isomorphism between $L^2(\lambda_\sigma)$ and the Fock space $\mathrm{Exp}L^2(\sigma)$.

Therefore, given a dense subspace $\mathcal{L} \subset L^2(\sigma)$, the set

$$\left\{e_\lambda(f,\cdot) := I_\lambda^{-1}e(f), \quad f \in \mathcal{L}\right\}$$

is a total set in $L^2(\lambda_\sigma)$. From the above considerations we have

$$e_\lambda(f,\{x_1,...,x_n\}) := \left(I_\lambda^{-1}e(f)\right)(\{x_1,...,x_n\}) = \prod_{i=1}^{n} f(x_i), \quad n \in \mathbb{N},$$

$e_\lambda(f,\emptyset) := 1$.

More generally, we may define $e_\lambda(f,\cdot)$ for any $\mathcal{B}(X)$-measurable function f.

Definition 3.3 *Given a $\mathcal{B}(X)$-measurable function $f : X \to \mathbb{C}$ we define (pointwisely) a $\mathcal{B}(\Gamma_0)$-measurable function $e_\lambda(f,\cdot) : \Gamma_0 \to \mathbb{C}$ by*

$$e_\lambda(f,\eta) := \prod_{x\in\eta} f(x), \quad \eta \in \Gamma_0.$$

The function $e_\lambda(f,\cdot)$ is called the (Lebesgue-Poisson) coherent state corresponding to the one-particle vector f.

This definition implies that if $f \in L^p(\sigma)$ $(p \geq 1)$, then $e_\lambda(f,\cdot) \in L^p(\lambda_\sigma)$ and

$$\|e_\lambda(f,\cdot)\|^p_{L^p(\lambda_\sigma)} = \exp\left(\|f\|^p_{L^p(\sigma)}\right). \tag{4}$$

Proposition 3.4 *The linear mapping $I_{\lambda\pi} := I_\pi^{-1} \circ I_\lambda : L^2(\lambda_\sigma) \to L^2(\pi_\sigma)$,*

$$\begin{aligned} I_{\lambda\pi}(G) &= \sum_{n=0}^{\infty} \left\langle C_n^\sigma, f^{(n)} \right\rangle, \\ & \quad f^{(n)}(x_1,...,x_n) := \frac{1}{n!} G\left(\{x_1,...,x_n\}\right), n \in \mathbb{N}, f^{(0)} := G(\emptyset). \end{aligned}$$

is a unitary isomorphism whose inverse mapping $I_{\pi\lambda} := I_\lambda^{-1} \circ I_\pi$ is given by

$$\begin{aligned} & I_{\pi\lambda}\left(\sum_{n=0}^{\infty} \left\langle C_n^\sigma, f^{(n)} \right\rangle\right) = G, \\ & G\left(\{x_1,...,x_n\}\right) := n! f^{(n)}(x_1,...,x_n), n \in \mathbb{N}, G(\emptyset) := f^{(0)}. \end{aligned}$$

Remark 3.5 In particular, $I_{\lambda\pi}$ maps the coherent state $e_\lambda(\varphi,\cdot)$, $\varphi \in \mathcal{U} \subset \mathcal{D}_\mathbb{C}$, to the normalized exponential $e_\pi(\varphi,\cdot)$:

$$\begin{aligned} \left(I_{\lambda\pi} e_\lambda(\varphi,\cdot)\right)(\gamma) &= \exp\left(\langle \gamma, \log(1+\varphi)\rangle - \langle \varphi \rangle_\sigma\right) \\ &= e^{-\langle \varphi \rangle_\sigma} \prod_{x\in\gamma} (1+\varphi(x)). \end{aligned}$$

The definition of normalized exponentials can be extended to functions $f \in L^1(\sigma)$. Indeed, since

$$\int_\Gamma \sum_{x\in\gamma} |f(x)| \, d\pi_\sigma(\gamma) = \int_\Gamma \langle \gamma, |f|\rangle \, d\pi_\sigma(\gamma) = \int_X |f(x)| \, d\sigma(x) < \infty,$$

the sum $\sum_{x\in\gamma} |f(x)|$ is π_σ-a.s. finite. Hence, the product $\prod_{x\in\gamma}(1+f(x))$ is π_σ-a.s. absolutely convergent. Thus, for $f \in L^1(\sigma)$ one can define the normalized exponential $e_\pi(f,\cdot)$ for π_σ-a.a. $\gamma \in \Gamma$ by the same expression:

$$e_\pi(f,\gamma) := e^{-\langle f \rangle_\sigma} \prod_{x\in\gamma} (1+f(x)).$$

Also the extension in L^2-sense has been considered see e.g. [Lie94].

Remark 3.6 The unitary isomorphism $I_{\lambda\pi} : L^2(\lambda_\sigma) \to L^2(\pi_\sigma)$ can be considered as another realization of the Fourier transform cf. Remark 3.2.

$$
\begin{tikzcd}
 & \begin{array}{c} L^2(\Gamma, \pi_\sigma) \\ e_\pi(\varphi) \end{array} \arrow[dl, "I_\pi"'] \\
\begin{array}{c} \mathrm{Exp} L^2(\sigma) \\ e(\varphi) \end{array} & \\
 & \begin{array}{c} L^2(\Gamma_0, \lambda_\sigma) \\ e_\lambda(\varphi) \end{array} \arrow[ul, "I_\lambda"] \arrow[uu, "I_{\lambda\pi}"']
\end{tikzcd}
$$

$$
\begin{aligned}
e(\varphi) &= \left(\frac{\varphi^{\otimes n}}{n!} \right)_{n \in \mathbb{N}_0} \\
e_\lambda(\varphi, \eta) &= \prod_{x \in \eta} \varphi(x) \\
e_\pi(\varphi, \gamma) &= \prod_{x \in \gamma} (1 + \varphi(x)) \exp(-\langle \varphi \rangle_\sigma) \\
&= \exp(\langle \gamma, \log(1 + \varphi) \rangle) \exp(-\langle \varphi \rangle_\sigma)
\end{aligned}
$$

Figure 1: Coherent states

4 Test and generalized functions in Poissonian analysis

We recall the definition and some properties of test and generalized functions on the Poisson space, in particular, characterization results for test and generalized functions (see, *e.g.*, [ADKS96], [KSS97], [KSWY98], [Us95] and the references therein).

4.1 Nuclear triples

Let $\mathcal{H}_{Re}$ be a real separable Hilbert space with inner product $(\cdot, \cdot)$ and the corresponding norm $|\cdot|$. The particular example we have in mind is the space $\mathcal{H}_{Re} = L^2_{Re}(\sigma)$. Given a separable nuclear space $\mathcal{N}$ densely and topologically embedded into $\mathcal{H}_{Re}$ we can construct the nuclear triple

$$\mathcal{N} \subset \mathcal{H}_{Re} \subset \mathcal{N}',$$

where $\mathcal{N}'$ is the dual space of $\mathcal{N}$ with respect to $\mathcal{H}_{Re}$. The dual pairing $\langle \cdot, \cdot \rangle$ of $\mathcal{N}$ and $\mathcal{N}'$ is then realized as an extension of the inner product on $\mathcal{H}_{Re}$:

$$\langle f, \xi \rangle = (f, \xi), \quad f \in \mathcal{H}_{Re}, \xi \in \mathcal{N}.$$

A convenient characterization of nuclear Fréchet spaces can be given in terms of projective limits of Hilbert spaces (cf. [Sch71]). In this way, the space $\mathcal{N}$ can be represented as

$$\mathcal{N} = \bigcap_{p\in\mathbb{N}} \mathcal{H}_p,$$

where $\{(\mathcal{H}_p, (\cdot,\cdot)_p) : p \in \mathbb{N}\}$ is a family of Hilbert spaces such that for all $p_1,\ p_2 \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that the embeddings $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_1}$ and $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_2}$ are of Hilbert-Schmidt class. The topology of $\mathcal{N}$ is given by the *projective limit topology*, *i.e.*, the coarsest topology on $\mathcal{N}$ such that the canonical embeddings $\mathcal{N} \hookrightarrow \mathcal{H}_p$ are continuous for all $p \in \mathbb{N}$. Denoting the Hilbertian norm on $\mathcal{H}_p$ by $|\cdot|_p$, $p \in \mathbb{N}$, without loss of generality we suppose that for each $p \in \mathbb{N}$ and $\xi \in \mathcal{N}$, $|\xi| \le |\xi|_p$, and that the system of norms is ordered, *i.e.*, $|\cdot|_p \le |\cdot|_q$ if $p \le q$. Applying the general duality theory the dual space $\mathcal{N}'$ (with respect to $\mathcal{H}_{Re}$) can be written as

$$\mathcal{N}' = \bigcup_{p\in\mathbb{N}} \mathcal{H}_{-p},$$

which is provided with the inductive limit topology given by the family of dual spaces $\{\mathcal{H}_{-p} := \mathcal{H}'_p : p \in \mathbb{N}\}$. The *inductive limit topology* (with respect to this family) is the finest topology on $\mathcal{N}'$ such that the embeddings $\mathcal{H}_{-p} \hookrightarrow \mathcal{N}'$ are continuous for all $p \in \mathbb{N}$. We denote the norm on $\mathcal{H}_{-p}$ by $|\cdot|_{-p}$.

Additionally, we introduce the notion of symmetric tensor product of nuclear spaces. Since there is no risk of confusion, we may preserve the notations $|\cdot|_p$ and $|\cdot|_{-p}$ for the norms on the symmetric tensor power $\mathcal{H}_p^{\hat{\otimes}n}$ and $\mathcal{H}_{-p}^{\hat{\otimes}n}$, $n \in \mathbb{N}$, respectively. Define

$$\mathcal{N}^{\hat{\otimes}n} := \operatorname*{prlim}_{p\in\mathbb{N}} \mathcal{H}_p^{\hat{\otimes}n}.$$

One can prove, see, *e.g.*, [Sch71], that $\mathcal{N}^{\hat{\otimes}n}$ is a nuclear space which is called the *n-th symmetric tensor power of* $\mathcal{N}$. The dual space $\mathcal{N}'^{\hat{\otimes}n}$ (with respect to $\mathcal{H}_{Re}^{\hat{\otimes}n}$) can be written as

$$\mathcal{N}'^{\hat{\otimes}n} := \operatorname*{indlim}_{p\in\mathbb{N}} \mathcal{H}_{-p}^{\hat{\otimes}n}.$$

As before we use the notation $\langle\cdot,\cdot\rangle$ for the bilinear dual pairing between $\mathcal{N}'^{\hat{\otimes}n}$ and $\mathcal{N}^{\hat{\otimes}n}$.

All the above quoted results remain true for complex spaces. In that case, we use the notations $|\cdot|_p$ and $|\cdot|_{-p}$ for the norms on $\mathcal{H}_{p,\mathbb{C}}^{\hat{\otimes}n}$ and $\mathcal{H}_{-p,\mathbb{C}}^{\hat{\otimes}n}$, $n \in \mathbb{N}$, respectively, and $\langle\cdot,\cdot\rangle$ for the dual pairing between $\mathcal{N}_{\mathbb{C}}'^{\hat{\otimes}n}$ and $\mathcal{N}_{\mathbb{C}}^{\hat{\otimes}n}$.

4.2 Test and generalized functions

In order to construct spaces of test and generalized functions we will use the nuclear triple

$$\mathcal{D} \subset \mathcal{N} \subset L^2_{Re}(\sigma) \subset \mathcal{N}' \subset \mathcal{D}'.$$

As before, we consider a family of Hilbert spaces $\{(\mathcal{H}_p, (\cdot,\cdot)_p) : p \in \mathbb{N}\}$ associated to the nuclear space $\mathcal{N}$. For $\Phi \in \mathcal{P}(\mathcal{D}')$, $\Phi = \sum_{n=0}^N \langle C_n^\sigma, \varphi^{(n)}\rangle$, $\varphi^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes} n}$, let us consider the family of Hilbertian norms

$$\|\Phi\|^2_{p,q,\pi} := \sum_{n=0}^{\infty} (n!)^2 2^{nq} \left|\varphi^{(n)}\right|_p^2, \quad p, q \in \mathbb{N}.$$

Define the Hilbert spaces $(\mathcal{H}_p)^1_{q,\pi}$ as the completion of $\mathcal{P}(\mathcal{D}')$ with respect to the norm $\|\cdot\|_{p,q,\pi}$. The space of test functions $(\mathcal{N})^1_\pi$ is then defined as the projective limit of $(\mathcal{H}_p)^1_{q,\pi}$:

$$(\mathcal{N})^1_\pi := \operatorname{pr}\lim_{p,q\in\mathbb{N}} (\mathcal{H}_p)^1_{q,\pi}.$$

Remark 4.1 In an equivalent way we have

$$(\mathcal{H}_p)^1_{q,\pi} = \left\{ F = \sum_{n=0}^{\infty} \langle C_n^\sigma, f^{(n)}\rangle \in L^2(\pi_\sigma) : \|F\|_{p,q,\pi} < \infty \right\}.$$

Remark 4.2 The topology of $(\mathcal{N})^1_\pi$ is uniquely determined by the topology of $\mathcal{N}$ and $(\mathcal{N})^1_\pi$ is a nuclear space continuously and densely embedded into $L^2(\pi_\sigma)$. Furthermore, $(\mathcal{N})^1_\pi$ is an algebra under the pointwise multiplication of functions. See, *e.g.*, [KSWY98], [KK99a].

The chaos decomposition provides a natural decomposition of the elements of the dual space $\mathcal{P}'_\pi(\mathcal{D}')$ of $\mathcal{P}(\mathcal{D}')$ (with respect to $L^2(\pi_\sigma)$). For any given $\psi^{(n)} \in \mathcal{D}'^{\hat{\otimes} n}_{\mathbb{C}}$ there exists a unique distribution in $\mathcal{P}'_\pi(\mathcal{D}')$, denoted by $\left\langle \psi^{(n)}, C_n^\sigma \right\rangle$, acting on smooth continuous polynomials $\Phi = \sum_{n=0}^N \langle C_n^\sigma, \varphi^{(n)}\rangle$ by

$$\ll \left\langle \psi^{(n)}, C_n^\sigma \right\rangle, \Phi \gg_\pi = n! \langle \psi^{(n)}, \varphi^{(n)}\rangle.$$

Here $\ll \cdot,\cdot \gg_\pi$ denotes the bilinear dual pairing between $\mathcal{P}'_\pi(\mathcal{D}')$ and $\mathcal{P}(\mathcal{D}')$ which extends the sesquilinear inner product on $L^2(\pi_\sigma)$:

$$\ll F, \Phi \gg_\pi = (F, \bar{\Phi})_{L^2(\pi_\sigma)}, \quad F \in L^2(\pi_\sigma), \Phi \in \mathcal{P}(\mathcal{D}'), \tag{5}$$

where $\bar{\Phi}$ is the complex conjugate function of Φ.

Therefore, any element $\Psi \in \mathcal{P}'_\pi(\mathcal{D}')$ has a unique decomposition of the form

$$\Psi = \sum_{n=0}^{\infty} \left\langle \psi^{(n)}, C_n^\sigma \right\rangle ,$$

where the sum converges weakly in $\mathcal{P}'_\pi(\mathcal{D}')$, and we have

$$\ll \Psi, \Phi \gg_\pi = \sum_{n=0}^{\infty} n! \langle \psi^{(n)}, \varphi^{(n)} \rangle,$$

for all $\Phi \in \mathcal{P}(\mathcal{D}')$ (see [KSS97] and [KSWY98] for more details and proofs). We have then $\mathcal{P}(\mathcal{D}') \subset L^2(\pi_\sigma) \subset \mathcal{P}'_\pi(\mathcal{D}')$ and the elements of $\mathcal{P}'_\pi(\mathcal{D}')$ are generalized functions on $\mathcal{D}'$.

Since $\mathcal{P}(\mathcal{D}') \subset (\mathcal{N})^1_\pi$, the dual space $(\mathcal{N})^{-1}_\pi$ of $(\mathcal{N})^1_\pi$ (with respect to $L^2(\pi_\sigma)$) may be regarded as a subspace of $\mathcal{P}'_\pi(\mathcal{D}')$. To describe the dual space $(\mathcal{N})^{-1}_\pi$ we consider the Hilbertian subspaces $(\mathcal{H}_{-p})^{-1}_{-q,\pi}$, p, $q \in \mathbb{N}$, of all elements $\Psi \in \mathcal{P}'_\pi(\mathcal{D}')$ for which the norm

$$\|\Psi\|^2_{-p,-q,\pi} := \sum_{n=0}^{\infty} 2^{-nq} \left|\psi^{(n)}\right|^2_{-p}$$

is finite. Each space $(\mathcal{H}_{-p})^{-1}_{-q,\pi}$ coincides with the dual space of $(\mathcal{H}_p)^1_{q,\pi}$ with respect to $L^2(\pi_\sigma)$. By the general duality theory we have

$$(\mathcal{N})^{-1}_\pi = \bigcup_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})^{-1}_{-q,\pi}$$

with the inductive limit topology.

In this way we obtain the following chain of spaces

$$\mathcal{P}(\mathcal{D}') \subset (\mathcal{N})^1_\pi \subset L^2(\pi_\sigma) \subset (\mathcal{N})^{-1}_\pi \subset \mathcal{P}'_\pi(\mathcal{D}'),$$

with the corresponding bilinear dual pairing given by $\ll \cdot, \cdot \gg_\pi$ as described in (5).

4.3 S-transform. Characterization of test and generalized functions

Gaussian analysis shows that it is useful to characterize test and generalized functions by integral transforms, see, *e.g.*, [BK88], [HKPS93]. The same is also true in Poissonian analysis. Here we recall the integral transform and characterization results for test and generalized functions in Poissonian analysis. For more details and proofs see, *e.g.*, [ADKS96], [KSS97], [KSWY98].

Definition 4.3 *The S_π-transform of a $\Psi \in (\mathcal{N})_\pi^{-1}$ is defined by*

$$(S_\pi \Psi)(\theta) := \ll \Psi, e_\pi(\theta, \cdot) \gg_\pi, \tag{6}$$

where $\theta \in \mathcal{N}_\mathbb{C}$ is such that $2^q |\theta|_p^2 < 1$. Here p, $q \in \mathbb{N}$ are such that $\Psi \in (\mathcal{H}_{-p})_{-q,\pi}^{-1}$.

Remark 4.4 Since the normalized exponentials $e_\pi(\theta, \cdot)$, $\theta \neq 0$, are not test functions in $(\mathcal{N})_\pi^1$, (6) has no meaning for an arbitrary $\theta \in \mathcal{N}_\mathbb{C}$. However, normalized exponentials $e_\pi(\theta, \cdot)$ are elements of $(\mathcal{H}_p)_{q,\pi}^1$ if $2^q |\theta|_p^2 < 1$. In this way (6) is well defined.

By the chaos decomposition, the S_π-transform of generalized functions may as well be described as follows.

Proposition 4.5 *If $\Psi = \sum_{n=0}^\infty \langle \psi^{(n)}, C_n^\sigma \rangle \in (\mathcal{N})_\pi^{-1}$, then*

$$(S_\pi \Psi)(\theta) = \sum_{n=0}^\infty \langle \psi^{(n)}, \theta^{\otimes n} \rangle \tag{7}$$

for each function $\theta \in \mathcal{N}_\mathbb{C}$ under the above conditions.

The S_π-transform restricted to subspaces of $(\mathcal{N})_\pi^{-1}$ can be extended to more general arguments θ than $\mathcal{N}_\mathbb{C}$. First, we consider the S_π-transform of functions $F \in L^2(\pi_\sigma) \subset (\mathcal{N})_\pi^{-1}$. Denote by $f^{(n)}$ the kernels of F under the chaos decomposition, *i.e.*, $F = \sum_{n=0}^\infty \langle C_n^\sigma, f^{(n)} \rangle$. Let $f \in L^2(\sigma)$ be given. Since the sum $\sum_{n=0}^\infty \frac{1}{n!} \langle C_n^\sigma, f^{\otimes n} \rangle$ converges in $L^2(\pi_\sigma)$, one can define the normalized exponential $e_\pi(f, \cdot) \in L^2(\pi_\sigma)$ by this sum. According to (6) and using (5), $S_\pi F$ is then given by the inner product on $L^2(\pi_\sigma)$, and thus $S_\pi F$ can be extended to the space $L^2(\sigma)$:

$$(S_\pi F)(f) = \int F(\omega) e_\pi(f, \omega) \, d\pi_\sigma(\omega) = \sum_{n=0}^\infty \langle f^{\otimes n}, f^{(n)} \rangle, \quad f \in L^2(\sigma). \tag{8}$$

Second, we consider the S_π-transform of test functions.

Definition 4.6 *For* $\Phi = \sum_{n=0}^{\infty}\langle C_n^\sigma, \varphi^{(n)}\rangle \in (\mathcal{N})_\pi^1$ *the* S_π*-transform of* Φ *is given by the sum*

$$(S_\pi\Phi)(\omega) := \sum_{n=0}^{\infty}\langle \omega^{\otimes n}, \varphi^{(n)}\rangle, \quad \omega \in \mathcal{D}'_{\mathbb{C}}.$$

For technical reasons we shall consider germs of holomorphic functions (at zero), *i.e.*, we just identify two functions if they coincide on an open set $0 \in \mathcal{U} \subset \mathcal{N}_{\mathbb{C}}$. The space of germs of holomorphic functions at zero will be denoted by $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$. The space $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ will be endowed with the inductive topology given by the family of norms

$$n_{p,l}(f) := \sup_{|\theta|_p \le 2^{-l}} |f(\theta)|, \quad p, l \in \mathbb{N}.$$

The next theorem is a characterization result for the space of generalized functions using the S_π-transform. Note that the series in (7) converges uniformly and absolutely on any closed ball $\{\varphi \in \mathcal{H}_{p,\mathbb{C}} : |\varphi|_p^2 \le r\}$, $r < 2^{-q}$. Thus, $S_\pi\Psi$ is holomorphic on a neighborhood of zero.

Theorem 4.7 *The* S_π*-transform defines a topological isomorphism of the space* $(\mathcal{N})_\pi^{-1}$ *onto the space* $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$*.*

To characterize test functions in $(\mathcal{N})_\pi^1$ by using the S_π-transform, first we need to introduce some spaces of holomorphic functions.

Definition 4.8 *The space* $\mathcal{E}_{\min}(\mathcal{N}'_{\mathbb{C}})$ *is the set of all functions* $\Phi : \mathcal{N}'_{\mathbb{C}} \to \mathbb{C}$ *such that the restriction of* Φ *to each* $\mathcal{H}_{-p,\mathbb{C}}$*,* $p \in \mathbb{N}$*, is entire and for every* $\varepsilon > 0$ *there exists a* $C > 0$ *such that*

$$\forall \omega \in \mathcal{H}_{-p,\mathbb{C}}, \quad |\Phi(\omega)| \le C \exp\left(\varepsilon\, |\omega|_{-p}\right).$$

We denote by $\mathcal{E}_{\min}(\mathcal{N}')$ *the set* $\{\Phi \restriction_{\mathcal{N}'} : \Phi \in \mathcal{E}_{\min}(\mathcal{N}'_{\mathbb{C}})\}$*.*

Theorem 4.9 *The* S_π*-transform is a topological isomorphism of the space* $(\mathcal{N})_\pi^1$ *onto the space* $\mathcal{E}_{\min}(\mathcal{N}'_{\mathbb{C}})$*.*

Moreover, we can give a complete internal description of test functions.

Theorem 4.10 *We have the following topological identity:*

$$(\mathcal{N})_\pi^1 = \mathcal{E}_{\min}(\mathcal{N}').$$

Remark 4.11 The previous theorem shows that the space $(\mathcal{N})^1_\pi$ is independent of the Poisson measure π_σ (see [KSWY98] for more details and proofs).

According to Proposition 4.13 below, the S_π-transform defines a unitary isomorphism between the space $L^2(\pi_\sigma)$ and the so-called Bargmann-Segal space which is the subject of the next subsection. There we follow closely [GKS97].

4.4 The Bargmann-Segal space

For the definition of the Bargmann-Segal space, first we introduce a Gaussian measure on the complex measurable space $(\mathcal{D}'_{\mathbb{C}}, \mathcal{C}_\sigma(\mathcal{D}'_{\mathbb{C}}))$. We denote by $\mu_{\frac{1}{2}\sigma}$ the centered Gaussian measure on $(\mathcal{D}', \mathcal{C}_\sigma(\mathcal{D}'))$ with covariance operator $\frac{1}{2}\mathrm{Id}_{L^2_{Re}(\sigma)}$, *i.e.*, with characteristic function given by

$$\int_{\mathcal{D}'} \exp(i\langle\omega,\varphi\rangle) d\mu_{\frac{1}{2}\sigma}(\omega) = \exp\left(-\frac{1}{4}|\varphi|^2_{L^2(\sigma)}\right), \quad \varphi \in \mathcal{D}.$$

A Gaussian measure ν_σ may be defined on $(\mathcal{D}'_{\mathbb{C}}, \mathcal{C}_\sigma(\mathcal{D}'_{\mathbb{C}}))$ by

$$d\nu_\sigma(\omega) := d\mu_{\frac{1}{2}\sigma}(\omega_1) \times d\mu_{\frac{1}{2}\sigma}(\omega_2),$$

where $\omega = \omega_1 + i\omega_2 \in \mathcal{D}'_{\mathbb{C}}$, $\omega_j \in \mathcal{D}'$, $j = 1, 2$. On the space $L^2(\nu_\sigma)$ we introduce *smooth holomorphic monomials of order* $n \in \mathbb{N}_0$ corresponding to kernels $\varphi^{(n)} \in \mathcal{D}_{\mathbb{C}}^{\hat{\otimes}n}$ by

$$\langle\omega^{\otimes n}, \varphi^{(n)}\rangle, \quad \omega \in \mathcal{D}'_{\mathbb{C}}.$$

These monomials are orthogonal, *i.e.*,

$$\left(\langle\cdot^{\otimes n}, \varphi^{(n)}\rangle, \langle\cdot^{\otimes m}, \phi^{(m)}\rangle\right)_{L^2(\nu_\sigma)} = \delta_{n,m} n! \left(\varphi^{(n)}, \phi^{(n)}\right)_{L^2(\sigma^{\otimes n})}$$

which allows us to extend these monomials to the measurable monomials $\langle\cdot^{\otimes n}, f^{(n)}\rangle \in L^2(\nu_\sigma)$, $f^{(n)} \in \hat{L}^2(X^n, \sigma^{\otimes n})$ with respect to which the above orthogonality property still holds.

In this context we have the following definition.

Definition 4.12 *The Bargmann-Segal space $E^2(\nu_\sigma)$ is the subspace of $L^2(\nu_\sigma)$ defined by*

$$E^2(\nu_\sigma) \ := \ \left\{ F \in L^2(\nu_\sigma) : F = \sum_{n=0}^{\infty} \langle\cdot^{\otimes n}, f^{(n)}\rangle, \right.$$

$$\left.\|F\|^2_{L^2(\nu_\sigma)} := \sum_{n=0}^\infty n! |f^{(n)}|^2_{L^2(\sigma^{\otimes n})} < \infty \right\}.$$

We note that the Bargmann-Segal space is the subspace of $L^2(\nu_\sigma)$ consisting of holomorphic functions on $L^2(\sigma)$, see, *e.g.*, [GKS97] for details.

Let us consider a function $F \in L^2(\pi_\sigma)$ with chaos decomposition of the form $F = \sum_{n=0}^\infty \langle C_n^\sigma, f^{(n)}\rangle$. According to (8) its S_π-transform is given by

$$(S_\pi F)(f) = \sum_{n=0}^\infty \langle f^{\otimes n}, f^{(n)}\rangle, \quad f \in L^2(\sigma).$$

Although the space $L^2(\sigma)$ has ν_σ-measure zero, there exists an extension (with respect to the norm on $L^2(\nu_\sigma)$) of $S_\pi F$ to $\mathcal{D}'_{\mathbb{C}}$:

$$\sum_{n=0}^\infty \langle \omega^{\otimes n}, f^{(n)}\rangle, \quad \omega \in \mathcal{D}'_{\mathbb{C}}.$$

Keeping the same notation for this extension, we see that $S_\pi F$ belongs to the Bargmann-Segal space.

Furthermore, we have the following result (cf., *e.g.*, [BK88], [GKS97]).

Proposition 4.13 *$S_\pi : L^2(\pi_\sigma) \to E^2(\nu_\sigma)$ is a unitary isomorphism.*

The composition of the unitary isomorphisms S_π and $I_{\lambda\pi}$ leads to a unitary isomorphism between the space $L^2(\lambda_\sigma)$ and the Bargmann-Segal space $E^2(\nu_\sigma)$:

$$S_\lambda := S_\pi \circ I_{\lambda\pi} : L^2(\lambda_\sigma) \to E^2(\nu_\sigma).$$

An explicit form of S_λ will be obtained below.

4.5 Test and generalized functions on Lebesgue-Poisson space

The unitary isomorphism between the Fock space $\mathrm{Exp}L^2(\sigma)$ and the space $L^2(\lambda_\sigma)$ leads to a natural construction of spaces of test and generalized functions on the Lebesgue-Poisson space.

As before, we consider the nuclear triple

$$\mathcal{D} \subset \mathcal{N} \subset L^2_{Re}(\sigma) \subset \mathcal{N}' \subset \mathcal{D}'.$$

$$\begin{tikzcd}
 & L^2(\Gamma, \pi_\sigma) \arrow[dl, "I_\pi"'] \arrow[dr, "S_\pi"] & \\
\mathrm{Exp}L^2(\sigma) & & E^2(\nu_\sigma) \\
 & L^2(\Gamma_0, \lambda_\sigma) \arrow[ul, "I_\lambda"] \arrow[ur, "S_\lambda"'] \arrow[uu, "I_{\lambda\pi}"'] &
\end{tikzcd}$$

Figure 2: Spaces

Through the unitary isomorphism $I_{\pi\lambda}$ we may define the family of Hilbert spaces

$$(\mathcal{H}_p)^1_{q,\lambda} := \left\{ G \in L^2(\lambda_\sigma) : \sum_{n=0}^{\infty} 2^{nq} \left| G^{(n)} \right|_p^2 < \infty, G^{(n)} := G \restriction_{\Gamma_X^{(n)}} \right\}$$

with the Hilbertian norm

$$\|G\|^2_{p,q,\lambda} := \sum_{n=0}^{\infty} 2^{nq} \left| G^{(n)} \right|_p^2, \quad p, q \in \mathbb{N}.$$

Note that $G^{(n)}$ can be an arbitrary element from $\mathcal{H}_{p,\mathbb{C}}^{\hat{\otimes} n}$. A space of test functions may be introduced on the Lebesgue-Poisson space by

$$(\mathcal{N})^1_\lambda := \bigcap_{p,q \in \mathbb{N}} (\mathcal{H}_p)^1_{q,\lambda}$$

with the projective limit topology.

As a direct consequence of this construction we have the following result.

Proposition 4.14 *The unitary isomorphism $I_{\pi\lambda}$ maps each space $(\mathcal{H}_p)^1_{q,\pi}$ onto $(\mathcal{H}_p)^1_{q,\lambda}$, p, $q \in \mathbb{N}$, and the nuclear space $(\mathcal{N})^1_\pi$ onto $(\mathcal{N})^1_\lambda$.*

Therefore, the mapping $I_{\pi\lambda}$ can be extended to the dual space $(\mathcal{H}_{-p})^{-1}_{-q,\pi}$ of $(\mathcal{H}_p)^1_{q,\pi}$. The extended mapping, also denoted by $I_{\pi\lambda}$, maps $(\mathcal{H}_{-p})^{-1}_{-q,\pi}$ onto $(\mathcal{H}_{-p})^{-1}_{-q,\lambda}$, the dual space of $(\mathcal{H}_p)^1_{q,\lambda}$ with respect to $L^2(\lambda_\sigma)$, *i.e.*,

$$I_{\pi\lambda} : (\mathcal{H}_{-p})^{-1}_{-q,\pi} \to (\mathcal{H}_{-p})^{-1}_{-q,\lambda}$$

and

$$(\mathcal{H}_{-p})^{-1}_{-q,\lambda} = \left\{ \Psi = \left(\Psi^{(n)}\right)_{n=0}^{\infty} : \Psi^{(n)} \in \mathcal{D}'^{\hat{\otimes} n}_{\mathbb{C}}, n \in \mathbb{N}_0, \sum_{n=0}^{\infty} \frac{2^{-nq}}{(n!)^2} \left|\Psi^{(n)}\right|^2_{-p} < \infty \right\}.$$

This implies, in particular, that the space $(\mathcal{H}_{-p})^{-1}_{-q,\lambda}$ has the structure of a Hilbert space with the Hilbertian norm defined by

$$\|\Psi\|^2_{-p,-q,\lambda} := \sum_{n=0}^{\infty} \frac{2^{-nq}}{(n!)^2} \left|\Psi^{(n)}\right|^2_{-p}.$$

From the general duality theory it follows that the dual space $(\mathcal{N})^{-1}_\lambda$ of $(\mathcal{N})^1_\lambda$ with respect to $L^2(\lambda_\sigma)$ is given by the inductive limit of $(\mathcal{H}_{-p})^{-1}_{-q,\lambda}$,

$$(\mathcal{N})^{-1}_\lambda := \bigcup_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})^{-1}_{-q,\lambda}.$$

The dual pairing between $(\mathcal{N})^{-1}_\lambda$ and $(\mathcal{N})^1_\lambda$ is realized as an extension of the inner product on $L^2(\lambda_\sigma)$:

$$\ll G, F \gg_\lambda = (G, \bar{F})_{L^2(\lambda_\sigma)}, \quad G \in (\mathcal{N})^1_\lambda, F \in L^2(\lambda_\sigma).$$

Moreover, for each $\Psi \in (\mathcal{N})^{-1}_\lambda$ and $G \in (\mathcal{N})^1_\lambda$ one has the following equality:

$$\ll \Psi, G \gg_\lambda = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \Psi^{(n)}, G^{(n)} \right\rangle.$$

This construction of test and generalized functions on the Lebesgue-Poisson space allows the use of the explicit forms of the S_π-transform in Poissonian analysis to obtain explicit forms for the unitary isomorphism $S_\lambda = S_\pi \circ I_{\lambda\pi}$. In addition, S_λ can be extended to the space of generalized functions $(\mathcal{N})^{-1}_\lambda$ in a way similar to Definition 4.3 and Remark 4.4.

Definition 4.15 *For any* $\Psi \in (\mathcal{N})_{\lambda}^{-1}$

$$\left(S_{\lambda}\Psi\right)(\theta) :=\ll \Psi, e_{\lambda}(\theta,\cdot) \gg_{\lambda},$$

where $\theta \in \mathcal{N}_{\mathbb{C}}$ *is such that* $2^q \left|\theta\right|_p^2 < 1$. *Here* p, $q \in \mathbb{N}$ *are chosen in such a way that* $\Psi \in (\mathcal{H}_{-p})_{-q,\lambda}^{-1}$.

Taking into account the equality of norms (4), one can deduce the following result.

Proposition 4.16 *If* $F \in L^2(\lambda_\sigma)$, *then*

$$\left(S_{\lambda}F\right)(f) = \int_{\Gamma_0} F(\eta) e_{\lambda}(f,\eta) d\lambda_{\sigma}(\eta), \quad f \in L^2(\sigma).$$

By Definition 4.6 and the definition of S_λ we obtain the following proposition.

Proposition 4.17 *For any* $G \in (\mathcal{N})_{\lambda}^{1}$ *the following equality holds:*

$$\left(S_{\lambda}G\right)(\omega) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \omega^{\otimes n}, G^{(n)} \rangle, \quad \omega \in \mathcal{D}'_{\mathbb{C}}.$$

5 Convolutions and algebraic structures on Poisson space

5.1 The Wick product

We consider a generalization of the Wick product in Gaussian analysis (see, *e.g.*, [KLS96]) to the Poissonian analysis setting following [KSWY98].

Definition 5.1 *For* $\Psi_i \in (\mathcal{N})_{\pi}^{-1}$, $i = 1, 2$, *the Wick product* $\Psi_1 \diamond \Psi_2$ *is defined by*

$$S_{\pi}(\Psi_1 \diamond \Psi_2) := S_{\pi}(\Psi_1) \cdot S_{\pi}(\Psi_2). \tag{9}$$

This definition is correct because the space $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ is an algebra under the pointwise multiplication of functions and thus, by Theorem 4.7, there exists a unique element $\Psi_1 \diamond \Psi_2 \in (\mathcal{N})_{\pi}^{-1}$ such that equality (9) holds.

Another characterization of Wick product can be given in terms of the chaos decomposition.

Proposition 5.2 *For any* $\Psi_i \in (\mathcal{N})^{-1}_\pi$ *of the form*

$$\Psi_i = \sum_{n=0}^\infty \left\langle \psi_i^{(n)}, C_n^\sigma \right\rangle, \quad \psi_i^{(n)} \in \mathcal{N}'^{\hat{\otimes} n}_{\mathbb{C}}, n \in \mathbb{N}_0, i = 1, 2,$$

the Wick product $\Psi_1 \diamond \Psi_2$ *is given by*

$$\Psi_1 \diamond \Psi_2 = \sum_{n=0}^\infty \left\langle \sum_{k=0}^n \psi_1^{(k)} \hat{\otimes} \psi_2^{(n-k)}, C_n^\sigma \right\rangle,$$

where $\psi_1^{(k)} \hat{\otimes} \psi_2^{(n-k)} = P_n \left(\psi_1^{(k)} \otimes \psi_2^{(n-k)} \right)$ *(see Section 3).*

Analogously, we can define Wick powers

$$\Psi^{\diamond n} := S_\pi^{-1}((S_\pi \Psi)^n), \quad \Psi \in (\mathcal{N})^{-1}_\pi$$

and Wick polynomials of finite order $\sum_{n=0}^N a_n \Psi^{\diamond n}$. Moreover, given a function $g : \mathbb{C} \to \mathbb{C}$ analytic on a neighborhood of the point $z_0 :=\ll \Psi, 1 \gg_\pi \in \mathbb{C}$, it is possible to define the *Wick composition* $g^\diamond(\Psi)$ by $g^\diamond(\Psi) := S_\pi^{-1}(g(S_\pi \Psi))$ (see, *e.g.*, [KSWY98], [KLS96]). In particular, we have defined the *Wick exponential* $\exp^\diamond \Psi$ on $(\mathcal{N})^{-1}_\pi$:

$$\exp^\diamond \Psi := S_\pi^{-1}(\exp(S_\pi \Psi)).$$

The space of generalized functions $(\mathcal{N})^{-1}_\pi$ endowed with the Wick product is a commutative algebra with unit element $e_\pi(0, \cdot) \equiv 1$.

Let us denote by $L^0(\Gamma, \mathcal{B}(\Gamma))$ the space of all (complex valued) $\mathcal{B}(\Gamma)$-measurable functions. For $\Lambda \in \mathcal{B}_c(X)$ we consider the subspace $L^0(\Gamma, \mathcal{B}_\Lambda(\Gamma))$ of all $\mathcal{B}_\Lambda(\Gamma)$-measurable functions F. Such functions F are called *cylinder functions* and Λ the *domain of cylindricity of* F. An equivalent description can be given as follows: $F \in L^0(\Gamma, \mathcal{B}_\Lambda(\Gamma))$ if and only if $F \in L^0(\Gamma, \mathcal{B}(\Gamma))$ and $F(\gamma) = F(\gamma_\Lambda)$, $\gamma \in \Gamma$. A special subspace is the set $\mathcal{FP}_{bc}(\Gamma)$ of all *cylinder polynomials on* Γ *with bounded coefficients*, *i.e.*, functions with chaos decomposition of the form

$$\sum_{n=0}^N \left\langle C_n^\sigma, \varphi^{(n)} \right\rangle, N \in \mathbb{N}_0,$$

where each kernel $\varphi^{(n)}$ belongs to $B_{bs}(X^n) \subset \hat{L}^2(X^n, \sigma^{\otimes n})$ ($B_{bs}(X^n) :=$ the space of all symmetric bounded $\mathcal{B}(X^n)$-measurable functions with bounded support). The space $\mathcal{FP}_{bc}(\Gamma)$ is a dense subset in $L^2(\pi_\sigma)$. Note that the Charlier polynomial $\langle C_n^\sigma, \varphi^{(n)}\rangle$, $\varphi^{(n)} \in B_{bs}(X^n)$, gives the same function in $L^2(\pi_\sigma)$ for kernels which differ from $\varphi^{(n)}$ on a set of measure zero.

The Wick product can also be introduced on the space $\mathcal{FP}_{bc}(\Gamma)$.

Corollary 5.3 *For any F_1, $F_2 \in \mathcal{FP}_{bc}(\Gamma)$,*

$$F_i = \sum_{n=0}^{N_i} \left\langle C_n^\sigma, \varphi_i^{(n)} \right\rangle, \quad \varphi_i^{(n)} \in B_{bs}(X^n), N_i \in \mathbb{N}_0, i = 1, 2, \tag{10}$$

the Wick product $F_1 \diamond F_2 \in \mathcal{FP}_{bc}(\Gamma)$ is given by

$$F_1 \diamond F_2 := \sum_{n=0}^{N_1+N_2} \left\langle C_n^\sigma, \sum_{k=0}^{n} \varphi_1^{(k)} \hat{\otimes} \varphi_2^{(n-k)} \right\rangle. \tag{11}$$

The space $\mathcal{FP}_{bc}(\Gamma)$ endowed with this product is a subalgebra of $(\mathcal{N})_\pi^{-1}$ with unit element. In particular, for $F \in \mathcal{FP}_{bc}(\Gamma)$ one can define the Wick exponential of F:

$$\exp^\diamond F := \sum_{n=0}^{\infty} \frac{1}{n!} F^{\diamond n},$$

where the sum converges in $(\mathcal{N})_\pi^{-1}$.

The unitary isomorphism I_π leads to the so-called *Borchers algebra* on the Fock space $\mathrm{Exp}L^2(\sigma)$.

Let us consider the dense subspace $\mathrm{Exp}_{fin}L^2(\sigma) \subset \mathrm{Exp}L^2(\sigma)$ of all finite vectors, *i.e.*, elements of the form $(f^{(0)}, f^{(1)}, ..., f^{(n)}, 0, 0, ...) \in \mathrm{Exp}L^2(\sigma)$. For $(f^{(n)})_{n=0}^\infty$, $(g^{(n)})_{n=0}^\infty \in \mathrm{Exp}_{fin}L^2(\sigma)$ the *Borchers product* $(f^{(n)})_{n=0}^\infty \odot (g^{(n)})_{n=0}^\infty$ is defined as the vector $(h^{(n)})_{n=0}^\infty \in \mathrm{Exp}_{fin}L^2(\sigma)$,

$$h^{(n)} := \sum_{k=0}^{n} f^{(k)} \hat{\otimes} g^{(n-k)} \in \hat{L}^2(X^n, \sigma^{\otimes n}), n \in \mathbb{N}, \quad h^{(0)} := f^{(0)} \cdot g^{(0)} \in \mathbb{C}.$$

With respect to this product, $\mathrm{Exp}_{fin}L^2(\sigma)$ has the structure of a commutative algebra with unit element $e(0) = (1, 0, 0, ...) \in \mathrm{Exp}_{fin}L^2(\sigma)$.

Proposition 5.4 *For all F_1, $F_2 \in \mathcal{FP}_{bc}(\Gamma)$ the following relation holds:*

$$I_\pi(F_1 \diamond F_2) = I_\pi(F_1) \odot I_\pi(F_2).$$

Proof: Note that I_π maps $\mathcal{FP}_{bc}(\Gamma)$ into $\mathrm{Exp}_{fin}L^2(\sigma)$. According to (11), for F_1, $F_2 \in \mathcal{FP}_{bc}(\Gamma)$ of the form (10) we have

$$I_\pi\left(F_1 \diamond F_2\right) = \left(\sum_{k=0}^{n} \varphi_1^{(k)} \hat{\otimes} \varphi_2^{(n-k)}\right)_{n=0}^{\infty},$$

which coincides with the Borchers product $I_\pi(F_1) \odot I_\pi(F_2)$. ■

5.2 The *-convolution

To introduce the $*$-convolution on the Lebesgue-Poisson space, which plays an important role in statistical physics, first we need to define some spaces.

A subset $A \subset \Gamma_0$ is called *bounded* if A is given by a finite union of bounded subsets of $\Gamma_X^{(n)}$, $n \in \mathbb{N}_0$, *i.e.*,

$$A = \bigsqcup_{n=0}^{N} A_n, \quad A_n \subset \Gamma_\Lambda^{(n)}, n = 0, 1, ..., N, \text{ for some } \Lambda \in \mathcal{B}_c(X), N \in \mathbb{N}_0.$$

Let us denote by $L^0(\Gamma_0, \mathcal{B}(\Gamma_0))$ the space of all (complex valued) $\mathcal{B}(\Gamma_0)$-measurable functions. We consider the subspace $B_{bs}(\Gamma_0)$ of all bounded measurable functions with bounded support, which is a dense subset in $L^2(\lambda_\sigma)$.

Definition 5.5 *Given $G_1, G_2 \in L^0(\Gamma_0, \mathcal{B}(\Gamma_0))$ we define the $*$-convolution by*

$$\left(G_1 * G_2\right)(\eta) := \sum_{\xi \subset \eta} G_1(\xi) G_2(\eta \backslash \xi), \quad \eta \in \Gamma_0.$$

With respect to this product the space $L^0(\Gamma_0, \mathcal{B}(\Gamma_0))$ is a commutative algebra with unit element $e_\lambda(0, \cdot) = \mathbf{1}_{\Gamma_X^{(0)}} \in B_{bs}(\Gamma_0)$. Indeed we have

$$\left(e_\lambda(0, \cdot) * G\right)(\eta) = \sum_{\xi \subset \eta} e_\lambda(0, \xi) G(\eta \backslash \xi) = e_\lambda(0, \emptyset) G(\eta) = G(\eta), \quad \eta \in \Gamma_0.$$

Thus, for $G \in L^0(\Gamma_0, \mathcal{B}(\Gamma_0))$ we may define by induction $G^{*n} \in L^0(\Gamma_0, \mathcal{B}(\Gamma_0))$, *i.e.*,

$$G^{*n}(\eta) := \sum_{(\eta_1, \dots, \eta_n) \in \mathcal{P}_n(\eta)} G(\eta_1) ... G(\eta_n), \quad \eta \in \Gamma_0, n \in \mathbb{N},$$

where $\mathcal{P}_n(\eta)$ denotes the set of all partitions of η in n parts, which may be empty.

Note that if G_1 and G_2 are two bounded measurable functions with bounded support contained in Γ_{Λ_i}, $i = 1, 2$, respectively, then $G_1 * G_2$ is a bounded measurable function with support contained in $\Gamma_{\Lambda_1 \cup \Lambda_2}$, *i.e.*, the space $B_{bs}(\Gamma_0)$ is closed under the $*$-convolution.

Lemma 5.6 *([Rue69], [FF91]) The following equality holds:*

$$\int_{\Gamma_0} \int_{\Gamma_0} G(\eta \cup \xi) H(\xi, \eta) d\lambda_\sigma(\eta) d\lambda_\sigma(\xi) = \int_{\Gamma_0} G(\eta) \sum_{\xi \subset \eta} H(\xi, \eta \backslash \xi) d\lambda_\sigma(\eta),$$

for all non-negative measurable functions $G : \Gamma_0 \longrightarrow \mathbb{R}$, $H : \Gamma_0 \times \Gamma_0 \longrightarrow \mathbb{R}$.

Proof: For simplicity, throughout this proof we will use the notation $\{x_i\}_{i=1}^n$ for $\mathrm{sym}_X^n(x_1, ..., x_n)$ and $d\sigma(x_i)_{i=1}^n$ for $d\sigma^{\otimes n}(x_1, ..., x_n)$. We obtain

$$\begin{aligned}
&\int_{\Gamma_0} \int_{\Gamma_0} G(\eta \cup \xi) H(\xi, \eta) d\lambda_\sigma(\eta) d\lambda_\sigma(\xi) \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^{k} \binom{k}{n} \int_{X^k} G(\{x_i\}_{i=1}^k) H(\{x_i\}_{i=1}^n, \{x_i\}_{i=n+1}^k) d\sigma(x_i)_{i=1}^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\Gamma_X^{(k)}} G(\eta) \sum_{\xi \subset \eta} H(\xi, \eta \backslash \xi) d\sigma^{(k)}(\eta).
\end{aligned}$$

■

As an immediate consequence we have the following useful estimate.

Corollary 5.7 *Let* G_1, $G_2 \in L^1(\lambda_\sigma)$ *be given. Then,* $G_1 * G_2 \in L^1(\lambda_\sigma)$ *and*

$$\|G_1 * G_2\|_{L^1(\lambda_\sigma)} \leq \|G_1\|_{L^1(\lambda_\sigma)} \|G_2\|_{L^1(\lambda_\sigma)}. \tag{12}$$

Moreover,

$$\int_{\Gamma_0} (G_1 * G_2)(\eta) d\lambda_\sigma(\eta) = \left(\int_{\Gamma_0} G_1(\eta) d\lambda_\sigma(\eta) \right) \left(\int_{\Gamma_0} G_2(\eta) d\lambda_\sigma(\eta) \right).$$

Proof: This result directly follows from Lemma 5.6 by replacing G by the function identically equal to 1 and H by $G_1 \cdot G_2$. ■

This result shows that the space $L^1(\lambda_\sigma)$ endowed with the $*$-convolution is a subalgebra of $L^0(\Gamma_0, \mathcal{B}(\Gamma_0))$. Moreover, by (12), we have

$$\|G^{*n}\|_{L^1(\lambda_\sigma)} \leq \|G\|_{L^1(\lambda_\sigma)}^n, \quad G \in L^1(\lambda_\sigma), n \in \mathbb{N}.$$

Hence, one may consider the sum

$$\exp^* G := e_\lambda(0,\cdot) + G + \frac{G^{*2}}{2!} + ... + \frac{1}{n!}G^{*n} + ...$$

which converges in $L^1(\lambda_\sigma)$ and one has

$$\|\exp^* G\|_{L^1(\lambda_\sigma)} \leq \exp\left(\|G\|_{L^1(\lambda_\sigma)}\right).$$

Remark 5.8 For $f \in L^1(\sigma)$ the $*$-exponential of $f\mathbf{1}_{\Gamma_X^{(1)}} \in L^1(\lambda_\sigma)$ can be described by

$$\exp^*\left(f\mathbf{1}_{\Gamma_X^{(1)}}\right) = e_\lambda(f,\cdot).$$

In general the $*$-convolution of two elements of $L^2(\lambda_\sigma)$ does not belong to $L^2(\lambda_\sigma)$. However, this property holds for a particular class of functions in $L^2(\lambda_\sigma)$.

Proposition 5.9 *Let $G_i \in L^2(\lambda_{2\sigma}) \subset L^2(\lambda_\sigma)$, $i = 1,2$ be given. Then $G_1 * G_2 \in L^2(\lambda_\sigma)$ and the following estimate holds:*

$$\|G_1 * G_2\|_{L^2(\lambda_\sigma)} \leq \|G_1\|_{L^2(\lambda_{2\sigma})} \|G_2\|_{L^2(\lambda_{2\sigma})}.$$

Proof: We have

$$\begin{aligned}
\|G_1 * G_2\|^2_{L^2(\lambda_\sigma)} &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma_X^{(n)}} |(G_1 * G_2)(\eta)|^2 \, d\sigma^{(n)}(\eta) \\
&\leq \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Gamma_X^{(n)}} 2^{|\eta|} \sum_{\xi \subset \eta} |G_1(\xi)|^2 \, |G_2(\eta \backslash \xi)|^2 \, d\sigma^{(n)}(\eta) \\
&= \int_{\Gamma_0} \int_{\Gamma_0} 2^{|\xi|} |G_1(\xi)|^2 \, 2^{|\eta|} |G_2(\eta)|^2 \, d\lambda_\sigma(\xi) d\lambda_\sigma(\eta) \\
&= \|G_1\|^2_{L^2(\lambda_{2\sigma})} \|G_2\|^2_{L^2(\lambda_{2\sigma})},
\end{aligned}$$

where we have used Lemma 5.6. ■

For coherent states on the Lebesgue-Poisson space the $*$-convolution has an especially simple form.

Proposition 5.10 *If f and g are two $\mathcal{B}(X)$-measurable functions, then*

$$e_\lambda(f,\cdot) * e_\lambda(g,\cdot) = e_\lambda(f+g,\cdot).$$

Proof: For each $\eta \in \Gamma_0$ one has

$$\begin{aligned}
(e_\lambda(f,\cdot) * e_\lambda(g,\cdot))(\eta) &= \sum_{\xi \subset \eta} \left(\prod_{x \in \xi} f(x) \right) \left(\prod_{x \in \eta \setminus \xi} g(x) \right) \\
&= \prod_{x \in \eta} (f(x) + g(x)) \\
&= e_\lambda(f+g, \eta).
\end{aligned}$$

■

The unitary isomorphism I_λ can also be regarded as an algebraic homomorphism.

Proposition 5.11 *For any G_1, $G_2 \in B_{bs}(\Gamma_0)$ we have*

$$I_\lambda(G_1 * G_2) = I_\lambda(G_1) \odot I_\lambda(G_2).$$

Proof: By the definition of the $*$-convolution for each $n \in \mathbb{N}$ and all $\{x_1, ..., x_n\} \in \Gamma_X^{(n)}$ we have

$$\begin{aligned}
&(G_1 * G_2)(\{x_1, ..., x_n\}) \\
&= \sum_{\iota \in S_n} \sum_{k=0}^{n} \frac{1}{k!} G_1(\{x_{\iota(1)}, ..., x_{\iota(k)}\}) \frac{1}{(n-k)!} G_2(\{x_{\iota(k+1)}, ..., x_{\iota(n)}\}),
\end{aligned}$$

where $G_j(\{x_{\iota(1)}, ..., x_{\iota(0)}\}) := G_j(\emptyset) =: G_j(\{x_{\iota(n+1)}, ..., x_{\iota(n)}\})$, $j = 1, 2$, and S_n is the permutation group over $\{1, ..., n\}$. ■

As a consequence of this result the unitary isomorphism $I_{\lambda\pi}$ is also an algebraic homomorphism.

Corollary 5.12 *For any G_1, $G_2 \in B_{bs}(\Gamma_0)$ one has*

$$I_{\lambda\pi}(G_1 * G_2) = I_{\lambda\pi}(G_1) \diamond I_{\lambda\pi}(G_2).$$

This corollary extended to normalized exponentials leads to the following result.

Proposition 5.13 *For all f, $g \in L^2(\sigma)$ it holds*

$$e_\pi(f,\cdot) \diamond e_\pi(g,\cdot) = e_\pi(f+g,\cdot).$$

Proof: It directly follows from the definition of normalized exponentials corresponding to functions in $L^2(\sigma)$ and Proposition 5.2. ■

Remark 5.14 As we can expect, the Wick exponential is related to the $*$-exponential. The relation between them can be established using the framework introduced in [KK99b]. There, the authors gave an integral representation of the unitary isomorphism $I_{\lambda\pi}$ which can be extended to a bounded operator defined on a particular subspace of $L^1(\lambda_\sigma)$ with values in $L^1(\pi_\sigma)$. If we denote this extension by $\bar{I}_{\lambda\pi}$, then one can prove that

$$\bar{I}_{\lambda\pi}\left(\exp^* G\right) = \exp^\diamond\left(I_{\lambda\pi} G\right)$$

for every $G \in B_{bs}(\Gamma_0)$.

Corollary 5.15 *For any G_1, $G_2 \in B_{bs}(\Gamma_0)$ we have*

$$S_\lambda\left(G_1 * G_2\right) = \left(S_\lambda G_1\right) \cdot \left(S_\lambda G_2\right).$$

With respect to the pointwise multiplication of functions on the Poisson space, the unitary isomorphism $I_{\lambda\pi}$ can also be regarded as an algebraic homomorphism. To prove it we need to introduce another product on the Lebesgue-Poisson space.

5.3 Other algebraic products on Lebesgue-Poisson space

Definition 5.16 *For functions G_1, $G_2 \in B_{bs}(\Gamma_0)$ the Poisson-convolution $G_1 \overset{P}{*} G_2 \in B_{bs}(\Gamma_0)$ is defined for $\eta \in \Gamma_0$ by*

$$\left(G_1 \overset{P}{*} G_2\right)(\eta) := \sum_{(\xi_1,\xi_2,\xi_3)\in\mathcal{P}_3(\eta)} \int_{\Gamma_0} G_1(\xi_1 \cup \xi_2 \cup \xi) G_2(\xi_2 \cup \xi_3 \cup \xi)\, d\lambda_\sigma(\xi),$$

where $\mathcal{P}_3(\eta)$ denotes the set of all partitions of η in three parts, which may be empty.

Definition 5.17 *For functions G_1, $G_2 \in B_{bs}(\Gamma_0)$ the Wiener-convolution $G_1 \overset{W}{*} G_2 \in B_{bs}(\Gamma_0)$ is defined for $\eta \in \Gamma_0$ by*

$$\left(G_1 \overset{W}{*} G_2\right)(\eta) := \int_{\Gamma_0} \sum_{\xi\subset\eta} G_1(\xi \cup \varsigma) G_2(\varsigma \cup (\eta\setminus\xi))\, d\lambda_\sigma(\varsigma).$$

$$\begin{tikzcd}
& \begin{array}{c} L^2(\Gamma,\pi_\sigma) \\ \diamond \end{array} \arrow[dl, "I_\pi"'] \arrow[dr, "S_\pi"] & \\
\begin{array}{c} \mathrm{Exp}L^2(\sigma) \\ \odot \end{array} & & \begin{array}{c} E^2(\nu_\sigma) \\ \cdot \end{array} \\
& \begin{array}{c} L^2(\Gamma_0,\lambda_\sigma) \\ * \end{array} \arrow[ul, "I_\lambda"] \arrow[ur, "S_\lambda"'] \arrow[uu, "I_{\lambda\pi}"'] &
\end{tikzcd}$$

Figure 3: Convolutions

The space $B_{bs}(\Gamma_0)$ equipped with one of the convolutions defined above is a commutative algebra with unit element $e_\lambda(0,\cdot)$.

Given a $\eta \in \Gamma_0$, let us consider the linear mapping $A_\eta^- : B_{bs}(\Gamma_0) \to B_{bs}(\Gamma_0)$ defined by

$$A_\eta^- G := G(\cdot \cup \eta), \quad G \in B_{bs}(\Gamma_0).$$

Relations between the algebraic products defined on $B_{bs}(\Gamma_0) \subset L^2(\lambda_\sigma)$ are stated in the proposition below. These relations can directly be checked using the definitions of the convolutions.

Proposition 5.18 *For all G_1, $G_2 \in B_{bs}(\Gamma_0)$ we have*

$$\left(G_1 \overset{W}{*} G_2\right)(\eta) = \int_{\Gamma_0} \left(A_\xi^- G_1 * A_\xi^- G_2\right)(\eta) d\lambda_\sigma(\xi)$$

and

$$\begin{aligned}
\left(G_1 \overset{P}{*} G_2\right)(\eta) &= \int_{\Gamma_0} \sum_{\xi \subset \eta} \left(A_{\varsigma\cup\xi}^- G_1 * A_{\varsigma\cup\xi}^- G_2\right)(\eta\backslash\xi) d\lambda_\sigma(\varsigma) \\
&= \sum_{\xi \subset \eta} \left(A_\xi^- G_1 \overset{W}{*} A_\xi^- G_2\right)(\eta\backslash\xi).
\end{aligned}$$

Remark 5.19 For more properties of the Wiener- and Poisson-convolutions see, *e.g.*, [Maa85], [LM90], [Lin90], [LP89] and [LP90], respectively.

The definition of the Poisson-convolution can be extended to several classes of functions. For example, the next result describes the Poisson-convolution for a special family of (Lebesgue-Poisson) coherent states.

Theorem 5.20 *Consider the linear hull of the set* $\{e_\lambda(f,\cdot) : f \in B_{bs}(X)\}$, *in symbols* $l.h.\,\{e_\lambda(f,\cdot) : f \in B_{bs}(X)\}$.

For any G_1, $G_2 \in l.h.\,\{e_\lambda(f,\cdot) : f \in B_{bs}(X)\}$ *the following equality holds:*

$$I_{\lambda\pi}(G_1 \overset{P}{*} G_2) = I_{\lambda\pi}(G_1) \cdot I_{\lambda\pi}(G_2). \tag{13}$$

Proof: Applying Proposition 5.10 for every $G_i = e_\lambda(f_i,\cdot)$ with $f_i \in B_{bs}(X)$, $i = 1, 2$ we obtain

$$\begin{aligned}
\left(G_1 \overset{P}{*} G_2\right)(\eta) &= \left(\int_{\Gamma_0} e_\lambda(f_1,\xi)e_\lambda(f_2,\xi)d\lambda_\sigma(\xi)\right) \\
&\quad \cdot \sum_{(\xi_1,\xi_2,\xi_3)\in\mathcal{P}_3(\eta)} e_\lambda(f_1,\xi_1\cup\xi_2)e_\lambda(f_2,\xi_2\cup\xi_3) \\
&= e^{\langle f_1 f_2\rangle_\sigma}\left(e_\lambda(f_1 f_2,\cdot) * e_\lambda(f_1,\cdot) * e_\lambda(f_2,\cdot)\right)(\eta) \\
&= e^{\langle f_1 f_2\rangle_\sigma} e_\lambda(f_1 + f_2 + f_1 f_2, \eta).
\end{aligned}$$

■

The equality (13) can be extended to other classes of functions.

Remark 5.21 There exists a unitary isomorphism between the space $L^2(\lambda_\sigma)$ and the Gaussian space $L^2(\mu_\sigma)$, where μ_σ is the centered Gaussian measure on $\mathcal{D}'$ with covariance operator $\mathrm{Id}_{L^2(\sigma)}$. This isomorphism is defined by the Wiener-Itô-Segal chaos decomposition and I_λ. As the Poisson-convolution, the Wiener-convolution of functions on the Lebesgue-Poisson space is carried under this isomorphism to the pointwise multiplication of functions on the Gaussian space $(\mathcal{D}', \mu_\sigma)$ (see, *e.g.*, [LM90]).

6 Some operators in Poissonian analysis

Throughout this section we present the definitions and the main properties of the annihilation and creation operators on the Fock, Poisson, and Lebesgue-Poisson spaces. We start by recalling these notions on the Fock space, see, *e.g.*, [BK88], [RS75].

6.1 Annihilation and creation operators on Fock space

Definition 6.1 *Let $h \in L^2_{Re}(\sigma)$ be given. On the subset of elements*

$$f^{(n)} = f_1 \hat{\otimes} ... \hat{\otimes} f_n \in \hat{L}^2(X^n, \sigma^{\otimes n}), \quad f_i \in L^2(\sigma), i = 1, ..., n \tag{14}$$

which is total in $\hat{L}^2(X^n, \sigma^{\otimes n})$ the annihilation operator $a^-(h)$ is defined by

$$\begin{aligned}\left(a^-(h)\right) f^{(n)} :=& \sum_{j=1}^{n} (h, f_j)_{L^2(\sigma)} f_1 \hat{\otimes} ... \hat{\otimes} f_{j-1} \hat{\otimes} f_{j+1} \hat{\otimes} ... \hat{\otimes} f_n \in \hat{L}^2(X^{n-1}, \sigma^{\otimes(n-1)}), \\ \left(a^-(h)\right) e(0) :=& \ 0. \end{aligned} \tag{15}$$

By linearity, we can extend (15) to the dense subspace in $\hat{L}^2(X^n, \sigma^{\otimes n})$ consisting of finite linear combinations of elements of the form (14) and, for this extension, the inequality

$$\left| \left(a^-(h)\right) f^{(n)} \right|_{L^2(\sigma^{\otimes(n-1)})} \leq \sqrt{n}\, |h|_{L^2(\sigma)} \left| f^{(n)} \right|_{L^2(\sigma^{\otimes n})}$$

holds (see, *e.g.*, [RS75]). Hence, we can extend $a^-(h)$ to a bounded operator $a^-(h) : \hat{L}^2(X^n, \sigma^{\otimes n}) \to \hat{L}^2(X^{n-1}, \sigma^{\otimes(n-1)})$ which allows us to extend $a^-(h)$ componentwise to $\mathrm{Exp}_{fin} L^2(\sigma)$. In this way we obtain a densely defined operator on $\mathrm{Exp} L^2(\sigma)$ also denoted by $a^-(h)$. So, the adjoint operator $a^+(h)$ exists.

Definition 6.2 *The operator $a^+(h)$, $h \in L^2_{Re}(\sigma)$, is called a creation operator.*

The action of the creation operator $a^+(h)$ on elements $f^{(n)} \in \hat{L}^2(X^n, \sigma^{\otimes n})$ with $n \in \mathbb{N}_0$ is given by

$$\left(a^+(h)\right) f^{(n)} := h \hat{\otimes} f^{(n)} \in \hat{L}^2(X^{n+1}, \sigma^{\otimes(n+1)}).$$

For this operator we have

$$\left| \left(a^+(h)\right) f^{(n)} \right|_{\mathrm{Exp}_{n+1} L^2(\sigma)} \leq \sqrt{n+1}\, |h|_{L^2(\sigma)} \left| f^{(n)} \right|_{\mathrm{Exp}_n L^2(\sigma)}.$$

For any contraction B on $L^2(\sigma)$ it is possible to define a contraction operator $\mathrm{Exp} B$ on $\mathrm{Exp} L^2(\sigma)$ which, on each space $\hat{L}^2(X^n, \sigma^{\otimes n})$, is given by $B \otimes ... \otimes B$ (n times). For any positive self-adjoint operator A on $L^2(\sigma)$ we have a contraction semigroup e^{-tA}, $t \geq 0$. The *second quantization operator* $d\mathrm{Exp} A$ is defined as the generator of the semigroup $\mathrm{Exp}\left(e^{-tA}\right)$, $t \geq 0$, *i.e.*, $\mathrm{Exp}\left(e^{-tA}\right) = \exp\left(-t d\mathrm{Exp} A\right)$, see, *e.g.*, [RS75].

6.2 Annihilation and creation operators on Poisson space

In this subsection we want to give a representation of the annihilation and creation operators by using the unitary isomorphism I_π. This situation was considered in [KSSU98], [NV95], and [Pri95].

To define annihilation operators on the Poisson space we consider the mapping a_π^- defined on $\mathcal{FP}_{bc}(\Gamma)$ by

$$\left(a_\pi^- F\right)(\gamma, x) := F(\gamma \cup \{x\}) - F(\gamma), \quad F \in \mathcal{FP}_{bc}(\Gamma), \gamma \in \Gamma, x \in X.$$

Proposition 6.3 *For any $F \in \mathcal{FP}_{bc}(\Gamma)$, $a_\pi^- F \in L^2(\Gamma \times X, \pi_\sigma \otimes \sigma)$.*

Proof: Consider the following class of functions on Γ:

$$\left\langle f^{(n)}, \gamma^{\odot n} \right\rangle := \sum_{\{x_1,\dots,x_n\} \subset \gamma} f^{(n)}(x_1, ..., x_n), \quad f^{(n)} \in B_{bs}(X^n), n \in \mathbb{N}, \gamma \in \Gamma.$$

The linear hull of this family of functions coincides with the linear hull of the set of Charlier monomials

$$\left\langle C_n^\sigma, f^{(n)} \right\rangle, \quad f^{(n)} \in B_{bs}(X^n), n \in \mathbb{N}.$$

In other words,

$$\mathcal{FP}_{bc}(\Gamma) = l.h. \left\{ \left\langle f^{(n)}, \cdot^{\odot n} \right\rangle, f^{(n)} \in B_{bs}(X^n), n \in \mathbb{N} \right\}.$$

For more details see Section 5 in [KK99b]. Thus it is enough to show that

$$\left\langle f^{(n)}, (\gamma \cup \{x\})^{\odot n} \right\rangle - \left\langle f^{(n)}, \gamma^{\odot n} \right\rangle \in L^2(\Gamma \times X, \pi_\sigma \otimes \sigma), \quad f^{(n)} \in B_{bs}(X^n), n \in \mathbb{N}.$$

This immediately follows from the fact that

$$\begin{aligned}
&\left\langle f^{(n)}, (\gamma \cup \{x\})^{\odot n} \right\rangle - \left\langle f^{(n)}, \gamma^{\odot n} \right\rangle \\
&= \sum_{\{x_1,\dots,x_n\} \subset \gamma \cup \{x\}} f^{(n)}(x_1, ..., x_n) - \sum_{\{x_1,\dots,x_n\} \subset \gamma} f^{(n)}(x_1, ..., x_n) \\
&= \sum_{\{x_1,\dots,x_{n-1}\} \subset \gamma} f^{(n)}(x, x_1, ..., x_{n-1}),
\end{aligned}$$

where the last function belongs to $L^2(\Gamma \times X, \pi_\sigma \otimes \sigma)$ because $f^{(n)} \in B_{bs}(X^n)$. ■

For this densely defined operator on $L^2(\pi_\sigma)$ its adjoint operator a_π^+ is well defined.

Proposition 6.4 *For any mapping $F : \Gamma \times X \to \mathbb{C}$ of the form*

$$F(\gamma, x) = \sum_{k=1}^{N} F_k(\gamma) f_k(x), \quad \gamma \in \Gamma, x \in X,$$

with $F_k \in \mathcal{FP}_{bc}(\Gamma)$, $f_k \in B_{bs}(X)$, $k = 1, ..., N$, $N \in \mathbb{N}$, we have

$$\left(a_\pi^+ F\right)(\gamma) := \sum_{x \in \gamma} F(\gamma \setminus \{x\}, x) - \int_X F(\gamma, x) d\sigma(x), \quad \gamma \in \Gamma.$$

Proof: Let $G \in \mathcal{FP}_{bc}(\Gamma)$ be given. Then

$$\begin{aligned} \left(a_\pi^- G, F\right)_{L^2(\pi_\sigma \otimes \sigma)} &= \int_X \int_\Gamma G(\gamma \cup \{x\}) \bar{F}(\gamma, x) d\pi_\sigma(\gamma) d\sigma(x) \\ &\quad - \int_X \int_\Gamma G(\gamma) \bar{F}(\gamma, x) d\pi_\sigma(\gamma) d\sigma(x). \end{aligned} \tag{16}$$

Applying the *Mecke identity*, see, *e.g.*, [Mec67, Theorem 3.1],

$$\int_\Gamma \sum_{x \in \gamma} H(\gamma, x) d\pi_\sigma(\gamma) = \int_X \int_\Gamma H(\gamma \cup \{x\}, x) d\pi_\sigma(\gamma) d\sigma(x),$$

to the first integral on the right-hand side of (16) we obtain

$$\int_\Gamma G(\gamma) \left[\sum_{x \in \gamma} \bar{F}(\gamma \setminus \{x\}, x) - \int_X \bar{F}(\gamma, x) d\sigma(x) \right] d\pi_\sigma(\gamma)$$

which proves the proposition. ■

For a $\varphi \in \mathcal{D}$ we consider the operator

$$a_\pi^-(\varphi) : \mathcal{FP}_{bc}(\Gamma) \to L^2(\pi_\sigma)$$

defined by

$$\begin{aligned} \left(a_\pi^-(\varphi) F\right)(\gamma) &:= \left(\left(a_\pi^- F\right)(\gamma, \cdot), \varphi\right)_{L^2(\sigma)} \\ &= \int_X \left(F(\gamma \cup \{x\}) - F(\gamma)\right) \varphi(x) d\sigma(x). \end{aligned}$$

Proposition 6.5 *For each $\varphi \in \mathcal{D}$ the image of the operator $a_\pi^-(\varphi)$ under the unitary isomorphism I_π is the annihilation operator $a^-(\varphi)$ on the Fock space* $\mathrm{Exp}L^2(\sigma)$*, i.e., $a_\pi^-(\varphi) = I_\pi^{-1} a^-(\varphi)\ I_\pi$.*

Proof: The operator $a_\pi^-(\varphi)$ can be extended to a larger class of functions. In particular, one can define it for normalized exponentials in the following way. By Remark 3.5 we obtain for each $\psi \in \mathcal{D}_\mathbb{C}$

$$\left(a_\pi^-(\varphi)e_\pi(\psi,\cdot)\right)(\gamma) = (\psi,\varphi)_{L^2(\sigma)}e_\pi(\psi,\gamma). \tag{17}$$

On the other hand, according to Definition 6.1, we have

$$\left(a^-(\varphi)\right)e(\psi) = (\psi,\varphi)_{L^2(\sigma)}e(\psi) \tag{18}$$

and $e(\psi) = I_\pi\left(e_\pi(\psi,\cdot)\right)$. Hence, if we apply I_π^{-1} to (18) we obtain the same result as (17). This is enough to prove this proposition, because the space spanned by the family of normalized exponentials $e_\pi(\psi,\cdot)$, $\psi \in \mathcal{D}_\mathbb{C}$, is a core of the annihilation operator. ■

Remark 6.6 As a direct consequence of (17) the following equality holds:

$$\left(a_\pi^-(\varphi)\langle C_n^\sigma,\psi^{\otimes n}\rangle\right)(\gamma) = n(\psi,\varphi)_{L^2(\sigma)}\langle C_{n-1}^\sigma(\gamma),\psi^{\otimes n-1}\rangle, \quad \varphi,\psi\in\mathcal{D},\gamma\in\Gamma. \tag{19}$$

Consider the adjoint operator $a_\pi^+(\varphi)$ of the operator $a_\pi^-(\varphi)$.

Proposition 6.7 *The action of $a_\pi^+(\varphi)$ on elements $F \in \mathcal{FP}_{bc}(\Gamma)$ is given by*

$$\left(a_\pi^+(\varphi)F\right)(\gamma) := \sum_{x\in\gamma}\left(F(\gamma\setminus\{x\})\varphi(x)\right) - F(\gamma)\int_X \varphi(x)d\sigma(x).$$

Remark 6.8 Through the orthogonality relation (2), we obtain from (19) the equality

$$\left(a_\pi^+(\varphi)\langle C_n^\sigma,\psi^{\otimes n}\rangle\right)(\gamma) = \langle C_{n+1}^\sigma(\gamma),\varphi\hat{\otimes}\psi^{\otimes n}\rangle, \quad \varphi,\psi\in\mathcal{D},\gamma\in\Gamma.$$

Proposition 6.9 *For all $\varphi\in\mathcal{D}$ the following equality holds*

$$a_\pi^+(\varphi) = I_\pi^{-1}a^+(\varphi)I_\pi.$$

Remark 6.10 Since $\left(a^+(\varphi)\right)^n e(0) = \varphi^{\otimes n} \in \hat{L}^2(X^n,\sigma^{\otimes n})$, one has

$$\left(\left(a_\pi^+(\varphi)\right)^n 1\right)(\gamma) = \langle C_n^\sigma(\gamma),\varphi^{\otimes n}\rangle, \quad n\in\mathbb{N}.$$

The operators a_π^+ and $a_\pi^+(\varphi)$, $\varphi \in \mathcal{D}$, are related in the following way. For $F := G \otimes \varphi$, $G \in \mathcal{FP}_{bc}(\Gamma)$, $\varphi \in \mathcal{D}$,

$$\begin{aligned}\left(a_\pi^+ F\right)(\gamma) &= \sum_{x\in\gamma} \left(G(\gamma \setminus \{x\})\varphi(x)\right) - G(\gamma)\int_X \varphi(x)d\sigma(x) \\ &= \left(a_\pi^+(\varphi)G\right)(\gamma), \quad \gamma \in \Gamma,\end{aligned}$$

i.e.,

$$a_\pi^+ \left(G \otimes \varphi\right) = a_\pi^+(\varphi)G.$$

Let A be a positive self-adjoint operator on $L^2(\sigma)$ with $\mathcal{D} \subset D(A)$. We denote by H_A^P the image of the second quantization operator $d\mathrm{Exp}A$ under the unitary isomorphism I_π^{-1}.

Theorem 6.11 *([AKR98a]) The symmetric bilinear form corresponding to the operator H_A^P has the form*

$$\left(H_A^P F, G\right)_{L^2(\pi_\sigma)} = \int_\Gamma \int_X \left(a_\pi^- F\right)(\gamma, x) A\left(\left(a_\pi^- G\right)(\gamma, \cdot)\right)(x) d\sigma(x) d\pi_\sigma(\gamma), \quad (20)$$

for all $F, G \in \mathcal{FP}(\mathcal{D}, \Gamma)$, where $\mathcal{FP}(\mathcal{D}, \Gamma)$ denotes the set of all elements of the form

$$F(\gamma) = p_F(\langle \gamma, \varphi_1\rangle, ..., \langle \gamma, \varphi_N\rangle), \varphi_1, ..., \varphi_N \in \mathcal{D}, N \in \mathbb{N}, \gamma \in \Gamma,$$

with p_F a polynomial on $\mathbb{R}^N$. The right-hand side of (20) is called the Poissonian pre-Dirichlet form with coefficient operator A.

6.3 Annihilation and creation operators on Lebesgue-Poisson space

Through the unitary isomorphism I_λ one can also give a representation of the annihilation and creation operators on the Lebesgue-Poisson space, see, *e.g.*, [FW93].

Let us consider the linear mapping a_λ^- defined on $B_{bs}(\Gamma_0)$ by

$$\left(a_\lambda^- G\right)(\eta, x) := G(\eta \cup \{x\}), \quad G \in B_{bs}(\Gamma_0), \eta \in \Gamma_0, x \in X. \quad (21)$$

Since each $G \in B_{bs}(\Gamma_0)$ has support contained in $\bigcup_{n=0}^{N} \Gamma_\Lambda^{(n)}$ for some $\Lambda \in \mathcal{B}_c(X)$ and $N \in \mathbb{N}_0$, we have

$$\begin{aligned}\int_{\Gamma_0} \int_X |G(\eta \cup \{x\})|^2 \, d\sigma(x) d\lambda_\sigma(\eta) &= \sum_{n=1}^{N} \frac{n}{n!} \int_{\Gamma_\Lambda^{(n)}} |G(\eta)|^2 \, d\sigma^{(n)}(\eta) \\ &\leq N \, \|G\|^2_{L^2(\lambda_\sigma)} .\end{aligned}$$

Therefore $a_\lambda^- G \in L^2(\Gamma_0 \times X, \lambda_\sigma \otimes \sigma)$.

Since a_λ^- is a densely defined operator on $L^2(\lambda_\sigma)$ its adjoint operator a_λ^+ is well defined.

Proposition 6.12 *Let $H : \Gamma_0 \times X \to \mathbb{C}$ be a function of the form*

$$H(\eta, x) = \sum_{k=1}^{N} H_k(\eta) f_k(x), \quad \eta \in \Gamma_0, x \in X,$$

with $H_k \in B_{bs}(\Gamma_0)$, $f_k \in B_{bs}(X)$, $k = 1, ..., N$, $N \in \mathbb{N}$. Then the action of the operator a_λ^+ on H is given by

$$\left(a_\lambda^+ H\right)(\eta) := \sum_{x \in \eta} H(\eta \backslash \{x\}, x), \quad \eta \in \Gamma_0.$$

Proof: By Lemma 5.6, for any $G \in B_{bs}(\Gamma_0)$ we have

$$\int_{\Gamma_0} \int_X G(\eta \cup \{x\}) H(\eta, x) d\sigma(x) d\lambda_\sigma(\eta) = \int_{\Gamma_0} G(\eta) \sum_{x \in \eta} H(\eta \backslash \{x\}, x) d\lambda_\sigma(\eta).$$

■

For a $h \in L^2_{Re}(\sigma)$ we consider the operator $a_\lambda^-(h)$ defined on $B_{bs}(\Gamma_0)$ by

$$\begin{aligned}\left(a_\lambda^-(h) G\right)(\eta) &:= \left(\left(a_\lambda^- G\right)(\eta, \cdot), h\right)_{L^2(\sigma)} \\ &= \int_X G(\eta \cup \{x\}) h(x) d\sigma(x), \quad \eta \in \Gamma_0.\end{aligned}$$

Proposition 6.13 *For each $h \in L^2_{Re}(\sigma)$ the image of the operator $a_\lambda^-(h)$ under the unitary isomorphism I_λ is the annihilation operator $a^-(h)$ on the Fock space* $\mathrm{Exp} L^2(\sigma)$*, i.e., $a_\lambda^-(h) = I_\lambda^{-1} a^-(h) \; I_\lambda$.*

Proof: To prove this proposition it is enough to show this equality of operators in a core of the annihilation operator. Let $h \in L^2_{Re}(\sigma)$ be given. For any $f \in L^2(\sigma)$ and $n \in \mathbb{N}$ we have

$$\left(a_\lambda^-(h)\left(e_\lambda(f,\cdot)\restriction_{\Gamma_X^{(n)}}\right)\right)(\eta) = \int_X e_\lambda(f,\eta\cup\{x\})h(x)d\sigma(x), \quad \eta \in \Gamma_X^{(n-1)},$$

where the integral on the right-hand side is equal to

$$e_\lambda(f,\eta)\int_X f(x)h(x)d\sigma(x) = (f,h)_{L^2(\sigma)}e_\lambda(f,\eta), \quad \eta \in \Gamma_X^{(n-1)}. \tag{22}$$

On the other hand by Definition 6.1

$$\left(a^-(h)\right)\frac{f^{\otimes n}}{n!} = \frac{1}{n!}\left(a^-(h)\right)f^{\otimes n} = (f,h)_{L^2(\sigma)}\frac{f^{\otimes n-1}}{(n-1)!}, \quad n \in \mathbb{N}, \tag{23}$$

and $I_\lambda\left(e_\lambda(f,\cdot)\restriction_{\Gamma_X^{(n)}}\right) = \frac{f^{\otimes n}}{n!}$. Hence, if we apply I_λ^{-1} to (23) we obtain the same result as (22). ■

As a direct consequence we have the following corollary.

Corollary 6.14 *For all $\varphi \in \mathcal{D}$ one has*

$$a_\pi^-(\varphi) = I_{\lambda\pi}a_\lambda^-(\varphi)I_{\pi\lambda}.$$

Proposition 6.15 *For all $h \in L^2_{Re}(\sigma)$ the operator $a_\lambda^+(h)$ is defined by*

$$\left(a_\lambda^+(h)G\right)(\eta) := \sum_{x\in\eta} G(\eta\backslash\{x\})h(x), \quad \eta \in \Gamma_0,$$

for any $G \in B_{bs}(\Gamma_0)$.

Proof: Let $h \in L^2_{Re}(\sigma)$ and $G \in B_{bs}(\Gamma_0)$ be given. Lemma 5.6 then yields

$$\begin{aligned}\int_{\Gamma_0}\left(a_\lambda^-(h)G\right)(\eta)H(\eta)d\lambda_\sigma(\eta) &= \int_{\Gamma_0}\int_X G(\eta\cup\{x\})h(x)H(\eta)d\sigma(x)d\lambda_\sigma(\eta)\\ &= \int_{\Gamma_0} G(\eta)\sum_{x\in\eta}h(x)H(\eta\backslash\{x\})d\lambda_\sigma(\eta).\end{aligned}$$

■

Corollary 6.16 *For any $h \in L^2_{Re}(\sigma)$ and $\varphi \in \mathcal{D}$ the following equalities hold:*

$$a^+_\lambda(h) = I_\lambda^{-1} a^+(h) I_\lambda, \quad a^+_\pi(\varphi) = I_{\lambda\pi} a^+_\lambda(\varphi) I_{\pi\lambda}.$$

Similar to the situation in Poissonian analysis, the operators a^+_λ and $a^+_\lambda(h)$ are related. Indeed, given a $G \in B_{bs}(\Gamma_0)$ and a real-valued function $h \in B_{bs}(X)$ one has

$$a^+_\lambda(h)G = a^+_\lambda(G \otimes h).$$

$$\begin{tikzcd}
& \begin{array}{c} L^2(\Gamma,\pi_\sigma) \\ a^\pm_\pi(\varphi) \end{array} \arrow[dl, "I_\pi"'] \\
\begin{array}{c} \mathrm{Exp}L^2(\sigma) \\ a^\pm(\varphi) \end{array} & \\
& \begin{array}{c} L^2(\Gamma_0,\lambda_\sigma) \\ a^\pm_\lambda(\varphi) \end{array} \arrow[ul, "I_\lambda"] \arrow[uu, "I_{\lambda\pi}"']
\end{tikzcd}$$

$$\begin{aligned}
(a^-_\pi(\varphi))F(\gamma) &= \int_X \Big[F(\gamma \cup \{x\}) \\
&\quad -F(\gamma)\Big]\varphi(x)\,d\sigma(x) \\
(a^+_\pi(\varphi))F(\gamma) &= \sum_{x\in\gamma} F(\gamma\backslash\{x\})\varphi(x) \\
&\quad -F(\gamma)\int_X \varphi(x)\,d\sigma(x) \\
(a^-_\lambda(\varphi))G(\eta) &= \int_X G(\eta\cup\{x\})\varphi(x)\,d\sigma(x) \\
(a^+_\lambda(\varphi))G(\eta) &= \sum_{x\in\eta} G(\eta\backslash\{x\})\varphi(x)
\end{aligned}$$

Figure 4: Annihilation and creation operators

Let A be a positive self-adjoint operator on $L^2(\sigma)$ with $\mathcal{D} \subset D(A)$. We denote by H_A^{LP} the image of the second quantization operator $d\mathrm{Exp}A$ under the unitary isomorphism I_λ^{-1}.

Proposition 6.17 *The symmetric bilinear form corresponding to the operator H_A^{LP} has the form*

$$\left(H_A^{LP}F, G\right)_{L^2(\lambda_\sigma)} = \int_{\Gamma_0}\int_X \left(a^-_\lambda F\right)(\eta, x) A\left(\left(a^-_\lambda G\right)(\eta,\cdot)\right)(x) d\sigma(x) d\lambda_\sigma(\eta),$$

for any F, $G \in I_{\pi\lambda}\left(\mathcal{FP}(\mathcal{D},\Gamma)\right)$.

6.4 Some operators in Lebesgue-Poisson analysis

Recall definition (21) of a_λ^-. A natural extension of this operator is the mapping A_λ^- defined on $B_{bs}(\Gamma_0)$ by

$$\left(A_\lambda^- G\right)(\eta,\xi) := G(\eta \cup \xi), \quad G \in B_{bs}(\Gamma_0), \eta, \xi \in \Gamma_0$$

(see, *e.g.*, [FF91], [FW93]). If we assume that $G \in B_{bs}(\Gamma_0)$ has support contained in $\bigcup_{n=0}^N \Gamma_\Lambda^{(n)}$ for some $\Lambda \in \mathcal{B}_c(X)$, $N \in \mathbb{N}_0$, then (through Lemma 5.6) the following estimate holds

$$\begin{aligned}
&\int_{\Gamma_0}\int_{\Gamma_0} |G(\eta\cup\xi)|^2 \, d\lambda_\sigma(\eta) d\lambda_\sigma(\xi) \\
&= \int_{\Gamma_0} |G(\eta)|^2 \sum_{\xi\subset\eta} e_\lambda(\mathbf{1}_\Lambda,\xi) e_\lambda(\mathbf{1}_\Lambda, \eta\backslash\xi) d\lambda_\sigma(\eta) \qquad (24)\\
&= \sum_{n=0}^N \frac{2^n}{n!} \int_{\Gamma_\Lambda^{(n)}} |G(\eta)|^2 \, d\sigma^{(n)}(\eta).
\end{aligned}$$

This shows that $A_\lambda^- G \in L^2(\Gamma_0 \times \Gamma_0, \lambda_\sigma \otimes \lambda_\sigma)$. For this densely defined operator its adjoint mapping A_λ^+ is well defined. By a direct application of Lemma 5.6 one has for each $F \in B_{bs}(\Gamma_0)$

$$\begin{aligned}
&\int_{\Gamma_0}\int_{\Gamma_0} \left(A_\lambda^- F\right)(\eta,\xi) G(\eta,\xi) d\lambda_\sigma(\eta) d\lambda_\sigma(\xi) \\
&= \int_{\Gamma_0}\int_{\Gamma_0} F(\eta\cup\xi) G(\eta,\xi) d\lambda_\sigma(\eta) d\lambda_\sigma(\xi) \\
&= \int_{\Gamma_0} F(\eta) \sum_{\xi\subset\eta} G(\eta\backslash\xi,\xi) d\lambda_\sigma(\eta),
\end{aligned}$$

where $G : \Gamma_0 \times \Gamma_0 \to \mathbb{C}$ is a mapping of the form

$$G(\eta,\xi) = \sum_{k=1}^N G_k(\eta) H_k(\xi), \quad \eta,\xi \in \Gamma_0,$$

with G_k, $H_k \in B_{bs}(\Gamma_0)$, $k = 1, ..., N$, $N \in \mathbb{N}$, *i.e.*,

$$\left(A_\lambda^+ G\right)(\eta) := \sum_{\xi\subset\eta} G(\eta\backslash\xi,\xi), \quad \eta \in \Gamma_0.$$

Let $H \in L^2_{Re}(\lambda_\sigma)$ be given. The operator $A^-_\lambda(H) : B_{bs}(\Gamma_0) \to L^2(\lambda_\sigma)$ defined by

$$\left(A^-_\lambda(H)G\right)(\eta) := \left(\left(A^-_\lambda G\right)(\eta, \cdot), H\right)_{L^2(\lambda_\sigma)} = \int_{\Gamma_0} G(\eta \cup \xi)H(\xi)d\lambda_\sigma(\xi),$$

for every $\eta \in \Gamma_0$, may be regarded as a generalization of the annihilation operator on the Lebesgue-Poisson space. The relation between this class of operators and the annihilation operators is clarified in the next result.

Theorem 6.18 *Let $h, g \in L^2_{Re}(\sigma)$, $G \in B_{bs}(\Gamma_0)$ be given. Then,*
(i) for

$$H^{(2)}_{h,g}(\eta) := \begin{cases} \frac{1}{2}\left(h(x)g(y) + h(y)g(x)\right), \eta = \{x, y\} \in \Gamma^{(2)}_X \\ 0, \eta \in \Gamma_0 \backslash \Gamma^{(2)}_X \end{cases}$$

we have

$$\frac{1}{2} a^-_\lambda(h)\left(a^-_\lambda(g)G\right) = A^-_\lambda(H^{(2)}_{h,g})G;$$

(ii)

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(a^-_\lambda(h)\right)^n G = A^-_\lambda(e_\lambda(h, \cdot))G,$$

where the sum converges in $L^2(\lambda_\sigma)$.

Proof: For $G \in B_{bs}(\Gamma_0)$ and $g \in L^2_{Re}(\sigma)$ one has $a^-_\lambda(g)G \in B_{bs}(\Gamma_0)$. Hence, we can define $a^-_\lambda(h)\left(a^-_\lambda(g)G\right)$ and

$$\begin{aligned} &\left(a^-_\lambda(h)\left(a^-_\lambda(g)G\right)\right)(\eta) \\ &= \frac{1}{2}\int_X \int_X G(\eta \cup \{x, y\})\left(h(x)g(y) + h(y)g(x)\right) d\sigma(y)d\sigma(x) \\ &= 2\int_{\Gamma_0} G(\eta \cup \xi)H^{(2)}_{h,g}(\xi)d\lambda_\sigma(\xi) = 2\left(A^-_\lambda(H^{(2)}_{h,g})G\right)(\eta). \end{aligned}$$

For $n \geq 3$, we obtain by induction

$$\begin{aligned} \left(\left(a^-_\lambda(h)\right)^n G\right)(\eta) &= \int_{X^n} G(\eta \cup \{x_1, ..., x_n\})h(x_1)...h(x_n)d\sigma^{\otimes n}(x_1, ..., x_n) \\ &= n!\int_{\Gamma_0} G(\eta \cup \xi)H^{(n)}_h(\xi)d\lambda_\sigma(\xi) = n!A^-_\lambda(H^{(n)}_h)G, \end{aligned}$$

where

$$H_h^{(n)}(\eta) := \begin{cases} \prod_{i=1}^n h(x_i), \eta = \{x_1, ..., x_n\} \in \Gamma_X^{(n)} \\ 0, \quad \eta \in \Gamma_0 \backslash \Gamma_X^{(n)} \end{cases}.$$

The following estimate

$$\begin{aligned} \left\| A_\lambda^-(H_h^{(n)})G \right\|_{L^2(\lambda_\sigma)}^2 &\leq \left\| H_h^{(n)} \right\|_{L^2(\lambda_\sigma)}^2 \left\| A_\lambda^- G \right\|_{L^2(\lambda_\sigma \otimes \lambda_\sigma)}^2 \\ &= \left\| H_h^{(n)} \right\|_{L^2(\lambda_\sigma)}^2 \sum_{k=0}^\infty \frac{2^k}{k!} \int_{\Gamma_X^{(k)}} |G(\eta)|^2 \, d\sigma^{(k)}(\eta) \\ &= \frac{1}{n!} \|h\|_{L^2(\sigma)}^{2n} \|G\|_{L^2(\lambda_{2\sigma})}^2 \end{aligned}$$

implies the convergence of the sum $\sum_{n=0}^\infty \frac{1}{n!} \left(a_\lambda^-(h) \right)^n G$ to $A_\lambda^-(e_\lambda(h, \cdot))G$ in $L^2(\lambda_\sigma)$. ■

By Lemma 5.6 we have for each $G \in B_{bs}(\Gamma_0)$

$$\begin{aligned} &\int_{\Gamma_0} \left(\int_{\Gamma_0} G(\eta \cup \xi) H(\xi) d\lambda_\sigma(\xi) \right) J(\eta) d\lambda_\sigma(\eta) \\ &= \int_{\Gamma_0} G(\eta) \sum_{\xi \subset \eta} H(\xi) J(\eta \backslash \xi) d\lambda_\sigma(\eta) \\ &= \int_{\Gamma_0} G(\eta) \left(H * J \right)(\eta) d\lambda_\sigma(\eta). \end{aligned}$$

The latter shows that for a real-valued function $H \in B_{bs}(\Gamma_0)$ the adjoint operator $A_\lambda^+(H)$ is defined by the expression

$$\left(A_\lambda^+(H) \right) G := G * H, \quad G \in B_{bs}(\Gamma_0).$$

The next result states a relation between the operators $A_\lambda^+(H)$ and the creation operators $a_\lambda^+(h)$.

Theorem 6.19 *Let $h, g \in \mathcal{D}$, $G \in B_{bs}(\Gamma_0)$ be given. Then,*
(i) for

$$H_{h,g}^{(2)}(\eta) := \begin{cases} \frac{1}{2} \left(h(x)g(y) + h(y)g(x) \right), \eta = \{x, y\} \in \Gamma_X^{(2)} \\ 0, \quad \eta \in \Gamma_0 \backslash \Gamma_X^{(2)} \end{cases}$$

one has

$$\frac{1}{2}a^+_\lambda(h)\left(a^+_\lambda(g)G\right) = A^+_\lambda(H^{(2)}_{h,g})G;$$

(ii)

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left(a^+_\lambda(h)\right)^n G = e_\lambda(h,\cdot) * G,$$

where the sum converges in $L^2(\lambda_\sigma)$.

Proof: The scheme of this proof is similar to the previous one. ■

Corollary 6.20 *For any* $\varphi \in \mathcal{D}$ *and* $F \in \mathcal{FP}_{bc}(\Gamma)$ *we have*

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left(a^+_\pi(\varphi)\right)^n F = e_\pi(\varphi,\cdot) \diamond F$$

in $L^2(\pi_\sigma)$.

Note that for H_1, $H_2 \in B_{bs}(\Gamma_0)$, H_1 a real-valued function, the following relation holds

$$A^+_\lambda(H_1)H_2 = H_1 * H_2 = A^+_\lambda H,$$

where $H(\eta,\xi) := H_1(\eta)H_2(\xi)$, η, $\xi \in \Gamma_0$.

Let us consider the linear operator D defined on $B_{bs}(\Gamma_0)$ by

$$DG := \int_{\Gamma_0} G(\cdot \cup \xi)d\lambda_\sigma(\xi).$$

Given a $G \in B_{bs}(\Gamma_0)$ with support contained in $\bigcup_{n=0}^N \Gamma^{(n)}_\Lambda$, $\Lambda \in \mathcal{B}_c(X)$, $N \in \mathbb{N}_0$, an estimate similar to (24) gives

$$\begin{aligned}\int_{\Gamma_0} |(DG)(\eta)|^2\, d\lambda_\sigma(\eta) &\leq \exp(\sigma(\Lambda))\int_{\Gamma_0}\int_{\Gamma_0} |G(\eta\cup\xi)|^2\, d\lambda_\sigma(\xi)d\lambda_\sigma(\eta) \\ &= \exp(\sigma(\Lambda))\sum_{n=0}^{N}\frac{2^n}{n!}\int_{\Gamma^{(n)}_X} |G(\eta)|^2\, d\sigma^{(n)}(\eta),\end{aligned}$$

showing that $DG \in L^2(\lambda_\sigma)$. Moreover, $DG \in B_{bs}(\Gamma_0)$ and the operator D verifies the following property.

Theorem 6.21 *The operator D is a linear isomorphism in $B_{bs}(\Gamma_0)$ and the inverse mapping is given by*

$$D^{-1}G := \int_{\Gamma_0} (-1)^{|\xi|} G(\cdot \cup \xi) d\lambda_\sigma(\xi), \quad G \in B_{bs}(\Gamma_0).$$

Proof: Assume that $G \in B_{bs}(\Gamma_0)$ has support contained in $S_G := \bigcup_{n=0}^N \Gamma_\Lambda^{(n)}$ for some $\Lambda \in \mathcal{B}_c(X)$, $N \in \mathbb{N}_0$. Then,

$$(DG)(\eta) = \mathbf{1}_{S_G}(\eta) \sum_{n=0}^N \frac{1}{n!} \int_{\Gamma_\Lambda^{(n)}} G(\eta \cup \xi) d\sigma^{(n)}(\xi), \quad \eta \in \Gamma_0,$$

and the measurable function DG has bounded support contained in S_G. Furthermore, we may majorized $|DG|$ by

$$\begin{aligned} |(DG)(\eta)| &\leq \sum_{n=0}^N \frac{1}{n!} \int_{\Gamma_\Lambda^{(n)}} |G(\eta \cup \xi)| \, d\sigma^{(n)}(\xi) \\ &\leq \sum_{n=0}^N \frac{C_G}{n!} \int_{\Gamma_\Lambda^{(n)}} d\sigma^{(n)}(\xi) \leq C_G \exp(\sigma(\Lambda)), \end{aligned}$$

where C_G is a constant with $|G| \leq C_G$. This proves that DG is a bounded function. Thus, $DG \in B_{bs}(\Gamma_0)$.

Given a $G \in B_{bs}(\Gamma_0)$ let us consider the element

$$\tilde{G} := \int_{\Gamma_0} (-1)^{|\xi|} G(\cdot \cup \xi) d\lambda_\sigma(\xi) \in B_{bs}(\Gamma_0).$$

Applying Lemma 5.6 to compute $D\tilde{G}$ we obtain

$$\left(D\tilde{G}\right)(\eta) = \int_{\Gamma_0} G(\eta \cup \xi) e_\lambda(0, \xi) d\lambda_\sigma(\xi) = G(\eta), \quad \eta \in \Gamma_0.$$

Analogously,

$$\int_{\Gamma_0} (-1)^{|\xi|} (DG)(\eta \cup \xi) d\lambda_\sigma(\xi) = G(\eta),$$

for all $\eta \in \Gamma_0$. ■

According to Lemma 5.6, note that

$$\int_{\Gamma_0} \int_{\Gamma_0} G(\eta \cup \xi) H(\eta) d\lambda_\sigma(\xi) d\lambda_\sigma(\eta) = \int_{\Gamma_0} G(\eta) \sum_{\xi \subset \eta} H(\eta \backslash \xi) d\lambda_\sigma(\eta),$$

implying that the adjoint operator of D, denoted by D^*, is defined by

$$(D^*G)(\eta) := \sum_{\xi\subset\eta} G(\eta\backslash\xi) = \sum_{\xi\subset\eta} G(\xi), \quad \eta\in\Gamma_0,$$

for each $G\in B_{bs}(\Gamma_0)$. More generally, for $G\in L^0(\Gamma_0,\mathcal{B}(\Gamma_0))$ we define

$$(D^*G)(\eta) := \sum_{\xi\subset\eta} G(\eta\backslash\xi) = (e_\lambda(1,\cdot)*G)(\eta), \quad \eta\in\Gamma_0.$$

Lemma 6.22 *The linear operator D can be extended to a bounded operator defined on $L^1(\lambda_{2\sigma})$ with values in $L^1(\lambda_\sigma)$. For any $G\in L^1(\lambda_{2\sigma})$ the extended operator is defined by*

$$(DG)(\eta) := \int_{\Gamma_0} G(\eta\cup\xi)d\lambda_\sigma(\xi), \quad \lambda_\sigma - a.a.\, \eta\in\Gamma_0. \tag{25}$$

Proof: It follows as a direct consequence of Lemma 5.6. For each function $G\in L^1(\lambda_{2\sigma})$ we obtain

$$\begin{aligned}\int_{\Gamma_0}\left|\int_{\Gamma_0} G(\eta\cup\xi)d\lambda_\sigma(\xi)\right| d\lambda_\sigma(\eta) &\le \int_{\Gamma_0}\int_{\Gamma_0} |G(\eta\cup\xi)|\, d\lambda_\sigma(\eta)d\lambda_\sigma(\xi)\\ &= \int_{\Gamma_0} 2^{|\eta|}\,|G(\eta)|\,d\lambda_\sigma(\eta)\end{aligned}$$

which is enough to show that the extension is given by (25). ■

Proposition 6.23 *Let $G_1, G_2\in L^1(\lambda_{2\sigma})$ be given. Then, we have the following equality:*

$$D(G_1*G_2) = DG_1 * DG_2.$$

Proof: To prove this proposition it is enough to show the claimed equality in a total set in $L^1(\lambda_{2\sigma})$. Since for every function $f\in B_{bs}(X)$ we have

$$De_\lambda(f,\cdot) = e^{\langle f\rangle_\sigma} e_\lambda(f,\cdot),$$

it follows for any $G_i = e_\lambda(f_i,\cdot)$, $f_i\in B_{bs}(X)$, $i=1,2$ the equalities

$$\begin{aligned} D(G_1*G_2) &= De_\lambda(f_1+f_2,\cdot)\\ &= \left(e^{\langle f_1\rangle_\sigma}e_\lambda(f_1,\cdot)\right) * \left(e^{\langle f_2\rangle_\sigma}e_\lambda(f_2,\cdot)\right)\\ &= DG_1 * DG_2.\end{aligned}$$

■

For the $*$-exponential defined above (see Subsection 5.2) we can add the following statement.

Corollary 6.24 *For any function $G \in L^1(\lambda_{2\sigma})$ we have*

$$\exp^*(DG) = D(\exp^* G).$$

Proof: Taking into account that $\exp^* G \in L^1(\lambda_{2\sigma})$ whenever $G \in L^1(\lambda_{2\sigma})$, the proof follows by Lemma 6.22. ■

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