

# Genericity of trivial Lyapunov spectrum for $L^p$ -cocycles derived from second order linear homogeneous differential equations

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## Abstract

We consider a probability space  $M$  on which an ergodic flow  $\varphi^t : M \rightarrow M$  is defined. We study a family of continuous-time linear cocycles, referred to as *kinetic*, that are associated with solutions of the second-order linear homogeneous differential equation  $\ddot{x} + \alpha(\varphi^t(\omega))\dot{x} + \beta(\varphi^t(\omega))x = 0$ . Here, the parameters  $\alpha$  and  $\beta$  evolve along the  $\varphi^t$ -orbit of  $\omega \in M$ . Our main result states that for a generic subset of kinetic continuous-time linear cocycles, where *generic* means a Baire second category with respect to an  $L^p$ -like topology on the infinitesimal generator, the Lyapunov spectrum is trivial.

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### 1. Introduction

We have known since the first half of 19th century and thanks to Liouville’s theorem, that severe restrictions exist when we try to apply analytic methods to integrate most functions. This result can be seen as a form of *differential Galois theory* and represent a significant obstacle to the explicit resolution of differential equations. To overcome this inevitability, we have two approaches: in one hand powerful numerical methods have been developed to approximate the solutions and on the other hand a qualitative theory of differential equations emerged from the pioneering works of Poincaré and Lyapunov. We will follow the latter approach in our study.

We intend to use our study to understand the asymptotic behavior of solutions of second order homogeneous linear differential equations. These equations will have coefficients that display  $L^p$  regularity and vary in time, while allowing an  $L^p$ -small perturbation. Namely, we want to describe the Lyapunov spectrum associated to it under  $L^p$ -generic conditions of their coefficients. We will consider families of equations of this type that are indexed in a flow which keeps invariant a probability in a measure space  $M$ . We have as motivation in the first instance a family of equations that describe the motion of the simple damped pendulum free from external forces, of type

$$\ddot{x}(t) + \alpha(\varphi^t(\omega))\dot{x}(t) + \beta(\varphi^t(\omega))x(t) = 0, \tag{1}$$

where  $\alpha$  and  $\beta$  are functions depending on  $\omega \in M$  evolving along a flow  $\varphi^t : M \rightarrow M$  for  $t \in \mathbb{R}$ . Clearly, if  $\alpha$  and  $\beta$  are first integrals related with  $\varphi^t$  (i.e. constant along the  $\varphi^t$ -orbits and also called *constants of motions*) equation (1) is easily solved by elementary methods of a first course on differential equations. Obtaining explicit solutions can be difficult when the parameters vary over time. This is the case when the *frictional force*  $\alpha$  and the *frequency of the oscillator*  $\beta$  change over time.

In [8] the second author dealt with a similar case but with periodic coefficients along periodic closed orbits and proved that small  $C^0$ -perturbations on the parameters allows us to obtain that asymptotic unstable solutions are precisely the uniformly hyperbolic saddle-type ones. In the present paper we intend to focus on a perturbative theory with a coarser topology, namely allowing perturbations in an  $L^p$ -type topology and assuming random (non-periodic) base dynamics.

Equations such as (1) are found in a wide range of fields including physics, engineering, and biology, as well as numerous mathematical applications such as solid state physics, structural stability, wave propagation in one-dimension, and the stability of synchronous electrical machines. Fixing *position and momentum*  $(x(0), \dot{x}(0))$  we intend to study the asymptotic behavior when  $t \rightarrow \infty$  of the pair  $(x(t), \dot{x}(t))$ , namely the asymptotic exponential growth rate given by the *Lyapunov exponent*. There is a significant amount of literature on the subject with comparable nuances, which can be found in [3,4,8,17,21,9] and the accompanying references. The field of Lyapunov exponents has experienced substantial growth over the past decade, and comprehensive treatment of its general aspects can be found in the following books: [6,14,18,28,31].

In [10], two of the authors studied the  $L^p$ -generic perspective on quite general linear differential systems, building on the work of Arbieto-Bochi [1] (see also the references therein). In [10] it was proved that the class of *accessible* (aka *twisting*) linear differential systems, a wider class that includes cocycles that evolve in  $GL(d, \mathbb{R})$ ,  $SL(d, \mathbb{R})$  and  $Sp(d, \mathbb{R})$ , have trivial Lyapunov spectrum  $L^p$ -generically. Since Millionshchikov’s research in the late 1960s, it has been established that the generic behavior changes when the sharper  $C^0$ -norm is taken into account (see [24]). Following the discrete approach taken in [11,12], a comprehensive study of the  $C^0$ -case was

conducted in [7]. Determining the  $C^0$ -generic asymptotic behavior of linear differential systems that arise from equations like (1) is still a work in progress. There has been recently a growing interest in understanding the discrete ‘dynamical cocycle’ from an  $L^p$ -perturbative viewpoint and  $L^p$ -generic properties [5,15,16]. Due to the fact that perturbing the cocycle provided by the derivative relies on perturbing the map, these dynamical cocycles are prone to introducing several other difficulties.

When we have two different Lyapunov exponents and intend to perturb the system slightly so that the Lyapunov exponents become equal the simplistic approach involves equally distributing of expansion/contraction rates for all directions. One way to achieve this is by rotating directions in a suitable manner to ensure that these rates become equally distributed. Rotating in a systematic way is indeed pivotal and was formulated in specific contexts within Soviet literature during the 1970s (see [25]) and in the 1980s by Mañé [22,23]. Issues related with the continuity of Lyapunov exponents are much more complicated when dealing with weaker topologies such as the  $L^p$  one. Our focus here is to dedicate significant effort towards comprehending the  $L^p$ -continuous dependence of the Lyapunov exponents when we perturb the system with an  $L^p$  topology (see §4.2). Furthermore, and also related to continuity, we know that usually a decrease in the distance between vector fields implies a decrease in the distance between their corresponding flow trajectories in compact times. Here, and once again, the  $L^p$  topology creates additional difficulties (see §4.1).

Overall, the primary outcome of this paper (Theorem 1) can be succinctly summarized as follows:

*For an  $L^p$ -generic choice of a kinetic linear cocycle associated to (1) no matter what position and momentum  $(x(0), \dot{x}(0))$  we chose as initial conditions because the asymptotic exponential behavior of almost every solution will be the same.*

This paper is organized as follows: in §2 we will present the basic definitions and state our main result (Theorem 1); section §3 is devoted to the perturbation framework; in section §4 we will address continuity issues of the Lyapunov exponents with respect to an  $L^p$  topology and, finally, in §5 we prove Theorem 1.

## 2. Definitions and main result

### 2.1. Linear cocycles

In this section we present some definitions that will be useful for the development of this work. Let  $(M, \mathcal{M}, \mu)$  be a probability space and let  $\varphi: \mathbb{R} \times M \rightarrow M$  be a *metric dynamical system* (or *flow*) in the sense that is a measurable map and

- (1)  $\varphi^t: M \rightarrow M$  given by  $\varphi^t(\omega) = \varphi(t, \omega)$  preserves the measure  $\mu$  for all  $t \in \mathbb{R}$ ;
- (2)  $\varphi^0 = \text{Id}_M$  and  $\varphi^{t+s} = \varphi^t \circ \varphi^s$  for all  $t, s \in \mathbb{R}$ .

Unless stated otherwise we will consider in the sequel that the flow is ergodic in the usual sense that there exist no invariant sets except null sets and their complements.

Let  $\mathcal{B}(X)$  be the Borel  $\sigma$ -algebra of a topological space  $X$ . A (continuous-time) linear *random dynamical system* (RDS) on  $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ , or a (continuous-time) *linear cocycle* over  $\varphi$  is a  $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{M}/\mathcal{B}(\text{GL}(2, \mathbb{R})))$ -measurable map

$$\Phi: \mathbb{R} \times M \rightarrow \text{GL}(2, \mathbb{R})$$

such that the mappings  $\Phi(t, \omega)$  form a cocycle over  $\varphi$ , i.e.,

- (1)  $\Phi(0, \omega) = \text{Id}$  for all  $\omega \in M$ ;
- (2)  $\Phi(t + s, \omega) = \Phi(t, \varphi^s(\omega)) \circ \Phi(s, \omega)$ , for all  $s, t \in \mathbb{R}$  and  $\omega \in M$ ,

and  $t \mapsto \Phi(t, \omega)$  is continuous for all  $\omega \in M$ . We recall that having  $\omega \mapsto \Phi(t, \omega)$  measurable for each  $t \in \mathbb{R}$  and  $t \mapsto \Phi(t, \omega)$  continuous for all  $\omega \in M$  implies that  $\Phi$  is measurable in the product measure space. These objects are also called *linear differential systems* in the literature (see §2.2).

### 2.2. Kinetic linear cocycles

As a source of motivation, we start by studying the non-autonomous linear differential equation that describes the motion of the damped harmonic oscillator, also known as the ‘simple pendulum’, along the path  $(\varphi^t(\omega))_{t \in \mathbb{R}}$ , where  $\omega \in M$ . Let  $K \subset \mathbb{R}^{2 \times 2}$  consisting of  $2 \times 2$  matrices of type

$$A = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}$$

for real numbers  $a, b$ . Denote by  $\mathcal{G}$  the set of measurable applications  $A : M \rightarrow \mathbb{R}^{2 \times 2}$  and by  $\mathcal{K} \subset \mathcal{G}$  the set of *kinetic* measurable applications  $A : M \rightarrow K$ . As usual we identify two applications on  $\mathcal{G}$  that coincide on a  $\mu$  full measure subset of  $M$ . Consider measurable maps  $\alpha : M \rightarrow \mathbb{R}$  and  $\beta : M \rightarrow \mathbb{R}$ . Recall the random differential equation given by (1) defined by:

$$\ddot{x}(t) + \alpha(\varphi^t(\omega))\dot{x}(t) + \beta(\varphi^t(\omega))x(t) = 0. \tag{2}$$

Considering the change of variables  $y(t) = \dot{x}(t)$  we may rewrite (2) as the following vectorial linear system

$$\dot{X} = A(\varphi^t(\omega)) \cdot X, \tag{3}$$

where  $X = X(t) = (x(t), y(t))^\top = (x(t), \dot{x}(t))^\top$  and  $A \in \mathcal{K}$  is given by

$$A(\omega) = \begin{pmatrix} 0 & 1 \\ -\beta(\omega) & -\alpha(\omega) \end{pmatrix}.$$

We define

$$\mathcal{G}^1 = \left\{ A \in \mathcal{G} : \int_M \|A(\omega)\| d\mu(\omega) < \infty \right\},$$

where  $\|\cdot\|$  denotes de standard Euclidean matrix norm. It follows from [2, Thm. 2.2.2] (see also Lemma 2.2.5 and Example 2.2.8 in this reference) that if  $A \in \mathcal{G}^1$ , then  $A$  generates a unique (up to indistinguishability) linear cocycle  $\Phi_A$  satisfying

$$\Phi_A(t, \omega) = \text{Id} + \int_0^t A(\varphi^s(\omega))\Phi_A(s, \omega) ds. \tag{4}$$

The solution  $\Phi_A(t, \omega)$  defined in (4) is called *the Carathéodory solution* or *weak solution*. Given an initial condition  $X(0) = v \in \mathbb{R}^2$ , we say that  $t \mapsto \Phi_A(t, \omega)v$  solves or is a solution of (3), or that (3) generates  $\Phi_A(t, \omega)$ . Note that  $\Phi_A(0, \omega)v = v$  for all  $\omega \in M$  and  $v \in \mathbb{R}^2$ . If the solution (4) is differentiable in time (i.e. with respect to  $t$ ) and satisfies for all  $t$

$$\frac{d}{dt} \Phi_A(t, \omega)v = A(\varphi^t(\omega))\Phi_A(t, \omega)v \quad \text{and} \quad \Phi_A(0, \omega)v = v, \tag{5}$$

then it is called a *classical solution* of (3).

Of course that  $t \mapsto \Phi_A(t, \omega)v$  is continuous for all  $\omega$  and  $v$ . Due to (5) we call  $A : M \rightarrow K$  a (kinetic) *infinitesimal generator* of  $\Phi_A$ . Sometimes, due to the relation between  $A$  and  $\Phi_A$ , we refer to both  $A$  and  $\Phi_A$  as a kinetic linear cocycle/RDS/differential system. If (3) has initial condition  $X(0) = v$  then  $\Phi_A(0, \omega)v = v$  and  $X(t) = \Phi_A(t, \omega)v$ .

### 2.3. The topology

We begin by defining an  $L^p$ -like topology generated by a metric that compares the infinitesimal generators on  $\mathcal{G}$ . By a standard use of Grönwall’s inequality arguments it is usual to obtain the approximation of flows (on compact times) by assuming the approximation of the corresponding infinitesimal generators. However,  $L^p$ -estimates,  $1 \leq p < \infty$ , are more demanding and it is not clear if such  $L^p$ -continuous dependence holds (cf. Lemma 4.3). Related to this problem see e.g. [27] where these issues are treated for  $L^1$ -approximation on initial conditions in quasi-linear elliptic-parabolic equations. With this in mind we define now the distance we are going to consider along this work. Given  $1 \leq p < \infty$  and  $A, B \in \mathcal{G}$  we set

$$\hat{\sigma}_p(A, B) := \begin{cases} \left( \int_M \|A(\omega) - B(\omega)\|^p d\mu(\omega) \right)^{\frac{1}{p}}, \\ \infty \text{ if the above integral does not exists,} \end{cases}$$

and define

$$\sigma_p(A, B) := \begin{cases} \frac{\hat{\sigma}_p(\Phi_A, \Phi_B)}{1 + \hat{\sigma}_p(\Phi_A, \Phi_B)}, & \text{if } \hat{\sigma}_p(A, B) < \infty \\ 1, & \text{if } \hat{\sigma}_p(A, B) = \infty \end{cases}.$$

Clearly,  $\sigma_p$  is a distance in  $\mathcal{G}$ . For standard results on  $L^p$  spaces see e.g. [29].

**Proposition 2.1.** *Consider  $1 \leq p < \infty$ . Then:*

- (i)  $\sigma_p(A, B) \leq \sigma_q(A, B)$  for all  $q \geq p$  and all  $A, B \in \mathcal{G}$ .
- (ii) If  $A \in \mathcal{G}^1$  then for any  $B \in \mathcal{G}$  satisfying  $\sigma_p(A, B) < 1$  we have  $B \in \mathcal{G}^1$ .

**Proof.** (i) For all  $1 \leq p \leq q < \infty$  it is a standard result on Lebesgue spaces that the  $L^q$ -topology is thinner than the  $L^p$  one. So, we have  $\hat{\sigma}_p(A, B) \leq \hat{\sigma}_q(A, B)$ . Hence,  $\sigma_p(A, B) \leq \sigma_q(A, B)$ .

(ii) Pick any  $1 \leq p < \infty$  and let  $A \in \mathcal{G}^1$  and  $B \in \mathcal{G}$  be such that  $\sigma_p(A, B) < 1$ . By (i) we have  $\sigma_1(A, B) < 1$ , thus  $\hat{\sigma}_1(A, B) < \infty$ , and we also have that  $\|B(\omega)\| \leq \|A(\omega)\| + \|B(\omega) - A(\omega)\|$  for all  $\omega \in M$ .  $\square$

The following result will be used to prove that the subspace of kinetic linear cocycles is a Baire space. Let  $\mathcal{K}^1 = \mathcal{K} \cap L^1(\mu) \subset \mathcal{G}^1$ .

**Lemma 2.2.** *The set  $\mathcal{K}^1$  is a closed subset of  $\mathcal{G}$  with respect to  $\sigma_p$  for all  $1 \leq p < \infty$ .*

**Proof.** Given any sequence  $(A_n)_n$ , with  $A_n \in \mathcal{K}^1$ , such that  $A_n \xrightarrow{\sigma_p} A_\infty$ , for some  $A_\infty \in \mathcal{G}$  we must prove that  $A_\infty \in \mathcal{K}^1$ . As  $A_n \in \mathcal{K}^1$  we get

$$A_n(\omega) = \begin{pmatrix} 0 & 1 \\ -\beta_n(\omega) & -\alpha_n(\omega) \end{pmatrix} \quad \text{and} \quad A_\infty = \begin{pmatrix} a(\omega) & b(\omega) \\ -\beta(\omega) & -\alpha(\omega) \end{pmatrix}$$

for maps  $\alpha_n, \beta_n \in L^1(\mu)$  and measurable maps  $a, b, \alpha$  and  $\beta$ . From (i) in Proposition 2.1 we have  $A_n \xrightarrow{\sigma_1} A_\infty$ . From (ii) in Proposition 2.1 we have that  $a, b, \alpha$  and  $\beta$  are also  $L^1$  maps, i.e.  $A_\infty \in \mathcal{G}^1$ . Hence,  $\int_M |a(\omega)| d\mu(\omega) = 0$  and  $\int_M |b(\omega) - 1| d\mu(\omega) = 0$ , which implies  $a(\omega) = 0$  and  $b(\omega) = 1$  for  $\mu$  a.e. (almost every)  $\omega$ . Therefore,  $A_\infty \in \mathcal{K}$ . In conclusion,  $A_\infty \in \mathcal{G}^1 \cap \mathcal{K} = \mathcal{K}^1$ .  $\square$

Since for all  $1 \leq p < \infty$ , the metric space  $(\mathcal{G}^1, \sigma_p)$  is complete and  $\mathcal{K}^1 \subset \mathcal{G}^1$  is  $\sigma_p$ -closed, we conclude the following.

**Corollary 2.3.** *For all  $1 \leq p < \infty$ ,  $(\mathcal{K}^1, \sigma_p)$  is a complete metric space and, therefore, a Baire space.*

### 2.4. Lyapunov exponents

Observe that if  $A \in \mathcal{K}^1$  then the cocycle  $\Phi_A$  satisfies the following *integrability condition*

$$\sup_{0 \leq t \leq 1} \log^+ \|\Phi_A(t, \omega)^{\pm 1}\| \in L^1(\mu), \tag{6}$$

where  $f^+ = \max\{0, f\}$ . Indeed, consider  $\omega$  in the full measure  $\varphi^t$ -invariant subset of  $M$  where  $t \mapsto A(\varphi^t(\omega))$  is locally integrable. By (4) we get

$$\|\Phi_A(t, \omega)^{\pm 1}\| \leq 1 + \int_0^t \|A(\varphi^s(\omega))\| \|\Phi_A(s, \omega)^{\pm 1}\| ds.$$

By Grönwall's inequality (see [2]) we have

$$\|\Phi_A(t, \omega)^{\pm 1}\| \leq \exp\left(\int_0^t \|A(\varphi^s(\omega))\| ds\right) \tag{7}$$

that is, for all  $t \geq 0$  we have  $\log^+ \|\Phi_A(t, \omega)^{\pm 1}\| \leq \int_0^t \|A(\varphi^s(\omega))\| ds$ . Therefore for all  $T > 0$

$$\sup_{0 \leq t \leq T} \log^+ \|\Phi_A(t, \omega)^{\pm 1}\| \leq \int_0^T \|A(\varphi^s(\omega))\| ds =: \psi(\omega, T). \tag{8}$$

By [2, Lemma 2.2.5] we have  $\psi(\cdot, T) \in L^1(\mu)$ . Thus  $\sup_{0 \leq t \leq 1} \log^+ \|\Phi_A(t, \omega)^{\pm 1}\| \in L^1(\mu)$ , getting the claim. Moreover, Fubini’s theorem allows us to obtain

$$\begin{aligned} \int_M \log^+ \|\Phi_A(t, \omega)^{\pm 1}\| d\mu(\omega) &\leq \int_M \int_0^t \|A(\varphi^s(\omega))\| ds d\mu(\omega) \\ &= \int_0^t \int_M \|A(\varphi^s(\omega))\| d\mu(\omega) ds = t \|A\|_1. \end{aligned}$$

If  $A \in \mathcal{G}^1$  then the integrability condition (6) holds and Oseledets theorem (see e.g. [26,2]) guarantees that for  $\mu$  a.e.  $\omega \in M$ , there exists a  $\Phi_A$ -invariant splitting called *Oseledets splitting* of the fiber  $\mathbb{R}^2_\omega = E^1_\omega \oplus E^2_\omega$  and real numbers called *Lyapunov exponents*  $\lambda_1(A, \omega) \geq \lambda_2(A, \omega)$ , such that:

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \|\Phi_A(t, \omega)v^i\| = \lambda_i(A, \omega), \tag{9}$$

for any  $v^i \in E^i_\omega \setminus \{\vec{0}\}$  and  $i = 1, 2$  where the norm in (9) represents the Euclidean norm of a vector in  $\mathbb{R}^2$ . Moreover, given any of these subspaces  $E^1_\omega$  and  $E^2_\omega$ , the angle between them along the orbit has subexponential growth, meaning that

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \sin\left(\angle\left(E^1_{\varphi^t(\omega)}, E^2_{\varphi^t(\omega)}\right)\right) = 0. \tag{10}$$

If the flow  $\varphi^t$  is ergodic, then the Lyapunov exponents and the dimensions of the associated subbundles are constant  $\mu$  a.e. point and we refer to them as  $\lambda_1(A)$  and  $\lambda_2(A)$ , with  $\lambda_1(A) \geq \lambda_2(A)$ . The Lyapunov spectrum of  $A$  is the set of its Lyapunov exponents. We say that  $A$  has *one-point Lyapunov spectrum* or *trivial Lyapunov spectrum* if for  $\mu$  a.e.  $\omega \in M$ ,  $\lambda_1(A, \omega) = \lambda_2(A, \omega)$ . Otherwise we say  $A$  has *simple Lyapunov spectrum*. For details on results on linear cocycles see [2] (in particular, Example 3.4.15). See also [19].

### 2.5. Statement of the main result

We are now in position to state our main contribution. In the present paper we establish the existence of a  $\sigma_p$ -residual of  $\mathcal{K}^1$  displaying one-point spectrum. More precisely we prove the following:

**Theorem 1.** *For all  $1 \leq p < \infty$  there exists a  $\sigma_p$ -residual subset  $\mathcal{R} \in \mathcal{K}^1$  such that any  $A \in \mathcal{R}$  has one-point spectrum.*

### 3. Interchanging Oseledets’ directions

This section is dedicated to constructing the essential tools for perturbations, enabling us to mix Oseledets directions. To begin, we will address several aspects related to the perturbations that we aim to create.

#### 3.1. Perturbations supported in compact sets

Take  $A \in \mathcal{G}$  and a non-periodic point  $\omega \in M$ . We will consider a perturbation  $B_\omega$  of  $A$  supported along a segment of the orbit of  $\omega$  with extremes  $\omega$  and  $\varphi^1(\omega)$ . For that denote

$$\varphi^{[0,1]}(\omega) = \{\varphi^t(\omega) : t \in [0, 1]\}. \tag{11}$$

Let  $P \in \mathcal{G}$  be given and define  $B : M \rightarrow \mathbb{R}^{2 \times 2}$  such that

$$B(\hat{\omega}) := \begin{cases} A(\hat{\omega}), & \text{if } \hat{\omega} \notin \varphi^{[0,1]}(\omega) \\ P(\hat{\omega}), & \text{if } \hat{\omega} \in \varphi^{[0,1]}(\omega) \end{cases} . \tag{12}$$

The map  $B$  is called a *perturbation of  $A$  by  $P$  supported on  $\varphi^{[0,1]}(\omega)$* .

**Lemma 3.1.** *Given  $A \in \mathcal{K}^1$ ,  $\omega \in M$ ,  $u, v \in \mathbb{R}^2 \setminus \{(0, 0)\}$ , there is  $\gamma \neq 0$ , and a perturbation  $B \in \mathcal{K}^1$  of  $A$  supported on  $\varphi^{[0,1]}(\omega)$  such that:*

- (i)  $\|B(\varphi^t(\omega))\| \leq 4\pi^2$  for all  $t \in [0, 1]$  and
- (ii)  $\Phi_B(1, \omega)u = \gamma v$ .

**Proof.** For  $\theta > 0$  we take an infinitesimal generator  $P_\theta : M \rightarrow \mathbb{R}^{2 \times 2}$  in  $\mathcal{K}^1$ , not depending on  $\omega \in M$ , given by

$$P_\theta = \begin{pmatrix} 0 & 1 \\ -\theta^2 & 0 \end{pmatrix}. \tag{13}$$

We define the perturbation  $B \in \mathcal{K}^1$  of  $A$  by  $P_\theta$  supported on  $\varphi^{[0,1]}(\omega)$  as in (12). Since  $\det P_\theta > 0$  and  $P_\theta$  is traceless the solution is a center (rotation-like behavior). Indeed, the infinitesimal generator in (13) generates a linear cocycle with fundamental classical solution (4) given, for all  $\hat{\omega} \in M$  and  $t \in \mathbb{R}$  by the ‘clockwise elliptical rotation’ defined by:

$$\Phi_{P_\theta}(t, \hat{\omega}) = \begin{pmatrix} \cos(\theta t) & \theta^{-1} \sin(\theta t) \\ -\theta \sin(\theta t) & \cos(\theta t) \end{pmatrix}.$$

Fixing  $u \in \mathbb{R}^2 \setminus \{(0, 0)\}$  and  $\omega \in M$  as in the statement of the lemma we consider now the evaluation map

$$\begin{aligned} \mathcal{E}_{u,\omega}: \quad ]0, 2\pi] &\longrightarrow \mathbb{RP}^1 \\ \theta &\longmapsto [\Phi_{P_\theta}(1, \omega)u] \end{aligned} \tag{14}$$

where  $\mathbb{RP}^1$  is the real projective space obtained considering the quotient of  $\mathbb{R}^2 \setminus \{(0, 0)\}$  under the equivalence relation  $a \sim \gamma a$  for any  $\gamma \in \mathbb{R} \setminus \{0\}$ . We denote by  $[a]$  the equivalence class of  $a \in \mathbb{R}^2$ .

Clearly  $\Phi_{P_\pi}(1, \omega)u = -u$  and  $\Phi_{P_{2\pi}}(1, \omega)u = u$  that is  $\mathcal{E}_{u,\omega}(\pi) = \mathcal{E}_{u,\omega}(2\pi)$ . Since  $\theta \mapsto \Phi_{P_\theta}(1, \omega)u$  is continuous, the map  $\mathcal{E}_{u,\omega}$  in (14) is also continuous. We conclude that the restriction of  $\mathcal{E}_{u,\omega}$  to the interval  $]\pi, 2\pi]$  is a surjective map. Indeed, it is just a direct application of Bolzano’s intermediate value theorem and the fact that  $\Phi_{P_\theta}(1, \omega)$  induces a rotation in  $\mathbb{RP}^1$ . Now given  $v \in \mathbb{R}^2 \setminus \{(0, 0)\}$  as in the statement of the lemma we can tune  $\theta$  in order that  $\Phi_B(1, \omega)u = \Phi_{P_\theta}(1, \omega)u = \gamma v$ , for some  $\gamma \neq 0$  fulfilling (ii). Since for all  $t \in [0, 1]$  we have  $B(\varphi^t(\omega)) = P_\theta(\varphi^t(\omega))$  and  $\|P_\theta\| = \max\{1, \theta^2\} \leq 4\pi^2$ , (i) holds.  $\square$

**Remark 3.1.** In the application of Lemma 3.1 further ahead in (20) we will consider  $u$  and  $v$  such that  $\mathbb{R}u := \{\gamma u : \gamma \in \mathbb{R}\} = E_\omega^1$  and  $\mathbb{R}v = E_{\varphi^1(\omega)}^2$  where  $E^i$  are the Oseledets directions associated to  $A$ . Therefore, we get  $\Phi_B(1, \omega)E_\omega^1 = E_{\varphi^1(\omega)}^2$ . Moreover, as the Euclidean norm and the maximum norm are equivalent Lemma 3.1 (i) and (7) imply  $\|\Phi_B(1, \omega)\|_{\max} \leq C \|\Phi_B(1, \omega)\| \leq Ce^{4\pi^2}$  for some  $C > 0$ .

### 3.2. Special flows

Consider a measure space  $\Sigma$ , an invertible map  $\mathcal{T} : \Sigma \rightarrow \Sigma$ , a  $\mathcal{T}$ -invariant probability measure  $\tilde{\mu}$  defined in  $\Sigma$  and a roof function  $h : \Sigma \rightarrow \mathbb{R}^+$  satisfying  $h(x) \geq H > 0$ , for some  $H > 0$  and all  $x \in \Sigma$ , and  $\int_\Sigma h(x)d\tilde{\mu}(x) < \infty$ . Define the quotient space  $M_h \subseteq \Sigma \times \mathbb{R}_+$  by

$$M_h = \{(x, r) \in \Sigma \times \mathbb{R}_+ : 0 \leq r \leq h(x)\}$$

with the identification between the pairs  $(x, h(x))$  and  $(\mathcal{T}(x), 0)$ . The flow defined on  $M_h$ , for  $t \geq 0$ , by

$$\varphi^t(\omega) = \varphi^t(x, r) = \left( \mathcal{T}^n(x), r + t - \sum_{i=0}^{n-1} h(\mathcal{T}^i(x)) \right),$$

where  $n$  is uniquely defined by

$$\sum_{i=0}^{n-1} h(\mathcal{T}^i(x)) \leq r + t < \sum_{i=0}^n h(\mathcal{T}^i(x))$$

and, for  $t < 0$  by

$$\varphi^t(x, r) = \begin{cases} (x, r + t), & \text{if } r + t \geq 0 \\ (\mathcal{T}^{-n}(x), r + t + \sum_{i=-n}^{-1} h(\mathcal{T}^i(x))), & \text{if } r + t < 0 \end{cases}$$

where  $n$  is uniquely defined by

$$-\sum_{i=-n}^{-1} h(\mathcal{T}^i(x)) \leq r + t < -\sum_{i=-n+1}^{-1} h(\mathcal{T}^i(x))$$

is called a *special flow* (see [13]). Furthermore, if  $\ell$  denotes the one dimensional Lebesgue measure the measure  $\mu = (\tilde{\mu} \times \ell) / \int h d\tilde{\mu}$  defined on  $M_h$  by

$$\int g d\mu = \frac{1}{\int h d\tilde{\mu}} \int \left( \int_0^{h(\omega)} g(\omega, t) dt \right) d\tilde{\mu}(\omega), \quad \forall g \in C^0(M_h)$$

is a probability measure and it is invariant by the special flow  $\varphi$ .

Although straightforward, the following result will constitute a crucial step in proving the second part of Proposition 3.3 ahead. We fix a special flow described by the quadruple  $(\varphi, \Sigma, \mathcal{T}, h)$  endowed with measure  $\mu = (\tilde{\mu} \times \ell) / \int h d\tilde{\mu}$ ,  $\tilde{\Sigma}_0 \subset \Sigma$ , with  $\tilde{\mu}(\tilde{\Sigma}_0) > 0$ , and set  $\Sigma_0 = \tilde{\Sigma}_0 \times \{0\}$ . Moreover, for  $b > a > 0$  similarly to (11) we denote

$$\varphi^{[a,b]}(\Sigma_0) = \{\varphi^t(\omega) : \omega \in \Sigma_0, t \in [a, b]\}.$$

Given  $A \in \mathcal{G}^1$ ,  $P \in \mathcal{G}$ ,  $\Sigma_0 \subseteq \Sigma$  and  $a > 0$ , we may extend the perturbation of  $A$  by  $P$  to be supported on the flowbox  $\varphi^{[a,a+1]}(\Sigma_0)$  in the following way: for  $\omega \in \varphi^{[a,a+1]}(\Sigma_0)$  we project  $\omega$  in  $\tilde{\omega} \in \Sigma_0$  and let  $B_{\tilde{\omega}}$  be a perturbation of  $A$  by  $P$  supported on  $\varphi^{[0,1]}(\varphi^a(\omega))$ , and define

$$B(\omega) := \begin{cases} A(\omega), & \text{if } \omega \notin \varphi^{[a,a+1]}(\Sigma_0) \\ B_{\tilde{\omega}}(\omega), & \text{if } \omega \in \varphi^{[a,a+1]}(\Sigma_0) \end{cases}.$$

To distinguish the situations we refer  $B$  as a *global* perturbation of  $A$  by  $P$  supported in  $\varphi^{[a,a+1]}(\Sigma_0)$ .

**Lemma 3.2.** *For all  $A \in \mathcal{G}^1$ ,  $a > 0$  and  $0 < \varepsilon < 1$ , there exists a measurable set  $\tilde{\Sigma}_0 \subset \Sigma$  with  $\tilde{\mu}(\tilde{\Sigma}_0) > 0$  such that for any global perturbation  $B \in \mathcal{G}^1$  of  $A$  supported in the flowbox  $\varphi^{[a,a+1]}(\Sigma_0)$  with  $\|B(\varphi^t(\omega))\| \leq 4\pi^2$  for all  $\omega \in \Sigma_0$  and  $t \in [a, a + 1]$ , we have that  $\sigma_1(A, B) < \varepsilon$ .*

**Proof.** Let  $A \in \mathcal{G}^1$ ,  $a > 0$  and  $0 < \varepsilon < 1$  be fixed. For any  $\tilde{\Sigma}_0 \subset \Sigma$  we have  $\mu(\varphi^{[a,a+1]}(\Sigma_0)) = \tilde{\mu}(\tilde{\Sigma}_0)$ . Let  $\varepsilon' = \frac{\varepsilon}{1-\varepsilon}$  and choose a measurable set  $\tilde{\Sigma}_0 \subset \Sigma$  such that

$$\int_{\varphi^{[a,a+1]}(\Sigma_0)} \|A(\omega)\| d\mu(\omega) < \frac{\varepsilon'}{2}, \tag{15}$$

and

$$0 < \tilde{\mu}(\tilde{\Sigma}_0) < \frac{\varepsilon'}{8\pi^2}. \tag{16}$$

Finally, we will check that  $\sigma_1(A, B) < \varepsilon$ . Indeed, we get:

$$\begin{aligned} \hat{\sigma}_1(A, B) &= \int_M \|A(\omega) - B(\omega)\| d\mu(\omega) = \int_{\varphi^{[a,a+1]}(\Sigma_0)} \|A(\omega) - B(\omega)\| d\mu(\omega) \\ &\leq \int_{\varphi^{[a,a+1]}(\Sigma_0)} \|A(\omega)\| + \|B(\omega)\| d\mu(\omega) \stackrel{(15)}{<} \frac{\varepsilon'}{2} + \mu(\varphi^{[a,a+1]}(\Sigma_0))4\pi^2 \stackrel{(16)}{<} \varepsilon', \end{aligned}$$

which implies  $\sigma_1(A, B) < \varepsilon$ .  $\square$

### 3.3. Lowering the maximal Lyapunov exponent

The next result asserts that is possible to lower the largest Lyapunov exponent of a kinetic cocycle with simple spectrum by a  $\sigma_p$ -small perturbation.

**Proposition 3.3.** *Given  $A \in \mathcal{K}^1$  and  $\varepsilon, \delta > 0$ , there exists  $B \in \mathcal{K}^1$  such that  $\sigma_1(A, B) < \varepsilon$  and*

$$\lambda_1(B) \leq \frac{\lambda_1(A) + \lambda_2(A)}{2} + \delta. \tag{17}$$

**Proof.** Let  $A \in \mathcal{K}^1$  and  $\varepsilon, \delta > 0$  be given. We assume that  $A$  has simple spectrum, i.e.  $\lambda_1(A) > \lambda_2(A)$ , otherwise the conclusion is trivial by considering  $B = A$ . We are going to perform perturbations supported on a segment of size 1 (cf. Lemma 3.1) in the orbit of points  $\omega$  of a subset  $\Sigma_0$  of positive measure, with the perturbation to be taken approximately at  $\varphi^{N/2}(\omega)$ , where  $N$  is large enough to approximate the Lyapunov exponents up to some given accuracy  $\eta$ , both for  $\omega$  and  $\varphi^{\frac{N}{2}+1}(\omega)$ . To avoid unpleasant overlapping situations, we codify  $\varphi$  by a special flow with a finite roof function. For a better understanding and since the flow  $\varphi$  is ergodic a simple application of Rudolph’s two symbol code representation (see [30]) allows us to consider a special flow with basis  $\Sigma \subset M$  and a roof function  $h: \Sigma \rightarrow \{N + 1, N + q\}$ , for some large  $N$  and some  $1 < q \in \mathbb{R} \setminus \mathbb{Q}$ . We recall that Rudolph’s theorem gives a step function of rationally independent high levels. Before returning to the basis of the special flow and considering  $h > N + 1$ , we will divide the orbit segment into three parts (this will be detailed later in (25)):

- (i) in the first part, from  $\omega$  to  $\varphi^{N/2}(\omega)$ , we iterate  $N/2$  and  $N$  is sufficiently large for the Lyapunov exponent to be *achieved* with accuracy  $\eta$  for  $\omega$  (see (18));
- (ii) in the second part, from  $\varphi^{N/2}(\omega)$  to  $\varphi^{N/2+1}(\omega)$ , we perform a time-1 perturbation (using Lemmas 3.1 and 3.2);
- (iii) in the third part, from  $\varphi^{N/2+1}(\omega)$  to  $\varphi^N(\omega)$ , we iterate  $N/2 - 1$  and  $N$  is sufficiently large for the Lyapunov exponent to be *achieved* with accuracy  $\eta$  for  $\varphi^{N/2+1}(\omega)$  (see (19)).

In this way, i.e. taking  $h > N + 1$ , we prevent overlapping of perturbations.

**Remark 3.2.** In order for the reader to understand the simplicity of the strategy we consider the non-trivial conservative case when  $\lambda_2(A) = -\lambda_1(A) < 0$ . During (i) we get a rate  $\lambda_1(A)$  along direction  $E_\omega^1$ . Then in (ii) we send  $E_\omega^1$  into  $E_{\varphi^{N/2+1}(\omega)}^2$ . Finally, during (iii) we get a rate  $\lambda_2(A) = -\lambda_1(A)$  along direction  $E_{\varphi^{N/2+1}(\omega)}^2$ . In the time interval  $[0, N]$  we get approximately  $\lambda_1(A) + \lambda_2(A) = 0$  for  $\lambda_1(B)$ . Overall, we obtain (17) which in this case will be  $\lambda_1(B) < \delta$ .

We have a decomposition of  $\mu = (\tilde{\mu} \times \ell) / \int h d\tilde{\mu}$ , where  $\ell$  is the time length and  $\tilde{\mu}$  is a measure on  $\Sigma$  invariant by the return map. By abuse of notation in the following we still denote the special flow by  $\varphi^t$ . Up to some rearrangement of the castle, given  $\eta > 0$  there is  $N \in \mathbb{N}$  such that by the Oseledets theorem there is a subset  $\tilde{\Sigma} \subseteq \Sigma$ , with  $\tilde{\mu}(\tilde{\Sigma}) > 0$ , such that for every  $\omega \in \tilde{\Sigma}$  we have:

$$\left| \lambda_i - \frac{1}{t} \log \left\| \Phi_A(t, \omega) |_{E_\omega^i} \right\| \right| < \eta/8 < \eta \tag{18}$$

for  $i = 1, 2$  and  $t \geq N/2$ . Notice that from the cocycle property, by taking  $N$  larger if necessary, for  $\omega \in \tilde{\Sigma}$  and setting  $\omega' = \varphi^{N/2+1}(\omega)$ , we also have

$$\left| \lambda_i - \frac{1}{N/2 - 1} \log \left\| \Phi_A(N/2 - 1, \omega') |_{E_{\omega'}^i} \right\| \right| < \eta. \tag{19}$$

Indeed, to obtain (19) we notice that

$$\begin{aligned} & \left| \lambda_i - \frac{1}{N/2 - 1} \log \left\| \Phi_A(N/2 - 1, \omega') |_{E_{\omega'}^i} \right\| \right| \\ &= \left| \lambda_i - \frac{1}{N/2 - 1} \log \left\| \Phi_A(N, \omega) |_{E_\omega^i} \Phi_A(-(N/2 + 1), \omega') |_{E_{\omega'}^i} \right\| \right| \\ &\leq \left| 2\lambda_i - \frac{N}{N/2 - 1} \frac{1}{N} \log \left\| \Phi_A(N, \omega) |_{E_\omega^i} \right\| \right| \\ &\quad + \left| -\lambda_i - \frac{N/2 + 1}{N/2 - 1} \frac{1}{N/2 + 1} \log \left\| \Phi_A(N/2 + 1, \omega)^{-1} |_{E_{\omega'}^i} \right\| \right|. \end{aligned}$$

The first term is less or equal than

$$\left| 2 - \frac{N}{N/2 - 1} \right| |\lambda_i| + \left| \lambda_i - \frac{1}{N} \log \left\| \Phi_A(N, \omega) |_{E_\omega^i} \right\| \right| \left| \frac{N}{N/2 - 1} \right|,$$

which can be made smaller than  $\eta/2$  for sufficiently large  $N$ , as well the second term because is less or equal than

$$\left| 1 - \frac{N/2 + 1}{N/2 - 1} \right| |\lambda_i| + \left| \lambda_i - \frac{1}{N/2 + 1} \log \left\| \Phi_A(N/2 + 1, \omega) |_{E_\omega^i} \right\| \right| \left| \frac{N/2 + 1}{N/2 - 1} \right|,$$

since

$$\begin{aligned} & \left| -\lambda_i - \frac{1}{N/2 - 1} \log \|\Phi_A(N/2 + 1, \omega)^{-1}|_{E_{\omega'}^i}\| \right| \\ &= \left| -\lambda_i + \frac{1}{N/2 - 1} \log \|\Phi_A(N/2 + 1, \omega)|_{E_{\omega'}^i}\| \right|. \end{aligned}$$

For every  $\tilde{\omega} \in \tilde{\Sigma}$  let  $B_{\tilde{\omega}}$  be the perturbation of  $A$  by  $P$ , depending on  $\mathbb{R}u = E_{\varphi^{N/2}(\tilde{\omega})}^1$  and  $\mathbb{R}v = E_{\varphi^{1+N/2}(\tilde{\omega})}^2$ , as in (13) given by Lemma 3.1, supported on  $\varphi^{[N/2, N/2+1]}(\tilde{\omega})$  such that we have

$$\Phi_{B_{\tilde{\omega}}}(1, \varphi^{N/2}(\tilde{\omega}))E_{\varphi^{N/2}(\tilde{\omega})}^1 = E_{\varphi^{1+N/2}(\tilde{\omega})}^2. \tag{20}$$

We may consider now a global perturbation  $B$  of  $A$  supported on the flowbox  $\varphi^{[N/2, N/2+1]}(\tilde{\Sigma})$  as before: for  $\omega \in \varphi^{[N/2, N/2+1]}(\tilde{\Sigma})$  we project  $\omega$  in  $\tilde{\omega} \in \tilde{\Sigma}$  and define

$$B(\omega) := \begin{cases} A(\omega), & \text{if } \omega \notin \varphi^{[N/2, N/2+1]}(\tilde{\Sigma}) \\ B_{\tilde{\omega}}(\omega), & \text{if } \omega \in \varphi^{[N/2, N/2+1]}(\tilde{\Sigma}) \end{cases}.$$

Notice that we have  $B \in \mathcal{K}^1$ . By passing to a subset of  $\tilde{\Sigma}$  if necessary, from Lemma 3.2 we may assume that  $\sigma_1(A, B) < \varepsilon$ .

We will see that we may choose a large  $N$  depending on  $\eta$  (depending on  $\delta$ ) such that for all  $\omega \in \tilde{\Sigma}$  we have

$$\frac{1}{N} \log \|\Phi_B(N, \omega)\| \leq \frac{\lambda_1(A, \omega) + \lambda_2(A, \omega)}{2} + \delta, \tag{21}$$

which will imply

$$\lambda_1(B, \omega) \leq \frac{\lambda_1(A, \omega) + \lambda_2(A, \omega)}{2} + \delta. \tag{22}$$

Having  $\tilde{\mu}(\tilde{\Sigma}) > 0$  implies that (21) holds for all  $\omega$  in a subset  $\tilde{M} \subseteq M_h$  with  $\mu(\tilde{M}) > 0$ . In addition, since we are assuming that  $\varphi^t$  is  $\mu$ -ergodic, we get

$$\lambda_1(B) \leq \frac{\lambda_1(A) + \lambda_2(A)}{2} + \delta$$

as required in (17).

Let us explain the effect of mixing Oseledets’ directions on the decay of the Lyapunov exponents. Fix  $\eta > 0$  to be chosen later and take  $\omega \in \tilde{\Sigma}$  with Oseledets’ directions  $E_{\omega}^1 \oplus E_{\omega}^2$  and  $N > 0$  sufficiently large in order to have (18) and (19) for  $\omega' = \varphi^{N/2+1}(\omega)$ . In other words

$$\Phi_A(N/2, \omega) \cdot v_{\omega}^1 \approx e^{\lambda_1 \frac{N}{2}}, \Phi_A(N/2, \omega) \cdot v_{\omega}^2 \approx e^{\lambda_2 \frac{N}{2}}, \tag{23}$$

and

$$\Phi_A(N/2 - 1, \omega') \cdot v_{\omega'}^1 \approx e^{\lambda_1(\frac{N}{2}-1)}, \Phi_A(N/2 - 1, \omega') \cdot v_{\omega'}^2 \approx e^{\lambda_2(\frac{N}{2}-1)}, \tag{24}$$

where  $\lambda_1 = \lambda_1(A)$  and  $\lambda_2 = \lambda_2(A)$  are the Lyapunov exponents of  $A$ ,  $v_\omega^i \in E_\omega^i$  and  $v_{\omega'}^i \in E_{\omega'}^i$  are unitary vectors, and  $\approx$  means  $\eta$ -close as in (18) and (19).

Notice that, for all  $\omega \in \tilde{\Sigma}$  we have:

$$\Phi_B(N, \omega) = \Phi_A(N/2 - 1, \omega') \cdot \Phi_{B_\omega}(1, \varphi^{N/2}(\omega)) \cdot \Phi_A(N/2, \omega). \tag{25}$$

So, for all  $\omega \in \tilde{\Sigma}$ , we may decompose the action of the map  $\Phi_B(N, \omega)$  in three parts:

- The first, between  $\omega$  and  $\varphi^{N/2}(\omega)$ , and the third, between  $\omega'$  and  $\varphi^N(\omega)$ , that, using the basis induced by the Oseledets directions with respect to the splitting  $E^1 \oplus E^2$ , may be written as

$$\Phi_A(N/2, \omega) = \begin{pmatrix} \phi_{1,1}(N/2, \omega) & 0 \\ 0 & \phi_{2,2}(N/2, \omega) \end{pmatrix}$$

and

$$\Phi_A(N/2, \omega') = \begin{pmatrix} \phi_{1,1}(N/2 - 1, \omega') & 0 \\ 0 & \phi_{2,2}(N/2 - 1, \omega') \end{pmatrix}.$$

Using (23) and (24) we get that  $\Phi_A(N/2, \omega)$  is an operator that can be represented approximately (with  $|\mu_i - \lambda_i| < \eta$ , where  $\mu_i$  ( $i = 1, 2$ ) are such that  $\phi_{1,1}(N/2, \omega) = e^{\mu_1 N/2}$  and  $\phi_{2,2}(N/2, \omega) = e^{\mu_2 N/2}$ ) by the matrix

$$\begin{pmatrix} e^{\lambda_1 N/2} & 0 \\ 0 & e^{\lambda_2 N/2} \end{pmatrix} \tag{26}$$

written, as usual, in the Oseledets basis  $\{v_\omega^1, v_\omega^2\}$ . Similarly,  $\Phi_A(N/2, \omega')$  can be represented approximately (with  $|\mu_i - \lambda_i| < \eta$ , where  $\mu_i$  ( $i = 1, 2$ ) are such that  $\phi_{1,1}(N/2 - 1, \omega') = e^{\mu_1(N/2-1)}$  and  $\phi_{2,2}(N/2 - 1, \omega') = e^{\mu_2(N/2-1)}$ ) by the matrix

$$\begin{pmatrix} e^{\lambda_1(N/2-1)} & 0 \\ 0 & e^{\lambda_2(N/2-1)} \end{pmatrix} \tag{27}$$

in the Oseledets basis  $\{v_{\omega'}^1, v_{\omega'}^2\}$ .

- The second piece, between  $\varphi^{N/2}(\omega)$  and  $\omega'$ , with matrix in the basis induced by the Oseledets directions with respect to the splitting  $E_{\varphi^{N/2}(\omega)}^1 \oplus E_{\varphi^{N/2}(\omega)}^2$  which we denote by:

$$\Phi_{B_\omega}(1, \varphi^{N/2}(\omega)) = \begin{pmatrix} 0 & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}. \tag{28}$$

With this interchange of directions the largest grow for  $\Phi_A(N, \omega)$  (given by  $e^{\lambda_1 N}$ ) can never be achieved for the perturbation  $\Phi_B(N, \omega)$ . Indeed, the entries of  $\Phi_B(N, \omega)$  are bounded by:

$$\max \left\{ e^{\mu_1(N/2-1)} e^{\mu_2 N/2}, e^{\mu_2(N/2-1)} e^{\mu_1 N/2}, e^{\mu_2 N/2} e^{\mu_2(N/2-1)} \right\} \times \max \{ a_{1,2}, a_{2,1}, a_{2,2} \}.$$

We now estimate  $\frac{1}{N} \log \|\Phi_B(N, \omega)\|$ . By (25) we know that  $\Phi_B(N, \omega)$  is the composition of the matrices (26), (28) and (27). For the sake of simplicity of presentation we estimate  $\|\Phi_B(N, \omega)\|$  using these three matrices which are in Oseledets’ basis. However, since the angle between the Oseledets fibers has subexponential decay as in (10), estimating  $\|\Phi_B(N, \omega)\|$  by considering the canonical basis only increases the technicalities which we avoid in the sequel. Taking this into consideration and also Remark 3.1 we can make sure that:

$$\|\Phi_B(N, \omega)\| \leq \max \left\{ e^{\frac{\mu_1 + \mu_2}{2} N - \mu_1}, e^{\frac{\mu_1 + \mu_2}{2} N - \mu_2} \right\} C e^{4\pi^2},$$

for some  $C > 0$ . Therefore,

$$\frac{1}{N} \log \|\Phi_B(N, \omega)\| \leq \max \left\{ \frac{\mu_1 + \mu_2}{2} - \frac{\mu_1}{N}, \frac{\mu_1 + \mu_2}{2} - \frac{\mu_2}{N} \right\} + \frac{\log C e^{4\pi^2}}{N}$$

and so

$$\frac{1}{N} \log \|\Phi_B(N, \omega)\| \leq \max \left\{ \frac{\lambda_1 + \lambda_2}{2} + \eta - \frac{\lambda_1 - \eta}{N}, \frac{\lambda_1 + \lambda_2}{2} + \eta - \frac{\lambda_2 - \eta}{N} \right\} + \frac{\log C e^{4\pi^2}}{N}.$$

Finally, taking  $\eta$  sufficiently small and  $N$  sufficiently large we obtain

$$\frac{1}{N} \log \|\Phi_B(N, \omega)\| \leq \frac{\lambda_1(A, \omega) + \lambda_2(A, \omega)}{2} + \delta.$$

In overall

$$\begin{aligned} \lambda_1(B, \omega) &= \lim_{t \rightarrow \infty} \frac{1}{t} \log \|\Phi_B(t, \omega)\| = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|\Phi_B(n, \omega)\| = \inf_n \frac{1}{n} \log \|\Phi_B(n, \omega)\| \\ &\leq \frac{1}{N} \log \|\Phi_B(N, \omega)\| \leq \frac{\lambda_1(A, \omega) + \lambda_2(A, \omega)}{2} + \delta \end{aligned}$$

and so (22) is proved.  $\square$

#### 4. Upper semi-continuity for the top Lyapunov exponent

##### 4.1. On $\sigma_p$ -continuous dependence results

In this subsection, initial results are presented with the goal of establishing upper-semicontinuity for the top Lyapunov exponent. We commence with a fundamental measure theoretical result that will play a pivotal role.

**Lemma 4.1.** *Let  $f \in L^1(\mu)$  be such that  $f \geq 0$ . For all  $\eta > 0$  there exists  $K > 0$  such that for all  $h \in L^1(\mu)$  with  $h \geq 0$  and  $\|h - f\|_1 < \eta$ , we have*

$$\int_{\{h > K\}} h \, d\mu(\omega) < 2\eta \quad \text{and} \quad \mu(\{h > K\}) < \frac{2\eta}{K}. \tag{29}$$

**Proof.** [1, Lemma 5].  $\square$

Throughout this subsection we consider  $A, B \in \mathcal{G}^1$ ,  $\epsilon > 0$  and  $T \in \mathbb{N}$ . Let us define

$$f(\omega) = \int_0^1 \|A(\varphi^s(\omega))\| ds \quad \text{and} \quad g(\omega) = \int_0^1 \|B(\varphi^s(\omega))\| ds. \tag{30}$$

Recall that by [2, Lemma 2.2.5] we have that  $f, g \in L^1(\mu)$ . We set  $\eta = \epsilon/(16(T + 2))$  and fix  $B$  satisfying  $\hat{\sigma}_p(A, B) < \eta$ , which also implies  $\|f - g\|_1 < \eta$ . We use now Lemma 4.1 for  $f$  and  $h = g$  which gives us  $K > 0$  (depending on  $\eta$  and  $A$ ) such that (29) holds. Let

$$E_f = \{f \leq K\} \quad \text{and} \quad E_g = \{g \leq K\}. \tag{31}$$

Then Lemma 4.1 gives that

$$\int_{E_h^c} h d\mu(\omega) < 2\eta \quad \text{and} \quad \mu(E_h^c) < \frac{2\eta}{K}, \quad \text{for } h = f, g. \tag{32}$$

Set

$$G = \bigcap_{i=0}^{T-1} \varphi^{-i}(E_f \cap E_g). \tag{33}$$

Then  $G^c = M \setminus G$  has small measure:

$$\mu(G^c) \leq \sum_{i=0}^T \mu(\varphi^{-i}(E_f \cup E_g)^c) \leq T \mu(E_f \cup E_g)^c \leq T \frac{4\eta}{K} < \frac{\epsilon}{4K}. \tag{34}$$

We can assert that the elements within set  $G$ , as defined in (33), possess favorable estimates. In fact, the following is true:

**Lemma 4.2.** *If  $\omega \in G$ , then for  $C = A, B$  we have  $\int_0^T \|C(\varphi^s(\omega))\| ds \leq TK$  and  $\|\Phi_C(T, \omega)\| \leq e^{TK}$ .*

**Proof.** Once  $\omega \in G$  we have  $\varphi^i(\omega) \in E_f$  for all  $i = 0, \dots, T$ , and

$$\int_0^T \|A(\varphi^s(\omega))\| ds = \sum_{i=0}^{T-1} \int_0^1 \|A(\varphi^{s+i}(\omega))\| ds \leq TK.$$

By (8) we get  $\log^+ \|\Phi_A(T, \omega)\| \leq \int_0^T \|B(\varphi^s(\omega))\| ds$  and we are done. The other case is similar.  $\square$

The following result gives us a ‘weak form’ of continuous dependence of the solutions on their infinitesimal generator. This will suffice for our purposes. The dependence will have an  $L^p$  flavor in the sense that the estimate on the separation of the two solutions will be computed in a set  $G \subset M$ . Due to this reason, we refer to it as ‘weak’.

**Lemma 4.3.** *For all  $1 \leq p < \infty$ ,*

$$\left( \int_G \|\Phi_A(1, \omega) - \Phi_B(1, \omega)\|^p d\mu(\omega) \right)^{\frac{1}{p}} \leq e^{2Kp} \hat{\sigma}_p(A, B).$$

**Proof.** For  $\omega \in G$  we have

$$\begin{aligned} \|\Phi_A(1, \omega) - \Phi_B(1, \omega)\| &\leq \int_0^1 \underbrace{\|A(\varphi^s(\omega))\|}_{\beta(s)} \|\Phi_A(s, \omega) - \Phi_B(s, \omega)\| ds \\ &\quad + \underbrace{\int_0^1 \|A(\varphi^s(\omega)) - B(\varphi^s(\omega))\| \|\Phi_B(s, \omega)\| ds}_{\alpha(t)}. \end{aligned}$$

Taking  $u(t) = \|\Phi_A(t, \omega) - \Phi_B(t, \omega)\|$ , we get  $u(t) \leq \int_0^t \beta(s)u(s) ds + \alpha(t)$ . Using Grönwall’s inequality we get  $u(t) \leq \alpha(t) \exp\left(\int_0^t \beta(s) ds\right)$ , which by Lemma 4.2 implies

$$\|\Phi_A(1, \omega) - \Phi_B(1, \omega)\| \leq \int_0^1 \|A(\varphi^s(\omega)) - B(\varphi^s(\omega))\| \|\Phi_B(s, \omega)\| ds e^K.$$

Since by Lemma 4.2 we have  $\|\Phi_B(s, \omega)\| \leq e^{sK}$ , using Jensen’s inequality we get

$$\begin{aligned} \|\Phi_A(1, \omega) - \Phi_B(1, \omega)\|^p &\leq \left( \int_0^1 \|A(\varphi^s(\omega)) - B(\varphi^s(\omega))\| \|\Phi_B(s, \omega)\| ds e^K \right)^p \\ &\leq \int_0^1 \|A(\varphi^s(\omega)) - B(\varphi^s(\omega))\|^p ds e^{2Kp}. \end{aligned}$$

By integrating on both sides and employing Fubini’s theorem and a change of coordinates (recalling that  $\varphi^s$  preserves  $\mu$ ) we get:

$$\int_G \|\Phi_A(1, \omega) - \Phi_B(1, \omega)\|^p d\mu(\omega) \leq e^{2Kp} \int_G \int_0^1 \|A(\varphi^s(\omega)) - B(\varphi^s(\omega))\|^p ds d\mu(\omega)$$

$$\begin{aligned}
 &\leq e^{2Kp} \int_0^1 \int_G \|A(\varphi^s(\omega)) - B(\varphi^s(\omega))\|^p d\mu(\omega) ds \\
 &= e^{2Kp} \int_0^1 \int_{\varphi^s(G)} \|A(\omega) - B(\omega)\|^p d\mu(\omega) ds \\
 &\leq e^{2Kp} \int_M \|A(\omega) - B(\omega)\|^p d\mu(\omega) \\
 &= e^{2Kp} \hat{\sigma}_p(A, B). \quad \square
 \end{aligned}$$

4.2. On the upper semi-continuity of the top Lyapunov exponent function

In this entire subsection we do not assume that  $\varphi^t$  is ergodic. We define the function

$$\begin{aligned}
 \mathcal{L}: \quad (\mathcal{G}^1, \sigma_p) &\longrightarrow \mathbb{R} \\
 A &\longmapsto \int_M \lambda_1(A, \omega) d\mu(\omega).
 \end{aligned}$$

Notice that under the ergodic hypothesis over the flow  $\varphi^t$  we would get  $\mathcal{L}(A) = \lambda_1(A)$ . In order to prove that  $\mathcal{L}$  is upper semi-continuous when  $\mathcal{G}^1$  is endowed with the  $\sigma_p$  topology defined in §2.3, we begin by given a preliminary result.

**Lemma 4.4.** For all  $t \in \mathbb{R}$ ,  $\omega \in M$  and  $A, B \in \mathcal{G}^1$ , we have

$$\log^+ \|\Phi_B(t, \omega)\| \leq \log^+ \|\Phi_A(t, \omega)\| + \|\Phi_B(t, \omega) - \Phi_A(t, \omega)\|.$$

**Proof.** The proof follows directly from the triangular inequality

$$\|\Phi_B(t, \omega)\| \leq \|\Phi_A(t, \omega)\| + \|\Phi_B(t, \omega) - \Phi_A(t, \omega)\|$$

and the fact that  $\log^+(x + y) \leq \log^+(x) + y$  for all  $x, y \geq 0$ .  $\square$

The upcoming result shares some resemblances with the reasoning in [1, Theorem 2]. However, we introduce several novelties, including: (i) the topology is considered in the infinitesimal generator of the object that supplies the Lyapunov exponent; (ii) this aspect results in several continuity dependency challenges, which we address using §4.1; and (iii) some arguments involve continuous-time concerns, necessitating various modifications.

**Proposition 4.5.** For all  $1 \leq p < \infty$ , the function  $\mathcal{L}$  is upper semicontinuous when we endow  $\mathcal{G}^1$  with the  $\sigma_p$ -topology, that is, for all  $A \in \mathcal{G}^1$  and  $\varepsilon > 0$  there is  $\delta > 0$  such that  $\sigma_p(A, B) < \delta$  implies  $\mathcal{L}(B) < \mathcal{L}(A) + \varepsilon$ .

**Proof.** By Proposition 2.1 we have  $\sigma_1(A, B) \leq \sigma_p(A, B)$ , for all  $1 \leq p < \infty$ , thus it is enough to consider  $p = 1$ . Let  $A \in \mathcal{G}^1$  and  $\varepsilon > 0$  be given. We assume first that for  $\mu$  a.e.  $\omega \in M$  we have

$$\lambda_1(A, \omega) \geq 0. \tag{35}$$

From the subadditive ergodic theorem, we know that the limit

$$\lambda_1(A, \omega) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \|\Phi_A(t, \omega)\|$$

holds a.e. and also in  $L^1$ . Hence, using (35), we get

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \int_M \log^- \|\Phi_A(t, \omega)\| d\mu(\omega) = 0,$$

where  $f^- := \min\{f, 0\}$ . Notice that

$$\begin{aligned} \mathcal{L}(A) &= \lim_{t \rightarrow +\infty} \frac{1}{t} \int_M \log \|\Phi_A(t, \omega)\| d\mu(\omega) = \lim_{n \rightarrow +\infty} \frac{1}{n} \int_M \log \|\Phi_A(n, \omega)\| d\mu(\omega) \\ &= \inf_{n \in \mathbb{N}} \frac{1}{n} \int_M \log \|\Phi_A(n, \omega)\| d\mu(\omega) \end{aligned} \tag{36}$$

so that it is possible to find  $T \in \mathbb{N}$  large enough in order to have

$$\frac{1}{T} \int_M \log^- \|\Phi_A(T, \omega)\| d\mu(\omega) > -\frac{\varepsilon}{8} \quad \text{and} \quad \frac{1}{T} \int_M \log \|\Phi_A(T, \omega)\| d\mu(\omega) < \mathcal{L}(A) + \frac{\varepsilon}{8}$$

and therefore, since  $f = f^- + f^+$ ,

$$\frac{1}{T} \int_M \log^+ \|\Phi_A(T, \omega)\| d\mu(\omega) \leq \mathcal{L}(A) + \frac{\varepsilon}{4}. \tag{37}$$

We apply Lemma 4.1 to  $f$  as in (30) and  $\eta = \varepsilon/(16(T + 2))$ , giving us  $K$  as in the statement. Set  $\delta' = \min\{\eta, \eta T e^{-K(T+2)}\}$  and  $\delta = \delta'/(1 + \delta')$ . Fix  $B \in \mathcal{G}^1$  satisfying  $\sigma_1(A, B) < \delta$ , which implies  $\hat{\sigma}_1(A, B) < \delta' \leq \eta$ . We use  $K, T, \eta$  and  $B$  to define the sets  $E_f, E_g$  and  $G$  as in (31) and (33) respectively. We are going to bound the expression

$$\frac{1}{T} \int_M \log^+ \|\Phi_B(T, \omega)\| d\mu(\omega) = \text{(I)} + \text{(II)},$$

where

$$\text{(I)} = \frac{1}{T} \int_{G^c} \log^+ \|\Phi_B(T, \omega)\| d\mu(\omega) \quad \text{and} \quad \text{(II)} = \frac{1}{T} \int_G \log^+ \|\Phi_B(T, \omega)\| d\mu(\omega).$$

For the first part (I), and by (8) we have

$$\begin{aligned} \frac{1}{T} \int_{G^c} \log^+ \|\Phi_B(T, \omega)\| d\mu(\omega) &\leq \frac{1}{T} \int_{G^c} \int_0^T \|B(\varphi^s(\omega))\| ds d\mu(\omega) \leq \frac{1}{T} \int_{G^c} \sum_{i=0}^{T-1} g(\varphi^i(\omega)) d\mu(\omega) \\ &= \frac{1}{T} \sum_{i=0}^{T-1} \int_{G^c} g(\varphi^i(\omega)) d\mu(\omega) = \frac{1}{T} \sum_{i=0}^{T-1} \int_{\varphi^i(G^c)} g(\omega) d\mu(\omega). \end{aligned}$$

For each  $i = 0, \dots, T - 1$  we have, by (32) and relation (34),

$$\begin{aligned} \int_{\varphi^i(G^c)} g d\mu(\omega) &\leq \int_{E_g^c} g d\mu(\omega) + \int_{E_g \cap \varphi^i(G^c)} g d\mu(\omega) < 2\eta + K\mu(E_g \cap \varphi^i(G^c)) \\ &\leq \frac{\varepsilon}{8} + K\mu(G^c) \leq \frac{\varepsilon}{2}. \end{aligned}$$

Putting all together we get

$$(I) = \frac{1}{T} \int_{G^c} \log^+ \|\Phi_B(T, \omega)\| d\mu(\omega) \leq \frac{\varepsilon}{2}. \tag{38}$$

Next we estimate the second part (II). Using Lemma 4.4 and (37) we have

$$\begin{aligned} (II) &\leq \frac{1}{T} \int_G \log^+ \|\Phi_A(T, \omega)\| d\mu(\omega) + \frac{1}{T} \int_G \|\Phi_B(T, \omega) - \Phi_A(T, \omega)\| d\mu(\omega) \\ &\leq \mathcal{L}(A) + \frac{\varepsilon}{4} + \frac{1}{T} \int_G \|\Phi_B(T, \omega) - \Phi_A(T, \omega)\| d\mu(\omega). \end{aligned} \tag{39}$$

To evaluate the integral on the right-hand side, we follow these steps: using the cocycle properties and Lemma 4.2 we have for all  $\omega \in G$  and  $i = 1, \dots, T - 1$ :

$$\begin{aligned} \|\Phi_B(i + 1, \omega) - \Phi_A(i + 1, \omega)\| &\leq \|\Phi_B(1, \varphi^i(\omega))\Phi_B(i, \omega) - \Phi_A(1, \varphi^i(\omega))\Phi_A(i, \omega)\| \\ &\leq \|\Phi_B(1, \varphi^i(\omega))\| \|\Phi_B(i, \omega) - \Phi_A(i, \omega)\| + \\ &\quad + \|\Phi_B(1, \varphi^i(\omega)) - \Phi_A(1, \varphi^i(\omega))\| \|\Phi_A(i, \omega)\| \\ &\leq e^K \|\Phi_B(i, \omega) - \Phi_A(i, \omega)\| \\ &\quad + e^{Ki} \|\Phi_B(1, \varphi^i(\omega)) - \Phi_A(1, \varphi^i(\omega))\|. \end{aligned}$$

Integrating over  $G$  we get by Lemma 4.3 with  $p = 1$  that

$$\int_G \|\Phi_B(i + 1, \omega) - \Phi_A(i + 1, \omega)\| d\mu(\omega) \leq e^K \int_G \|\Phi_B(i, \omega) - \Phi_A(i, \omega)\| d\mu(\omega) + e^{Ki+2K} \delta'$$

By induction, we obtain for all  $i = 1, \dots, T$

$$\int_G \|\Phi_B(i, \omega) - \Phi_A(i, \omega)\| d\mu(\omega) \leq (i + 2)e^{K(i+2)}\delta'.$$

In particular if we take  $i = T$  we get

$$\int_G \|\Phi_B(T, \omega) - \Phi_A(T, \omega)\| d\mu(\omega) \leq (T + 2)e^{K(T+2)}\delta' \leq \frac{\varepsilon}{4}T. \tag{40}$$

Finally, from (39) and (40) we get

$$(\text{II}) = \frac{1}{T} \int_G \log^+ \|\Phi_A(T, \omega)\| d\mu(\omega) \leq \mathcal{L}(A) + \frac{\varepsilon}{2}. \tag{41}$$

To complete we consider (38) and (41) to conclude that

$$\begin{aligned} \mathcal{L}(B) &\leq \frac{1}{T} \int_M \log \|\Phi_B(T, \omega)\| d\mu(\omega) \\ &= \frac{1}{T} \int_{G^c} \log^+ \|\Phi_B(T, \omega)\| d\mu(\omega) + \frac{1}{T} \int_G \log^+ \|\Phi_B(T, \omega)\| d\mu(\omega) \\ &\leq \frac{\varepsilon}{2} + \left(\mathcal{L}(A) + \frac{\varepsilon}{2}\right) = \mathcal{L}(A) + \varepsilon. \end{aligned}$$

Let us prove now the general case. Again, let  $A \in \mathcal{G}^1$  and  $\varepsilon > 0$  be given. For  $\alpha > 0$  we define the  $\varphi^t$ -invariant set

$$L_\alpha = \{\omega \in M : \lambda_1(A, \omega) < -\alpha\}.$$

Consider  $\alpha$  large enough such that

$$\int_{L_\alpha} \log^+ \|\Phi_A(1, \omega)\| d\mu(\omega) < \frac{\varepsilon}{8} \quad \text{and} \quad \int_{L_\alpha} \lambda_1(A, \omega) d\mu(\omega) > -\frac{\varepsilon}{4}. \tag{42}$$

In particular we get

$$\int_{L_\alpha^c} \lambda_1(A, \omega) d\mu(\omega) < \mathcal{L}(A) + \frac{\varepsilon}{4}. \tag{43}$$

Consider the (non kinetic) infinitesimal generator defined by  $A + \alpha Id$ . We claim that the weak solution of (4) considering the generator  $A + \alpha Id$  is given by  $\Phi_{A+\alpha Id} = e^{\alpha t} \Phi_A(t, \omega)$ . Indeed, it suffices to prove that

$$e^{\alpha t} \Phi_A(t, \omega) = \text{Id} + \int_0^t (A + \alpha Id)(\varphi^s(\omega)) e^{\alpha s} \Phi_A(s, \omega) ds \tag{44}$$

that is

$$e^{\alpha t} \Phi_A(t, \omega) = \text{Id} + \int_0^t e^{\alpha s} A(\varphi^s(\omega)) \Phi_A(s, \omega) ds + \alpha \int_0^t e^{\alpha s} \Phi_A(s, \omega) ds.$$

Using integrating by parts  $u = e^{\alpha s}$ ,  $dv = A(\varphi^s(\omega)) \Phi_A(s, \omega) ds$ ,  $du = \alpha e^{\alpha t} ds$  and  $v = \Phi_A(s, \omega)$  we get

$$e^{\alpha t} \Phi_A(t, \omega) = \text{Id} + e^{\alpha t} \Phi_A(t, \omega) - \text{Id} - \int_0^t \alpha e^{\alpha s} \Phi_A(s, \omega) ds + \alpha \int_0^t e^{\alpha s} \Phi_A(s, \omega) ds,$$

and the claim (44) proved. Now, we define its ‘maximal Lyapunov exponent’ by

$$\lambda_1(A + \alpha Id, \omega) := \lim_{t \rightarrow \infty} \frac{1}{t} \log^+ \|e^{\alpha t} \Phi_A(t, \omega)\|.$$

Notice that  $\lambda_1(A + \alpha Id, \omega) = \lambda_1(A, \omega) + \alpha$  and if  $\omega \in L_\alpha^c$  we have  $\lambda_1(A + \alpha Id, \omega) \geq 0$ . Define now

$$\begin{aligned} \tilde{\mathcal{L}}: \quad (\mathcal{G}^1, \sigma_p) &\longrightarrow \mathbb{R} \\ A &\longmapsto \int_M \lambda_1(A + \alpha Id, \omega) d\mu(\omega). \end{aligned}$$

Proceeding similarly to the previous computations for  $\mathcal{L}$ , we have that  $\tilde{\mathcal{L}}$  is upper semicontinuous if we restrict  $A + \alpha Id$  to  $L_\alpha^c$ . Additionally, observe that Lemma 4.3 remains applicable when substituting  $\Phi_A$  and  $\Phi_B$  with their respective scaled versions  $e^\alpha \Phi_A$  and  $e^\alpha \Phi_B$ . Hence there is  $\delta > 0$  such that if  $\sigma_p((A + \alpha Id)|_{L_\alpha^c}, (B + \alpha Id)|_{L_\alpha^c}) < \delta$  we have

$$\int_{L_\alpha^c} \lambda_1(B + \alpha Id, \omega) d\mu(\omega) \leq \int_{L_\alpha^c} \lambda_1(A + \alpha Id, \omega) d\mu(\omega) + \frac{\varepsilon}{4},$$

that is,

$$\int_{L_\alpha^c} \lambda_1(B, \omega) d\mu(\omega) \leq \int_{L_\alpha^c} \lambda_1(A, \omega) d\mu(\omega) + \frac{\varepsilon}{4}. \tag{45}$$

On the other hand, since  $L_\alpha$  is invariant, we have

$$\int_{L_\alpha} \lambda_1(\Phi_B, \omega) d\mu(\omega) = \inf_n \frac{1}{n} \int_{L_\alpha} \log^+ \|\Phi_B(n, \omega)\| d\mu(\omega) \leq \int_{L_\alpha} \log^+ \|\Phi_B(1, \omega)\| d\mu(\omega) \tag{46}$$

Consider  $T = 1$  and  $\eta$ ,  $K$  and  $G$  as before, and replace  $\eta$  by  $\eta' = \eta/4$ . From Lemma 4.4, (42) and (40) we get

$$\begin{aligned}
 \int_{L_\alpha \cap G} \log^+ \|\Phi_B(1, \omega)\| d\mu(\omega) &\leq \int_{L_\alpha \cap G} \log^+ \|\Phi_A(1, \omega)\| d\mu(\omega) \\
 &\quad + \int_{L_\alpha \cap G} \|\Phi_B(1, \omega) - \Phi_A(1, \omega)\| d\mu(\omega) \\
 &\leq \frac{\varepsilon}{16} + \frac{\varepsilon}{16} = \frac{\varepsilon}{8}.
 \end{aligned}
 \tag{47}$$

Similarly as to (38), we obtain

$$\int_{L_\alpha \cap G^c} \log^+ \|\Phi_B(1, \omega)\| d\mu(\omega) \leq \frac{\varepsilon}{8},$$

which together with (47) leads to

$$\int_{L_\alpha} \log^+ \|\Phi_B(1, \omega)\| d\mu(\omega) \leq \frac{\varepsilon}{4}.
 \tag{48}$$

Finally, from (45), (46), (43) and (48) we have

$$\begin{aligned}
 \mathcal{L}(B) &= \int_{L_\alpha} \lambda_1(B, \omega) d\mu(\omega) + \int_{L_\alpha^c} \lambda_1(B, \omega) d\mu(\omega) \\
 &\leq \int_{L_\alpha} \lambda_1(A, \omega) d\mu(\omega) + \frac{\varepsilon}{4} + \int_{L_\alpha} \log^+ \|\Phi_B(T, \omega)\| d\mu(\omega) \\
 &\leq \left( \mathcal{L}(A) + \frac{\varepsilon}{2} \right) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \mathcal{L}(A) + \varepsilon. \quad \square
 \end{aligned}$$

### 5. Proof of Theorem 1

The approach in [11,12,7,8] to decrease Lyapunov exponents in  $C^0$  cocycles equipped with the  $C^0$  norm (or essential bounded cocycles equipped with the  $L^\infty$  norm) cannot be employed in our present context. This is because  $L^p$  norms capture information pertaining to a neighborhood of an orbit segment and not just the segment itself, unlike the  $C^0$  norm. Hence, we need to adopt a different methodology.

We recall that since we are assuming that  $\varphi^t$  is ergodic, the Lyapunov exponents of a given  $A \in \mathcal{K}^1$  are constant  $\mu$  a.e. and referred as  $\lambda_1(A) \geq \lambda_2(A)$ . We define the *jump map* by:

$$\begin{aligned}
 \mathcal{J}: \quad (\mathcal{K}^1, \sigma_p) &\longrightarrow \mathbb{R} \\
 A &\longmapsto \frac{\lambda_1(A) - \lambda_2(A)}{2}.
 \end{aligned}$$

Next result is a straightforward consequence of Proposition 3.3 and is crucial to obtain the proof of Theorem 1.

**Lemma 5.1.** Consider  $1 \leq p < \infty$  and let  $A \in \mathcal{K}^1$  and  $\varepsilon, \delta > 0$  be given. There exists  $B \in \mathcal{K}^1$  with  $\sigma_p(A, B) < \varepsilon$  such that

$$\mathcal{L}(B) < \delta - \mathcal{J}(A) + \mathcal{L}(A).$$

**Proof.** From (i) in Proposition 2.1 and Proposition 3.3 there is  $B \in \mathcal{K}^1$  with  $\sigma_1(A, B) \leq \sigma_p(A, B) < \varepsilon$  such that

$$\mathcal{L}(B) < \frac{\lambda_1(A) + \lambda_2(A)}{2} + \delta = \delta - \mathcal{J}(A) + \mathcal{L}(A). \quad \square$$

**Theorem 5.2.** Consider  $1 \leq p < \infty$  and the complete metric space  $(\mathcal{K}^1, \sigma_p)$ . If  $A \in \mathcal{K}^1$  is a continuity point of  $\mathcal{L}$ , then  $\mathcal{J}(A) = 0$ .

**Proof.** We take a sequence of kinetic linear differential systems  $A_n \in \mathcal{K}^1$  converging to  $A \in \mathcal{K}^1$  in the  $\sigma_p$ -sense. Since  $A$  is a continuity point we must have  $\lim \mathcal{L}(A_n) = \mathcal{L}(A)$ . By Lemma 5.1, given  $\varepsilon_n \rightarrow 0$  and  $\delta > 0$ , there exists  $B_n \in \mathcal{K}^1$ , with  $\sigma_p(A_n, B_n) < \varepsilon_n$ , such that

$$\mathcal{L}(B_n) < \delta - \mathcal{J}(A_n) + \mathcal{L}(A_n).$$

Considering limits on  $n$  we get

$$\lim_{n \rightarrow \infty} \mathcal{L}(B_n) < \delta - \lim_{n \rightarrow \infty} \mathcal{J}(A_n) + \mathcal{L}(A).$$

Since  $A$  is a continuity point of  $\mathcal{L}$  we obtain that  $\mathcal{J}(A_n) = 0$  for all  $n$  sufficiently large and thus  $\mathcal{J}(A) = 0$ .  $\square$

We are now in condition to finish the proof of our main result.

**Proof.** (of Theorem 1) By Proposition 4.5 the function  $\mathcal{L}$  is upper semicontinuous and by Theorem 5.2 the continuity points of  $\mathcal{L}$  have trivial spectrum (*jump* equal to zero). It is well-known that the set of points of continuity of an upper semicontinuous function is a residual subset (see [20]). Thus, there exists an  $\sigma_p$ -residual subset  $\mathcal{R} \subset \mathcal{K}^1$  such that if  $A \in \mathcal{R}$ , then  $\mathcal{J}(A) = 0$ , that is  $\lambda_1(A) = \lambda_2(A)$ .  $\square$

From Corollary 2.3 we have that  $(\mathcal{K}^1, \sigma_p)$  is a Baire space, hence a  $\sigma_p$ -residual is  $\sigma_p$ -dense. Therefore, we obtain a  $\sigma_p$ -prevalence of trivial Lyapunov spectrum among kinetic systems.

**Data availability**

No data was used for the research described in the article.

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