Idempotent Generated Endomorphisms of an Independence Algebra

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Abstract

The aim of this note is to give a direct proof for the following result proved by Fountain and Lewin: *Let $\mathcal{A}$ be an independence algebra of finite rank and let a be a singular endomorphism of $\mathcal{A}$. Then $a = e_1 \dots e_n$ where $e_i^2 = e_i$ and $rank(a) = rank(e_i)$.*

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We assume the reader to have a basic knowledge of independence algebras. As reference we suggest [3]. Throughout this note $\mathcal{A}$ will be an independence algebra of finite rank ($rank(\mathcal{A}) \geq 2$) and with universe A. By $End(\mathcal{A})$ and $PEnd(\mathcal{A})$ we will denote, respectively, the monoid of endomorphisms and of partial endomorphisms of $\mathcal{A}$, and $Aut(\mathcal{A})$ will be the automorphism group of $\mathcal{A}$. Moreover, $End(\mathcal{A}) \setminus Aut(\mathcal{A})$ is the monoid of singular endomorphisms and will be denoted by $Sing(\mathcal{A})$. For $a \in PEnd(\mathcal{A})$ and $X \subseteq A$, denote by $\Delta(a), \nabla(a)$, respectively, the domain and the image of a, and denote by $a|X$ the restriction of a to X. The following proof is inspired by [1] and goes as follows:

(1) for every $a \in Sing(\mathcal{A})$ there exists an idempotent $e \in End(\mathcal{A})$ and a partial monomorphism a' such that $a = ea'$ (with $\nabla(e) = \langle E \rangle = \Delta a'$ and $\nabla(a') = \langle Ea \rangle$, for some independent set E);

(2) there exist idempotents $e_1, \dots, e_n \in PEnd(\mathcal{A})$ such that $rank(e_i) = rank(a)$ and $(E)e_1 \dots e_n = Ea$;

(3) for every $f \in Sym(Ea)$ (the symmetric group on Ea), there exist idempotents $f_1, \dots, f_m \in PEnd(\mathcal{A})$ such that $rank(f_i) = rank(a)$ and $f = (f_1 \dots f_m)|Ea$;

(4) finally, using (2), $a'|E : E \to Ea$ can be factorized as $a'|E = (e_1 \dots e_n f)|E$, with $f \in Sym(Ea)$, and hence, by (3), we have $a'|E = (e_1 \dots e_n f_1 \dots f_m)|E$, so that $a = ea = ea' = ee_1 \dots e_n f_1 \dots f_m$.

We start by proving (1). Let $a \in Sing(\mathcal{A})$. Since $End(\mathcal{A})$ is regular there exists an idempotent $e \in End(\mathcal{A})$ such that $ea = a$ (and $rank(e) = rank(a)$). Moreover, if E is a basis for $\nabla(e)$, we have

$$|Ea| \leq |E| = rank(e) = rank(a) = rank(ea) = rank(\langle E \rangle a) = rank(\langle Ea \rangle) \leq |Ea|.$$

Therefore $rank(\langle Ea\rangle) = |Ea|$ and hence Ea is independent. Since $|E| = |Ea|$ and Ea is independent, then $a|E$ is $1-1$ and $a|\langle E\rangle$ is a monomorphism. It is proved that $a = e(a|\langle E\rangle)$ and (1) of the scheme above follows.

For the remains of this note a, e, E and Ea are the objects introduced above and are fixed. We now prove (2). Let $l = rank(a)$ and let $K_l = \{B \subseteq A \mid |B| = l \text{ and } B \text{ is independent}\}$. Consider in K_l the following relation: $(B_1, B_2) \in \rho$ if and only if $|B_1 \setminus B_2| \leq 1$.

Lemma 1 *For some natural number $n \geq 0$ we have $(E, Ea) \in \rho^n$.*

Proof: Let $\uparrow E = \{C \in K_l \mid (E, C) \in \rho^m, \text{ for some natural } m\}$. Now let $C \in \uparrow E$ such that for all $D \in \uparrow E$ we have $|C \cap Ea| \geq |D \cap Ea|$. We claim that $C = Ea$. In fact, if by contradiction $C \neq Ea$, then there exists $d \in C \setminus Ea$ (as $|C| = |Ea|$). Therefore, for some $c \in Ea$, we have $c \notin \langle C \setminus \{d\}\rangle$ (since we cannot have $Ea \subseteq \langle C \setminus \{d\}\rangle$ as $rank(\langle C \setminus \{d\}\rangle) < rank(\langle Ea\rangle)$). Thus $C_0 = (C \setminus \{d\}) \cup \{c\} \in K_l$ and $(C, C_0) \in \rho$. Since $(E, C) \in \rho^k$ and $(C, C_0) \in \rho$ it follows that $(E, C_0) \in \rho^{k+1}$ so that $C_0 \in \uparrow E$ and $|C_0 \cap Ea| > |C \cap Ea|$, a contradiction. It is proved that $C = Ea$ and hence $Ea \in \uparrow E$. ∎

We observe that the key fact used in the proof of this lemma is a well known property of matroids (see [4], Exercise 1., p.15).

Now let $E = E_1 \rho E_2 \rho \ldots \rho E_{n-1} \rho E_n = Ea$. We claim that, for every $i = 1, \ldots, n-1$, there exist two idempotents $e_{i,1}, e_{i,2} \in PEnd(\mathcal{A})$ such that $(E_i)e_{i,1}e_{i,2} = E_{i+1}$. In fact let $D = E_i \cap E_{i+1}$ so that $E_i = D \cup \{x\}$ and $E_{i+1} = D \cup \{y\}$. Then either $y \notin \langle D \cup \{x\}\rangle$ or $y \in \langle D \cup \{x\}\rangle$.

In the first case, $D' = D \cup \{x, y\}$ is independent. Thus we can define a mapping $f : D' \to D \cup \{y\}$ by $xf = yf = y$ and $df = d$, for all $d \in D$. This mapping f induces a partial idempotent endomorphism $e_{i,1} : \langle D'\rangle \to \langle D \cup \{y\}\rangle$ such that $(E_i)e_{i,1} = (D \cup \{x\})e_{i,1} = D \cup \{y\} = E_{i+1}$. (Let $e_{i,2} = e_{i,1}$).

In the second case, $rank(\langle D \cup \{x, y\}\rangle) = rank(\langle D \cup \{x\}\rangle) < rank(\mathcal{A})$ and hence there exists $z \notin \langle D \cup \{x, y\}\rangle$. Therefore $D \cup \{x, z\}$ and $D \cup \{y, z\}$ are independent sets and hence there exist two idempotents $e_{i,1}, e_{i,2} \in PEnd(\mathcal{A})$ such that $e_{i,1}|D = id_D = e_{i,2}|D$, $xe_{i,1} = ze_{i,1} = z$ and $ze_{i,2} = ye_{i,2} = y$. It is obvious that $(D \cup \{x\})e_{i,1}e_{i,2} = D \cup \{y\}$ and hence $(E_i)e_{i,1}e_{i,2} = E_{i+1}$. We have constructed idempotents $e_{1,1}, e_{1,2}, \ldots, e_{n,1}, e_{n,2} \in PEnd(\mathcal{A})$ such that $(E)e_{1,1}e_{1,2} \ldots e_{n-1,1}e_{n-1,2} = Ea$ and hence (2) is proved.

Observation 2 *Observe that in both cases considered above, and for all $i = 1, \ldots, n-1$, we have $\nabla(e_{i,1}) \leq \Delta(e_{i,2})$ and $\nabla(e_{i,1}e_{i,2}) \leq \Delta(e_{i+1,1})$.*

Lemma 3 *Let $Sym(Ea)$ be the symmetric group on Ea and let $(xy) \in Sym(Ea)$ be a transposition. Then there exist idempotents $f_1, f_2, f_3 \in PEnd(\mathcal{A})$ such that $f = (f_1 f_2 f_3)|Ea$ and $rank(f_i) = rank(a)$.*

Proof: Since $|Ea| < rank(\mathcal{A})$ there exists $z \notin \langle Ea\rangle$ and hence $Ea \cup \{z\}$ is independent. Let f_1, f_2, f_3 be idempotents of domain $\langle Ea \cup \{z\}\rangle$ defined as follows: $xf_1 = z = zf_1$ and $uf_1 = u$, for the remaining elements of Ea; $xf_2 = x = yf_2$ and $uf_2 = u$, for the remaining elements of Ea; $zf_3 = y = yf_3$ and $uf_1 = u$, for the remaining elements of Ea. Hence $(x)f_1f_2f_3 = (z)f_2f_3 = (z)f_3 = y$ and $(y)f_1f_2f_3 = (y)f_2f_3 = (x)f_3 = x$. Moreover, for all $b \in Ea \setminus \{x, y\}$, $(b)f_1f_2f_3 = b$. It is proved that $(f_1f_2f_3)|Ea = (xy)$ and it is clear that $rank(f_i) = rank(a)$. ∎

Lemma 4 *Let $f \in Sym(Ea)$. Then there exist idempotents $f_1, \ldots, f_m \in PEnd(\mathcal{A})$ such that $f = (f_1 \ldots f_m)|Ea$ and $rank(f_i) = rank(a)$.*

Proof: Clearly $(Ea)f = (Ea)(x_1y_1)\ldots(x_my_m)$, since every permutation of a finite set can be decomposed in to a product of transpositions. Now the result follows by repeated application of the previous lemma. ■

Observation 5 *Observe that, for all the partial idempotents f_i considered in the proof of the two previous lemmas, we have $\Delta(f_i) = \langle Ea \cup \{z\}\rangle$ and $\nabla(f_i) = \langle Ea\rangle$. Thus $\nabla(f_i) < \nabla(f_{i+1})$.*

Lemma 6 *Let $i_0, i_1 \in PEnd(\mathcal{A})$ and suppose that $\nabla(i_0) \leq \Delta(i_1)$. Then there exist $\beta_0, \beta_1 \in End(\mathcal{A})$ such that $(\beta_0\beta_1)|\Delta(i_0) = i_0i_1$, $rank(\beta_j) = rank(e_j)$ and if i_j is idempotent, then β_j is idempotent ($j \in \{0,1\}$).*

Proof: Let $i_0 : \langle B_0\rangle \to \langle C_0\rangle$ be a partial endomorphism (B_0, C_0 are independent sets). Then we can extend B_0, C_0, respectively, to B, C, bases of $\mathcal{A}$ and define $\beta_0 : \mathcal{A} \to \mathcal{A}$ such that $\beta_0|B_0 = i_0$ and $(B \setminus B_0)\beta_0 = \{c\} \subseteq C_0$. In the same way we extend i_1 to $\beta_1 \in End(\mathcal{A})$. Therefore $\beta_0|\Delta(i_0) = i_0$ and $\nabla(\beta_0) = \nabla(i_0)$. Thus $(\beta_0 i_1)|\Delta(i_0) = i_0i_1$ and $(\beta_0\beta_1)|\Delta(i_0) = i_0i_1$. ■

Theorem 7 *(Fountain and Lewin [2]) Every $a \in Sing(\mathcal{A})$ is the product of idempotents $e_1, \ldots, e_n \in End(\mathcal{A})$ such that $rank(a) = rank(e_i)$.*

Proof: Let $a \in Sing(\mathcal{A})$. Then there exists an idempotent e (with $\nabla(e) = \langle E\rangle$) and a partial monomorphism $a' = a|\langle E\rangle$ such that $a = ea'$. Moreover a' maps the basis E of $\nabla(e)$ into a basis Ea of $\nabla(a)$. We proved that for some idempotents $e_1, \ldots, e_k \in PEnd(\mathcal{A})$ we have $(E)e_1 \ldots e_k = Ea$. Let $h = e_1 \ldots e_k$ and consider $f \in Sym(Ea)$ defined by $(xh)f = xa$, for all $x \in E$. It is obvious that $(e_1 \ldots e_k f)|E = a|E$ and hence $(e_1 \ldots e_k\phi)|\langle E\rangle = a|\langle E\rangle = a'$, where $\phi \in Aut(\langle Ea\rangle)$ is (the automorphism) induced by f. Therefore $ee_1 \ldots e_k\phi = ea' = a$. It is also proved that for some idempotents $f_1, \ldots, f_n \in PEnd(\mathcal{A})$ we have $f = (f_1 \ldots f_n)|Ea$. Thus $\phi = (f_1 \ldots f_n)|\langle Ea\rangle$ and hence

$$a = ea' = ee_1 \ldots e_k\phi = ee_1 \ldots e_k(f_1 \ldots f_n)|\langle Ea\rangle = ee_1 \ldots e_k f_1 \ldots f_n$$

(since $\nabla(ee_1 \ldots e_k) = \langle Ea\rangle$). To finish the proof we only need to show that the idempotent partial endomorphisms used above can be replaced by idempotent total endomorphisms. This follows from Observations 2 and 5 and the previous lemma. ■

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