Chaos Decomposition and Gap Renormalization of Brownian Self-Intersection Local Times

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Abstract

We study the chaos decomposition of self-intersection local times and their regularization, with a particular view towards Varadhan's renormalization for the planar Edwards model.

Keywords: Edwards model, self-intersection local time, Varadhan renormalization, white noise analysis

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1 Introduction

The self-intersection local time of d-dimensional Brownian motion, informally, is given as

$$L = \int_0^T dt_2 \int_0^{t_2} dt_1 \, \delta \left(\mathbf{B}(t_2) - \mathbf{B}(t_1) \right). \tag{1}$$

We shall see that, while "reasonably well defined" for $d = 1$, these local times become more and more singular as the dimension d increases. Intersections have thus been the object of extensive study by authors such as Dvoretzky, Erdös, Kakutani [6], [7], [8], Varadhan [35], Westwater [30], [31], [32], Le Gall [20], [21], Rosen [24], [25], [26], Dynkin [9], [10], [11], Watanabe [29], Yor [33], [34], Imkeller et al. [18], Albeverio et al. [1], [2]. For fractional Brownian motion there are papers e.g. by Rosen [27], Hu & Nualart [17], Grothaus et al. [15].

Apart from its intrinsic mathematical interest the self-intersection local time has played a role in constructive quantum field theory, and is a standard model in polymer physics for the self-repulsion ("excluded volume effect") of chain polymers in solvents [28].

Replacement of the Dirac delta function in (1) by a Gaussian

$$\delta_\varepsilon(x) := \frac{1}{(2\pi\varepsilon)^{d/2}} e^{-\frac{|x|^2}{2\varepsilon}}, \quad \varepsilon > 0,$$

leads to regularized local times

$$L_\varepsilon := \int_0^T dt_2 \int_0^{t_2} dt_1 \, \delta_\varepsilon(\mathbf{B}(t_2) - \mathbf{B}(t_1))$$

and for $d = 1$ one can show L^2 convergence w.r.t. white noise or Wiener measure space. But already for $d = 2$ this fails since the expectation of L_ε will diverge in the limit, asymptotically

$$\mathbb{E}(L_\varepsilon) \approx -\frac{T}{2\pi} \ln \varepsilon.$$

In this case it is sufficient to subtract the expectation, i.e. the centered regularized local time does have a well-defined L^2 limit:

$$L_{\varepsilon,c} := L_\varepsilon - \mathbb{E}(L_\varepsilon) \to L_c.$$

Apart from the Gaussian regularization above, others have been considered to remove the singularity at $t_1 = t_2$ in the integral (1). The "staircase regularization" avoids the line $t_1 = t_2$ as in see e.g. Bolthausen [5] (Fig. 1).

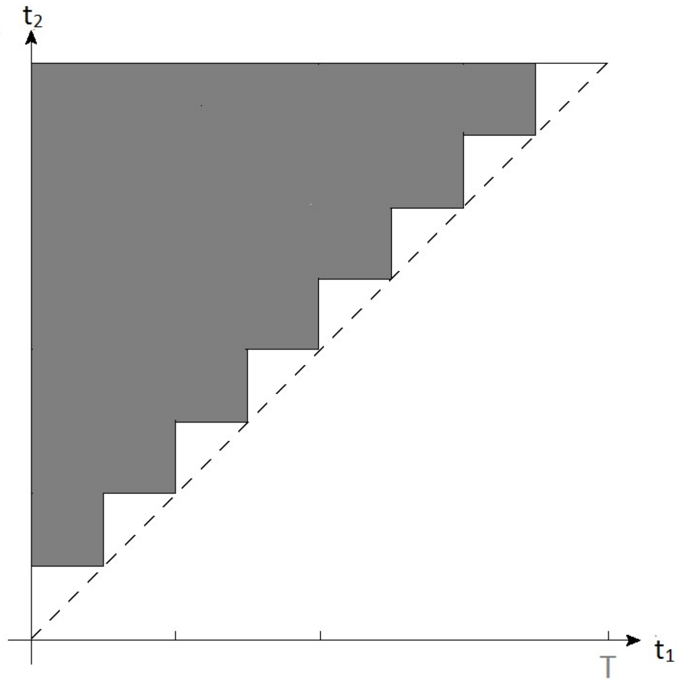

Fig. 1: Domain of integration for the staircase-regularized local time.

The widely used "gap regularization" does the same by omitting the strip $t_2 - t_1 < \Lambda$ in the integral. In the modelling of chain polymers the gap size Λ will be a "microscopic" quantity, i.e. of the order of the inter-monomer distance, more precisely the "Kuhn" or "persistence" length. It plays an important role in renormalization group calculations [28]: critical parameters are obtained from the postulate that macroscopic quantities do not depend on microscopic length scales.

2 Tools from White Noise Analysis [16]

Based on a d-tuple of independent Gaussian white noises $\boldsymbol{\omega} = (\omega_1, ..., \omega_d)$ one defines a d-dimensional Brownian motion $\mathbf{B}$ through

$$\mathbf{B}(t) \equiv \langle \boldsymbol{\omega}, \mathbb{1}_{[0,t]} \rangle = \int_0^t ds\, \boldsymbol{\omega}(s).$$

We shall use a multi-index notation

$$\mathbf{n} = (n_1, \dots, n_d), \quad n = \sum_{i=1}^d n_i, \quad \mathbf{n}! = \prod_{i=1}^d n_i!$$

and for d-tuples of Schwartz test functions $\mathbf{f} = (f_1, \dots, f_d) \in S(\mathbb{R}, \mathbb{R}^d)$,

$$\langle \mathbf{f}, \mathbf{f} \rangle = \sum_{i=1}^d \int dt\, f_i^2(t)$$

$$\langle F_\mathbf{n}, \mathbf{f}^{\otimes \mathbf{n}} \rangle = \int d^n t\, F_\mathbf{n}(t_1, \dots, t_n) \bigotimes_{i=1}^d f_i^{\otimes n_i}(t_1, \dots, t_n)$$

and similarly for $\langle : \boldsymbol{\omega}^{\otimes \mathbf{n}} :, F_\mathbf{n} \rangle$ where for d-tuples of white noise the Wick product $: \cdot :$ [16] generalizes to

$$: \boldsymbol{\omega}^{\otimes \mathbf{n}} := \bigotimes_{i=1}^d : \omega_i^{\otimes n_i} :$$

The vector valued white noise ω has the characteristic function

$$C(\mathbf{f}) := \mathbb{E}(e^{i\langle \boldsymbol{\omega}, \mathbf{f} \rangle}) = \int_{S^*(\mathbb{R}, \mathbb{R}^d)} d\mu(\boldsymbol{\omega}) e^{i\langle \boldsymbol{\omega}, \mathbf{f} \rangle} = e^{-\frac{1}{2}\langle \mathbf{f}, \mathbf{f} \rangle},$$

where $\langle \boldsymbol{\omega}, \mathbf{f} \rangle = \sum_{i=1}^d \langle \omega_i, f_i \rangle$ and $f_i \in S(\mathbb{R}, \mathbb{R})$.

Writing

$$(L^2) := L^2(S^*(\mathbb{R}, \mathbb{R}^d), d\mu)$$

there is the Itô-Segal-Wiener isomorphism with the Fock space of symmetric square integrable functions:

$$(L^2) \simeq \left(\bigoplus_{k=0}^\infty \mathrm{Sym}\, L^2(\mathbb{R}^k, k! d^k t) \right)^{\otimes d}.$$

This implies the chaos expansion

$$\varphi(\boldsymbol{\omega}) = \sum_{\mathbf{n}\in\mathbb{N}_0^d} \langle : \boldsymbol{\omega}^{\otimes\mathbf{n}} :, F_{\mathbf{n}} \rangle \text{ for } \varphi \in (L^2)$$

with kernel functions $F_{\mathbf{n}}$ in Fock space.

Generalized functionals are constructed via a Gel'fand triple

$$(S) \subset (L^2) \subset (S)^*.$$

The generalized functionals in $(S)^*$ are conveniently characterized by their action on exponentials. In particular we use the

$$: \exp(\langle \boldsymbol{\omega}, \mathbf{f} \rangle) := C(\mathbf{f}) \exp(\langle \boldsymbol{\omega}, \mathbf{f} \rangle) \in (S)$$

to make the

Definition 1 *The transformation defined for all test functions $\mathbf{f} \in S(\mathbb{R}, \mathbb{R}^d)$ via the bilinear dual product on $(S)^* \times (S)$ by*

$$(S\Phi)(\mathbf{f}) = \langle\!\langle \Phi, : \exp(\langle \cdot, \mathbf{f} \rangle) : \rangle\!\rangle$$

is called the S-transform of $\Phi \in (S)^*$.

The multilinear expansion of $S(\Phi)$

$$(S\Phi)(\mathbf{f}) = \sum_{\mathbf{n}\in\mathbb{N}_0^d} \langle \boldsymbol{\varphi}_{\mathbf{n}}, \mathbf{f}^{\otimes\mathbf{n}} \rangle$$

extends the chaos expansion to $\Phi \in (S)^*$, with distribution valued kernels $\varphi_{\mathbf{n}}$, such that

$$\langle\!\langle \Phi, F \rangle\!\rangle = \sum_{\mathbf{n}\in\mathbb{N}_0^d} \mathbf{n}! \langle \boldsymbol{\varphi}_{\mathbf{n}}, F_{\mathbf{n}} \rangle.$$

Definition 2 *We shall indicate the projection onto chaos of order $n \geq k$ by a superscript (k):*

$$\langle\!\langle \Phi^{(k)}, F \rangle\!\rangle = \sum_{\mathbf{n}: n\geq k} \mathbf{n}! \langle \boldsymbol{\varphi}_{\mathbf{n}}, F_{\mathbf{n}} \rangle.$$

Proposition 3 *[14]*

$$\delta^{(2N)}(\mathbf{B}(t_2)-\mathbf{B}(t_1)) \in (S)^*$$

with even kernel functions

$$\boldsymbol{\psi}_{\mathbf{2n}}(u_1,\ldots,u_{2n};t_1,t_2) = \frac{1}{\mathbf{n}!}(2\pi)^{-d/2}\left(\frac{1}{|t_2-t_1|}\right)^{\frac{d}{2}+n}\left(-\frac{1}{2}\right)^n \prod_{k=1}^{2n} \mathbb{1}_{[t_1,t_2]}(u_k).$$

All the kernel functions with odd indices vanish.

Setting

$$\begin{aligned} v &:= \max(u_1,\ldots,u_{2n}) \\ u &:= \min(u_1,\ldots,u_{2n}) \end{aligned}$$

one computes [14] the kernel functions of the truncated local time $L^{(2N)}$ for $2N > d-2$ by integration over $0 < t_1 < t_2 < T$:

$$\begin{aligned} \varphi_{\mathbf{2n}}(u_1,\ldots,u_{2n}) &= \frac{(2\pi)^{-d/2}}{\mathbf{n}!}\left(-\frac{1}{2}\right)^n \int_0^T dt_2 \int_0^{t_2} dt_1\,(t_2-t_1)^{-n-d/2}\mathbb{1}_{[t_1,t_2]}^{\otimes 2n}(u_1,\ldots,u_{2n}) \\ &= (-1)^n\left((n+\frac{d}{2}-1)(n+\frac{d}{2}-2)(2\pi)^{d/2}\,2^n\,\mathbf{n}!\right)^{-1}\cdot\Theta(u)\Theta(T-v)\cdot \\ &\quad \cdot(T^{-n-\frac{d}{2}+2}-v^{-n-\frac{d}{2}+2}-(T-u)^{-n-\frac{d}{2}+2}+(v-u)^{-n-\frac{d}{2}+2}) \end{aligned}$$

except for $2n = d = 2$ where

$$\varphi_2(u_1,u_2) = -\frac{1}{4\pi}\left(\ln v + \ln(T-u) - \ln(v-u) - \ln T\right)\cdot\Theta(u)\Theta(T-v).$$

The Heaviside function Θ here is the indicator function of the positive half line.

3 Regularizations

Replacement of the Dirac delta function by a Gaussian

$$\delta_\varepsilon(x) = \frac{1}{(2\pi\varepsilon)^{d/2}}e^{-\frac{|x|^2}{2\varepsilon}}, \quad \varepsilon > 0,$$

leads to regularized local times

$$L_\varepsilon = \int_0^T dt_2 \int_0^{t_2} dt_1\, \delta_\varepsilon(\mathbf{B}(t_2) - \mathbf{B}(t_1)) \tag{2}$$

with kernel functions [14]

$$\begin{aligned}
& \varphi_{\varepsilon,\mathbf{2n}}(u_1,\ldots,u_{2n}) \\
= \; & \frac{(2\pi)^{-d/2}}{\mathbf{n}!}\left(-\frac{1}{2}\right)^n \int_0^T dt_2 \int_0^{t_2} dt_1\,(\varepsilon + |t_2 - t_1|)^{-n-d/2}\, \mathbb{1}^{\otimes 2n}_{[t_1,t_2]}(u_1,\ldots,u_{2n}) \\
= \; & (-1)^n \left((n+\frac{d}{2}-1)(n+\frac{d}{2}-2)(2\pi)^{d/2}\, 2^n\, \mathbf{n}!\right)^{-1} \cdot \Theta(u)\Theta(T-v)\cdot \\
& \cdot((T+\varepsilon)^{-n-\frac{d}{2}+2} - (v+\varepsilon)^{-n-\frac{d}{2}+2} - (T-u+\varepsilon)^{-n-\frac{d}{2}+2} + (v-u+\varepsilon)^{-n-\frac{d}{2}+2}),
\end{aligned}$$

$$\begin{aligned}
& \varphi_{\varepsilon,2}(u_1,u_2) \\
= \; & -\frac{1}{4\pi}\left(\ln(v+\varepsilon) + \ln(T-u+\varepsilon) - \ln(v-u+\varepsilon) - \ln(T+\varepsilon)\right)\cdot\Theta(u)\Theta(T-v).
\end{aligned}$$

3.1 Gap Regularization of Kernel Functions

In renormalization group studies of self-repelling Brownian motion another regularization is often used, see e.g. [28] and the references there; it suppresses intersection in small time intervals $t_2 - t_1$ between intersections by setting informally

$$L(\Lambda) := \int_{\substack{0<t_1<t_2<T \\ t_2-t_1>\Lambda}} d^2t \;\; \delta(\mathbf{B}(t_2) - \mathbf{B}(t_1)).$$

The expectation of $L(\Lambda)$ is equal to

$$\mathbb{E}\left(L(\Lambda)\right) = \int_\Lambda^T dt_2 \int_0^{t_2-\Lambda} dt_1\, \boldsymbol{\psi}_0(t_1,t_2) = (2\pi)^{-d/2}\int_\Lambda^T dt_2 \int_0^{t_2-\Lambda} dt_1\,(t_2-t_1)^{-d/2}.$$

We note that for in particular $d = 2$

$$\mathbb{E}\left(L(\Lambda)\right) = -\frac{T}{2\pi}\ln\Lambda + O(1). \tag{3}$$

Recall [14] that for $\Lambda = 0$ the kernel functions $\varphi_{\mathbf{2n}}$ of the truncated local time $L^{(2N)}$ are obtained by integrating the kernel functions $\psi_{\mathbf{2n}}(u_1,\ldots,u_{2n};t_1,t_2)$

over the rectangle $0 < t_1 < u$ and $v < t_2 < T$, shaded grey in see Fig. 2. For $\Lambda > 0$ the integration is further restricted to the light domain with $t_2 - t_1 > \Lambda$. This restriction is non-trivial when $v - u < \Lambda$.

$$L^{(2N)} - L^{(2N)}(\Lambda)$$

thus has kernel functions $\rho_{\mathbf{2n}}(u, v)$ with support on

$$\{0 < u < v < T\} \cap \{v - u < \Lambda\}\,.$$

Fig. 2 is pertinent to the case where $\Lambda < v$ and $u < T - \Lambda$. In this case the kernel functions $\rho_{\mathbf{2n}}(u, v)$ are obtained by integrating the $\psi_{\mathbf{2n}}(u_1, \ldots, u_{2n}; t_1, t_2)$ with respect to the t_i over $v - \Lambda < t_1 < u$ and $v < t_2 < t_1 + \Lambda$. Excepting the kernel function $\rho_2(u, v)$ for $d = 2$, one finds,

$$\begin{aligned} \rho_{\mathbf{2n}}(u,v) &= \int_{v-\Lambda}^{u} dt_1 \int_{v}^{t_1+\Lambda} dt_2\, \psi_{\mathbf{2n}}(u_1, \ldots, u_{2n}; t_1, t_2) \\ &= \frac{(2\pi)^{-d/2}}{\mathbf{n}!} \left(-\frac{1}{2}\right)^n \frac{1}{d/2+n-1} \cdot \\ &\quad \cdot \left(\frac{1}{d/2+n-2} \left((v-u)^{-d/2-n+2} - \Lambda^{-d/2-n+2} \right) + (v-u-\Lambda)\Lambda^{-d/2-n+1} \right) \end{aligned} \tag{4}$$

and for $2n = d = 2$

$$\rho_2(u,v) = \frac{1}{4\pi} \left(\ln(v-u) - \ln\Lambda + \frac{\Lambda - v + u}{\Lambda} \right). \tag{5}$$

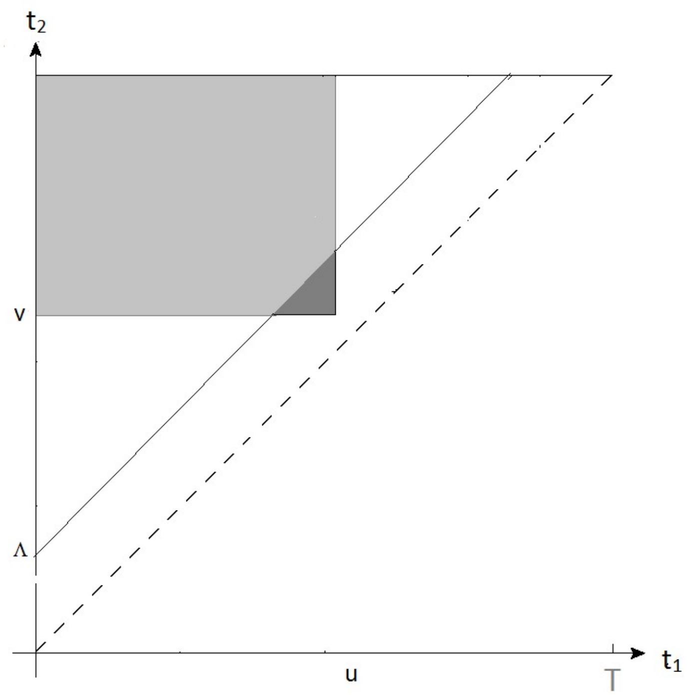

Fig. 2: Domain of integration for kernels of the local time, light grey for the regularized local time, dark grey for the subtraction ρ as in (4).

Using $\tau = t_2 - t_1$ we obtain the following estimate

$$\begin{aligned} |\rho_{2\mathbf{n}}(u,v)| &= \left| \int_{v-\Lambda}^{u} dt_1 \int_{v}^{t_1+\Lambda} dt_2\, \psi_{2\mathbf{n}}(u_1,\ldots,u_{2n};t_1,t_2) \right| \\ &= \frac{1}{2^n \mathbf{n}!}(2\pi)^{-d/2} \int_{v-\Lambda}^{u} dt_1 \int_{v-t_1}^{\Lambda} d\tau \left(\frac{1}{\tau}\right)^{\frac{d}{2}+n} \\ &\leq \frac{1}{2^n \mathbf{n}!}(2\pi)^{-d/2} \frac{1}{d/2+n-1} \int_{v-\Lambda}^{u} dt_1\ (v-t_1)^{-d/2-n+1} \\ &\leq \frac{1}{2^n \mathbf{n}!}(2\pi)^{-d/2} \frac{1}{d/2+n-1} \frac{1}{d/2+n-2}\,(v-u)^{-d/2-n+2} \end{aligned} \tag{6}$$

while for $d = 2n = 2$ one readily finds from (5)

$$|\rho_2(u,v)| \leq \frac{1}{4\pi}\,|\ln(v-u)| \tag{7}$$

when $v - u < \Lambda \ll 1$.

For very small or very large u, v, i.e. $0 < u < v < \Lambda$ or $T - \Lambda < u < v < T$, respectively, the range of integrations in (4) and (6) for $\psi_{\mathbf{2n}}(u_1, \ldots, u_{2n}; t_1, t_2)$ will be $0 < t_1 < u$ and $v < t_2 < t_1 + \Lambda$, or $v < t_2 < T$ and $t_2 - \Lambda < t_1 < u$ respectively, see Fig. 3.

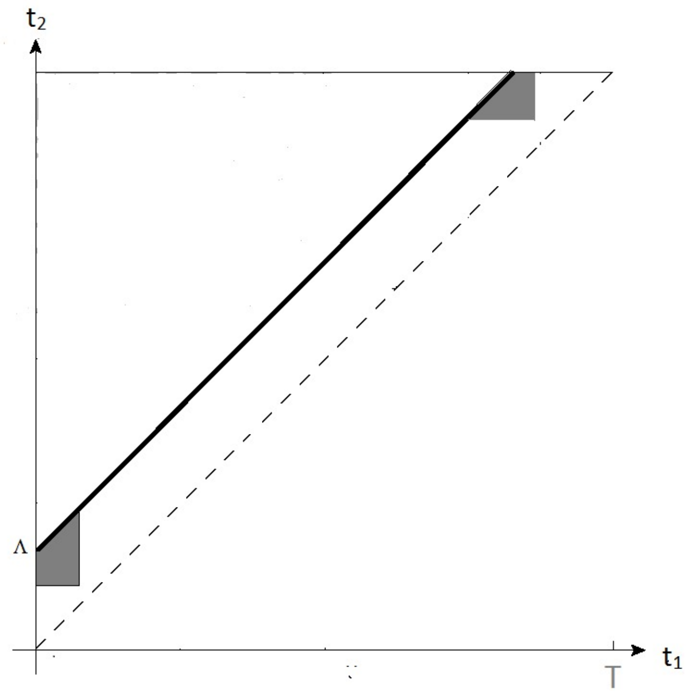

Fig. 3: Modified integration domains for ρ when u, v are close to zero or to T respectively.

Computations as in (4) are again straightforward, we note here only that the estimate of (6) is true also in these two cases.

Theorem 4 *For $N > 0$ and $0 < \Lambda \ll T$ the chaos expansion of the gap-regularized local time $L^{(2N)}(\Lambda)$ has the kernel functions*

$$\varphi_{\Lambda,\mathbf{2n}}(u_1, \ldots, u_{2n}) = \varphi_{\mathbf{2n}}(u_1, \ldots, u_{2n}) - \Theta(\Lambda - (v - u))\rho_{\mathbf{2n}}(u_1, \ldots, u_{2n}), \quad (8)$$

and zero otherwise, while

$$L^{(2N)} - L^{(2N)}(\Lambda)$$

has the kernel functions $\rho_{\mathbf{2n}}$ for $0 < u < v < T$, $n \geq N$.

The Heaviside function Θ exhibits the support property of the $\rho_{2\mathbf{n}}$, i.e. in the gap regularization the kernel functions of the local time are only modified when all arguments u_k are close to each other.

With these results one can in particular estimate the rate of convergence for the centered self-intersection local time in $d = 2$. Apart from the term $n = 1$ the sum

$$\left\| L^{(2)} - L^{(2)}(\Lambda) \right\|_{(L^2)}^2 = \sum_{\mathbf{n}:n\geq 1} (2\mathbf{n})! \, \|\rho_{2\mathbf{n}}\|^2_{L^2([0,T]^{2n})}$$

can be estimated as follows:

$$\begin{aligned}\sum_{\mathbf{n}:n>1} (2\mathbf{n})! \, \|\rho_{2\mathbf{n}}\|^2_{L^2([0,T]^{2n})} \quad \leq \quad & (2\pi)^{-d} \sum_{\mathbf{n}:n>1} \frac{(2\mathbf{n})!}{(\mathbf{n}!)^2} \left(\frac{1}{2}\right)^{2n} \left(\frac{1}{d/2+n-1}\frac{1}{d/2+n-2}\right)^2 \\ & \int_{\substack{0 \\ v-u<\Lambda}}^{T} d^{2n}u_k \left(\frac{1}{v-u}\right)^{d+2n-4}.\end{aligned}$$

We can integrate out the $2n-2$ variables u_k with $u < u_k < v$ that lie between the smallest and the largest and obtain in this way

$$\begin{aligned}\sum_{\mathbf{n}:n>1} (2\mathbf{n})! \, \|\rho_{2\mathbf{n}}\|^2_{L^2([0,T]^{2n})} \quad \leq \quad & (2\pi)^{-2} \sum_{\mathbf{n}:n>1} \frac{(2\mathbf{n})!}{(\mathbf{n}!)^2} \left(\frac{1}{2}\right)^{2n} \left(\frac{1}{n}\frac{1}{n-1}\right)^2 \\ & \cdot 2n(2n-1) \int_0^T dv \int_{\substack{0 \\ v-u<\Lambda}}^{v} du \\ \leq \quad & \frac{\Lambda T}{2\pi^2} \sum_{\mathbf{n}:n>1} \frac{(2\mathbf{n})!}{2^{2n}\,(\mathbf{n}!)^2} \frac{1}{n} \frac{2n-1}{(n-1)^2}.\end{aligned}$$

The series is convergent (Stirling's formula). From (7) it is straightforward to estimate the remaining term with $n = 1$:

$$\|\rho_2\|^2_{L^2([0,T]^2)} \leq \left(\frac{1}{4\pi}\right)^2 \int_0^T dt \int_0^\Lambda d\tau \, \ln^2 \tau = 0(T\Lambda \ln^2 \Lambda).$$

So we have shown

Theorem 5 *For $d = 2$*

$$\left\| L^{(2)} - L^{(2)}(\Lambda) \right\|_{(L^2)}^2 = 0(T\Lambda \ln^2 \Lambda) \ \textit{as} \ \Lambda \searrow 0.$$

A similar improvement of the rate of convergence has been found in the Gaussian regularization in [3].

4 Varadhan Renormalization

The model proposed by Edwards [12] for self-repelling Brownian motion suppresses self-crossings, modifying the Brownian path (or white noise) measure by a density function, informally

$$\varrho = Z^{-1}\ \exp\left(-gL\right)$$

with $g > 0$

$$Z = \mathbb{E}\left(\exp\left(-gL\right)\right).$$

There is no problem for $d = 1$ since L is a positive random variable and hence $\exp\left(-gL\right) < 1$. For $d = 2$ however we should replace L by the centered L_c and this then is no more positive, so that $\exp\left(-gL_c\right)$ is unbounded. The point of Varadhan's theorem is to show that this happens only on small sets so that

Theorem 6 *(Varadhan [35]) For $d = 2$*

$$\varrho = Z^{-1}\ \exp\left(-gL_c\right)$$

with

$$Z = \mathbb{E}\left(\exp\left(-gL_c\right)\right)$$

is integrable.

Varadhan defines the centered local time as the limit of Gaussian approximations as in (2) and uses the Chebyshev inequality to show that $\exp\left(-gL_c\right)$ is integrable for $0 < g < \frac{\pi}{T}$.

A similar slightly stronger result can be obtained using Varadhan's technique with the approximation

$$L^{(2)}(\Lambda) \to L^{(2)} = L_c.$$

Fix $0 < \Lambda < 1$. By (3) there exists a positive constant k such that

$$L^{(2)}(\Lambda) \geq -\mathbb{E}(L(\Lambda)) \geq -k - \frac{T}{2\pi}|\ln(\Lambda)|.$$

For any constant $N \geq k + \frac{T}{2\pi}|\ln(\Lambda)|$ one has

$$\begin{aligned} \mathbb{P}(L_c \leq -N) &= \mathbb{P}(L_c - L^{(2)}(\Lambda) \leq -N - L^{(2)}(\Lambda)) \\ &\leq \mathbb{P}\left(|L^{(2)}(\Lambda) - L_c| \geq N - k - \frac{T}{2\pi}|\ln(\Lambda)|\right). \end{aligned}$$

An application of Chebyshev's inequality, using Theorem 5 then yields

$$\mathbb{P}(L_c \leq -N) \leq \frac{\mathbb{E}(|L^{(2)}(\Lambda) - L_c|^2)}{\left(N - k - \frac{T}{2\pi}|\ln(\Lambda)|\right)^2} \leq K \frac{\Lambda \ln^2 \Lambda}{\left(N - k - \frac{T}{2\pi}|\ln(\Lambda)|\right)^2}.$$

In particular, for

$$\Lambda = \exp\left(-\alpha(N-k)\right), \quad 0 < \alpha < \frac{2\pi}{T}$$

one obtains

$$\mathbb{P}(L_c \leq -N) \leq \frac{K\alpha^2}{\left(1 - \frac{T}{2\pi}\alpha\right)^2} \exp\left(-\alpha(N-k)\right).$$

Hence, $\exp(-gL_c)$ is integrable for $g < \frac{2\pi}{T}$.

For the Gaussian regularization an analogous result can be found in [3].

5 Concluding Remarks

[13], [14], [17], [18], [19], [22], [23], in the present context of regularizations and the rate of convergence see in particular [3]. Yet another regularization of the self-intersection local time is suggested by (8), namely

$$L_{\Lambda}^{(2N)} := \sum_{\mathbf{n}: n \geq N} \langle : \boldsymbol{\omega}^{\otimes 2\mathbf{n}} :, \varphi_{2\mathbf{n}}^{(\Lambda)} \rangle$$

with kernel functions

$$\varphi_{2\mathbf{n}}^{(\Lambda)}(u_1, \ldots, u_{2n}) := \Theta(v - u - \Lambda)\varphi_{2\mathbf{n}}(u_1, \ldots, u_{2n})$$

where simply the range $v - u < \Lambda$ is cut out. Details of this regularization will be discussed elsewhere.

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