Intersection local times of independent Brownian motions as generalized white noise functionals

Sergio Albeverio
Institut für Angewandte Mathematik,
Universität Bonn, D 53115 Bonn;
Ruhr-Universität Bochum;
BiBoS, Universität Bielefeld, D 33501 Bielefeld;
SFB 256 and SFB 237; CERFIM (Locarno); USI (Mendrisio)

Maria João Oliveira
Univ. Aberta; Grupo de Física Matemática,
Av. Prof. Gama Pinto, 2, P 1649-003 Lisboa

Ludwig Streit
BiBoS, Universität Bielefeld, D 33501 Bielefeld;
CCM, Universidade da Madeira, P 9000 Funchal

Abstract

A "chaos expansion" of the intersection local time functional of two independent Brownian motions in $\mathbb{R}^d$ is given. The expansion is in terms of normal products of white noise (corresponding to multiple Wiener integrals). As a consequence of the local structure of the normal products, the kernel functions in the expansion are explicitly given and exhibit clearly the dimension dependent singularities of the local time functional. Their L^p-properties are discussed. An important tool for deriving the chaos expansion is a computation of the "S-transform" of the corresponding regularized intersection local times and a control about their singular limit.

AMS Classification (1991): 28C20, 46F25, 60J55, 60J65, 82D60

1 Introduction

The intersection properties of Brownian motion paths have been investigated since the forties [24], and since then a large number of results on intersection local times of Brownian motion accumulated, see, *e.g.*, [1]–[12], [14]–[17], [19]–[20], [22]–[45] or the recent reference [15]. As for applications in physics, the Edwards model of long polymer molecules by Brownian motion paths uses the local time L to model the "excluded volume" effect: different parts of the molecule should not be located at the same point in space, while Symanzik [38] introduced L as a tool in constructive quantum field theory.

One can consider intersections of sample paths with themselves or, *e.g.*, with other independent Brownian motions [43], one can study simple [8] or n-fold intersections [10] and one can ask all of these questions for linear, planar, spatial or – in general – d-dimensional Brownian motion: as one expects, self-intersections become increasingly scarce as the dimension d increases.

Intersection functionals of independent Brownian motions are used in models handling different types of polymers, see, *e.g.*, [3], [5], [36]. They also occur in models of quantum fields, see, *e.g.*, [2].

For low dimensions the framework of white noise analysis permits definition of the intersection local time of two independent Brownian motions $B^{(i)}$ in terms of an integral over Donsker's δ-function

$$L \equiv \int d^2t \, \delta(B^{(1)}(t_1) - B^{(2)}(t_2)),$$

intended to sum up the contributions from each pair of "times" t_1, t_2 for which the Brownian motions $B^{(i)}$ arrive at the same point.

A rigorous definition, such as, *e.g.*, through a sequence of Gaussians approximating the δ-function, will lead to increasingly singular objects and will necessitate various "renormalizations" as the dimension d increases. For $4 \leq d < 6$ the expectation will diverge in the limit and must be subtracted; as a side effect L will then no more be positive (see, *e.g.*, [22], [39] for the corresponding case of a single Brownian motion). For $d \geq 6$ further subtractions will make L into a well defined generalized function of Brownian motion.

In this note we are particularly interested in the chaos expansion of L. Contrary to [8], [15], [16], [31] we expand in terms of "normal products" [13] of white noise, an expansion which corresponds to that in terms of multiple Wiener integrals when one considers the Wiener process as the fundamental random variable. As a consequence of the local structure of these normal products, the kernel functions of the expansion are remarkably simple and exhibit clearly the dimension dependent singularities of L. We calculate them in closed form and discuss their L^p properties. For comparison we also calculate the regularized kernel functions corresponding to the Gaussian δ-sequence mentioned above.

We are confident that these results will provide a useful tool for further investigations, in particular as far as nonlinear properties of L are concerned.

In Section 2 we provide some background material from white noise analysis, Section 3 contains the results and their proofs.

2 Elements of white noise analysis

In this section we briefly recall some notions of white noise analysis, which we will use in Section 3. For more details see, *e.g.*, [18], [21].

On the Hilbert space $L^2_{2d}(\mathbb{R}) \equiv L^2(\mathbb{R}, \mathbb{R}^{2d})$, $d \in \mathbb{N}$, of vector valued square integrable functions, we consider the densely embedded nuclear space $S_{2d}(\mathbb{R})$ of vector valued Schwartz test functions. The topology on $S_{2d}(\mathbb{R})$ may be given in terms of a system of increasing Hilbertian norms

$$|\vec{\xi}|_p^2 = \sum_{i=1}^{2d} |\xi_i|_p^2, \ \vec{\xi} = (\xi_1, \cdots, \xi_{2d}) \in S_{2d}(\mathbb{R}), \ \xi_i \in S(\mathbb{R}), \ i = 1, \cdots, 2d, \ p \in \mathbb{N}_0.$$

Taking the dual space $S'_{2d}(\mathbb{R})$ of $S_{2d}(\mathbb{R})$ with respect to $L^2_{2d}(\mathbb{R})$ we obtain the space of vector valued tempered distributions. The bilinear dual pairing between $S'_{2d}(\mathbb{R})$ and $S_{2d}(\mathbb{R})$, $< \cdot, \cdot >_{2d}$ (also simply denoted by $< \cdot, \cdot >$), is connected to the sesquilinear inner product on $L^2_{2d}(\mathbb{R})$ by

$$< \mathbf{g}, \vec{\xi} >_{2d} = \sum_{i=1}^{2d} \int dt \, g_i(t) \xi_i(t), \ \mathbf{g} = (g_1, \cdots, g_{2d}) \in L^2_{2d}(\mathbb{R}), \vec{\xi} \in S_{2d}(\mathbb{R}).$$

To construct two independent d-dimensional Brownian motions we consider a $2d$-tuple of independent Gaussian white noises

$$\vec{\omega} \equiv (\vec{\omega}_1, \vec{\omega}_2), \text{ where } \vec{\omega}_i = (\omega_{i,1}, \cdots, \omega_{i,d}),$$

and introduce the notation

$$\vec{n} = (n_1, \cdots, n_d) \in \mathbb{N}^d, \ n = \sum_{i=1}^d n_i, \ \vec{n}! = \prod_{i=1}^d n_i!.$$

For every test function $\vec{\xi} = (\vec{\xi_1}, \vec{\xi_2})$ on $S_{2d}(\mathrm{I\!R})$, $\vec{\xi_i} \in S_d(\mathrm{I\!R})$, the vector valued white noise $\vec{\omega}$ has the characteristic function

$$C(\vec{\xi}) \equiv E\Big(e^{i\sum_{k=1}^{2} <\vec{\omega}_k, \vec{\xi}_k>_d}\Big) = e^{-\frac{1}{2}<\vec{\xi},\vec{\xi}>_{2d}},$$

defining a unique measure μ on $S'_{2d}(\mathrm{I\!R})$. The Hilbert space

$$(L^2) \equiv L^2(S'_{2d}(\mathrm{I\!R}), d\mu)$$

is canonically isomorphic to the Fock space of symmetric square integrable functions,

$$(L^2) \simeq \Big(\bigoplus_{k=0}^{\infty} \mathrm{Sym} L^2(\mathrm{I\!R}^k, k! d^k t)\Big)^{\otimes 2d},$$

which implies the chaos expansion for a general element of (L^2),

$$\begin{aligned} f(\vec{\omega}) = & \sum_{\vec{m}\in\mathbb{N}^d} \sum_{\vec{k}\in\mathbb{N}^d} \Big\langle :\vec{\omega}_1^{\otimes\vec{m}} : \otimes : \vec{\omega}_2^{\otimes\vec{k}} :, f_{\vec{m},\vec{k}}\Big\rangle \\ \equiv & \sum_{\vec{m}} \sum_{\vec{k}} \int_{\mathrm{I\!R}^{m+k}} d^{m+k}t\, f_{\vec{m},\vec{k}}(t_1^1, \cdots, t_{m+k}^{2d}) \prod_{i=1}^{d} \Big(:\omega_{1,i}^{\otimes m_i} : (t_1^i, \cdots, t_{m_i}^i) \cdot \\ & \qquad \cdot : \omega_{2,i}^{\otimes k_i} : (t_1^{d+i}, \cdots, t_{k_i}^{d+i})\Big) \end{aligned}$$

with kernel functions $f_{\vec{m},\vec{k}}$ in Fock space, *i.e.*, square integrable functions of $m+k$ arguments and symmetric in each m_i-tuple $x_1^i, \cdots, x_{m_i}^i$ and each k_i-tuple $x_1^{d+i}, \cdots, x_{k_i}^{d+i}$.

Using the second quantization $\Gamma(A)$ of A (operating on (L^2)), where A is defined by

$$(A\mathbf{g})_i(t) = \Big(-\frac{d^2}{dt^2} + t^2 + 1\Big) g_i(t),$$

we can introduce the space of generalized functions (S) as the projective limit of the Hilbert spaces $(S)_k \equiv D(\Gamma(A^k))$. Taking its dual space $(S)^*$ (w.r.t. (L^2)), we obtain the Gel'fand triple

$$(S) \subset (L^2) \subset (S)^*.$$

The bilinear dual product on $(S)^* \times (S)$ will be denoted by $\ll \cdot, \cdot \gg$.

The Hida distributions are conveniently characterized by their action on exponentials. In particular, using the Wick exponential

$$\begin{aligned} : \exp < \vec{\omega}, \vec{\xi} > : \equiv & \sum_{\vec{m}} \sum_{\vec{k}} \frac{1}{\vec{m}!\vec{k}!} \Big\langle : \vec{\omega}_1^{\otimes\vec{m}} : \otimes : \vec{\omega}_2^{\otimes\vec{k}} :, \vec{\xi}_1^{\otimes\vec{m}} \otimes \vec{\xi}_2^{\otimes\vec{k}} \Big\rangle \\ = & C(\vec{\xi}) e^{<\vec{\omega},\vec{\xi}>}, \quad \vec{\xi} = (\vec{\xi_1}, \vec{\xi_2}) \in S_{2d}(\mathrm{I\!R}), \end{aligned}$$

we can define the S-transform of an element Φ from $(S)^*$ by

$$S\Phi(\vec{\xi}) \equiv \ll \Phi, : \exp < \cdot, \vec{\xi} >: \gg .$$

The multilinear expansion of $S(\Phi)$

$$S\Phi(\vec{\xi}) = \sum_{\vec{m}} \sum_{\vec{k}} \langle G_{\vec{m},\vec{k}}, \vec{\xi}_1^{\otimes \vec{m}} \otimes \vec{\xi}_2^{\otimes \vec{k}} \rangle$$

extends the chaos expansion to $\Phi \in (S)^*$, with distribution valued kernels $G_{\vec{m},\vec{k}}$, such that

$$\ll \Phi, \varphi \gg = \sum_{\vec{m}} \sum_{\vec{k}} \vec{m}!\vec{k}! < G_{\vec{m},\vec{k}}, \varphi_{\vec{m},\vec{k}} >,$$

for every generalized function $\varphi \in (S)$ with kernel functions $\varphi_{\vec{m},\vec{k}}$.

The S-transform of Hida distributions are U-functionals in the sense of the following definition.

Definition 2.1. *A mapping $G : S_{2d}(\mathbb{R}) \to \mathbb{C}$ is called a U-functional if*

(i) for all $\mathbf{f}, \mathbf{g} \in S_{2d}(\mathbb{R})$, $\mathbb{R} \ni \lambda \mapsto G(\lambda \mathbf{f} + \mathbf{g}) \in \mathbb{C}$ has an entire extension to $\lambda \in \mathbb{C}$,

(ii) there exists constants C, $K > 0$ and $p \in \mathbb{N}_0$ such that for all $\mathbf{g} \in S_{2d}(\mathbb{R})$, $z \in \mathbb{C}$

$$|G(z\mathbf{g})| \leq Ce^{K|z|^2|A^p\mathbf{g}|^2}.$$

This fact provides a characterization result for Hida distributions (see [21]):

Theorem 2.2. *A mapping $G : S_{2d}(\mathbb{R}) \to \mathbb{C}$ is the S-transform of an element in $(S)^*$ if and only if it is a U-functional.*

From the above result we can deduce two useful corollaries. The first one concerns the convergence of sequences of generalized functionals and the second one the Bochner integration of families of Hida distributions.

Corollary 2.3. *Given a sequence of U-functionals $(G_n)_{n\in\mathbb{N}}$ such that*

(i) for all $\mathbf{g} \in S_{2d}(\mathbb{R})$, $(G_n(\mathbf{g}))_{n\in\mathbb{N}}$ is a Cauchy sequence,

(ii) there exist constants C, $K > 0$ and $p \in \mathbb{N}_0$ such that for every $\mathbf{g} \in S_{2d}(\mathbb{R})$, $z \in \mathbb{C}$, for almost all $n \in \mathbb{N}$

$$|G_n(z\mathbf{g})| \leq Ce^{K|z|^2|A^p\mathbf{g}|^2},$$

then $(S^{-1}G_n)_{n\in\mathbb{N}}$ *converges strongly in* $(S)^*$ *to a unique Hida distribution.*

Corollary 2.4. *Let* $(\Omega, \mathcal{B}, m)$ *be a measure space and* $\lambda \mapsto \Phi_\lambda$ *a mapping from* Ω *to* $(S)^*$*. If the* S*-transform of* Φ_λ *verifies the following two properties:*

(i) for every $\mathbf{g} \in S_{2d}(\mathbb{R})$ *the mapping* $\lambda \mapsto S\Phi_\lambda$ *is measurable,*

(ii) there exists $p \in \mathbb{N}_0$ *so that for all* $\lambda \in \Omega$*,* $\mathbf{g} \in S_{2d}(\mathbb{R})$*,* $z \in \mathbb{C}$

$$|S\Phi_\lambda(z\mathbf{g})| \leq C_\lambda e^{K_\lambda |z|^2 |A^p \mathbf{g}|^2},$$

where $\lambda \mapsto C_\lambda$ *is integrable with respect to* m *and* $\lambda \mapsto K_\lambda$ *is bounded* m*-a.e.,*

then Φ_λ *is Bochner integrable on the dual space* $(S)_{-q}$ *of* $(S)_q$ *for* q *large enough. In particular,*

$$\int_\Omega \Phi_\lambda \, dm(\lambda) \in (S)^*$$

and

$$S\left(\int_\Omega \Phi_\lambda \, dm(\lambda)\right)(\mathbf{g}) = \int_\Omega S\Phi_\lambda(\mathbf{g}) \, dm(\lambda).$$

3 The chaos expansion

We begin by defining two (independent) d-dimensional Brownian motions $B^{(i)}$ through

$$B^{(i)}(t_i) \equiv < \vec{\omega}_i, 1_{[0,t_i]} > = \int_0^{t_i} \vec{\omega}_i(t)\, dt, \ \ t_i \in [0,T]\,, T > 0, \ \ i = 1,2$$

as a function of (independent) white noises $\vec{\omega}_i$. Accordingly we have the following

Theorem 3.1. *For every positive integer* d *and* $\varepsilon > 0$*, the intersection time of two independent stochastic processes*

$$\begin{aligned} L_\varepsilon &\equiv \int_{I^2} d^2t\, \delta_\varepsilon\Big(B^{(1)}(t_1) - B^{(2)}(t_2)\Big) \\ &\equiv \int_{I^2} d^2t \left(\frac{1}{2\pi\varepsilon}\right)^{d/2} e^{-\frac{(B^{(1)}(t_1)-B^{(2)}(t_2))^2}{2\varepsilon}}, \quad I \equiv \left]0, T\right], \end{aligned}$$

is a functional with S*-transform given by*

$$SL_\varepsilon(\mathbf{f}) = \int_{I^2} d^2t \left(\frac{1}{2\pi(\varepsilon + t_1 + t_2)}\right)^{d/2} e^{-\frac{\left(\int_0^{t_1} \mathbf{f}(t)\,dt - \int_0^{t_2} \mathbf{f}(t)\,dt\right)^2}{2(\varepsilon+t_1+t_2)}}, \tag{3.1}$$

for all $\mathbf{f} \in S_d(\mathbb{R})$.

Proof: To prove this result we will apply Corollary 2.4 to the S-transform of the integrand

$$\Phi_\varepsilon(\vec{\omega}_1, \vec{\omega}_2) \equiv \left(\frac{1}{2\pi\varepsilon}\right)^{d/2} e^{-\frac{(B^{(1)}(t_1)-B^{(2)}(t_2))^2}{2\varepsilon}}$$

with respect to Lebesgue measure $d(t_1, t_2)$ on I^2.

Since

$$\begin{aligned} S\Phi_\varepsilon(\mathbf{f}) &= \prod_{i=1}^{d} \frac{1}{\sqrt{2\pi(\varepsilon + t_1 + t_2)}} e^{-\frac{\left(\int_0^{t_1} f_i(t)\,dt - \int_0^{t_2} f_i(t)\,dt\right)^2}{2(\varepsilon+t_1+t_2)}} \\ &= \left(\frac{1}{2\pi(\varepsilon + t_1 + t_2)}\right)^{d/2} e^{-\frac{\left(\int_0^{t_1} \mathbf{f}(t)\,dt - \int_0^{t_2} \mathbf{f}(t)\,dt\right)^2}{2(\varepsilon+t_1+t_2)}} \end{aligned}$$

(see, *e.g.*, [18]), the measurability follows immediately. For the boundedness condition we observe that for any complex number z and $\mathbf{f} \in S_d(\mathbb{R})$,

$$\begin{aligned} |S\Phi_\varepsilon(z\mathbf{f})| &= \left(\frac{1}{2\pi(\varepsilon+t_1+t_2)}\right)^{d/2} \left| e^{-\frac{z^2\left(\int_0^{t_1} \mathbf{f}(t)\,dt - \int_0^{t_2} \mathbf{f}(t)\,dt\right)^2}{2(\varepsilon+t_1+t_2)}} \right| \\ &\leq \left(\frac{1}{2\pi(\varepsilon+t_1+t_2)}\right)^{d/2} e^{\frac{|z|^2 \left\| \int_0^{t_1} \mathbf{f}(t)\,dt - \int_0^{t_2} \mathbf{f}(t)\,dt \right\|^2}{2(\varepsilon+t_1+t_2)}} \\ &\leq \left(\frac{1}{2\pi(\varepsilon+t_1+t_2)}\right)^{d/2} e^{\frac{|z|^2 (t_1+t_2)^2 \sum_{i=1}^d \sup_t |f_i(t)|^2}{2(\varepsilon+t_1+t_2)}}, \end{aligned} \tag{3.2}$$

where, in the last expression obtained, the coefficient in front of the exponential is integrable and the term $\frac{(t_1+t_2)^2}{2(\varepsilon+t_1+t_2)}$, which appears in the exponential, is bounded on I^2 as a function of t_1 and t_2.

Hence, by the characterization result mentioned above, it follows that the S-transform of the functional L_ε at a point $\mathbf{f} \in S_d(\mathbb{R})$ is given by

$$\begin{aligned} SL_\varepsilon(\mathbf{f}) &= \int_{I^2} d^2t \prod_{i=1}^{d} \frac{1}{\sqrt{2\pi(\varepsilon + t_1 + t_2)}} e^{-\frac{(F_{1i}(t_1)-F_{2i}(t_2))^2}{2(\varepsilon+t_1+t_2)}} \\ &= \int_{I^2} d^2t \left(\frac{1}{2\pi(\varepsilon + t_1 + t_2)}\right)^{d/2} e^{-\frac{(\vec{F}_1(t_1)-\vec{F}_2(t_2))^2}{2(\varepsilon+t_1+t_2)}}, \end{aligned}$$

where we have introduced the notation

$$\vec{F}_i(t_i) \equiv (F_{ij}(t_i))_{j=1,\cdots,d} \equiv \Big(\int_0^{t_i} f_j(t)\,dt\Big)_{j=1,\cdots,d}, \quad i = 1, 2.$$

□

Using the notation

$$\underline{u}\equiv\begin{cases}\min\limits_{1\le i\le m} u_i & \text{if } m\ge 1\\ 0 & \text{if } m=0\end{cases}\qquad \underline{v}\equiv\begin{cases}\min\limits_{1\le i\le k} v_i & \text{if } k\ge 1\\ 0 & \text{if } k=0\end{cases}$$

$$\bar{u}\equiv\begin{cases}\max\limits_{1\le i\le m} u_i & \text{if } m\ge 1\\ 0 & \text{if } m=0\end{cases}\qquad \bar{v}\equiv\begin{cases}\max\limits_{1\le i\le k} v_i & \text{if } k\ge 1\\ 0 & \text{if } k=0\end{cases}$$

we have the following result.

Theorem 3.2. *For every positive integer d and $\varepsilon>0$, the intersection time L_ε has the chaos expansion*

$$L_\varepsilon=\sum_{\vec{m}}\sum_{\vec{k}}\Big\langle :\vec{\omega}_1^{\otimes\vec{m}}:\otimes:\vec{\omega}_2^{\otimes\vec{k}}:, G_{\vec{m},\vec{k}}\Big\rangle,$$

where the kernel functions $G_{\vec{m},\vec{k}}$ of L_ε are given by

$$\begin{aligned}G_{\vec{m},\vec{k}}(u_1,\cdots,u_m,\,v_1,\cdots,v_k)&=(-1)^k\Big(\frac{1}{2\pi}\Big)^{d/2}\Big(-\frac{1}{2}\Big)^{\frac{m+k}{2}}\frac{1}{\Big(\frac{\vec{m}+\vec{k}}{2}\Big)!}\binom{\vec{m}+\vec{k}}{\vec{m}}\cdot\\&\frac{1}{\Big(\frac{d+m+k}{2}-1\Big)\Big(\frac{d+m+k}{2}-2\Big)}\theta(\underline{u})\theta(\underline{v})\theta(T-\bar{u})\theta(T-\bar{v})\cdot\\&\cdot\bigg[\Big(\frac{1}{\varepsilon+\bar{u}+\bar{v}}\Big)^{\frac{d+m+k}{2}-2}-\Big(\frac{1}{\varepsilon+T+\bar{v}}\Big)^{\frac{d+m+k}{2}-2}\\&-\Big(\frac{1}{\varepsilon+T+\bar{u}}\Big)^{\frac{d+m+k}{2}-2}+\Big(\frac{1}{\varepsilon+2T}\Big)^{\frac{d+m+k}{2}-2}\bigg]\end{aligned}$$

for $m+k\neq 0$, $\vec{m}+\vec{k}$ even, and zero otherwise, with the exception of the case $d=2$ and $m+k=2$ where

$$\begin{aligned}G_{\vec{m},\vec{k}}(u_1,v_1)=&\frac{1}{2\pi}\Big[\ln(\varepsilon+T+v_1)-\ln(\varepsilon+u_1+v_1)-\ln(\varepsilon+2T)\\&+\ln(\varepsilon+T+u_1)\Big]\cdot\theta(u_1)\theta(v_1)\theta(T-u_1)\theta(T-v_1),\end{aligned}$$

for $m=k=1$;

$$\begin{aligned}G_{\vec{m},0}(u_1,u_2)=&\frac{1}{4\pi}\Big[\ln(\varepsilon+\bar{u})-\ln(\varepsilon+T+\bar{u})-\ln(\varepsilon+T)+\ln(\varepsilon+2T)\Big]\cdot\\&\cdot\theta(\underline{u})\theta(T-\bar{u}),\end{aligned}$$

for $m=2$ *and* $k=0$;

$$G_{0,\vec{k}}(v_1,v_2)=\frac{1}{4\pi}\Big[\ln(\varepsilon+\bar{v})-\ln(\varepsilon+T+\bar{v})-\ln(\varepsilon+T)+\ln(\varepsilon+2T)\Big]\cdot \\ \cdot\theta(\underline{v})\theta(T-\bar{v}),$$

for $k=2$ *and* $m=0$.

Here θ *is the indicator function of the positive half-line* $\mathbb{R}^+$.

Moreover, the expectation of L_ε *with respect to the white noise measure* μ *is given by*

$$E_\mu(L_\varepsilon)=\Big(\frac{1}{2\pi}\Big)^{d/2}\frac{1}{\Big(\frac{d}{2}-1\Big)\Big(\frac{d}{2}-2\Big)}\Bigg(\Big(\frac{1}{\varepsilon+2T}\Big)^{\frac{d}{2}-2}-2\Big(\frac{1}{\varepsilon+T}\Big)^{\frac{d}{2}-2}+\Big(\frac{1}{\varepsilon}\Big)^{d-4}\Bigg),$$

for $d\neq 2$ *and* $d\neq 4$;

$$E_\mu(L_\varepsilon)=\frac{1}{2\pi}\Big[(\varepsilon+2T)\ln(\varepsilon+2T)-2(\varepsilon+T)\ln(\varepsilon+T)+\varepsilon\ln(\varepsilon)\Big],$$

for $d=2$; *and*

$$E_\mu(L_\varepsilon)=\Big(\frac{1}{2\pi}\Big)^2\Big[2\ln(\varepsilon+T)-\ln(\varepsilon+2T)-\ln(\varepsilon)\Big],$$

for $d=4$.

Proof: For every fixed $\mathbf{f}\in S_d(\mathbb{R})$,

$$\begin{aligned} SL_\varepsilon(\mathbf{f}) &= \int_{I^2} d^2t\,\prod_{i=1}^{d}\frac{1}{\sqrt{2\pi(\varepsilon+t_1+t_2)}}e^{-\frac{(F_{1i}(t_1)-F_{2i}(t_2))^2}{2(\varepsilon+t_1+t_2)}} \\ &= \Big(\frac{1}{\pi}\Big)^{d/2}\int_{I^2} d^2t\,\sum_{n\geq 0}\Bigg\{(-1)^n\Big(\frac{1}{2(\varepsilon+t_1+t_2)}\Big)^{n+d/2} \\ &\quad \sum_{\substack{n_1,\cdots,n_d\\ n_1+\cdots+n_d=n}}\frac{1}{n_1!\cdots n_d!}\prod_{i=1}^{d}\sum_{\substack{m_i,k_i\\ m_i+k_i=2n_i}}\binom{m_i+k_i}{m_i}F_{1i}^{m_i}(-F_{2i})^{k_i}\Bigg\} \\ &= \Big(\frac{1}{\pi}\Big)^{d/2}\int_{I^2} d^2t\,\sum_{n\geq 0}\sum_{\substack{n_1,\cdots,n_d\\ n_1+\cdots+n_d=n}}\sum_{\substack{m_i,k_i,i=1,\cdots,d\\ m_i+k_i=2n_i}} \\ &\quad \Bigg\{\prod_{i=1}^{d}(-1)^{\frac{m_i+k_i}{2}}\Big(\frac{1}{2(\varepsilon+t_1+t_2)}\Big)^{\frac{1}{2}+\frac{m_i+k_i}{2}}\frac{1}{\Big(\frac{m_i+k_i}{2}\Big)!}\Bigg\}\cdot \\ &\quad \cdot\Bigg\{\prod_{i=1}^{d}\binom{m_i+k_i}{m_i}F_{1i}^{m_i}(-F_{2i})^{k_i}\Bigg\} \end{aligned}$$

$$= \Big(\frac{1}{\pi}\Big)^{d/2} \int_{I^2} d^2t \sum_{\substack{\vec{m},\vec{k} \\ \vec{m}+\vec{k}\, even}} (-1)^{\frac{m+k}{2}} \Big(\frac{1}{2(\varepsilon + t_1 + t_2)}\Big)^{\frac{d+m+k}{2}}$$

$$\frac{1}{\Big(\frac{\vec{m}+\vec{k}}{2}\Big)!} \begin{pmatrix} \vec{m}+\vec{k} \\ \vec{m} \end{pmatrix} \vec{F}_1^{\vec{m}} (-\vec{F}_2)^{\vec{k}}.$$

This is now ready to be compared with the general form of the chaos expansion

$$L_\varepsilon = \sum_{\vec{m}} \sum_{\vec{k}} \Big\langle :\vec{\omega}_1^{\otimes \vec{m}}: \otimes :\vec{\omega}_2^{\otimes \vec{k}}:, G_{\vec{m},\vec{k}} \Big\rangle.$$

By comparison, we find that $G_{\vec{m},\vec{k}}$ vanishes for $\vec{m}+\vec{k}$ not even, and for $\vec{m}+\vec{k}$ even, $\vec{m}+\vec{k} \neq 0$, the kernel functions are equal to

$$\begin{aligned} G_{\vec{m},\vec{k}} \quad (u_1, \cdots, u_m, v_1, \cdots, v_k) &= C'_{\vec{m},\vec{k}} \int_{I^2} d^2t \, (-1)^k \Big(\frac{1}{\varepsilon + t_1 + t_2}\Big)^{\frac{d+m+k}{2}} \cdot \\ &\quad \cdot \Big(1_{[0,t_1]}^{\otimes m} \otimes 1_{[0,t_2]}^{\otimes k}\Big)(u_1, \cdots, u_m, v_1, \cdots, v_k) \\ &= \begin{cases} C_{\vec{m},\vec{k}} \int_{\bar{u}}^{T} dt_1 \int_{\bar{v}}^{T} dt_2 \Big(\frac{1}{\varepsilon + t_1 + t_2}\Big)^{\frac{d+m+k}{2}} & \text{if } \underline{u}, \underline{v} \geq 0 \text{ and } \bar{u}, \bar{v} \leq T \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

where we have used the notation,

$$\begin{aligned} &C'_{\vec{m},\vec{k}} \equiv \Big(\tfrac{1}{2\pi}\Big)^{d/2} \Big(-\tfrac{1}{2}\Big)^{\frac{m+k}{2}} \frac{1}{\Big(\frac{\vec{m}+\vec{k}}{2}\Big)!} \begin{pmatrix} \vec{m}+\vec{k} \\ \vec{m} \end{pmatrix}, \\ &C_{\vec{m},\vec{k}} \equiv (-1)^k C'_{\vec{m},\vec{k}}, \\ &1_{[0,t_i]}^{\otimes 0} \equiv 1. \end{aligned} \tag{3.3}$$

This means, if $\underline{u}, \underline{v} \geq 0$ and $\bar{u}, \bar{v} \leq T$,

$$\begin{aligned} G_{\vec{m},\vec{k}} \, (u_1, \cdots, u_m, v_1, \cdots, v_k) &= \frac{C_{\vec{m},\vec{k}}}{\Big(\frac{d+m+k}{2} - 1\Big)\Big(\frac{d+m+k}{2} - 2\Big)} \Bigg[\Big(\frac{1}{\varepsilon + \bar{u} + \bar{v}}\Big)^{\frac{d+m+k}{2} - 2} \\ &- \Big(\frac{1}{\varepsilon + T + \bar{v}}\Big)^{\frac{d+m+k}{2} - 2} - \Big(\frac{1}{\varepsilon + T + \bar{u}}\Big)^{\frac{d+m+k}{2} - 2} + \Big(\frac{1}{\varepsilon + 2T}\Big)^{\frac{d+m+k}{2} - 2} \Bigg], \end{aligned}$$

provided that

$$(i) \qquad \frac{d+m+k}{2} \neq 1, \text{ and}$$

$$(ii) \qquad \frac{d+m+k}{2} \neq 2.$$

Since $\vec{m}+\vec{k}$ is even and $m+k\neq 0$, we always have inequality (i), while inequality (ii) is valid except when $d=2$ and $m+k=2$. In the exceptional situation, three separate cases are possible:

$$m=k=1, \quad m=2 \text{ and } k=0, \quad k=2 \text{ and } m=0.$$

In the first case, we have

$$\begin{aligned}
G_{\vec{m},\vec{k}}(u_1,v_1) &= \frac{1}{2\pi}\int_{u_1}^{T} dt_1 \int_{v_1}^{T} dt_2 \Big(\frac{1}{\varepsilon+t_1+t_2}\Big)^2 \\
&= \frac{1}{2\pi}\Big[\ln(\varepsilon+T+v_1)-\ln(\varepsilon+u_1+v_1) \\
&\quad -\ln(\varepsilon+2T)+\ln(\varepsilon+T+u_1)\Big],
\end{aligned}$$

if $0\le u_1,v_1\le T$. In the second one,

$$\begin{aligned}
G_{\vec{m},0}(u_1,u_2) &= -\frac{1}{4\pi}\int_{\bar{u}}^{T} dt_1 \int_{0}^{T} dt_2 \Big(\frac{1}{\varepsilon+t_1+t_2}\Big)^2 \\
&= \frac{1}{4\pi}\Big[\ln(\varepsilon+\bar{u})-\ln(\varepsilon+T+\bar{u})-\ln(\varepsilon+T)+\ln(\varepsilon+2T)\Big],
\end{aligned}$$

if $\underline{u}\ge 0$, $\bar{u}\le T$. In the last case, we have

$$G_{0,\vec{k}}(v_1,v_2) = \frac{1}{4\pi}\Big[\ln(\varepsilon+\bar{v})-\ln(\varepsilon+T+\bar{v})-\ln(\varepsilon+T)+\ln(\varepsilon+2T)\Big],$$

if $\underline{v}\ge 0$, $\bar{v}\le T$.

Considering the case $m=k=0$, we obtain

$$\begin{aligned}
(E_\mu(L_\varepsilon)) = G_{00} &= \Big(\frac{1}{2\pi}\Big)^{d/2}\int_{I^2} d^2t \Big(\frac{1}{\varepsilon+t_1+t_2}\Big)^{d/2} \\
&= \Big(\frac{1}{2\pi}\Big)^{d/2}\frac{1}{\big(\frac{d}{2}-1\big)\big(\frac{d}{2}-2\big)}\bigg(\Big(\frac{1}{\varepsilon+2T}\Big)^{\frac{d}{2}-2}-2\Big(\frac{1}{\varepsilon+T}\Big)^{\frac{d}{2}-2}+\Big(\frac{1}{\varepsilon}\Big)^{d-4}\bigg),
\end{aligned}$$

except for $d=2$ or $d=4$ where we have, respectively,

$$\begin{aligned}
E_\mu(L_\varepsilon) &= \frac{1}{2\pi}\int_{I^2} d^2t\,\frac{1}{\varepsilon+t_1+t_2} \\
&= \frac{1}{2\pi}\Big[(\varepsilon+2T)\ln(\varepsilon+2T)-2(\varepsilon+T)\ln(\varepsilon+T)+\varepsilon\ln(\varepsilon)\Big],
\end{aligned}$$

and

$$\begin{aligned}
E_\mu(L_\varepsilon) &= \Big(\frac{1}{2\pi}\Big)^2\int_{I^2} d^2t \Big(\frac{1}{\varepsilon+t_1+t_2}\Big)^2 \\
&= \Big(\frac{1}{2\pi}\Big)^2\Big[2\ln(\varepsilon+T)-\ln(\varepsilon+2T)-\ln(\varepsilon)\Big].
\end{aligned}$$

To conclude the proof it remains to observe that for each d, $\vec{m} \neq 0$ or $\vec{k} \neq 0$, $G_{\vec{m},\vec{k}}$ is a symmetric function of $u_1, \cdots, u_m, v_1, \cdots, v_k$ and is L^p-integrable on $\mathbb{R}^{m+k}$ for each $p \geq 1$. In fact, as $G_{\vec{m},\vec{k}}$ is continuous on $[0,T]^2$, the integral

$$\int_{\mathbb{R}^{m+k}} d^m(u_1, \cdots, u_m) d^k(v_1, \cdots, v_k) |G_{\vec{m},\vec{k}}(u_1, \cdots, u_m, v_1, \cdots, v_k)|^p =$$
$$\int_{I^{m+k}} d^m(u_1, \cdots, u_m) d^k(v_1, \cdots, v_k) |G_{\vec{m},\vec{k}}(u_1, \cdots, u_m, v_1, \cdots, v_k)|^p,$$

is always finite, independently of the particular form that $G_{\vec{m},\vec{k}}$ takes.

The proof follows by the uniqueness of the chaos decomposition. □

Theorem 3.3. *For each t_1 and t_2 strictly positive real numbers the Bochner integral*

$$\delta\Big(B^{(1)}(t_1) - B^{(2)}(t_2)\Big) \equiv \left(\frac{1}{2\pi}\right)^d \int_{\mathbb{R}^d} d^d\lambda\, e^{i\lambda\cdot(B^{(1)}(t_1)-B^{(2)}(t_2))}$$

is a generalized white noise functional with S-transform given by

$$S\Big(\delta(B^{(1)}(t_1) - B^{(2)}(t_2))\Big)(\mathbf{f}) = \left(\frac{1}{2\pi(t_1+t_2)}\right)^{d/2} e^{-\frac{\left(\int_0^{t_1} \mathbf{f}(t)\,dt - \int_0^{t_2} \mathbf{f}(t)\,dt\right)^2}{2(t_1+t_2)}}, \tag{3.4}$$

for every $\mathbf{f} \in S_d(\mathbb{R})$.

Proof: Applying once again Corollary 2.4 we will check that the S-transform of the integrand

$$\Psi(\vec{\omega}_1, \vec{\omega}_2) \equiv e^{i\lambda\cdot(B^{(1)}(t_1)-B^{(2)}(t_2))} \tag{3.5}$$

verifies the conditions of its applicability with respect to Lebesgue measure on $\mathbb{R}^d$.

Recalling that the S-transform at a point $\mathbf{f}$ is the expectation of the functional Ψ with argument shifted by $\mathbf{f}$ (see, *e.g.*, [18]), it follows from the independence of the Brownian motions that

$$\begin{aligned} S\Psi(\mathbf{f}) &= E\Big(e^{i\lambda\cdot<\vec{\omega}_1+\mathbf{f},1_{[0,t_1]}>}\Big) E\Big(e^{-i\lambda\cdot(<\vec{\omega}_2+\mathbf{f},1_{[0,t_2]}>)}\Big) \\ &= e^{i\lambda\cdot(\int_0^{t_1} \mathbf{f}(t)\,dt - \int_0^{t_2} \mathbf{f}(t)\,dt) - \frac{1}{2}\|\lambda\|^2(t_1+t_2)}. \end{aligned}$$

The measurability is obvious and for the boundedness condition we observe that

$$\begin{aligned} |S\Psi(z\mathbf{f})| &= \left|e^{i\lambda\cdot z(\int_0^{t_1} \mathbf{f}(t)\,dt - \int_0^{t_2} \mathbf{f}(t)\,dt) - \frac{1}{2}\|\lambda\|^2(t_1+t_2)}\right| \\ &= e^{-\frac{1}{2}\|\lambda\|^2(t_1+t_2)} \left|e^{i\lambda\cdot z(\int_0^{t_1} \mathbf{f}(t)\,dt - \int_0^{t_2} \mathbf{f}(t)\,dt)}\right| \\ &\leq e^{-\frac{1}{4}\|\lambda\|^2(t_1+t_2)} e^{-\frac{1}{4}\|\lambda\|^2(t_1+t_2) + \|\lambda\|\cdot|z|\cdot\left|\int_0^{t_1} \mathbf{f}(t)\,dt - \int_0^{t_2} \mathbf{f}(t)\,dt\right|}, \end{aligned}$$

where the value of the second exponential is dominated by

$$e^{\frac{|z|^2}{t_1+t_2}\left\| \int_0^{t_1} \mathbf{f}(t)\,dt - \int_0^{t_2} \mathbf{f}(t)\,dt \right\|^2},$$

because

$$\begin{aligned}
&-\frac{1}{4}\|\lambda\|^2(t_1+t_2) + \|\lambda\|\cdot|z|\cdot\Big\| \int_0^{t_1} \mathbf{f}(t)\,dt - \int_0^{t_2} \mathbf{f}(t)\,dt \Big\| \\
&= -\left(\frac{|z|}{\sqrt{t_1+t_2}} \Big\| \int_0^{t_1} \mathbf{f}(t)\,dt - \int_0^{t_2} \mathbf{f}(t)\,dt \Big\| - \frac{\|\lambda\|}{2}\sqrt{t_1+t_2} \right)^2 \\
&+ \frac{|z|^2}{t_1+t_2} \Big\| \int_0^{t_1} \mathbf{f}(t)\,dt - \int_0^{t_2} \mathbf{f}(t)\,dt \Big\|^2 .
\end{aligned}$$

Thus, we have

$$\begin{aligned}
|S\Psi(z\mathbf{f})| &\le e^{-\frac{1}{4}\|\lambda\|^2(t_1+t_2)} e^{\frac{|z|^2}{t_1+t_2}\left\| \int_0^{t_1} \mathbf{f}(t)\,dt - \int_0^{t_2} \mathbf{f}(t)\,dt \right\|^2} \\
&\le e^{-\frac{1}{4}\|\lambda\|^2(t_1+t_2)} e^{|z|^2(t_1+t_2)\sum_{i=1}^d \sup_t |f_i(t)|^2},
\end{aligned}$$

where the first exponential is integrable on $\mathbb{R}^d$ and the second one is constant as function of λ. The result mentioned above can be applied and the S-transform of the δ-function is obtained by integration over λ. □

Indicating the projection onto chaos of order $n \ge N$ by a superscript (N), we have the following result on intersection local times L, respectively their subtracted counterparts $L^{(k)}$.

Theorem 3.4. *For any pair of integers d and $N \ge 0$ such that $2N > d-4$, the Bochner integral*

$$L^{(N)} \equiv \int_{I^2} d^2t\, \delta^{(N)}\Big(B^{(1)}(t_1) - B^{(2)}(t_2)\Big)$$

is a Hida distribution.

Proof: Denoting the truncated exponential series by

$$\exp_N(x) = \sum_{n=N}^{\infty} \frac{x^n}{n!},$$

it follows from (3.4) that the S-transform of $\delta^{(N)}(B^{(1)}(t_1) - B^{(2)}(t_2))$ is given by

$$\begin{aligned}
&S\Big(\delta^{(N)}\,(B^{(1)}(t_1) - B^{(2)}(t_2))\Big)(\mathbf{f}) \\
&\quad= \left(\frac{1}{2\pi(t_1+t_2)}\right)^{d/2} \exp_N\left(-\frac{\left(\int_0^{t_1} \mathbf{f}(t)\,dt - \int_0^{t_2} \mathbf{f}(t)\,dt\right)^2}{2(t_1+t_2)} \right).
\end{aligned} \tag{3.6}$$

Estimating the argument of $\exp_N$ in (3.6) by $a \cdot x \equiv (t_1 + t_2) \cdot \frac{1}{2} \sum_{i=1}^d \sup_t |f_i(t)|^2$, and using the estimate

$$\exp_N(ax) \leq a^N e^{(N+a)x}$$

for non-negative numbers a and x, we see that

$$\left| S\Big(\delta^{(N)}(B^{(1)}(t_1) - B^{(2)}(t_2))\Big)(z\mathbf{f}) \right| \leq \Big(\frac{1}{2\pi}\Big)^{d/2} (t_1 + t_2)^{N-\frac{d}{2}} e^{\frac{|z|^2}{2}(N+t_1+t_2) \sum_{i=1}^d \sup_t |f_i(t)|^2}.$$

Since $N + t_1 + t_2$ is bounded as function of t_1, t_2, and $(t_1 + t_2)^{N-\frac{d}{2}}$ is integrable on I^2 if and only if $d - 2N < 4$, the proof follows from Corollary 2.4 with respect to Lebesgue measure on I^2. □

In particular, the intersection local time is well defined for $d < 4$.

Corollary 3.5. *For $d < 4$ the intersection local time*

$$L \equiv \int_{I^2} d^2t\, \delta\Big(B^{(1)}(t_1) - B^{(2)}(t_2)\Big)$$

is a Hida distribution. In this case, its S-transform is given by

$$SL(\mathbf{f}) = \Big(\frac{1}{2\pi}\Big)^{d/2} \int_{I^2} d^2t \Big(\frac{1}{t_1 + t_2}\Big)^{d/2} e^{-\frac{\left(\int_0^{t_1} \mathbf{f}(t)\,dt - \int_0^{t_2} \mathbf{f}(t)\,dt\right)^2}{2(t_1+t_2)}},$$

for all $\mathbf{f} \in S_d(\mathrm{I\!R})$.

Proof: The S-transform of the functional Ψ introduced in (3.5) obeys the conditions of Corollary 2.4 with respect to Lebesgue measure $d(\lambda, (t_1, t_2))$ on $\mathrm{I\!R}^d \times I^2$. □

For $d = 4$ or $d = 5$ it is sufficient to center L or rather δ:

$$L^{(1)} = \int_{I^2} d^2t\, \delta^{(1)}\Big(B^{(1)}(t_1) - B^{(2)}(t_2)\Big).$$

Informally speaking, the local time becomes well defined if we subtract its (divergent) expectation.

The chaos expansion for the local times $L^{(N)}$ is then given as follows.

Theorem 3.6. *For each natural numbers d and $N \geq 0$ such that $2N > d - 4$, $L^{(N)}$ has the chaos expansion*

$$L^{(N)} = \sum_{\vec{m}} \sum_{\vec{k}} \Big\langle :\vec{\omega}_1^{\otimes \vec{m}}: \otimes :\vec{\omega}_2^{\otimes \vec{k}}:, G_{\vec{m},\vec{k}} \Big\rangle,$$

where the kernel functions $G_{\vec{m},\vec{k}}$ of $L^{(N)}$ vanish for $m+k<2N$ or $\vec{m}+\vec{k}$ not even, and for $m+k\neq 0$, $m+k\geq 2N$, $\vec{m}+\vec{k}$ even $G_{\vec{m},\vec{k}}$ are given by

$$G_{\vec{m},\vec{k}}\ (u_1,\cdots,u_m,v_1,\cdots,v_k) = (-1)^k\Big(\frac{1}{2\pi}\Big)^{d/2}\Big(-\frac{1}{2}\Big)^{\frac{m+k}{2}}\frac{1}{\big(\frac{\vec{m}+\vec{k}}{2}\big)!}\binom{\vec{m}+\vec{k}}{\vec{m}}\cdot$$
$$\cdot\frac{1}{\big(\frac{d+m+k}{2}-1\big)\big(\frac{d+m+k}{2}-2\big)}\theta(\underline{u})\theta(\underline{v})\theta(T-\bar{u})\theta(T-\bar{v})\cdot$$
$$\cdot\Big[\Big(\frac{1}{\bar{u}+\bar{v}}\Big)^{\frac{d+m+k}{2}-2}-\Big(\frac{1}{T+\bar{v}}\Big)^{\frac{d+m+k}{2}-2}-\Big(\frac{1}{T+\bar{u}}\Big)^{\frac{d+m+k}{2}-2}+\Big(\frac{1}{2T}\Big)^{\frac{d+m+k}{2}-2}\Big],$$

with the exception of the case $d=2$ and $m+k=2$ where

$$G_{\vec{m},\vec{k}}(u_1,v_1)=\frac{1}{2\pi}\Big[\ln(T+v_1)-\ln(u_1+v_1)-\ln(2T)+\ln(T+u_1)\Big]\cdot$$
$$\cdot\theta(u_1)\theta(v_1)\theta(T-u_1)\theta(T-v_1),$$

for $m=k=1$;

$$G_{\vec{m},0}(u_1,u_2)=\frac{1}{4\pi}\Big[\ln(\bar{u})-\ln(T+\bar{u})+\ln(2)\Big]\cdot\theta(\underline{u})\theta(T-\bar{u}),$$

for $m=2$ and $k=0$; and

$$G_{0,\vec{k}}(v_1,v_2)=\frac{1}{4\pi}\Big[\ln(\bar{v})-\ln(T+\bar{v})+\ln(2)\Big]\cdot\theta(\underline{v})\theta(T-\bar{v}),$$

for $k=2$ and $m=0$.

Moreover

$$E_\mu(L)=\begin{cases}\big(\frac{1}{2\pi}\big)^{d/2}\frac{1}{\left(\frac{d}{2}-1\right)\left(\frac{d}{2}-2\right)}\left(\big(\frac{1}{2T}\big)^{\frac{d}{2}-2}-2\big(\frac{1}{T}\big)^{\frac{d}{2}-2}\right) & \text{if } d\neq 2, d<4\\ \frac{T}{\pi}\ln 2 & \text{if } d=2\end{cases}.$$

Proof: By Corollary 2.4 the S-transform of the (truncated) local time is given as an integral over (3.6). Hence, for each $\mathbf{f}\in S_d(\mathrm{I\!R})$,

$$SL^{(N)}(\mathbf{f})=\Big(\frac{1}{2\pi}\Big)^{d/2}\int_{I^2}d^2t\,\Big(\frac{1}{t_1+t_2}\Big)^{d/2}\exp_N\left(-\frac{(F_{1i}(t_1)-F_{2i}(t_2))^2}{2(t_1+t_2)}\right)$$
$$=\Big(\frac{1}{\pi}\Big)^{d/2}\int_{I^2}d^2t\,\sum_{n=N}^{\infty}\Big\{(-1)^n\Big(\frac{1}{2(t_1+t_2)}\Big)^{n+d/2}$$

$$\sum_{\substack{n_1,\cdots,n_d\\ n_1+\cdots+n_d=n}} \frac{1}{n_1!\cdots n_d!}\prod_{i=1}^{d}\sum_{\substack{m_i,k_i\\ m_i+k_i=2n_i}} \binom{m_i+k_i}{m_i} F_{1i}^{m_i}(-F_{2i})^{k_i}\Big\}$$

$$= \Big(\frac{1}{\pi}\Big)^{d/2}\int_{I^2} d^2t \sum_{n=N}^{\infty}\sum_{\substack{n_1,\cdots,n_d\\ n_1+\cdots+n_d=n}}\sum_{\substack{m_i,k_i,i=1,\cdots,d\\ m_i+k_i=2n_i}}$$

$$\Big\{\prod_{i=1}^{d}(-1)^{\frac{m_i+k_i}{2}}\Big(\frac{1}{2(t_1+t_2)}\Big)^{\frac{1}{2}+\frac{m_i+k_i}{2}}\frac{1}{\Big(\frac{m_i+k_i}{2}\Big)!}\Big\}\cdot\Big\{\prod_{i=1}^{d}\binom{m_i+k_i}{m_i}F_{1i}^{m_i}(-F_{2i})^{k_i}\Big\}$$

$$= \Big(\frac{1}{\pi}\Big)^{d/2}\int_{I^2} d^2t \sum_{\substack{\vec m,\vec k\\ \vec m+\vec k\ even\\ m+k\geq 2N}} (-1)^{\frac{m+k}{2}}\Big(\frac{1}{2(t_1+t_2)}\Big)^{\frac{d+m+k}{2}}\frac{1}{\Big(\frac{\vec m+\vec k}{2}\Big)!}\binom{\vec m+\vec k}{\vec m}\vec F_1^{\vec m}(-\vec F_2)^{\vec k}.$$

Comparing with the general form of the chaos expansion

$$L^{(N)} = \sum_{\vec m}\sum_{\vec k}\Big\langle :\vec\omega_1^{\otimes\vec m}:\otimes:\vec\omega_2^{\otimes\vec k}:, G_{\vec m,\vec k}\Big\rangle,$$

we find that $G_{\vec m,\vec k}$ is equal to zero for $m+k<2N$ and for $\vec m+\vec k$ not even. For $\vec m+\vec k$ even, $\vec m+\vec k\neq 0$, $m+k\geq 2N$ the kernel functions are equal to

$$\begin{aligned}G_{\vec m,\vec k}(u_1,\cdots,u_m,v_1,\cdots,v_k) &= C'_{\vec m,\vec k}\textstyle\int_{I^2} d^2t\,(-1)^k\Big(\frac{1}{t_1+t_2}\Big)^{\frac{d+m+k}{2}}\cdot\\ &\qquad\cdot\Big(1_{]0,t_1]}^{\otimes m}\otimes 1_{]0,t_2]}^{\otimes k}\Big)(u_1,\cdots,u_m,v_1,\cdots,v_k)\\ &= \begin{cases} C_{\vec m,\vec k}\displaystyle\int_{\bar u}^{T} dt_1\int_{\bar v}^{T} dt_2\Big(\frac{1}{t_1+t_2}\Big)^{\frac{d+m+k}{2}} & \text{for } \underline{u},\underline{v}>0 \text{ and } \bar u,\bar v\leq T\\ 0 & \text{otherwise}\end{cases},\end{aligned}$$

for $C_{\vec m,\vec k}$ and $C'_{\vec m,\vec k}$ defined in (3.3). This means, for $\underline{u},\underline{v}>0$ and $\bar u,\bar v\leq T$,

$$G_{\vec m,\vec k}\,(u_1,\cdots,u_m,v_1,\cdots,v_k) = \frac{C_{\vec m,\vec k}}{\Big(\frac{d+m+k}{2}-1\Big)\Big(\frac{d+m+k}{2}-2\Big)}\cdot$$
$$\cdot\Big[\Big(\frac{1}{\bar u+\bar v}\Big)^{\frac{d+m+k}{2}-2}-\Big(\frac{1}{T+\bar v}\Big)^{\frac{d+m+k}{2}-2}-\Big(\frac{1}{T+\bar u}\Big)^{\frac{d+m+k}{2}-2}+\Big(\frac{1}{2T}\Big)^{\frac{d+m+k}{2}-2}\Big],$$

provided that

$$(i')\qquad \frac{d+m+k}{2}\neq 1,\ \text{and}$$
$$(ii')\qquad \frac{d+m+k}{2}\neq 2.$$

As $\vec{m}+\vec{k}$ is even, $0 \neq m+k \geq 2N$ and $2N > d-4$, we always have inequality (i'), and the equality in (ii') can only occur if $d=2$ and $m+k=2$. Analogously, there exist three possible situations when $d=2$ and $m+k=2$:

$$m=k=1, \quad m=2 \text{ and } k=0, \quad k=2 \text{ and } m=0.$$

For the first case,

$$\begin{aligned} G_{\vec{m},\vec{k}}(u_1,v_1) &= \frac{1}{2\pi}\int_{u_1}^{T} dt_1 \int_{v_1}^{T} dt_2 \Big(\frac{1}{t_1+t_2}\Big)^2 \\ &= \frac{1}{2\pi}\Big[\ln(T+v_1) - \ln(u_1+v_1) - \ln(2T) + \ln(T+u_1)\Big], \end{aligned}$$

for $0 < u_1, v_1 \leq T$; for the second one,

$$\begin{aligned} G_{\vec{m},0}(u_1,u_2) &= -\frac{1}{4\pi}\int_{\bar{u}}^{T} dt_1 \int_{0}^{T} dt_2 \Big(\frac{1}{t_1+t_2}\Big)^2 \\ &= \frac{1}{4\pi}\Big[\ln(\bar{u}) - \ln(T+\bar{u}) + \ln(2)\Big], \end{aligned}$$

for $\underline{u} > 0$, $\bar{u} \leq T$; and, for the last case,

$$G_{0,\vec{k}}(v_1,v_2) = \frac{1}{4\pi}\Big[\ln(\bar{v}) - \ln(T+\bar{v}) + \ln(2)\Big],$$

for $\underline{v} > 0$, $\bar{v} \leq T$.

Considering the special case $m=k=0$, we have for $d=2$

$$(E_\mu(L)) = G_{00} = \frac{1}{2\pi}\int_{I^2} d^2t \, \frac{1}{t_1+t_2} = \frac{T}{\pi}\ln 2,$$

and for the remaining dimensions – $d<4$, $d \neq 2$:

$$\begin{aligned} (E_\mu(L)) = G_{00} &= \Big(\frac{1}{2\pi}\Big)^{d/2} \int_{I^2} d^2t \, \Big(\frac{1}{t_1+t_2}\Big)^{d/2} \\ &= \Big(\frac{1}{2\pi}\Big)^{d/2} \frac{1}{\big(\frac{d}{2}-1\big)\big(\frac{d}{2}-2\big)} \Bigg(\Big(\frac{1}{2T}\Big)^{\frac{d}{2}-2} - 2\Big(\frac{1}{T}\Big)^{\frac{d}{2}-2}\Bigg). \end{aligned}$$

Since for each $\vec{m}$, $\vec{k}$, $G_{\vec{m},\vec{k}}$ is a symmetric function of $u_1,\cdots,u_m,v_1,\cdots,v_k$ the proof is finished. □

Theorem 3.7. *Whenever $m \neq 0$ or $k \neq 0$, the functions $G_{\vec{m},\vec{k}}$ introduced in the above theorem belongs to $L^p(\mathbb{R}^{m+k})$ for all p such that $p < 2 - \frac{d-4}{\frac{d+m+k}{2}-2}$. For* *the particular case $d = 2$ and $m + k = 2$, $G_{\vec{m},\vec{k}}$ are in $L^p(\mathbb{R}^2)$ for each $p \geq 1$, independently of the particular form that $G_{\vec{m},\vec{k}}$ takes.*

Proof: For each $m \neq 0$ and $k \neq 0$ such that $m + k \neq 2$ if $d = 2$,

$$\begin{aligned}
&\int_{\mathbb{R}^{m+k}} d^m(u_1, \cdots, u_m) d^k(v_1, \cdots, v_k) |G_{\vec{m},\vec{k}}(u_1, \cdots, u_m, v_1, \cdots, v_k)|^p \\
&= \left| \frac{C_{\vec{m},\vec{k}}}{\left(\frac{d+m+k}{2} - 1\right)\left(\frac{d+m+k}{2} - 2\right)} \right|^p \int_{I^{m+k}} d^m(u_1, \cdots, u_m) d^k(v_1, \cdots, v_k) \cdot \\
&\left| \left(\frac{1}{\bar{u}+\bar{v}}\right)^{\frac{d+m+k}{2}-2} - \left(\frac{1}{T+\bar{v}}\right)^{\frac{d+m+k}{2}-2} - \left(\frac{1}{T+\bar{u}}\right)^{\frac{d+m+k}{2}-2} + \left(\frac{1}{2T}\right)^{\frac{d+m+k}{2}-2} \right|^p,
\end{aligned}$$

where the integral can be estimated by

$$\begin{aligned}
&\int_{I^{m+k}} d^m(u_1, \cdots, u_m) d^k(v_1, \cdots, v_k) \\
&\left| \left(\frac{1}{\bar{u}+\bar{v}}\right)^{\frac{d+m+k}{2}-2} - \left(\frac{1}{T+\bar{v}}\right)^{\frac{d+m+k}{2}-2} - \left(\frac{1}{T+\bar{u}}\right)^{\frac{d+m+k}{2}-2} + \left(\frac{1}{2T}\right)^{\frac{d+m+k}{2}-2} \right|^p \\
&= mk \int_0^T d\bar{u} \int_0^T d\bar{v} \left| \left(\frac{1}{\bar{u}+\bar{v}}\right)^{\frac{d+m+k}{2}-2} - \left(\frac{1}{T+\bar{v}}\right)^{\frac{d+m+k}{2}-2} - \left(\frac{1}{T+\bar{u}}\right)^{\frac{d+m+k}{2}-2} \right. \\
&\qquad \left. + \left(\frac{1}{2T}\right)^{\frac{d+m+k}{2}-2} \right|^p \int_0^{\bar{u}} d^{m-1}u_i \int_0^{\bar{v}} d^{k-1}v_j \\
&\leq 4^p mk \int_0^T d\bar{u} \int_0^T d\bar{v} \left(\left(\frac{1}{\bar{u}+\bar{v}}\right)^{p\frac{d+m+k}{2}-2p} + \left(\frac{1}{T+\bar{v}}\right)^{p\frac{d+m+k}{2}-2p} + \left(\frac{1}{T+\bar{u}}\right)^{p\frac{d+m+k}{2}-2p} \right. \\
&\qquad \left. + \left(\frac{1}{2T}\right)^{p\frac{d+m+k}{2}-2p} \right) \cdot \bar{u}^{m-1} \bar{v}^{k-1} \\
&\leq 4^{p+1} mk \int_0^T d\bar{u} \int_0^T d\bar{v} \left(\frac{1}{\bar{u}+\bar{v}}\right)^{p\frac{d+m+k}{2}-2p} (\bar{u}+\bar{v})^{m+k-2} \\
&= 4^{p+1} mk \int_0^T d\bar{u} \int_0^T d\bar{v} \left(\frac{1}{\bar{u}+\bar{v}}\right)^{p(\frac{d+m+k}{2}-2)-m-k+2}.
\end{aligned}$$

Hence, the integral is finite provided $p(\frac{d+m+k}{2} - 2) - m - k + 2 < 2$.

When $\vec{k} = 0$, $\vec{m}$ even, $m > 2$ or $\vec{m} = 0$, $\vec{k}$ even, $k > 2$, it follows for the first situation and in an analogous way

$$\begin{aligned}
\int_{\mathbb{R}^m} d^m(u_1, ..., u_m)\ |G_{\vec{m},0}(u_1, ..., u_m)|^p &= \left| \frac{C_{\vec{m},0}}{\left(\frac{d+m}{2} - 1\right)\left(\frac{d+m}{2} - 2\right)} \right|^p \int_{I^m} d^m(u_1, ..., u_m) \cdot \\
&\left| \left(\frac{1}{\bar{u}}\right)^{\frac{d+m}{2}-2} - \left(\frac{1}{T+\bar{u}}\right)^{\frac{d+m}{2}-2} - \left(\frac{1}{T}\right)^{\frac{d+m}{2}-2} + \left(\frac{1}{2T}\right)^{\frac{d+m}{2}-2} \right|^p,
\end{aligned}$$

where the integral can be estimated by

$$\int_{I^m} d^m(u_1, \cdots, u_m) \left| \left(\frac{1}{\bar{u}}\right)^{\frac{d+m}{2}-2} - \left(\frac{1}{T+\bar{u}}\right)^{\frac{d+m}{2}-2} - \left(\frac{1}{T}\right)^{\frac{d+m}{2}-2} + \left(\frac{1}{2T}\right)^{\frac{d+m}{2}-2} \right|^p$$

$$= m\int_0^T d\bar{u}\,\Big|\Big(\frac{1}{\bar{u}}\Big)^{\frac{d+m}{2}-2} - \Big(\frac{1}{T+\bar{u}}\Big)^{\frac{d+m}{2}-2} - \Big(\frac{1}{T}\Big)^{\frac{d+m}{2}-2} + \Big(\frac{1}{2T}\Big)^{\frac{d+m}{2}-2}\Big|^p \int_0^{\bar{u}} d^{m-1}u_i$$

$$\leq 4^p m\int_0^T d\bar{u}\,\Big(\Big(\frac{1}{\bar{u}}\Big)^{p\frac{d+m}{2}-2p} + \Big(\frac{1}{T+\bar{u}}\Big)^{p\frac{d+m}{2}-2p} + \Big(\frac{1}{T}\Big)^{p\frac{d+m}{2}-2p} + \Big(\frac{1}{2T}\Big)^{p\frac{d+m}{2}-2p}\Big)\cdot \bar{u}^{m-1}$$

$$\leq 4^p m\Big(\Big(\frac{1}{2}\Big)^{p\frac{d+m}{2}-2p-1} + 2\Big)\int_0^T d\bar{u}\,\Big(\frac{1}{\bar{u}}\Big)^{p(\frac{d+m}{2}-2)-m+1};$$

the above integral is finite whenever $p(\frac{d+m}{2}-2)-m+1<1$.

The situation $\vec{m}=0$, $\vec{k}$ even, $k>2$ is treated analogously. Also for the particular case $d=2$ and $m+k=2$ the treatment for the logarithmic kernel functions is analogous. □

Corollary 3.8. *For each $d<4$, the kernel functions $G_{\vec{m},\vec{k}}$ are square integrable on $\mathbb{R}^{m+k}$, whenever $m\neq 0$ or $k\neq 0$.*

Applying Corollary 2.3 related to the convergence of sequences on the space of Hida distributions, we deduce the following result.

Theorem 3.9. *For each d and $N\geq 0$ such that $2N>d-4$, the (truncated) functional $L_\varepsilon^{(N)}$ converges strongly to the (truncated) local time $L^{(N)}$ on $(S)^*$ when ε tends to zero.*

Proof: From (3.1) the S-transform of $L_\varepsilon^{(N)}$ is given by

$$SL_\varepsilon^{(N)}(\mathbf{f}) = \int_{I^2} d^2t\,\left(\frac{1}{2\pi(\varepsilon+t_1+t_2)}\right)^{d/2}\exp_N\left(-\frac{\left(\int_0^{t_1}\mathbf{f}(t)\,dt - \int_0^{t_2}\mathbf{f}(t)\,dt\right)^2}{2(\varepsilon+t_1+t_2)}\right). \tag{3.7}$$

We remark that $L_\varepsilon^{(N)}$ is a Hida distribution; in fact, this follows from

$$\begin{aligned}&\Big|S\Big(\delta_\varepsilon^{(N)}\,(B^{(1)}(t_1)-B^{(2)}(t_2))\Big)(z\mathbf{f})\Big|\\ &\quad=\left(\frac{1}{2\pi(\varepsilon+t_1+t_2)}\right)^{d/2}\left|\exp_N\left(-\frac{|z|^2\left(\int_0^{t_1}\mathbf{f}(t)\,dt-\int_0^{t_2}\mathbf{f}(t)\,dt\right)^2}{2(\varepsilon+t_1+t_2)}\right)\right|\\ &\quad\leq\Big(\frac{1}{2\pi}\Big)^{d/2}(t_1+t_2)^{N-\frac{d}{2}}e^{\frac{|z|^2}{2}(N+t_1+t_2)\sum_{i=1}^d\sup_t|f_i(t)|^2},\end{aligned}$$

and from the characterization result for Hida distributions (Corollary 2.4).

By analogous estimates as in (3.2), for each complex number z and $\mathbf{f}\in S_d(\mathbb{R})$ we have

$$|SL_\varepsilon^{(N)}(z\mathbf{f})|\leq\int_{I^2}d^2t\,\left(\frac{1}{2\pi(\varepsilon+t_1+t_2)}\right)^{d/2}\exp_N\left(\frac{|z|^2(t_1+t_2)^2\sum_{i=1}^d\sup_t|f_i(t)|^2}{2(\varepsilon+t_1+t_2)}\right)$$

$$\leq \int_{I^2} d^2t \left(\frac{1}{2\pi(t_1+t_2)}\right)^{d/2} \exp_N\left(\frac{|z|^2(t_1+t_2)^2\sum_{i=1}^d \sup_t |f_i(t)|^2}{2(t_1+t_2)}\right)$$
$$\leq \left(\frac{1}{2\pi}\right)^{d/2} e^{\frac{N+2T}{2}|z|^2\sum_{i=1}^d \sup_t |f_i(t)|^2} \int_{I^2} d^2t \left(\frac{1}{t_1+t_2}\right)^{\frac{d}{2}-N}.$$

Using the expression for the S-transform of the functionals $L_\varepsilon^{(N)}$ (see (3.7)) and $L^{(N)}$ which, by (3.4), Theorem 3.4 and its proof, is equal to

$$SL^{(N)}(\mathbf{f}) = \int_{I^2} d^2t \left(\frac{1}{2\pi(t_1+t_2)}\right)^{d/2} \exp_N\left(-\frac{\left(\int_0^{t_1}\mathbf{f}(t)\,dt - \int_0^{t_2}\mathbf{f}(t)\,dt\right)^2}{2(t_1+t_2)}\right),$$

it follows from the above inequality with $z = 1$ and by a dominated convergence criterion, that $SL_\varepsilon^{(N)}(\mathbf{f})$ converges to $SL^{(N)}(\mathbf{f})$ when ε tends to zero. Applying the corollary mentioned above, we obtain the required convergence. □

Acknowledgements: L. S. is grateful for the hospitality of the Département de Mathématiques, Faculté des Sciences de Tunis, and for financial support from DAAD. M. J. O. would like to express her gratitude for the generous hospitality and support of BiBoS.

References

[1] Albeverio, S., Bolthausen, E., Zhou, X. Y., A modified discrete Edwards model in three dimensions. In preparation.

[2] Albeverio, S., Fenstad, J. E., Høegh-Krohn, R., Lindstrøm, T., *Nonstandard Methods in Stochastic Analysis and Mathematical Physics.* Academic Press, New York, 1986.

[3] Albeverio, S., Hu, Y. Z., Röckner, M., Zhou, X. Y., Stochastic quantization of the two-dimensional polymer measure. *Appl. Math. Optim.* **40** (1999), 341–354.

[4] Albeverio, S., Hu, Y. Z., Zhou, X. Y., A remark on non-smoothness of the self-intersection local time of planar Brownian motion. *Statist. Probab. Lett.* **32** (1997), 57–65.

[5] Albeverio, S., Röckner, M., Zhou, X. Y., Stochastic quantization of the three-dimensional polymer measure. *Inf. Dim. Anal. Quant. Prob.* (1999), to appear.

[6] Albeverio, S., Zhou, X. Y., Intersections of random walks and Wiener sausages in four dimensions. *Acta Appl. Math.* **45** (1996), 195–237.

[7] Bass, R. F., Khoshnevisan, D., Intersection local times and Tanaka formulas. *Ann. Inst. H. Poincaré Probab. Statist.* **29** (1993), 419–451.

[8] Dvoretzky, A., Erdös, P., Kakutani, S., Taylor, S. J., *Triple points of the Brownian motion in 3-space.* "Proc. Cambridge Philos. Soc." **53** (1957), 856–862.

[9] Dvoretzky, A., Erdös, P., Kakutani, S., Double points of paths of Brownian motion in the plane. *Bull. Res. Council Israel* Sect. F 3 (1954), 364–371.

[10] Dvoretzky, A., Erdös, P., Kakutani, S., Double points of paths of Brownian motion in n-space. *Acta Sci. Math. Szeged* **12** (1950), 75–81.

[11] Dynkin, E. B., Regularized self-intersection local times of planar Brownian motion. *Ann. Probab.* **16** (1988), 58–73.

[12] Dynkin, E. B., Self-intersection gauge for random walks and for Brownian motion. *Ann. Probab.* **16** (1988), 1–57.

[13] Dynkin, E. B., Polynomials of the occupation field and related random fields. *J. Funct. Anal.* **58** (1984), 20–52.

[14] de Faria, M., Drumond, C., Streit, L., The renormalization of self intersection local times I: The chaos expansion. Grupo de Física Matemática da Universidade de Lisboa, preprint IFM 1/98, 1998.

[15] de Faria, M., Hida, T., Streit, L., Watanabe, H., Intersection local times as generalized white noise functionals. *Acta Appl. Math.* **46** (1997), 351–362.

[16] Geman, D., Horowitz. J., Rosen, J., A local time analysis of intersections of Brownian paths in the plane. *Ann. Probab.* **12** (1984), 86–107.

[17] He, S. W., Yang, W. Q., Yao, R. Q., Wang, J. G., Local times of self-intersection for multidimensional Brownian motion. *Nagoya Math. J.* **138** (1995), 51–67.

[18] Hida, T., Kuo, H. H., Potthoff, J., Streit, L., *White Noise. An Infinite Dimensional Calculus.* Kluwer-Academic, Dordrecht, 1993.

[19] Imkeller, P., Pérez-Abreu, V., Vives, J., Chaos expansions of double intersection local times of Brownian motion in $\mathbb{R}^d$ and renormalization. *Stochastic Process. Appl.* **56** (1995), 1–34.

[20] Imkeller, P., Yan, J. A., Multiple intersection local time of planar Brownian motion as a particular Hida distribution. *J. Funct. Anal.* **140** (1996), 256–273.

[21] Kondratiev, Yu., Leukert, P., Potthoff, J., Streit, L., Westerkamp, W., Generalized functionals in Gaussian spaces: the characterization theorem revisited. *J. Funct. Anal.* **141** (1996), 301–318.

[22] Le Gall, J. F., Sur le temps local d'intersection du mouvement brownien plan et la méthode de renormalisation de Varadhan. Sém. Prob. XIX, *Lecture Notes in Mathematics* **1123**, Springer, Berlin, 1985, 314–331.

[23] Le Gall, J. F., Sur la saucisse de Wiener et les points multiples du mouvement brownien. *Ann. Probab.* **14** (1986), 1219–1244.

[24] Lévy, P., Le mouvement brownien plan. *Amer. J. Math.* **62** (1940), 487–550.

[25] Lyons, T. J., The critical dimension at which quasi-every Brownian motion is self-avoiding. Adv. in Appl. Probab. (Spec. Suppl. 1986), 87–99.

[26] Mendonça, S., Streit, L., Multiple intersection local times in terms of white noise. CCM, preprint, 1999.

[27] Meyer, P. A., Yan, J. A., Les fonctions caractéristiques des distributions sur l'espace de Wiener. Sém. Prob. XXV, *Lecture Notes in Mathematics* **1485**, Springer, Berlin, 1991, 61–78.

[28] Nualart, D., Vives, J., Smoothness of Brownian local times and related functionals. *Potential Anal.* **1** (1992), 257–263.

[29] Nualart, D., Vives, J., Chaos expansion and local times. *Publications Mathematiques* **36**(2), 827-836.

[30] Penrose, M. D., On the existence of self-intersections for quasi-every Brownian path in space. *Ann. Probab.* **17** (1989), 482–502.

[31] Pérez-Abreu, V., Chaos expansions: a review. "Proc. V Latin Amer. Congr. Prob. and Math. Stat., São Paulo 1993", *RESHAS* **1**(2–3) (1994), 335–359.

[32] Rosen, J., A local time approach to the self-intersections of Brownian paths in space. *Comm. Math. Phys.* **88** (1983), 327–338.

[33] Rosen, J., Tanaka's formula and renormalisation for intersections of planar Brownian motion. *Ann. Probab.* **14** (1986), 1425–1251.

[34] Rosen, J., A renormalized local time for multiple intersections of planar Brownian motion. Sém. de Prob. XX, *Lecture Notes in Mathematics* **1204**, Springer, Berlin, 1986, 515–531.

[35] Shieh, N. R., White noise analysis and Tanaka formula for intersections of planar Brownian motion. *Nagoya Math. J.* **122** (1991), 1–17.

[36] Stoll, A., Invariance principle for Brownian local time and polymer measures. *Math. Scand.* **64** (1989), 133–160.

[37] Streit, L. and Westerkamp, W., A generalization of the characterization theorem for generalized functionals of white noise, in "Dynamics of Complex and Irregular Systems" (Ph. Blanchard et al., eds.), World Scientific, Singapore, 1993.

[38] Symanzik, K., Euclidean quantum field theory, in "Local Quantum Theory" (R. Jost, ed.), Academic Press, New York, 1969.

[39] Varadhan, S. R. S., Appendix to "Euclidean quantum field theory" by K. Symanzik, in "Local Quantum Theory" (R. Jost. ed.), Academic Press, New York, 1969.

[40] Watanabe, H., The local time of self-intersections of Brownian motions as generalized Brownian functionals. *Lett. Math. Phys.* **23** (1991), 1–9.

[41] Werner, W., Sur les singularités des temps locaux d'intersection du mouvement brownien plan. *Ann. Inst. H. Poincaré Probab. Statist.* **29** (1993), 391–418.

[42] Westwater, J., On Edward's model for long polymer chains. *Comm. Math. Phys.* **72** (1980), 131–174.

[43] Wolpert, R., Wiener path intersection and local time. *J. Funct. Anal.* **30** (1978), 329–340.

[44] Yor, M., Compléments aux formules de Tanaka-Rosen. Sém. de Prob. XIX, *Lecture Notes in Mathematics* **1123**, Springer, Berlin, 1985, 332–348.

[45] Yor, M., Renormalisation et convergence en loi pour les temps locaux d'intersection du mouvement brownien dans $\mathbb{R}^3$. Séminaire de Probabilité, *Lecture Notes in Mathematics* **1123**, Springer, Berlin, 1985, 350–365.