A CHARACTERIZATION OF ADEQUATE SEMIGROUPS BY FORBIDDEN SUBSEMIGROUPS

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ABSTRACT. A semigroup is *amiable* if there is exactly one idempotent in each $\mathcal{R}^*$-class and in each $\mathcal{L}^*$-class. A semigroup is *adequate* if it is amiable and if its idempotents commute. We characterize adequate semigroups by showing that they are precisely those amiable semigroups which do not contain isomorphic copies of two particular nonadequate semigroups as subsemigroups.

1. Introduction

For a semigroup S, the usual Green's equivalence relations $\mathcal{L}$ and $\mathcal{R}$ are defined by $x\mathcal{L}y$ if and only if $S^1x = S^1y$ and $x\mathcal{R}y$ if and only if $xS^1 = yS^1$ for all $x, y \in S$ where $S^1 = S$ if S is a monoid and otherwise $S^1 = S \cup \{1\}$, that is S with an identity element 1 adjoined. Naturally linked to these relations are the classes of semigroups defined as follows:

- A semigroup is *regular* if there is an idempotent in each $\mathcal{L}$-class and in each $\mathcal{R}$-class;
- A semigroup is *inverse* if there is a unique idempotent in each $\mathcal{L}$-class and in each $\mathcal{R}$-class. A regular semigroup is inverse if and only if its idempotents commute ([6, Theorem 5.1.1]).

The starred Green's relation $\mathcal{L}^*$ is defined by $x\mathcal{L}^*y$ if and only if $x\mathcal{L}y$ in some semigroup containing S as a subsemigroup, and a similar definition gives $\mathcal{R}^*$. These are characterized, respectively, by $x\mathcal{L}^*y$ if and only if, for all $a, b \in S^1$, $xa = xb \Leftrightarrow ya = yb$ and by $x\mathcal{R}^*y$ if and only if, for all $a, b \in S^1$, $ax = bx \Leftrightarrow ay = by$. Naturally linked to these relations are the classes of semigroups defined as follows:

- A semigroup is *abundant* if there is an idempotent in each $\mathcal{L}^*$-class and in each $\mathcal{R}^*$-class [4].
- A semigroup is *adequate* if it is abundant and its idempotents commute [3].
- A semigroup is *amiable* if there is a unique idempotent in each $\mathcal{L}^*$-class and in each $\mathcal{R}^*$-class [1]. Every adequate semigroup is amiable [3].

Since $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$, abundant semigroups generalize regular semigroups, and amiable (and hence, adequate) semigroups generalize inverse semigroups. Of course, the classes of regular and inverse semigroups are among the most intensively studied classes of semigroups. Many of the fundamental results in these classes have been generalized to abundant and adequate semigroups, for which there is also an extensive literature.

It has been known since Fountain's first paper [3] that the class of adequate semigroups is properly contained in the class of amiable semigroups, because he constructed an infinite amiable, but not adequate, semigroup. Later M. Kambites asked whether these two classes coincide on finite semigroups and a negative answer was provided in [1]. The aim of this paper is to characterize adequate semigroups inside the class of amiable semigroups. We therefore hope that our main result will provide a useful tool for generalizing the extensive literature on inverse and adequate semigroups to the setting of amiable semigroups.

We say that a semigroup S *avoids* a semigroup T if S does not contain an isomorphic copy of T as a subsemigroup. The main result of this paper is the following.

Main Theorem. *Let S be an amiable semigroup. Then S is adequate if and only if S avoids both of the semigroups defined by the presentations*

$$\mathcal{F} = \langle a, b \mid a^2 = a, b^2 = b \rangle \tag{F}$$

and

$$\mathcal{M} = \langle a, b \mid a^2 = a, b^2 = b, aba = bab = ab \rangle . \tag{M}$$

The semigroup F defined by the presentation $\mathcal{F}$ is Fountain's original example of an amiable semigroup which is not adequate ([3], Example 1.4). Except for changes in notation, the example is given as follows. Let

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Set $F_0 = \{2^n A, 2^n B, 2^n C, 2^n D \mid n \geq 0\}$. Then F_0 is a semigroup under the usual matrix multiplication. It is easy to see that A and B are the only idempotents of F_0. The $\mathcal{L}^*$-classes are $\{2^n A, 2^n D \mid n \geq 0\}$, $\{2^n B, 2^n C \mid n \geq 0\}$, and the $\mathcal{R}^*$-classes are $\{2^n A, 2^n C \mid n \geq 0\}$, $\{2^n B, 2^n D \mid n \geq 0\}$. Hence F_0 is amiable, but it is not adequate since $AB = C \neq D = BA$.

Note that for $n \geq 0$, $2^n C = C^{n+1} = (AB)^{n+1}$, $2^n D = D^{n+1} = (BA)^{n+1}$, $2^n A = C^n A = (AB)^n A$ and $2^n B = D^n B = (BA)^n B$. Therefore $F_0 = \{(AB)^n A, (BA)^n B, (AB)^m, (BA)^m \mid n \geq 0, m \geq 1\}$ with no two of the listed elements coinciding. On the other hand, $F = \{(ab)^n a, (ba)^n b, (ab)^m, (ba)^m \mid n \geq 0, m \geq 1\}$ with no two of the listed elements coinciding. Therefore Fountain's F_0 has $\mathcal{F}$ as a presentation.

The semigroup M defined by the presentation $\mathcal{M}$ is the first known *finite* example of an amiable semigroup which is not adequate [1]. Setting $c = ab$ and $d = ba$, it is easy to see from the relations that $M = \{a, b, c, d\}$ with the following multiplication table:

M	a	b	c	d
a	a	c	c	c
b	d	b	c	d
c	c	c	c	c
d	d	c	c	c

TABLE 1. The smallest amiable semigroup which is not adequate.

The $\mathcal{L}^*$-classes are $\{a, d\}$, $\{b\}$ and $\{c\}$. The $\mathcal{R}^*$-classes are $\{b, d\}$, $\{a\}$ and $\{c\}$. Thus M is amiable but it is evidently not adequate since $ab = c \neq d = ba$.

In fact, the original motivation for this paper was a conjecture offered in [1], that every finite amiable semigroup which is not adequate contains an isomorphic copy of M. The conjecture was based on a computer search in which it was found that the conjecture holds up to order 37. The confirmation of this conjecture is a trivial corollary of our Main Theorem.

The preceding discussion has shown that the avoidance condition of the Main Theorem is certainly necessary, since F and M contain non-commuting idempotents and hence cannot be subsemigroups of adequate semigroups. The next section is devoted to the proof of the sufficiency. In the last section, we pose some problems.

2. The Proof

In what follows we make frequent use of the fact that for idempotent elements (more generally, for regular elements) s, t of an abundant semigroup, $s\mathcal{L}^*t$ if and only if $s\mathcal{L}t$ and similarly, $s\mathcal{R}^*t$ if and only if $s\mathcal{R}t$ [4].

For each x in an amiable semigroup, we denote by x_ℓ the unique idempotent in the $\mathcal{L}^*$-class of x and we denote by x_r the unique idempotent in the $\mathcal{R}^*$-class of x. (In the literature, these are sometimes denoted by x^* and x^+, respectively.) We can view amiable semigroups as algebras of type $\langle 2, 1, 1\rangle$ where the binary operation is the semigroup multiplication and the unary operations are $x \mapsto x_\ell$ and $x \mapsto x_r$. Thus we may think of amiable semigroups as forming a quasivariety axiomatized by, for instance, associativity together with these eight quasi-identities

$$\begin{array}{cc} x_\ell x_\ell = x_\ell & x_r x_r = x_r \\ x x_\ell = x & x_r x = x \\ xy = xz \Rightarrow x_\ell y = x_\ell z & yx = zx \Rightarrow y x_r = z x_r \end{array}$$

$$(\ xx = x \ \wedge \ yy = y \ \wedge \ x\mathcal{L}y\ ) \Rightarrow x = y$$
$$(\ xx = x \ \wedge \ yy = y \ \wedge \ x\mathcal{R}y\ ) \Rightarrow x = y\,.$$

Here $x\mathcal{L}y$ abbreviates the conjunction $(\ xy = x \ \wedge \ yx = y\ )$, and similarly $x\mathcal{R}y$ abbreviates $(\ xy = y \ \wedge \ yx = x\ )$. We will use these quasi-identities in what follows without explicit reference.

Lemma 2.1. *For all x, y in an amiable semigroup,*

$$(x_\ell y)_\ell = (xy)_\ell\,. \tag{2.1}$$

Proof. It can be easily seen from the definition that the relation $\mathcal{L}^*$ is a right congruence. Since $x\mathcal{L}^*x_\ell$, we have $xy\mathcal{L}^*x_\ell y$ and so $(x_\ell y)_\ell = (xy)_\ell$. □

Lemma 2.2. *Let S be an amiable semigroup and let $a, b \in S$ be noncommuting idempotents. The following are equivalent: (i) $aba = ab$, (ii) $bab = ab$, (iii) $abab = ab$. When these conditions hold, the subsemigroup of S generated by a and b is isomorphic to M.*

Proof. The equivalence of (i) and (ii) is ([1], Lemma 2). If (i) holds, then clearly $abab = abb = ab$ and so (iii) holds. Now assume (iii). Then $aba \cdot aba = ababa = aba$, and so aba is an idempotent. We have $aba \cdot ab = ab$ and $ab \cdot aba = aba$, and so $aba\mathcal{R}ab$. Since S is amiable, $aba = ab$, that is (i) holds. The remaining assertion is ([1], Theorem 3). □

We can interpret Lemma 2.2 in terms of quasi-identities.

Lemma 2.3. *The class of all amiable semigroups which avoid M is a subquasivariety of the quasivariety of all amiable semigroups. It is characterized by the defining quasi-identities of* *amiable semigroups together with any one of the following:*

$$(xx = x \ \wedge \ yy = y \ \wedge \ xyx = xy) \Rightarrow xy = yx\,, \tag{2.2}$$

$$(xx = x \ \wedge \ yy = y \ \wedge \ xyx = yx) \Rightarrow xy = yx\,, \tag{2.3}$$

$$(xx = x \ \wedge \ yy = y \ \wedge \ xyxy = xy) \Rightarrow xy = yx\,. \tag{2.4}$$

***Proof*.** If a semigroup S contains a copy of M, then (2.2) is not satisfied since $aba = ca = c = ab$. Conversely, if (2.2) is not satisfied in S, then there exist idempotents a and b with $aba = ab$. By Lemma 2.2, a and b generate a copy of M. The proofs for the other two cases are similar. □

Lemma 2.4. *Let S be an amiable semigroup which avoids M and let $c \in S$ be an idempotent. Then for all $x \in S$,*

$$x(xc)_\ell = xc\,, \tag{2.5}$$

$$x_\ell(xc)_\ell = x_\ell c\,. \tag{2.6}$$

***Proof*.** Since $(xc)_\ell$ is the unique idempotent in the $\mathcal{L}^*$-class of xc and $xc = (xc)c$, we have from the definition of $\mathcal{L}^*$ that $(xc)_\ell = (xc)_\ell c$, and so $c(xc)_\ell c = c(xc)_\ell$. By (2.2), $c(xc)_\ell = (xc)_\ell c = (xc)_\ell$. Thus $xc = xc(xc)_\ell = x(xc)_\ell$, which establishes (2.5), and then (2.6) follows from (2.5). □

Lemma 2.5. *Let S be an amiable semigroup which avoids M, let $c, x \in S$ and assume c is an idempotent. If $cx = xc$, then $cx_\ell = x_\ell c$.*

***Proof*.** Obviously $xc = cx$ implies $xcx_\ell = cxx_\ell$. Hence, since $xx_\ell = x$ we get $xcx_\ell = cx$, and therefore $xc = xcx_\ell$. By the definition of $\mathcal{L}^*$ and since $x\mathcal{L}^* x_\ell$, we get $x_\ell c = x_\ell c x_\ell$, with both c and x_ℓ idempotents. By (2.2), we get $cx_\ell = x_\ell c$ as required. □

Lemma 2.6. *Let S be an amiable semigroup which avoids M, let $a, b \in S$ be idempotents and suppose there exists a positive integers m, n, with $m > n$ such that $(ab)^m = (ab)^n$. Then $(ab)^{n+1} = (ab)^n$.*

***Proof*.** Consider the monogenic subsemigroup of S generated by ab. Since $(ab)^m = (ab)^n$ it is finite. Hence it has an idempotent element $(ab)^k$, for some $k \in \mathbb{N}$, with $k \leq m$.

Now $b(ab)^k b = b(ab)^k$ which implies by (2.2) that $b(ab)^k = (ab)^k b = (ab)^k$. Hence $(ab)^{k+1} = a(ab)^k = (ab)^k$.

Since $(ab)^k = (ab)^{k+j}$, for all $j \in \mathbb{N}$, $k \leq m$ and $(ab)^m = (ab)^n$, the result follows. □

Lemma 2.7. *Let S be an amiable semigroup which avoids M, let $a, b \in S$ be idempotents and let $x \in S$ be such that $ax = x = xb$, $xab = abx = xabx$ and xab is an idempotent. Then*

$$[xa]_\ell \cdot b \;=\; b \cdot [xa]_\ell\,, \tag{2.7}$$

$$xab \;=\; a \cdot x_\ell\,. \tag{2.8}$$

***Proof*.** We begin by showing that xab commutes with a, b and x, and therefore it commutes with xa. From the hypothesis it is easy to see that $a \cdot xab \cdot a = xab \cdot a$ and $b \cdot xab \cdot b = b \cdot xab$ and so by (2.2) and (2.3) we conclude that xab commutes with a and b. It is immediate from the hypothesis that xab commutes with x.

Now, the first equation goes as follows. Since xab is an idempotent, that is, $xa{\cdot}b = xa{\cdot}b\,xab$, we have

$$[xa]_\ell \cdot b = [xa]_\ell \cdot b\,xab = [xa]_\ell \cdot xab\,b = [xa]_\ell \cdot xab\,.$$

As shown xab commutes with xa. By Lemma 2.5, xab also commutes with $[xa]_\ell$. Thus

$$\begin{aligned}[xa]_\ell \cdot b = xab \cdot [xa]_\ell = ab \cdot x[xa]_\ell \\ \overset{(2.5)}{=} ab \cdot xa = xab\,a = a\,xab = xab\,.\end{aligned}$$

Hence $b \cdot [xa]_\ell \cdot b = b\,xab = xab\,b = xab = [xa]_\ell \cdot b$. Applying (2.3), we have

$$[xa]_\ell \cdot b = b \cdot [xa]_\ell$$

as required.

Next, we compute

$$xab \overset{(2.5)}{=} x[xa]_\ell \cdot b \overset{(2.7)}{=} xb \cdot [xa]_\ell = x[xa]_\ell \overset{(2.5)}{=} xa\,.$$

Hence $x \cdot xab = xab = x \cdot a$, and so $x_\ell \cdot xab = x_\ell \cdot a$. Since xab commutes with x, it also commutes with x_ℓ by Lemma 2.5. Thus

$$x_\ell \cdot a = xab\,x_\ell = ab \cdot xx_\ell = ab \cdot x = xab\,.$$

Now $a \cdot x_\ell \cdot a = a\,xab = xab = x_\ell \cdot a$. By (2.3), $a \cdot x_\ell = x_\ell \cdot a$ from which we get the intended result. □

Lemma 2.8. *Let S be an amiable semigroup which avoids M, let $a, b \in S$ be idempotents and suppose $(ab)^{n+1} = (ab)^n$ for some integer $n > 0$. Then $ab = ba$.*

Proof. If $n = 1$, then the desired result follows from (2.4). If $n > 1$ we will show that then our hypothesis lead to the conclusion that $(ab)^n = (ab)^{n-1}$. Applying repetitively the same argument we reduce to the case $n = 1$ and thus prove our lemma.

Assume that $n > 1$ and let $x = (ab)^{n-1}$. It is easy to verify that a, b and x are in the conditions of the previous lemma. Now, if $n = 2$, then (2.8) is $(ab)^2 = a \cdot [ab]_\ell = ab$ by (2.5). If $n > 2$, then we multiply both sides of (2.8) by $(ab)^{n-2}$ on the left. Since $(ab)^{n-2}(ab)^n = (ab)^n$, we get

$$(ab)^n = (ab)^{n-2}a \cdot [(ab)^{n-2}a \cdot b]_\ell = (ab)^{n-2}a \cdot b = (ab)^{n-1}\,,$$

using (2.5). Therefore we have shown that the assumption that $(ab)^{n+1} = (ab)^n$ implies $(ab)^n = (ab)^{n-1}$ as required. □

Corollary 2.9. *Let S be an amiable semigroup which avoids M, let $a, b \in S$ be idempotents and suppose there exist positive integers m, n, with $m > n$ such that $(ab)^m = (ab)^n$. Then $ab = ba$.*

Proof. This follows from Lemmas 2.6 and 2.8. □

Now let S denote an amiable semigroup which is not adequate and which avoids M. We fix noncommuting idempotents $a, b \in S$ and let H denote the subsemigroup generated by a and b. The elements of H are

$$H = \{(ab)^m, (ba)^m, (ab)^n a, (ba)^n b \mid m \geq 1, n \geq 0\}\,. \tag{2.9}$$

(Note that since S is not necessarily a monoid, we interpret $(ab)^0 a$ to be equal to a and similarly $(ba)^0 b = b$.) Our goal is to show that H is an isomorphic copy of F. Comparing the elements of H with those of F, we see that it is sufficient to show the elements listed in (2.9) are all distinct.

Lemma 2.10. *The elements of H listed in* (2.9) *are all distinct.*

Proof. We show that each possible case of two elements of H coinciding will lead to a contradiction. Because a and b can be interchanged, half of the cases follow from the rest by symmetry. We sometimes use this observation implicitly in the arguments that follow when we refer to already proven cases.

Case 1: If $(ab)^m = (ab)^n$ for some $m > n > 0$, then by Corollary 2.9, $ab = ba$, a contradiction.

Case 2: If $(ab)^m = (ab)^n a$ for some $m > 0$, $n \geq 0$, then $(ba)^{m+1} = b(ab)^m a = b(ab)^n a \cdot a = (ba)^{n+1}$ which, by Case 1, leads to a contradiction if $m \neq n$. Also $(ab)^{n+1} = (ab)^n a \cdot b = (ab)^m b = (ab)^m$ which yields a contradiction by Case 1 if $m \neq n + 1$.

Case 3: If $(ab)^m = (ba)^n$ for some $m, n > 0$, then $(ab)^m = a(ab)^m = a(ba)^n = (ab)^n a$, which contradicts Case 2.

Case 4: If $(ab)^m = (ba)^n b$ for some $m > 0$, $n \geq 0$, we get a contradiction in the same way we got for Case 2.

Case 5: If $(ab)^m a = (ba)^n b$ for some $m, n \geq 0$, then $(ab)^{m+1} = (ab)^m a \cdot b = (ba)^n b \cdot b = (ba)^n b$, which contradicts Case 4.

Case 6: If $(ab)^m a = (ab)^n a$ for some $m > n \geq 0$, then $(ab)^{m+1} = (ab)^{n+1}$ which contradicts Case 1.

By the symmetry in a and b, this exhausts all possible cases of elements of H coinciding. The proof is complete. □

By Lemma 2.10, the semigroup F defined by the presentation $\langle a, b \mid a^2 = a, b^2 = b \rangle$ is the subsemigroup H generated by a and b. This completes the proof of the Main Theorem.

3. Open Problems

A semigroup is *left* abundant if each $\mathcal{R}^*$-class contains an idempotent, *left* amiable if each $\mathcal{R}^*$-class contains exactly one idempotent and *left* adequate if it is left abundant and the idempotents commute. There are more left amiable semigroups which are not left adequate than just F and M. For instance, every right regular band (that is, every idempotent semigroup in which $\mathcal{L}$ is the equality relation) which is not a semilattice is left amiable but not left adequate.

Problem 3.1. *Extend the Main Theorem to characterize left amiable semigroups which are not left adequate.*

In [1] we suggested the problem of characterizing the free objects in the quasivariety of amiable semigroups. Perhaps the following would be more tractable.

Problem 3.2. *Determine the free objects in the quasivariety of amiable semigroups which avoid M.*

By our Main Theorem, if there is a nonadequate free object in this quasivariety, then it would contain a copy of F.

Problem 3.3. *Determine if the class of amiable semigroups which avoid F forms a quasi-variety, and if so, find explicit characterizing identities.*

The ultimate goal regarding amiable semigroups is the following.

Problem 3.4. *Determine to what extent the structure theory of adequate semigroups can be extended to amiable semigroups.*

Finally, the "tilde" Green's relation $\widetilde{\mathcal{L}}$ on a semigroup S is defined by $a\widetilde{\mathcal{L}}b$ if and only if, for each idempotent $e \in S$, $ae = a$ if and only if $be = b$. The relation $\widetilde{\mathcal{R}}$ is defined dually. We have $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}$ and similarly for the dual relations. A semigroup is *semiabundant* if there is an idempotent in each $\widetilde{\mathcal{L}}$-class and in each $\widetilde{\mathcal{R}}$-class, and a semiabundant semigroup is *semiadequate* if its idempotents commute [5]. A semigroup is *semiamiable* if there is a unique idempotent in each $\widetilde{\mathcal{L}}$-class and in each $\widetilde{\mathcal{R}}$-class. Every semiadequate semigroup is semiamiable. There are one-sided versions of all of these notions as well. It is natural to suggest the following.

Problem 3.5. *Extend the Main Theorem to characterize (left) semiadequate semigroups among (left) semiamiable semigroups.*

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