# Covers of groups definable in o-minimal structures

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**Abstract**

We develop in this paper the theory of covers for locally definable groups in o-minimal structures.

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# 1 Introduction

Throughout this paper, $\mathcal{N}$ will be an $\aleph_1$-saturated o-minimal structure and definable will mean $\mathcal{N}$-definable (possibly with parameters). We will assume the reader's familiarity with the basic theory of o-minimal structures and definable groups in such structures. See for example [vdd] and [p] respectively.

When studying definable groups one often makes use of certain groups which are not definable and are called in the literature $\bigvee$-definable groups (see [pst]). Roughly, these are groups whose underlying sets are unions of definable sets and the graphs of the group operations are unions of definable sets. In a real closed field these sets, when equipped with a natural topology, are called in [dk] locally semi-algebraic spaces. For this reason, we prefer to call $\bigvee$-definable groups locally definable groups since the groups that we will study here will be equipped with a topology such that in the semi-algebraic case they are locally semi-algebraic spaces. Furthermore, as we shall see in Section 2 when we introduce the exact definitions, what we call here a locally definable group is a small modification of what is called in [pst] a $\bigvee$-definable group. In [pst] $\bigvee$-definable groups are defined with a restriction on the size of the parameter set and with no restriction on the size of the cover by definable subsets. Here we require that locally definable groups have a countable subcover by definable subsets. This is not a big restriction since all the important examples are of this form and this constrain allows us to prove many results which otherwise would be impossible to verify.

Let us mention a few examples where locally definable groups have occurred in connection with the theory of definable groups. In [e1], we prove the Lie-Kolchin-Mal'cev theorem for solvable definable groups. This theorem says that given a solvable definable group $G$, the commutator subgroup $G^{(1)}$ of $G$ and the smallest definable subgroup $d(G^{(1)})$ of $G$ containing $G^{(1)}$ are nilpotent. The commutator subgroup $G^{(1)}$ is a locally definable subgroup of $G$. In [pst], Peterzil and Starchenko show that if $G$ is a solvable definable group which is definably compact (the o-minimal analogue of semi-algebraically complete), then $G$ is abelian by finite. The proof of this result given in [pst] uses the groups of definable homomorphisms between definable abelian groups. The group of definable homomorphisms between two definable abelian groups is a locally definable group. In [pps], Peterzil, Pillay and Starchenko use locally definable groups to show that if a definable group is not nilpotent by finite, then the group structure interprets a field. In [ps] (see also [s]), Peterzil and Steinhorn construct certain definably compact, abelian definable groups which are not the direct product of one-dimensional definably compact, abelian definable groups. In a sense, these definable groups are constructed by first giving their o-minimal universal covers and their o-minimal fundamental groups. These o-minimal universal covers and these o-minimal fundamental groups are locally definable groups.

In the paper [e2] we developed the general theory of locally definable groups. The goal of this paper was to generalise the last example and show that the o-minimal universal cover of a definable group or a locally definable group is a locally definable group. In this way, in Section 3 the theory of locally definable covering homomorphisms is developed. The o-minimal universal covering homomorphism $\widetilde{p} : \widetilde{G} \longrightarrow G$ of a definably connected locally definable group $G$ is defined as the inverse limit of the locally definable covering homomorphisms $h : H \longrightarrow G$ of $G$ with $H$ definably connected. The o-minimal fundamental group $\pi(G)$ of $G$ is defined as the kernel of $\widetilde{p}$. Thus we have a short exact sequence

$$1 \to \pi(G) \to \widetilde{G} \stackrel{\widetilde{p}}{\to} G \to 1.$$

It is not immediately clear that this short exact sequence is in the category of locally definable groups. Note also that in arbitrary o-minimal structures, the o-minimal fundamental group as defined above is a completely new object even for definable groups. Nevertheless we show that $\pi(G)$ is always abelian and the following theorem holds (see Theorem 3.15).

**Theorem 1.1** *Let $G$ be a definably connected, abelian locally definable group. Suppose that, for each $k > 0$, $G$ is $k$-divisible and the subgroup of $k$-torsion points of $G$ has dimension zero. Then the following hold:*

*(i) the o-minimal universal covering group $\widetilde{G}$ of $G$ is divisible and torsion free;*

*(ii) the o-minimal fundamental group $\pi(G)$ of $G$ is torsion-free abelian group;*

*(iii) the $k$-torsion subgroup of $G$ is isomorphic to $\pi(G)/k\pi(G)$, for each $k > 0$.*

In section 4, we assume that $\mathcal{N}$ is an o-minimal expansion of a real closed field and show that, for a definably connected locally definable group $G$, the o-minimal fundamental group $\pi(G)$ is isomorphic to the o-minimal fundamental group $\pi_1(G)$ defined using definable paths and definable homotopies as in [bo].

# 2 Locally definable groups

In this section we define locally definable groups and present some of the properties of these groups that we will need in the paper.

**Definition 2.1** A set $Z$ is a *locally definable set over* $A$ where $A \subseteq N$ and $|A| < \aleph_1$ if there is a collection $\{Z_i : i \in I\}$ of definable subsets of $N^n$, all definable over $A$ such that: (i) $Z = \cup\{Z_i : i \in I\}$; (ii) there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $Z = \cup\{Z_i : i \in I_0\}$; (iii) for every $i, j \in I$ there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$.

Given two locally definable sets $X$ and $Z$ over $A$, we say that $X$ is *locally definable subset of* $Z$ *over* $A$ if $X$ is a subset of $Z$.

A map $\alpha : Z \longrightarrow X$ between locally definable sets over $A$ is called a *locally definable map over* $A$ if for every definable subset $V \subseteq Z$ defined over $A$, the restriction $\alpha_{|V}$ is a definable map over $A$.

By saturation, the set $Z$ does not depend on the choice of the collection $\{Z_i : i \in I\}$ in the sense that if $Z = \cup\{Y_j : j \in J\}$ with each $Y_j$ definable over $B$, $|B| < \aleph_1$, then the following hold: (i) every $Y_j$ is contained in some $Z_i$ and (ii) there is $J_0 \subseteq J$ with $|J_0| < \aleph_1$ and $Z = \cup\{Y_j : j \in J_0\}$. *For this reason we will always assume from now on that* $|I| < \aleph_1$.

Let $\alpha : Z \longrightarrow X$ be a locally definable map over $A$ between locally definable sets over $A$ and let $Y$ be a locally definable subset of $X$ over $A$. Then $\alpha(Z)$ is a locally definable set over $A$ and $\alpha^{-1}(Y)$ is a locally definable subset of $Z$ over $A$.

If $Z = \cup\{Z_i : i \in I\}$ is a locally definable set over $A$, we define the *dimension* of $Z$ by $\dim Z = \max\{\dim Z_i : i \in I\}$.

Recall that, by [pst], a group $G = (G, \cdot)$ is a $\bigvee$-*definable group* over $A \subseteq N$, where $|A| < \aleph_1$, if there is a collection $\{Z_i : i \in I\}$ of definable subsets of $N^n$, all definable over $A$ such that: (i) $G = \cup\{Z_i : i \in I\}$; (ii) for every $i, j \in I$ there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$ and (iii) the restriction of the group multiplication to $Z_i \times Z_j$ is a definable map into $N^n$. We modify this definition slightly in the following way.

**Definition 2.2** A group $(G, \cdot)$ is a *locally definable group over* $A$, with $A \subseteq N$ and $|A| < \aleph_1$, if there is a collection $\{Z_i : i \in I\}$ of definable subsets of $N^n$, all definable over $A$ such that: (i) $G = \cup\{Z_i : i \in I\}$; (ii) there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $G = \cup\{Z_i : i \in I_0\}$; (iii) for every $i, j \in I$ there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$ and (iv) the restriction of the group multiplication to $Z_i \times Z_j$ is a definable map into $N^n$.

Given two locally definable groups $H$ and $G$ over $A$, we say that $H$ is *locally definable subgroup of* $G$ *over* $A$ if $H$ is a subgroup of $G$.

A homomorphism $\alpha : G \longrightarrow H$ between locally definable groups over $A$ is called a *locally definable homomorphism over* $A$ if for every definable subset $Z \subseteq G$ defined over $A$, the restriction $\alpha_{|Z}$ is a definable map over $A$.

As in the $\bigvee$-definable case, the locally definable group $G$ over $A$ does not depend on the choice of the collection $\{Z_i : i \in I\}$ in the sense that if $G = \cup\{Y_j : j \in J\}$ with each $Y_j$ definable over $B$, $|B| < \aleph_1$, then by saturation and the definition the following hold: (i) every $Y_j$ is contained in some $Z_i$ and (ii) there is $J_0 \subseteq J$ with $|J_0| < \aleph_1$ and $G = \cup\{Y_j : j \in J_0\}$. *For this reason we will always assume from now on that* $|I| < \aleph_1$. This condition will be used in the proof of one of our main results, namely Theorem 3.6.

Also note that if $\alpha : H \longrightarrow G$ is a locally definable homomorphism over $A$ between locally definable groups over $A$ and if $K$ is a locally definable subgroup of $G$ over $A$, then $\alpha(H)$ is a locally definable group over $A$ and $\alpha^{-1}(K)$ is a locally definable subgroup of $H$ over $A$.

Before we proceed any further we recall the two main examples of locally definable groups.

**Example 2.3** The following are the two main examples of locally definable groups over $A$, with $A \subseteq N$ and $|A| < \aleph_1$.

(1) The locally definable groups over $A$ of dimension zero: Let $\{Z_i : i \in I\}$ be a collection of finite subsets of $N^k$ all of which defined over $A$ such that for all $i, j \in I$ there is $k \in I$ with $Z_i \cup Z_j \subseteq Z_k$ and $(\mathcal{Z}, \cdot)$ is an abstract group, where $\mathcal{Z} = \cup\{Z_i : i \in I\}$, and there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $\mathcal{Z} = \cup\{Z_i : i \in I_0\}$. Then $(\mathcal{Z}, \cdot)$ is a locally definable group over $A$ of dimension zero.

(2) The locally definable groups over $A$ which are the subgroups of (type) definable groups: Let $(G,\cdot)$ be a (type) definable group over $B \subseteq A$; let $\{Z_i : i \in I\}$ be a collection of definable subsets of $G$ all of which defined over $A$ such that for all $i, j \in I$ there is $k \in I$ with $Z_i \cup Z_j \subseteq Z_k$, $(\mathcal{Z},\cdot)$ is a subgroup of $(G,\cdot)$, where $\mathcal{Z} = \cup\{Z_i : i \in I\}$, and there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $\mathcal{Z} = \cup\{Z_i : i \in I_0\}$. Then $(\mathcal{Z},\cdot)$ is a locally definable group over $A$.

The proof of [pst] Proposition 2.2 also show the following theorem.

**Theorem 2.4** *Let $G \subseteq N^k$ be a locally definable group over $A$. Then there is a uniformly definable family $\{V_s : s \in S\}$ of definable subsets of $G$ defined over $A$ and containing the identity element of $G$ and there is a unique topology $\tau$ on $G$ such that: (i) $\{V_s : s \in S\}$ is a basis for the $\tau$-open neighbourhoods of the identity element of $G$; (ii) $(G,\tau)$ is a topological group and (iii) every generic element of $G$ has an open definable neighbourhood $U \subseteq N^k$ such that $U \cap G$ is $\tau$-open and the topology which $U \cap G$ inherits from $\tau$ agrees with the topology it inherits from $N^k$.*

In Theorem 2.4, by a uniformly definable family $\{V_s : s \in S\}$ of definable subsets of $G$ defined over $A$ we mean that $S$ is definable over $A$ and there is a definable subset of $N^k \times S$ over $A$ such that the fiber over $s$ is $V_s$ for each $s \in S$.

As in [pst] Lemma 2.6 we see that the following result holds.

**Theorem 2.5** *Let $G$ be a locally definable group over $A$ and $H$ a locally definable subgroup of $G$ over $A$. Then the following holds: (i) the $\tau$-topology on $H$ is the subspace topology induced by the $\tau$-topology on $G$; (ii) $H$ is closed in $G$ in the $\tau$-topology and (iii) $H$ is open in $G$ in the $\tau$-topology if and only if $\dim H = \dim G$.*

The proof of [pst] Lemma 2.8 gives the following theorem.

**Theorem 2.6** *Any locally definable homomorphism between locally definable groups is continuous with respect to the $\tau$-topology.*

Theorems 2.4, 2.5 and 2.6 are called property (TOP) for locally definable groups since they generalise the corresponding property for definable groups.

*From now on, whenever we use topological notions on a locally definable group we are referring to the $\tau$-topology.*

The following easy result will be useful later.

**Lemma 2.7** *Let $G$ be a locally definable group over $A$. Then there exists a collection $\{X_i : i \in I\}$ of open definable subsets of $G$ over $A$ such that: (i) $G = \cup\{X_i : i \in I\}$; (ii) there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $G = \cup\{X_i : i \in I_0\}$; (iii) for every $i, j \in I$ there is $k \in I$ such that $X_i \cup X_j \subseteq X_k$.*

**Proof.** Let $\{Z_i : i \in I\}$ be the collection of definable subsets of $G$ over $A$ such that: (i) $G = \cup\{Z_i : i \in I\}$; (ii) there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $G = \cup\{Z_i : i \in I_0\}$; (iii) for every $i, j \in I$ there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$. Let $U$ be an open definable neighbourhood in $G$ of $e_G$ over $A$. For each $i \in I$, let $X_i$ be the open definable subset $UZ_i$ of $G$ over $A$. Since, for each $i \in I$, we have $Z_i \subseteq X_i$, then the collection $\{X_i : i \in I\}$ satisfies the lemma. $\square$

We will now introduce the notion of compatible locally definable subsets of a locally definable group. This notion will be very useful later.

**Definition 2.8** Let $G$ be a locally definable group over $A$ and let $H$ be a locally definable subgroup (resp., subset) of $G$ over $A$. We say that $H$ is a *compatible locally definable subgroup (resp., subset)* if for every open definable subset $U$ of $G$ over $A$, the set $H \cap U$ is a definable subset of $G$ over $A$.

For example, if $H$ is a definable subgroup (resp., subset) of $G$ over $A$, then $H$ is a compatible locally definable subgroup (resp., subset) of $G$. We now prove some lemmas on the notion of compatible locally definable subsets which will be used quite often later.

**Lemma 2.9** *Let $G$ be a locally definable group over $A$ and let $X$ be a locally definable subset of $G$ over $A$. Then $X$ is compatible if and only if for every definable subset $Z$ of $G$, not necessarily over $A$, the intersection $Z \cap X$ is a definable subset of $G$.*

**Proof.** Suppose that $X$ is a compatible locally definable subset of $G$ over $A$ and let $Z$ be a definable subset of $G$. By Lemma 2.7 and saturation, there are open definable subsets $U_1, \ldots, U_l$ of $G$ over $A$ such that $Z \subseteq U_1 \cup \cdots \cup U_l$. But then $Z \cap X = Z \cap (U_1 \cap X \cup \cdots \cup U_l \cap X)$ is definable since each $U_i \cap X$ is definable. The converse is clear. $\square$

**Lemma 2.10** *Let $G$, $H$ and $K$ be locally definable groups over $A$. The following hold:*

*(i) $G$ is a compatible locally definable subgroup of $G$ over $A$;*

*(ii) if $K$ is a compatible locally definable subgroup of $G$ over $A$ and $H$ is a locally definable subgroup of $G$ over $A$ containing $K$, then $K$ is a compatible locally definable subgroup of $H$ over $A$;*

*(iii) if $K$ is a compatible locally definable subgroup of $H$ over $A$ and $H$ is a compatible locally definable subgroup of $G$ over $A$, then $K$ is a compatible locally definable subgroup of $G$ over $A$.*

**Proof.** (i) is obvious. For (ii), let $U$ be an open definable subset of $H$ over $A$. Then, by Lemma 2.7 and saturation, there is an open definable subset $V$ of $G$ over $A$ such that $U \subseteq V$. But then $U \cap K = U \cap (V \cap K)$ is definable over $A$. For (iii), let $U$ be an open definable subset of $G$ over $A$. Then $U \cap H$ is an open definable subset of $H$ over $A$. Hence $U \cap K = (U \cap H) \cap K$ is definable over $A$. $\square$

**Lemma 2.11** *Let $\alpha : G \longrightarrow H$ be a locally definable map over $A$ between locally definable groups over $A$. If $Y$ is a compatible locally definable subset of $H$ over $A$, then $\alpha^{-1}(Y)$ is a compatible locally definable subset of $G$ over $A$.*

**Proof.** Let $Z$ be an open definable subset of $G$ over $A$. Then $\alpha(Z)$ is a definable subset of $H$ over $A$ and, since there is a uniformly definable basis $\{V_s : s \in S\}$ for the $\tau$-open neighbourhoods of the identity element of $H$, there is an open definable subset $X$ of $H$ over $A$ such that $\alpha(Z) \subseteq X$. But clearly $Z \cap \alpha^{-1}(Y) = \alpha_{|Z}^{-1}(X \cap Y)$. Thus, since $Y$ is compatible, $X \cap Y$ is definable. Hence $\alpha_{|Z}^{-1}(X \cap Y)$ is definable since $\alpha_{|Z}$ is definable. $\square$

**Lemma 2.12** *Suppose that $f, g : G \longrightarrow H$ are locally definable maps over $A$ between locally definable groups over $A$. Then $\{x \in G : f(x) = g(x)\}$ is a compatible locally definable subset of $G$ over $A$.*

**Proof.** Consider the locally definable map $\alpha : G \longrightarrow H$ given by $\alpha(x) = f(x)g(x)^{-1}$. By Lemma 2.11, $\{x \in G : f(x) = g(x)\} = \alpha^{-1}(e_H)$ is a compatible locally definable subset of $G$ over $A$. $\Box$

Lemmas 2.10 and 2.11 will be used quite often in the paper without mentioning it.

**Lemma 2.13** *Let $G$ be a locally definable group over $A$. If $H$ is a compatible locally definable subgroup of $G$ over $A$ and $X$ an open definable subset of $G$ over $A$, then the equivalence relation on $X$ given by $x \simeq y$ if and only if $xH = yH$ is definable over $A$.*

**Proof.** Let $\theta : X \times X \longrightarrow \theta(X \times X)$ be the map given by $\theta(x, y) = x^{-1}y$. Then, by definition of locally definable groups and saturation, $\theta$ is a definable map over $A$ and $\theta(X \times X)$ is an open definable subset over $A$. Since $H$ is a compatible locally definable subgroup of $G$ over $A$, the set $Z = \theta(X \times X) \cap H$ is a definable subset of $H$ over $A$. But, for all $x \in X$, we have $xH \cap X = xZ \cap X$. Thus the equivalence relation on $X$ given by $x \simeq y$ if and only if $xH = yH$ is definable since $x \simeq y$ if and only if there is $z \in Z$ such that $y = xz$. $\Box$

The next result is the generalization of [pst] Lemma 2.15 (i).

**Proposition 2.14** *Let $G$ be a locally definable group over $A$ and let $H$ be a compatible locally definable subgroup of $G$ over $A$. Then the following are equivalent: (i) $H$ is open in $G$; (ii) $\dim H = \dim G$ and (iii) $(G : H) < \aleph_1$.*

**Proof.** By Theorem 2.5, $H$ is open in $G$ if and only if $\dim H = \dim G$. On the other hand, if $(G : H) < \aleph_1$, then by compactness we clearly have $\dim H = \dim G$.

Suppose that $\dim H = \dim G$. We must show that $(G : H) < \aleph_1$, i.e., we must show that there is a locally definable subset $\{z_s : s \in S\}$ of $G$ over $A$ such that $G = \cup\{z_s H : s \in S\}$.

Let $Z$ be an open definable subset of $G$ over $A$. We must show that $Z$ is covered by finitely many cosets of $H$ all defined over $A$. By Lemma 2.13, the equivalence relation on $Z$ given by $x \simeq y$ if and only if $xH = yH$ is definable over $A$. But since $xH = yH$ if and only if $xH \cap Z = yH \cap Z$, we see that the equivalence classes of $\simeq$ in $Z$ have dimension $\dim H = \dim G$. Therefore, there are finitely many equivalence classes of $\simeq$ in $Z$ for otherewise, by [vdd] Chapter IV (1.5), the definable set $Z$ would have dimension greater than $\dim G$, which is a contradiction. So there are finitely many elements $u_1, \dots, u_{r_Z}$ of $Z$ defined over $A$ such that $Z \subseteq \cup\{u_l H : l = 1, \dots, r_Z\}$.

Let $\{V_j : j \in J\}$ be the collection of open definable subsets of $G$ over $A$ given by Lemma 2.7. Let $S = \{(j, l) : j \in J,\ l = 1, \dots, r_{V_j}\}$ and for $s = (j, l) \in S$, let $z_s$ be the element $u_l$ obtained as above with $Z = V_j$. Then by Lemma 2.7, $G = \cup\{z_s H : s \in S\}$. Also $\{z_s : s \in S\}$ is a locally definable subset of $G$ over $A$ since each $z_s$ is defined over $A$ and $\{z_s : s \in S\}$ is the union of the collection of all finite subsets of $\{z_s : s \in S\}$. $\Box$

The following corollary of the proof of Proposition 2.14 will be used quite often.

**Corollary 2.15** *Let $G$ be a locally definable group over $A$ and let $H$ be a compatible locally definable subgroup of $G$ over $A$. If $(G : H) < \aleph_1$, then there is a locally definable subset $\{z_s : s \in S\}$ of $G$ over $A$ such that $G = \cup\{z_s H : s \in S\}$ (disjoint union).*

The following definition is the analogue of [pst] Definition 2.12.

**Definition 2.16** Let $G$ be a locally definable group over $A$. We say that a set $Z \subseteq G$ is *definably connected* if there is no definable subset $U \subseteq G$ over $A$ such that $U \cap Z$ is a nonempty proper subset of $Z$ which is closed and open in the topology induced on $Z$ by $G$.

The next remark can be proved in exactly the same way as [pst] Lemmas 2.13 and 2.14.

**Remark 2.17** Let $G$ be a locally definable group over $A$. Then the following hold:

(1) Every definable open subset $Z \subseteq G$ over $A$ can be partitioned into finitely many definably connected definable subsets of $G$ over $A$.

(2) There is a locally definable subgroup $G'$ of $G$ over $A$ which is definably connected and such that $\dim G' = \dim G$.

As pointed out in [pst], the definably connected locally definable subgroups given by Remark 2.17 (2) are not unique. In fact, let $\mathcal{N}$ be a non standard model of the theory of the ordered additive group of real numbers, $G = (N^2, +)$, $G' = \{(x, y) \in N^2 : \text{there exists } n \in \mathbb{N} \text{ such that } -n < x < n\}$ and $G'' = \{(x, y) \in N^2 : \text{there exists } n \in \mathbb{N} \text{ such that } -n < y < n\}$. Then $G'$ and $G''$ are two distinct definably connected locally definable subgroups of $G$ over $\mathbb{Z}$.

Nevertheless, we have the following generalization of [pst] Lemma 2.15 (iii).

**Proposition 2.18** *Let $G$ be a locally definable group over $A$. Then there is a unique definably connected compatible locally definable normal subgroup $G^0$ of $G$ over $A$ with dimension* $\dim G$. *Moreover, the following hold:*

*(i) $G^0$ contains all definably connected locally definable subgroups of $G$ over $A$;*

*(ii) $G^0$ is the smallest compatible locally definable subgroup of $G$ over $A$ such that $(G : G^0) < \aleph_1$*

*(iii) there is a locally definable subset $\{x_s : s \in S\}$ of $G$ over $A$ such that $G = \cup\{x_s G^0 : s \in S\}$ (disjoint union).*

**Proof.** By Lemma 2.7, let $\{Z_k : k \in K\}$ be a collection of open definable subsets of $G$ over $A$ such that: (i) $G = \cup\{Z_k : k \in K\}$; (ii) there is $K_0 \subseteq K$ with $|K_0| < \aleph_1$ and $G = \cup\{Z_k : k \in K_0\}$; (iii) for every $i, j \in K$ there is $k \in K$ such that $Z_i \cup Z_j \subseteq Z_k$. By definition of locally definable groups, we may assume that each $Z_k$ contains the identity 1 of $G$. By Remark 2.17 (1), each $Z_k$ can be partitioned into finitely many definably connected components. For each such $Z_k$, let $Z_k^0$ be the definably connected component of $Z_k$ which contains 1.

We claim that $G^0 = \cup\{Z_k^0 : k \in K\}$ is a compatible locally definable subgroup of $G$ over $A$. Indeed, given $i, j \in K$, we have $Z_i \cup Z_j \subseteq Z_k$ for some $k \in K$, hence $Z_i^0 \cup Z_j^0 \subseteq Z_k$. But $Z_i^0 \cup Z_j^0$ is a definably connected set which contains 1, hence it must be contained in $Z_k^0$. Similarly, $Z_i^0 \cdot Z_j^0$ and $(Z_i^0)^{-1}$

are contained in some $Z_k^0$. Thus $G^0$ is a locally definable subgroup of $G$ over $A$ which, by construction, is obviously compatible, definably connected and $\dim G^0 = \dim G$.

By Proposition 2.14, we have $(G : G^0) < \aleph_1$ and so, by Corollary 2.15, $G = \cup\{z_s G^0 : s \in S\}$ (disjoint union) for some locally definable subset $\{z_s : s \in S\}$ of $G$ over $A$. Thus to show that $G^0$ is normal, it is enough to show that for each $z_s$ with $s \in S$, $z_s G^0 (z_s)^{-1} = G^0$. But this is obvious since, for every $Z_i^0$, the open definably connected definable set $z_s Z_i^0 (z_s)^{-1}$ over $A$ containing the identity of $G$ is necessarily contained in a set of the form $Z_j^0$.

As $G = \cup\{z_s G^0 : s \in S\}$ (disjoint union) for some locally definable subset $\{z_s : s \in S\}$ of $G$ over $A$, we see that $G^0$ contains all definably connected locally definable subgroups of $G$ over $A$.

By Proposition 2.14, if $H$ is a compatible locally definable subgroup of $G$ such that $(G : H) < \aleph_1$, then $\dim H = \dim G$. Let $H^0$ be the compatible, definably connected locally definable subgroup of $H$ over $A$ obtained from $H$ in the same way as we obtained $G^0$ from $G$. Then, by Lemma 2.10 (iii), $H^0$ is a compatible definably connected locally definable subgroup of $G$ over $A$ such that $\dim H^0 = \dim G$ and so, by (i), $H^0 \subset G^0$. Hence, by Proposition 2.14, $H^0$ is open in $G^0$. Therefore, again by Proposition 2.14, we have $(G^0 : H^0) < \aleph_1$ and so, by Corollary 2.15, $G^0 = \cup\{z_s H^0 : s \in S\}$ (disjoint union) for some locally definable subset $\{z_s : s \in S\}$ of $G^0$ over $A$. But both $H^0$ and $G^0$ are definably connected, so $|S| = 1$, $H^0 = G^0$ and $G^0 \subseteq H$. □

# 3 Locally definable covering groups

In this section we present the theory of locally definable covering groups. We will often follow the topological case on Fulton's book [f] and the definable case treated in [eo].

## 3.1 Locally definable covering homomorphisms

**Definition 3.1** A locally definable homomorphism $p : H \longrightarrow G$ over $A$ between locally definable groups over $A$ is called a *locally definable covering homomorphism* if $p$ is surjective and there is a family $\{U_l : l \in L\}$ of open definable subsets of $G$ over $A$ such that $G = \cup\{U_l : l \in L\}$ and, for each $l \in L$, the compatible locally definable subset $p^{-1}(U_l)$ of $H$ over $A$ is a disjoint union of open definable subsets of $H$ over $A$, each of which is mapped homeomorphically by $p$ onto $U_l$.

We call $\{U_l : l \in L\}$ a *$p$-admissible family of definable neighbourhoods* over $A$.

If $p : H \longrightarrow G$ is a locally definable covering homomorphism over $A$, then for each $x \in G$, the fibre $p^{-1}(x)$ is a compatible locally definable subset of $H$ over $A$ of dimension zero. Furthermore, $p : H \longrightarrow G$ is an open surjection. In fact, let $V$ be an open definable subset of $H$ over $A$ and, for each $l \in L$, let $\{U^l_s : s \in S_l\}$ be the collection of open disjoint definable subsets of $H$ over $A$ such that $p^{-1}(U_l) = \cup\{U^l_s : s \in S_l\}$ and $p_{|U^l_s} : U^l_s \longrightarrow U_l$ is a definable homeomorphism over $A$ for every $s \in S_l$. Since $|A| < \aleph_1$, by saturation, there is $\{W_1, \ldots, W_m\} \subseteq \{U^l_s : l \in L,\ s \in S_l\}$ such that $V \subseteq \cup\{W_i : i = 1, \ldots, m\}$. But then $V = \cup\{V \cap W_i : i = 1, \ldots, m\}$ and $p(V) = \cup\{p(V \cap W_i) : i = 1, \ldots, m\}$ is open.

If $q : K \longrightarrow H$ is another locally definable covering homomorphism over $A$, then $p \circ q : K \longrightarrow G$ is also a locally definable covering homomorphism over $A$. Indeed, if $\{V_r : r \in R\}$ is a $q$-admissible family of open neighbourhoods of $H$ over $A$, then $\{p(V_r \cap U^l_s) : r \in R,\ l \in L,\ s \in S_l\}$ is a $p \circ q$-admissible family of open neighbourhoods of $G$ over $A$.

The next remark follows from Lemma 2.12 and the argument in the proof of [f] Lemma 11.5.

**Lemma 3.2** *Let $p : H \longrightarrow G$ be a locally definable covering homomorphism over $A$ and let $f, g : K \longrightarrow H$ be two continuous locally definable maps over $A$ between locally definable groups over $A$ such that $p \circ f = p \circ g$. If $K$ is definably connected and $f(z) = g(z)$ for some $z \in K$, then $f = g$.*

**Proof.** Let $\{U_l : l \in L\}$ be a $p$-admissible family of open neighbourhoods of $G$ over $A$. Let $\{U^l_s : s \in S_l\}$ be the collection of open disjoint definable subsets of $H$ over $A$ such that $p^{-1}(U_l) = \cup\{U^l_s : s \in S_l\}$ and $p_{|U^l_s} : U^l_s \longrightarrow U_l$ is a definable homeomorphism over $A$ for every $s \in S_l$.

By continuity, there is a cover of $K$ by a family $\{V^l_k : l \in L,\ k \in M_l\}$ of open definably connected definable subsets of $K$ over $A$ such that we have $(p \circ f)(V^l_k) = (p \circ g)(V^l_k) \subseteq U_l$ for every $l \in L$ and $k \in M_l$. Note that if $v \in V^l_k$

and $f(v) \in U_s^l$, then $f_{|V_k^l} = (p_{|U_s^l})^{-1} \circ (p \circ f)_{|V_k^l}$. Thus, by Lemma 2.12, the set $\{z \in K : f(z) = g(z)\}$ is an open and closed compatible locally definable subset of $K$ over $A$. Since $K$ is definably connected, we have $f = g$. $\Box$

**Definition 3.3** Consider a locally definable covering homomorphism $p : H \longrightarrow G$ over $A$. The group $\mathrm{Aut}(H/G)$ of *locally definable covering transformations over* $A$ is the group of all locally definable homeomorphisms $\phi : H \longrightarrow H$ over $A$ such that $p \circ \phi = p$.

Note that $\mathrm{Aut}(H/G)$ acts on $p^{-1}(e_G)$ and, if $H$ is definably connected, then by Lemma 3.2, $\phi \in \mathrm{Aut}(H/G)$ is uniquely determined by $\phi(e_H)$.

**Proposition 3.4** *Let $h : H \longrightarrow G$ be a locally definable covering homomorphism over $A$. Suppose that $H$ is definably connected. Then*

$$\mathrm{Ker} h \simeq \mathrm{Aut}(H/G)$$

*and* $\mathrm{Aut}(H/G)$ *is abelian.*

**Proof.** For $y \in \mathrm{Ker}h$, let $l_y : H \longrightarrow H$ be the locally definable homeomorphism over $A$ given by $l_y(z) = yz$. Clearly $l_y \in \mathrm{Aut}(H/G)$ and this correspondence determines an injective homomorphism $\mathrm{Ker}h \longrightarrow \mathrm{Aut}(H/G)$. We now show that this homomorphism is also surjective. Take $\phi \in \mathrm{Aut}(H/G)$ and fix $z \in H$. As $h = h \circ \phi$, there is $y \in \mathrm{Ker}h$ such that $l_y(z) = yz = \phi(z)$. But then by Lemma 3.2, we have $\phi = l_y$.

We now show that $\mathrm{Ker}h \subseteq Z(H)$, from which it follows that $\mathrm{Aut}(H/G)$ is abelian. Let $y \in \mathrm{Ker}h$. Then we have a locally definable map $\sigma_y : H \longrightarrow \mathrm{Ker}h$ over $A$ given by $\sigma_y(x) = xyx^{-1}$ for every $x \in H$. Since $\mathrm{Ker}h$ has dimension zero, by Lemma 2.11, $(\sigma_y)^{-1}(y)$ is an open and closed compatible locally definable subset of $H$ over $A$ containing $e_H$. Since $H$ is definably connected, we have $H = (\sigma_y)^{-1}(y)$ and the result follows. $\Box$

The analogue of the previous and the next result for definable covering homomorphisms is proved in [eo]. The arguments are quite similar. But before we need the following lemma.

**Lemma 3.5** *Let $H$ be a locally definable group over $A$ and $V$ an open compatible locally definable subset of $H$ over $A$ such that* $\dim H \setminus V < \dim H$. *Then there is a set $\{y_s \in H : s \in S\}$ whose elements are defined over $A$ such that $H = \cup\{y_s V : s \in S\}$*

**Proof.** Let $\mathcal{K}$ be the prime model of $\mathrm{Th}_A(\mathcal{N})$ and suppose that $\{X_i : i \in I\}$ (resp., $\{V_j : j \in J\}$) is the collection of all open definable subsets of $H$ (resp., $V$) over $A$. Let $\mathcal{M}$ be a sufficiently saturated model of $\mathrm{Th}_A(\mathcal{N})$, $i \in I$, $a \in X_i(M)$ and let $c \in X_i(M)$ be a generic point over $K$ such that $\mathrm{tp}(c/Ka)$ is finitely satisfiable in $K$. Then $c$ is a generic point of $X_i(M)$ over $Ka$ (see the proof of [p] Lemma 2.4). Since $\dim H \setminus V < \dim H$, we also have that $\dim X_i \setminus (X_i \cap V) < \dim X_i$. Thus $(X_i \cap V)(M)$ is a large definable subset of $X_i(M)$ over $A$, i.e., every generic point of $X_i(M)$ over $A$ is contained in $(X_i \cap V)(M)$. But we also have that $\dim H \setminus Va^{-1} < \dim H$. So $X_i(M) \cap V(M)a^{-1}$ is a large definable subset of $X_i(M)$ over $A \cup \{a\}$ (to see that $X_i \cap Va^{-1}$ is a definable set, we use the fact that by Lemma 2.9, $X_i a \cap V$ is a definable set). Therefore, by definition, $c \in V_j(M)a^{-1}$ and $a \in c^{-1}V_j(M)$ for some $j \in J$. Since $\mathrm{tp}(c/Ka)$ is finitely satisfiable over $K$, there is $b \in X_i(K)$ such that $a \in b^{-1}V_j(M)$ for some $j \in J$. Therefore, by the compactness theorem, for each $i \in I$, there are $b_1, \dots, b_{r_i} \in X_i(K)$ and $j_1, \dots, j_{r_i} \in J$ such that, for every $a \in X_i(M)$, we have $a \in \cup\{(b_l)^{-1}V_{j_l}(M) : l = 1, \dots, r_i\}$. □

**Theorem 3.6** *Let $h : H \longrightarrow G$ be a surjective locally definable homomophism over $A$ between locally definable groups over $A$. If* $\mathrm{Ker}h$ *has dimension zero, then $h : H \longrightarrow G$ is a locally definable covering homomorphism over $A$.*

**Proof.** We start by proving that there exists a locally definable map $\alpha : G \longrightarrow H$ over $A$ such that $h \circ \alpha = 1_G$.

Let $\{X_i : i \in I\}$ (resp., $\{Y_j : j \in J\}$) be the collection of all open definable subsets of $H$ (resp., $G$) over $A$. Fix $j \in J$. Since $\{h(X_i) : X_i \subseteq h^{-1}(Y_j)\}$ is a cover of $Y_j$ by definable subsets over $A$, by saturation, there is a finite subset $I(j)$ of $I$ such that $\{h(X_i) : i \in I(j)\}$ is a cover of $Y_j$ by definable subsets over $A$. Let $U_j = \cup\{X_i : i \in I(j)\}$. Then $h^{-1}(y) \cap U_j$ is finite for every $y \in Y_j$. Moreover, since by o-minimality, for each $i \in I(j)$, there is a uniform bound for $|h^{-1}(y) \cap X_i|$, there is a uniform bound for $|h^{-1}(y) \cap U_j|$. So there is a definable map $s_j : Y_j \longrightarrow U_j$ over $A$ such that $h_{|U_j} \circ s_j = 1_{Y_j}$.

Let $\kappa \leq \omega$ be an enumeration of $J$ and without loss of generality we may assume that $\{Y_j : j \in \kappa\}$ is an increasing sequence. We define the locally definable map $\alpha : G \longrightarrow H$ of the claim in the following way. Put $\alpha_{|Y_0} = s_0$; suppose that $\alpha_{|Y_j}$ has been defined, then we define $\alpha_{|Y_{j+1}}$ by $\alpha_{|Y_{j+1}}(y) = \alpha_{|Y_j}(y)$ for $y \in Y_j$ and $\alpha_{|Y_{j+1}}(y) = s_{j+1}(y)$ for $y \in Y_{j+1} \setminus Y_j$. Clearly, by construction, we have $h \circ \alpha = 1_G$.

Let $\alpha : G \longrightarrow H$ be the locally definable map over $A$ given above. For each $a \in \mathrm{Ker}h$, let $\alpha_a : G \longrightarrow H$ be the locally definable map given by $\alpha_a(x) = a\alpha(x)$. Then for each $x \in G$ we have $h^{-1}(x) = \{\alpha_a(x) : a \in \mathrm{Ker}h\}$. For $a \in \mathrm{Ker}h$, let $S_a = \alpha_a(G)$. Then $S_a \cap S_b = \emptyset$ if and only if $a \neq b$. In fact, $y \in S_a \cap S_b$ if and only if there are $u, v \in G$ such that $\alpha_a(u) = y = \alpha_b(v)$ if and only if $u = h(y) = v$ if and only if $a = b$.

By o-minimality there is an open compatible locally definable subset $U$ of $G$ such that $\dim G \setminus U < \dim G$ and $\alpha : U \longrightarrow H$ is continuous. But then $\alpha_a : U \longrightarrow H$ is continuous for each $a \in \mathrm{Ker}h$. By the above and Lemma 2.11, $V = h^{-1}(U)$ is an open compatible locally definable subset of $H$ which is a disjoint union of open locally definable subsets $V_a = \alpha_a(U)$ with $a \in \mathrm{Ker}h$ such that $h_{|V_a} : V_a \longrightarrow U$ is a locally definable homeomorphism over $A$ for every $a \in \mathrm{Ker}h$. Since $h(H \setminus h^{-1}(U)) \subseteq G \setminus U$ and $\dim\mathrm{Ker}h = 0$, we also have $\dim H \setminus V < \dim H$.

By Lemma 3.5 there is a set $\{y_s \in H : s \in S\}$ over $A$ such that $H = \cup\{y_s V : s \in S\}$. Hence, it follows that $G = \cup\{h(y_s)U : l = s \in S\}$. So, for each $s \in S$, we have that $h^{-1}(h(y_s)U)$ is the disjoint union of the open locally definable sets $y_s V_a$ with $a \in \mathrm{Ker}h$. Furthermore, $h_{|y_s V_a} : y_s V_a \longrightarrow h(y_s)U$ is a locally definable homeomorphism for every $a \in \mathrm{Ker}h$. Hence, $h : H \longrightarrow G$ is a locally definable covering homomorphism over $A$. □

**Corollary 3.7** *Suppose that $h : H \longrightarrow G$ is a locally definable covering homomophism over $A$. If $G$ is definably connected, then $h_{|H^0} : H^0 \longrightarrow G$ is a locally definable covering homomorphism over $A$.*

**Proof.** Clearly, $\mathrm{Ker}h_{|H^0}$ has dimension zero. So by Theorem 3.6 we need to show that $h_{|H^0} : H^0 \longrightarrow G$ is surjective. Note that by Proposition 2.18 (iii), there is a locally definable subset $\{x_s : s \in S\}$ of $H$ over $A$ such that $H =$

$\cup\{x_sH^0 : s \in S\}$ (disjoint union). Let $s,t \in S$ and suppose that $h(x_tH^0) \cap h(x_sH^0) \neq \emptyset$. Then there are $u,v \in H^0$ such that $h(x_t)h(u) = h(x_s)h(v)$. Thus $h(x_s) = h(x_t)h(uv^{-1}) \in h(x_tH^0)$ and $h(x_t) = h(x_s)h(vu^{-1}) \in h(x_sH^0)$. Hence, $h(x_sH^0) = h(x_tH^0)$. So there is a subset $S_0 \subset S$ such that $G = \cup\{h(x_s)h(H^0) : s \in S_0\}$ (disjoint union). Since $h$ is an open mapping, $h(H^0)$ and so each $h(x_s)h(H^0)$ is open in $G$. Thus each $h(x_s)h(H^0)$ is open and closed in $G$. If $h_{|H^0} : H^0 \longrightarrow G$ is not surjective, then $|S_0| > 1$ and this implies that $G$ is not definably connected. In fact, there is a definable subset $U$ of $G$, which can only be covered by more than one of the sets $h(x_s)h(H^0)$ with $s \in S_0$. By saturation, there are $s_1,\dots,s_l \in S_0$ and definable subsets $U_i \subseteq h(x_{s_i})h(H^0)$ with $i = 1,\dots,l$, such that $U$ is the disjoint union $(U \cap U_1) \cup \cdots \cup (U \cap U_l) = (U \cap h(x_{s_1})h(H^0)) \cup \cdots \cup (U \cap h(x_{s_l})h(H^0))$ of open and closed subsets of $U$. $\Box$

## 3.2 The universal covering homomorphism

The goal now is to show that for a definably connected locally definable group $G$ there exists a universal covering homomorphism $\widetilde{p} : \widetilde{G} \longrightarrow G$ of $G$. Universal here means that if $h : H \longrightarrow G$ is a locally definable covering homomophism (over some $A$ with $|A| < \aleph_1$), then there exists a covering homomorphism $\widetilde{p}_{(H,h)} : \widetilde{G} \longrightarrow H$ such that $\widetilde{p} = h \circ \widetilde{p}_{(H,h)}$.

We denote by $\mathrm{Cov}(G)$ the category whose objects are locally definable covering homomorphisms $p : H \longrightarrow G$ (over some $A$ with $|A| < \aleph_1$) and whose morphisms are surjective locally definable homomorphisms $r : H \longrightarrow K$ (over some $A$ with $|A| < \aleph_1$) such that $q \circ r = p$, where $q : K \longrightarrow G$ is a locally definable covering homomorphism (over some $A$ with $|A| < \aleph_1$). Let $p : H \longrightarrow G$ and $q : K \longrightarrow G$ be locally definable covering homomorphisms. If $r : H \longrightarrow K$ is a morphism in $\mathrm{Cov}(G)$, then by Theorem 3.6, $r : H \longrightarrow K$ is a locally definable covering homomorphism. Here is another immediate consequence of Theorem 3.6.

**Lemma 3.8** *If $p : H \longrightarrow G$ and $q : K \longrightarrow G$ are locally definable covering homomorphisms over $A$, then $p \times_G q : H \times_G K \longrightarrow G$, $\pi_H : H \times_G K \longrightarrow H$ and $\pi_K : H \times_G K \longrightarrow K$, where $\pi_H$ and $\pi_K$ are the obvious projections, are locally definable covering homomorphisms over $A$ such that $p \times_G q = p \circ \pi_H = q \circ \pi_K$.*

Lemma 3.8 shows that $\mathrm{Cov}(G)$ is an inverse system. Given $G$ a definably connected locally definable group, we denote by $\mathrm{Cov}^0(G)$ the full subcategory of $\mathrm{Cov}(G)$ whose objects are locally definable covering homomorphisms $p : H \longrightarrow G$ with $H$ definably connected. Corollary 3.7 and Lemma 3.8 shows that $\mathrm{Cov}^0(G)$ is an inverse system.

**Definition 3.9** Let $G$ be a definably connected locally definable group. The inverse limit $\widetilde{p} : \widetilde{G} \longrightarrow G$ of the inverse system $\mathrm{Cov}^0(G)$ is called the *(o-minimal) universal covering homomorphism of* $G$.

The kernel of the universal covering homomorphism $\widetilde{p} : \widetilde{G} \longrightarrow G$ of $G$ is called the *(o-minimal) fundamental group of* $G$ and is denoted by $\pi(G)$.

By definition of inverse limit, the elements of $\widetilde{G}$ are sequences $x = (x_{(K,k)})$ with $x_{(K,k)} \in K$, $k : K \longrightarrow G \in \mathrm{Cov}^0(G)$ such that $l(x_{(L,k\circ l)}) = x_{(K,k)}$ for all $l : L \longrightarrow K \in \mathrm{MorCov}^0(G)$. In $\widetilde{G}$ the product $z = xy$ of $x = (x_{(K,k)})$ and $y = (y_{(K,k)})$ is given by the sequence $(z_{(K,k)})$ with $z_{(K,k)} \in K$, $k : K \longrightarrow G \in \mathrm{Cov}^0(G)$ such that $z_{(K,k)}$ is the product $x_{(K,k)}y_{(K,k)})$ of the elements $x_{(K,k)}$ and $y_{(K,k)}$ in $K$ with the same $k : K \longrightarrow G \in \mathrm{Cov}^0(G)$ as for $z_{(K,k)}$. If $h : H \longrightarrow G \in \mathrm{Cov}^0(G)$, then there exists a covering homomorphism $\widetilde{p}_{(H,h)} : \widetilde{G} \longrightarrow H$ such that $\widetilde{p} = h\circ\widetilde{p}_{(H,h)}$ and sending the sequence $x = (x_{(K,k)})$ into the element $x_{(H,h)} \in H$.

By construction we have the following theorem.

**Theorem 3.10** *Let $G$ be a definably connected locally definable group. Then we have the following short exact sequence*

$$1 \to \pi(G) \to \widetilde{G} \xrightarrow{\widetilde{p}} G \to 1.$$

We do not know if in general this short exact sequence exists in the category of locally definable groups.

## 3.3 The o-minimal fundamental group

In this subsection we develop the theory of o-minimal fundamental groups of definably connected locally definable groups.

**Proposition 3.11** *Let $G$ be a definably connected locally definable group. Then $\pi(G)$ is abelian.*

**Proof.** As we saw after Definition 3.9, a point $x \in \widetilde{G}$ is a sequence $x = (x_{(K,k)})$ with $x_{(K,k)} \in K$ and $k : K \longrightarrow G \in \mathrm{Cov}^0(G)$ such that for every morphism $l : L \longrightarrow K$ in $\mathrm{Cov}^0(G)$ we have $l(x_{(L,k\circ l)}) = x_{(K,k)}$. Moreover, the product $xy$ in $\widetilde{G}$ of $x = (x_{(K,k)})$ and $y = (y_{(K,k)})$ is given by $(x_{(K,k)}y_{(K,k)})$ with $x_{(K,k)}, y_{(K,k)} \in K$ and $k : K \longrightarrow G \in \mathrm{Cov}^0(G)$. Thus if $x \in \pi(G)$, then $\widetilde{p}_{(K,k)}(x) = x_{(K,k)} \in \mathrm{Ker}(k : K \longrightarrow G)$ for every $k : K \longrightarrow G \in \mathrm{Cov}^0(G)$ since $\widetilde{p} = k \circ \widetilde{p}_{(K,k)}$. So, by Proposition 3.4, $\pi(G)$ is abelian. □

Let $h : H \longrightarrow G$ be a locally definable covering homomorphism. Suppose that $H$ and $G$ are definably connected and let $\widetilde{q} : \widetilde{H} \longrightarrow H$ and $\widetilde{p} : \widetilde{G} \longrightarrow G$ be the o-minimal universal covering homomorphisms. Then we have an isomorphism

$$\widetilde{h} : \widetilde{H} \longrightarrow \widetilde{G}$$

such that for every $K \xrightarrow{k} H \in \mathrm{Cov}^0(H)$ and $K \xrightarrow{h\circ k} G \in \mathrm{Cov}^0(G)$ we have $\widetilde{q}_{(K,k)} = \widetilde{p}_{(K,h\circ k)} \circ \widetilde{h}$, i.e., for $x = (x_{(K,k)}) \in \widetilde{G}$, the coordinate $\widetilde{h}(x)_{(K,h\circ k)}$ of $\widetilde{h}(x)$ is equal to the coordinate $x_{(K,k)}$ of $x$. Since the collection of all $K \xrightarrow{h\circ k} G \in \mathrm{Cov}^0(G)$ for $K \xrightarrow{k} H \in \mathrm{Cov}^0(H)$ is cofinal in $\mathrm{Cov}^0(G)$, the coordinates $\widetilde{h}(x)_{(K,h\circ k)}$ determine the element $\widetilde{h}(x)$. Let us verify why $\widetilde{h} : \widetilde{H} \longrightarrow \widetilde{G}$ is an isomorphism. If $z = (z_{(L,l)}) \in \widetilde{H}$ and $x = (x_{(K,k)}) \in \widetilde{G}$ are elements such that $\widetilde{h}(z) = x$, then $z_{(L,l)} = \widetilde{q}_{(L,l)}(z) = \widetilde{p}_{(L,h\circ l)}(x) = x_{(L,h\circ l)}$. But $x_{(L,h\circ l)} = \pi_L(x_{(K\times_G L,(h\circ l)\circ\pi_L)})$ and so, we have $\pi_L(x_{(K\times_G L,(h\circ l)\circ\pi_L)}) = z_{(L,l)}$ for all $k : K \longrightarrow G \in \mathrm{Cov}^0(G)$ and $l : L \longrightarrow H \in \mathrm{Cov}^0(H)$. Consequently, $\widetilde{h} : \widetilde{H} \longrightarrow \widetilde{G}$ is indeed an isomorphism.

The isomorphism $\widetilde{h} : \widetilde{H} \longrightarrow \widetilde{G}$ restricts to an injective homomorphism $h_* : \pi(H) \longrightarrow \pi(G)$.

**Proposition 3.12** *Let $h : H \longrightarrow G$ be a locally definable covering homomorphism. Suppose that $H$ and $G$ are definably connected. Then the following hold:*

*(i) $h_* : \pi(H) \longrightarrow \pi(G)$ is an injective homomorphism;*

*(ii) $\pi(G)/h_*(\pi(H)) \simeq \mathrm{Aut}(H/G)$.*

**Proof.** As we saw above, $h_* : \pi(H) \longrightarrow \pi(G)$ is injective. We prove (ii). First we define a group homomorphism $\theta : \pi(G) \longrightarrow \mathrm{Aut}(H/G)$, with

$\theta(x) : H \longrightarrow H : w \mapsto \theta(x)(w) = \phi_{\widetilde{p}_{(H,h)}(x)}(w)$, where $\phi_{\widetilde{p}_{(H,h)}(x)}$ is the unique locally definable covering transformation such that $\phi_{\widetilde{p}_{(H,h)}(x)}(e_H) = \widetilde{p}_{(H,h)}(x)$. Since $\widetilde{p}_{(H,h)}(x) \in \mathrm{Ker} h$, we have that $\phi_{\widetilde{p}_{(H,h)}(x)}$ is indeed in $\mathrm{Aut}(H/G)$. Since $\widetilde{p}_{(H,h)}(xy) = \widetilde{p}_{(H,h)}(x)\widetilde{p}_{(H,h)}(y)$, it follows that $\theta$ is a homomorphism.

Let $\widetilde{q} : \widetilde{H} \longrightarrow H$ be the o-minimal universal covering homomorphism of $H$. Then we have $\widetilde{q} = \widetilde{p}_{(H,h)} \circ \widetilde{h}$ where $\widetilde{h}$ is an isomorphism. So by the definition of $h_* : \pi(H) \longrightarrow \pi(G)$, the kernel of $\theta$ is $h_*(\pi(H))$. It remains to show that $\theta$ is surjective. So let $\phi \in \mathrm{Aut}(H/G)$. Since $\widetilde{p}_{(H,h)} : \widetilde{G} \longrightarrow H$ is surjective, there is $x \in G$ such that $\widetilde{p}_{(H,h)}(x) = \phi(e_H)$. But $\widetilde{p}(x) = h \circ \widetilde{p}_{(H,h)}(x) = h \circ \phi(e_H) = h(e_H) = e_G$. So $x \in \pi(G)$ and $\theta(x) = \phi$. $\square$

Let $h : H \longrightarrow G$ be a locally definable covering homomorphism. Suppose that $H$ and $G$ are definably connected. Then by Proposition 3.12 we have a short exact sequence

$$1 \to h_*(\pi(H)) \to \pi(G) \xrightarrow{\theta} \mathrm{Aut}(H/G) \to 1.$$

If $k : K \longrightarrow H$ is another locally definable covering homomorphism with $K$ definably connected, then we have an obvious commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & h_*(\pi(H)) & \to & \pi(G) & \xrightarrow{\theta} & \mathrm{Aut}(H/G) & \to & 1 \\ & & \uparrow & & \uparrow^{\mathrm{id}} & & \uparrow^{\theta^K_H} & & \\ 1 & \to & (h \circ k)_*(\pi(K)) & \to & \pi(G) & \xrightarrow{\theta} & \mathrm{Aut}(K/G) & \to & 1 \end{array}$$

of homomorphisms.

**Corollary 3.13** *Let $G$ be a definably connected locally definable group. Then $\pi(G)$ is the direct limit of the family of group homomorphisms*

$$\{\theta^K_H : \mathrm{Aut}(K/G) \longrightarrow \mathrm{Aut}(H/G) \mid K \longrightarrow H \in \mathrm{MorCov}^0(G)\}.$$

For the proof of the main theorem of this subsection we will require the following lemma.

**Lemma 3.14** *Let $G$ be a definably connected, abelian locally definable group over $A$ and let $k > 0$. Suppose that $G$ is $k$-divisible and the subgroup of $k$-torsion points of $G$ has dimension zero. If $h : H \longrightarrow G$ is a locally definable covering homomorphism over $A$ with $H$ definably connected, then $H$ is $k$-divisible and the subgroup of $k$-torsion points of $H$ has dimension zero.*

**Proof.** Consider the map $p_k : G \longrightarrow G : x \mapsto kx$ which is a locally definable homomorphism over $A$. By the assumptions on $G$ and Theorem 3.6, this is also a locally definable covering homomorphism over $A$. We also have a commutative diagram

$$\begin{array}{ccc} H & \stackrel{p_k}{\rightarrow} & H \\ \downarrow^h & & \downarrow^h \\ G & \stackrel{p_k}{\rightarrow} & G. \end{array}$$

Thus the kernel of $p_k : H \longrightarrow H$ is contained in the kernel of $p_k \circ h : H \longrightarrow G$ which is a locally definable covering homomorphism over $A$. So the kernel of $p_k : H \longrightarrow H$ has dimension zero. It remains to show that $H = p_k(H)$. First we note that $H = \cup\{zp_k(H) : z \in \mathrm{Ker}h\}$. In fact, $h_{|p_k(H)} : p_k(H) \longrightarrow G$ is surjective, thus if $x \in H$, then there is $y \in p_k(H)$ such that $h(y) = h(x)$. So $z = xy^{-1} \in \mathrm{Ker}h$ and $x = zy \in zp_k(H)$.

Let $z_1, z_2 \in \mathrm{Ker}h$. If $z_1p_k(H) \cap z_2p_k(H) \neq \emptyset$, then there are $x_1, x_2 \in p_k(H)$ such that $z_1x_1 = z_2x_2$. So $z_2 = z_1(x_1x_2^{-1}) \in z_1p_k(H)$ and $z_1 = z_2(x_2x_1^{-1}) \in z_2p_k(H)$. Therefore, $z_1p_k(H) = z_2p_k(H)$. So there is a subset $S_0 \subset \mathrm{Ker}h$ such that $H = \cup\{zp_k(H) : z \in S_0\}$ (disjoint union). Since $\dim p_k(H) = \dim H$, by Theorem 2.5, $p_k(H)$ and so each $zp_k(H)$ is open in $H$. Thus each $zp_k(H)$ is open and closed in $H$. If $p_k : H \longrightarrow H$ is not surjective, then $|S_0| > 1$ and this implies that $H$ is not definably connected. In fact, there is a definable subset $U$ of $H$, which can only be covered by more than one of the sets $zp_k(H)$ with $z \in S_0$. By saturation, there are $z_1, \ldots, z_l \in S_0$ and definable subsets $U_i \subseteq z_ip_k(H)$ with $i = 1, \ldots, l$, such that $U$ is the disjoint union $(U \cap U_1) \cup \cdots \cup (U \cap U_l) = (U \cap z_1p_k(H)) \cup \cdots \cup (U \cap z_lp_k(H))$ of open and closed subsets of $U$. $\square$

The following result is the generalization of [eo] Theorem 2.1.

**Theorem 3.15** *Let $G$ be a definably connected, abelian locally definable group. Suppose that, for each $k > 0$, $G$ is $k$-divisible and the subgroup of $k$-torsion points of $G$ has dimension zero. Then the following hold:*

*(i) the o-minimal universal covering group $\widetilde{G}$ of $G$ is divisible and torsion free;*

*(ii) the o-minimal fundamental group $\pi(G)$ of $G$ is a torsion-free abelian group;*

*(iii) the $k$-torsion subgroup of $G$ is isomorphic to $\pi(G)/k\pi(G)$, for each $k > 0$.*

**Proof.** We consider, for each $k > 0$, the map $p_k : G \longrightarrow G : x \mapsto kx$ which is a locally definable homomorphism. By the assumptions on $G$ and Theorem 3.6, this is also a locally definable covering homomorphism.

By definition, for every $k > 0$, the isomorphism $\widetilde{p_k} : \widetilde{G} \longrightarrow \widetilde{G}$ is given by multiplication by $k$. In fact, by Lemma 3.14, if $h : H \longrightarrow G \in \mathrm{Cov}^0(G)$, then

$$\begin{array}{ccc} H & \xrightarrow{p_k} & H \\ \downarrow^h & & \downarrow^h \\ G & \xrightarrow{p_k} & G \end{array}$$

is a commutative diagram of locally definable covering homomorphisms. So, $\widetilde{p}_{(H,p_k \circ h)}(kx) = k\widetilde{p}_{(H,p_k \circ h)}(x) = kx_{(H,p_k \circ h)} = kx_{(H,h \circ p_k)} = p_k(x_{(H,h \circ p_k)}) = x_{(H,h)}$ and the coordinate $(kx)_{(H,p_k \circ h)}$ of $kx$ is equal to the coordinate $x_{(H,h)}$ of $x$ just like the coordinate $\widetilde{p_k}(x)_{(H,p_k \circ h)}$ of $\widetilde{p_k}(x)$ is equal to the coordinate $x_{(H,h)}$ of $x$. Thus $kx = \widetilde{p_k}(x)$ and (i) holds.

By Proposition 3.12 (i), $(p_k)_* : \pi(G) \longrightarrow \pi(G)$ is an injective homomorphism for all $k > 0$. Since $(p_k)_* = \widetilde{p}_{k|\pi(G)}$, the homomorphism $(p_k)_* : \pi(G) \longrightarrow \pi(G)$ is given by $(p_k)_*(x) = kx$. Therefore, $\pi(G)$ is a torsion-free abelian group and, by Propositions 3.4 and 3.12 (ii), the $k$-torsion subgroup of $G$ is isomorphic to $\pi(G)/k\pi(G)$, for each $k > 0$. $\square$

We conjecture that every definably connected locally definable abelian group $G$, for each $k > 0$, $G$ is $k$-divisible and the subgroup of $k$-torsion points of $G$ has dimension zero. For definable groups this is proved in [s]. However, the methods used there do not generalise to the locally definable case.

# 4 In o-minimal expansions of fields

In this section we assume that $\mathcal{N}$ is an o-minimal expansion of a field. In this case, given a definably connected locally definable group $G$, we can define an o-minimal fundamental group $\pi_1(G)$ of $G$ using definable paths and definable homotopies adapting the definable case treated in [bo]. We will show that $\pi(G)$ and $\pi_1(G)$ are isomorphic.

So let $G$ be a definably connected locally definable group. A map $f : X \longrightarrow G$ where $X$ is a definable set, is a *definable map* if there is a definable subset $Y$ of $G$ such that $f : X \longrightarrow Y$ is a definable map. A *definable path* in $G$ is a continuous definable map $\alpha : [0,1] \longrightarrow G$.

The *o-minimal fundamental group* $\pi_1(G)$ *of* $G$ is defined in the usual way except that we use definable paths $\alpha : [0,1] \longrightarrow G$ in $G$ such that $\alpha(0) = \alpha(1) = e_G$ and definable homotopies. If $f : G \longrightarrow H$ is a continuous locally definable map between definably connected locally definable groups with $f(e_G) = e_H$, then the induced map $f_* : \pi_1(G) \longrightarrow \pi_1(H) : [\sigma] \mapsto [f \circ \sigma]$ is a group homomorphism and we have the usual functorial properties. See [bo] for the theory of o-minimal fundamental groups in the category of definable sets with continuous definable maps.

We now generalize the theory of [eo] Section 2 to the category of locally definable groups. Since the arguments are similar we will omit the details. We start with the following analogue of [eo] Lemma 2.3.

**Proposition 4.1** *Let $G$ be a definably connected locally definable group. Then $\pi_1(G)$ is abelian.*

Let $p : H \longrightarrow G$ be a locally definable covering homomorphism (over some $A$ with $|A| < \aleph_1$). Let $Z$ be a definable set and let $f : Z \longrightarrow G$ be a definable continuous map. A *lifting of* $f$ is a continuous definable map $\widetilde{f} : Z \longrightarrow H$ such that $p \circ \widetilde{f} = f$.

**Lemma 4.2 (Unicity of liftings)** *Let $p : H \longrightarrow G$ be a locally definable covering homomorphism over $A$. Let $Z$ be a definably definably connected definable set and let $f : Z \longrightarrow G$ be a definable continuous map. If $\widetilde{f}_1, \widetilde{f}_2 : Z \longrightarrow H$ are two liftings of $f$, then $\widetilde{f}_1 = \widetilde{f}_2$ provided there is a $z \in Z$ such that $\widetilde{f}_1(z) = \widetilde{f}_2(z)$.*

**Proof.** As in the proof of Lemma 3.2, both sets $\{w \in Z : \widetilde{f}_1(w) = \widetilde{f}_2(w)\}$ and $\{w \in Z : \widetilde{f}_1(w) \neq \widetilde{f}_2(w)\}$ are definable and open, the first one is nonempty. $\Box$

**Lemma 4.3 (Path and homotopy lifting)** *Suppose that $p : K \longrightarrow G$ be a locally definable covering homomorphism over $A$. Then the following hold.*

*(1) Let $\gamma$ be a definable path in $G$ and $y \in K$. If $p(y) = \gamma(0)$, then there is a unique definable path $\widetilde{\gamma}$ in $K$, lifting $\gamma$, such that $\widetilde{\gamma}(0) = y$.*

*(2) Suppose that $H : [a,b] \times [0,1] \longrightarrow G$ is a definable homotopy between the definable paths $\gamma$ and $\sigma$ in $G$. Let $\widetilde{\gamma}$ be a definable path in $K$ lifting $\gamma$. Then there is a unique definable lifting $\widetilde{H}$ of $H$, which is a definable homotopy between $\widetilde{\gamma}$ and $\widetilde{\sigma}$, where $\widetilde{\sigma}$ is a definable path in $K$ lifting $\sigma$.*

**Proof.** These results as their definable analogues in [eo] are consequences of path and the homotopy lifting. In our category, the path and the homotopy liftings can be proved as in [eo] by observing that, by saturation, a definable subset of $G$ is covered by finitely many open definable subsets of $G$. $\square$

**Notation:** Referring to Lemma 4.3, if $\gamma$ is a definable path in $G$ and $y \in K$, we denote by $y * \gamma$ the final point $\widetilde{\gamma}(1)$ of the definable lifting $\widetilde{\gamma}$ of $\gamma$ with initial point $\widetilde{\gamma}(0) = y$.

The following consequence of Lemma 4.3 is proved in exactly the same way as its definable analogue in [eo] Corollary 2.9.

**Remark 4.4** Suppose that $p : H \longrightarrow G$ is a locally definable covering homomorphism over $A$ and let $y \in H$ be such that $p(y) = e_G$. Suppose that $H$ and $G$ are definably connected. Then the following hold.

(1) If $\sigma$ is a definable path in $G$ from $e_G$ to $e_G$, then $y = y * \sigma$ if and only if $[\sigma] \in p_*(\pi_1(H))$.

(2) If $\sigma$ and $\sigma'$ are two definable paths in $G$ from $e_G$ to $x$, then $y * \sigma = y * \sigma'$ if and only if $[\sigma \cdot \sigma'^{-1}] \in p_*(\pi_1(H))$.

If $G$ is a locally definable group over $A$, we say that $G$ is *definably path connected* if for every $u, v \in G$ there is a definable path $\alpha : [0,1] \longrightarrow G$ such that $\alpha(0) = u$ and $\alpha(1) = v$.

**Lemma 4.5** *Let $G$ be a locally definable group over $A$. Then the following hold.*

*(1) For every definably connected open definable subset $U$ of $G$ over $A$ there is a definable subset $\Sigma$ of $U \times U \times [0,1] \times U$ over $A$ such that for every $(u,v) \in U \times U$, the fiber $\Sigma_{(u,v)} = \{(t,w) \in [0,1] \times U : (u,v,t,w) \in \Sigma\}$ is the graph of a definable path in $G$ over $A \cup \{u,v\}$ from $u$ to $v$.*

*(2) $G$ is definably connected if and only if $G$ is definably path connected.*

**Proof.** By Lemma 2.7, there is $G$ is covered by the family $\{U_i : i \in I\}$ of all open definably connected definable subsets of $G$ over $A$. By [vdd] Chapter VI, Proposition 3.2, property (1) holds for each $U_i$. Since $U = U_{i_1} \cup \cdots \cup U_{i_l}$, property (1) holds for $U$.

Fix $i_0 \in I$, $x_i \in U_i$ over $A$ and let $I_0 = \{i \in I$ : there is a definable path in $G$ from $x_{i_0}$ to $x_i\}$. By (1), $\cup\{U_i : i \in I_0\}$ is a nonempty open and closed compatible locally definable subset of $G$ over $A$ which is definably path connected. Thus $G$ is definably connected if and only if $G = \cup\{U_i : i \in I_0\}$. □

The first part of the next proposition is also a consequence of Lemma 4.3 and is proved in exactly the same way as its definable analogue in [eo] Corollary 2.8. We give here a different proof of the second part.

**Proposition 4.6** *Let $h : H \longrightarrow G$ be a locally definable covering homomorphism. Suppose that $H$ and $G$ are definably connected. Then the following hold:*

*(i) $h_* : \pi_1(H) \longrightarrow \pi_1(G)$ is an injective homomorphism;*

*(ii) $\pi_1(G)/h_*(\pi_1(H)) \simeq \mathrm{Aut}(H/G)$.*

**Proof.** As we mentioned (i) is similar to [eo] Corollary 2.8. We prove (ii). First we define a group homomorphism $\psi : \pi_1(G) \longrightarrow \mathrm{Aut}(H/G)$, with $\psi([\sigma]) : H \longrightarrow H : w \mapsto \psi([\sigma])(w) = \phi_{[\sigma]}(w)$, where $\phi_{[\sigma]}$ is the unique locally definable covering transformation such that $\phi_{[\sigma]}(e_H) = e_H * \sigma$. By definition, $e_H * \sigma$ is in $\mathrm{Ker}h$ and so $\phi_{[\sigma]}$ is in $\mathrm{Aut}(H/G)$. By Remark 4.4 (2), the map $\phi_{[\sigma]}$ is well defined and depends only on the class $[\sigma]$ and not on the particular representative of this class. Since $e_H * (\sigma \cdot \sigma') = (e_H * \sigma) * \sigma'$, it follows that $\psi$ is a homomorphism. By Remark 4.4 (1), the kernel of $\psi$ is $h_*(\pi_1(H))$. It remains to show that $\psi$ is surjective. So let $\phi \in \mathrm{Aut}(H/G)$. Since $H$ is definably connected, by Lemma 4.5 (2), there is a definable path $\alpha$ in $H$ from $e_H$ to $\phi(e_H)$. Let $\sigma = h \circ \alpha$. Then $[\sigma] \in \pi_1(G)$ and $\psi([\sigma]) = \phi$. □

As in [eo] Theorem 2.1 we have the following result.

**Theorem 4.7** *Let $G$ be a definably connected, abelian locally definable group. Suppose that, for each $k > 0$, $G$ is $k$-divisible and the subgroup of $k$-torsion points of $G$ has dimension zero. Then the following hold:*

*(i) the o-minimal fundamental group $\pi_1(G)$ of $G$ is a torsion-free abelian group, and*

*(ii) the $k$-torsion subgroup of $G$ is isomorphic to $\pi_1(G)/k\pi_1(G)$, for each $k > 0$.*

**Proof.** We consider, for each $k > 0$, the map $p_k : G \longrightarrow G : x \mapsto kx$ which is a locally definable homomorphism. By the assumptions on $G$ and Theorem 3.6, this is also a locally definable covering homomorphism. By Proposition 4.6 (i), $(p_k)_* : \pi_1(G) \longrightarrow \pi_1(G)$ is an injective homomorphism for all $k > 0$. But as in [eo] Lemma 2.4, we see that $(p_k)_* : \pi_1(G) \longrightarrow \pi_1(G)$ is the homomorphism given by $(p_k)_*([\sigma]) = k[\sigma]$. Therefore, $\pi_1(G)$ is a torsion-free abelian group and, by Propositions 3.4 and 4.6 (ii), the $k$-torsion subgroup of $G$ is isomorphic to $\pi_1(G)/k\pi_1(G)$, for each $k > 0$. $\square$

Theorem 4.7 is also a consequence of Theorem 3.15 and the following result.

**Theorem 4.8** *Let $G$ be a definably connected locally definable group. Then $\pi(G)$ and $\pi_1(G)$ are isomorphic.*

**Proof.** If $h : H \longrightarrow G \in \mathrm{Cov}^0(G)$, then by Proposition 4.6 we have a short exact sequence

$$1 \to h_*(\pi_1(H)) \to \pi_1(G) \xrightarrow{\psi} \mathrm{Aut}(H/G) \to 1.$$

If $k : K \longrightarrow H$ is another locally definable covering homomorphism with $K$ definably connected, then we have an obvious commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & h_*(\pi_1(H)) & \to & \pi_1(G) & \xrightarrow{\psi} & \mathrm{Aut}(H/G) & \to & 1 \\ & & \uparrow & & \uparrow^{\mathrm{id}} & & \uparrow^{\psi_H^K} & & \\ 1 & \to & (h \circ k)_*(\pi_1(K)) & \to & \pi_1(G) & \xrightarrow{\psi} & \mathrm{Aut}(K/G) & \to & 1 \end{array}$$

of homomorphisms. Therefore, $\pi_1(G)$ is the direct limit of the family of group homomorphisms

$$\{\psi_H^K : \mathrm{Aut}(K/G) \longrightarrow \mathrm{Aut}(H/G) \mid K \longrightarrow H \in \mathrm{MorCov}^0(G)\}.$$

But by Proposition 3.4 we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Aut}(H/G) & \leftarrow & \mathrm{Ker}h & \rightarrow & \mathrm{Aut}(H/G) \\ \uparrow^{\theta_H^K} & & \uparrow^{k} & & \uparrow^{\psi_H^K} \\ \mathrm{Aut}(K/G) & \leftarrow & \mathrm{Ker}(h \circ k) & \rightarrow & \mathrm{Aut}(K/G) \end{array}$$

where the horizontal arrows are isomorphisms. So $\pi(G)$ is isomorphic to $\pi_1(G)$. $\square$

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