

# Molaei's Generalized Groups are Completely Simple Semigroups

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## Abstract

In [2] Molaei introduces generalized groups, a class of algebras of interest to physics, and proves some results about them.

The aim of this note is to prove that Generalized Groups are the Completely Simple Semigroups.

*Mathematics subject classification:* 20M20.

## 1 The Main Theorem

Let  $\mathbf{G}$  be a groupoid, that is,  $\mathbf{G} = (G, f)$  where  $X$  is a non-empty set and  $f$  is a binary operation  $f : G \times G \rightarrow G$ . As usual, for  $a, b \in G$ , we denote  $f(a, b)$  by  $ab$ . Suppose that  $\mathbf{G}$  satisfies the following axioms:

$$\text{(M1)} \quad (\forall_{x,y,z \in G}) \quad (xy)z = x(yz);$$

$$\text{(M2)} \quad (\forall_{x \in G}) (\exists_{e(x) \in G}^1) \quad xe(x) = x = e(x)x;$$

$$\text{(M3)} \quad (\forall_{x \in G}) (\exists_{x^{-1} \in G}) \quad xx^{-1} = e(x) = x^{-1}x.$$

In [2] Molaei introduces these groupoids and calls them *Generalized Groups*. However, since the name Generalized Group already appears in literature defining a different algebraic structure, we are going to call the groupoids satisfying (M1)-(M3) *Molaei's Generalized groups*. It is well known that a groupoid satisfying (M1) is called a *Semigroup*.

We start by giving an example of a Molaei's generalized group. Let  $I, L$  be two nonempty sets and let  $G$  be a group. Moreover, let  $P = (p_{l,i})_{l \in L, i \in I}$  be a  $L \times I$  matrix with entries in  $G$ . In  $I \times G \times L$  consider the following product: for all  $(i, g, l), (j, h, m) \in I \times G \times L$ ,

$$(i, g, l)(j, h, m) = (i, gp_{m,j}h, m).$$

It is easy to verify that  $I \times G \times L$  with this product is a semigroup which will be denoted by  $\mathcal{M}(G; I, L; P)$ .

Now let  $(i, g, l) \in \mathcal{M}(G; I, L; P)$  and consider the element  $(i, p_{l,i}^{-1}, l)$  of  $\mathcal{M}(G; I, L; P)$ . We claim that  $(i, p_{l,i}^{-1}, l)$  is the unique element  $e \in \mathcal{M}(G; I, L; P)$  such that  $(i, g, l)e = e(i, g, l) = (i, g, l)$  and hence  $\mathcal{M}(G; I, L; P)$  satisfies (M1). In fact

$$(i, g, l)(i, p_{l,i}^{-1}, l) = (i, gp_{l,i}p_{l,i}^{-1}, l) = (i, g, l)$$

and

$$(i, p_{l,i}^{-1}, l)(i, g, l) = (i, p_{l,i}^{-1}p_{l,i}g, l) = (i, g, l).$$

Now suppose that  $(i, g, l)(j, h, m) = (i, g, l)$ . Then  $(i, gp_{l,j}h, m) = (i, g, l)$  and hence  $l = m$ ,  $p_{l,j}^{-1} = h$ . In the same way,  $(j, h, m)(i, g, l) = (i, g, l)$  implies  $j = i$  and  $p_{m,i}^{-1} = h$ . Thus  $l = m$ ,  $i = j$  and  $p_{l,j}^{-1} = p_{l,i}^{-1} = p_{m,j}^{-1} = h$ . Thus  $(j, h, m) = (i, p_{l,i}^{-1}, l)$ . It is proved that  $(i, p_{l,i}^{-1}, l) = e((i, g, l))$  and hence  $\mathcal{M}(G; I, L; P)$  satisfies (M1).

To prove that  $\mathcal{M}(G; I, L; P)$  satisfies (M2), let  $(i, g, l) \in \mathcal{M}(G; I, L; P)$  and consider the element  $(i, p_{l,i}^{-1}g^{-1}p_{l,i}^{-1}, l)$  of  $\mathcal{M}(G; I, L; P)$ . It is obvious that

$$(i, p_{l,i}^{-1}g^{-1}p_{l,i}^{-1}, l)(i, g, p) = (i, g, p)(i, p_{l,i}^{-1}g^{-1}p_{l,i}^{-1}, l) = (i, p_{l,i}^{-1}, l) = e((i, g, p))$$

and hence  $\mathcal{M}(G; I, L; P)$  is a generalized group. In the main theorem we prove that every generalized group is isomorphic to some semigroup  $\mathcal{M}(G; I, L; P)$ . But before stating the main theorem we introduce a definition.

A semigroup  $S$  is said to be *Completely Simple* if

(C1)  $SaS = S$ , for all  $a \in S$ ;

(C2) If  $e, f \in S$  are idempotents (that is,  $e^2 = e$  and  $f^2 = f$ ) and  $e = ef = fe$ , then  $e = f$ .

**Theorem 1.1** *Let  $S$  be a semigroup. The following are equivalent:*

(1)  $S$  is isomorphic to some  $\mathcal{M}(G; I, L; P)$ ;

(2)  $S$  is completely simple;

(3)  $S$  is a Molaie's generalized group.

**Proof:** We have proved above that every semigroup  $\mathcal{M}(G; I, L; P)$  is a Molaie's generalized group. Therefore (1) implies (3).

That (2) implies (1) follows from Rees Theorem, the most famous theorem in Semigroup Theory. An account of this theorem can be found in [1] (Chapter 3).

It remains to prove that (3) implies (2). Observe first that for all idempotent  $e \in S$  we have  $e(e) = e$  (because  $ee = ee = e$ ). Now suppose that we have two idempotents  $e, f \in S$  such that  $e = ef = fe$ . Then, by (M2),  $e(e) = f$  and hence  $e = f$  (because  $e(e) = e$ , for all idempotents in  $S$ ). Condition (C2) is proved.

Now observe that  $Se(x)S = SxS$ , for all  $x \in S$ . In fact  $x = e(x)e(x)x$  (because, by (M2),  $e(x)e(x)x = e(x)x = x$ ) and hence  $SxS \subseteq Se(x)S$ . Conversely,

$$e(x)xx^{-1} = xx^{-1}xx^{-1} = x(x^{-1}x)x^{-1} = xe(x)x^{-1} = xx^{-1} = e(x).$$

Thus  $e(x) = e(x)xx^{-1}$  and hence  $Se(x)S \subseteq SxS$ . Thus, for all  $x \in S$ , we have  $Se(x)S = SxS$ .

To prove (C1) we want to prove that  $S \subseteq SyS$ , for all  $y \in S$ . Thus let  $x, y \in S$ . We claim that  $x \in SyS$ . In fact, let  $z = e(x)ye(x)$ . It is obvious that  $ze(x) = e(x)z = z$  and hence, by (M2),  $e(z) = e(x)$ . Now

$$x \in Se(x)S = Se(z)S = SzS = Se(x)ye(x)S \subseteq SyS.$$

It is proved that  $S \subseteq SyS$ , for all  $y \in S$ , and hence  $S$  is completely simple. The theorem follows. ■

Finally we observe that [2] there is a problem with the proof of Theorem 2.3. Let  $S$  be a Molaei's generalized group and  $x \in S$ . Then let  $R(X) = \{y \in S \mid xy = e(x) = yx\}$ . In the referred theorem it is proved that  $|R(x)| = 1$  but the proof uses the unproved assumption that  $e(x)e(x^{-1}) = e(x^{-1})e(x)$ .

**Lemma 1.2** *Let  $S$  be a Molaei's generalized group. Then the following are equivalent*

- (a)  $e(x) = e(x^{-1})$ , for every  $x \in S, x^{-1} \in R(x)$ .
- (b)  $(x^{-1})^{-1} = x$ , for every  $x \in S, x^{-1} \in R(x)$  and  $(x^{-1})^{-1} \in R(x^{-1})$ .
- (c)  $x^{-1}xx^{-1} = x^{-1}$ , for every  $x \in S, x^{-1} \in R(x)$ .
- (d)  $|R(x)| = 1$ , for every  $x \in S$ .
- (e)  $e(x)e(x^{-1}) = e(x^{-1})e(x)$ , for every  $x \in S, x^{-1} \in R(x)$ .

**Proof:** We start by proving that (a) implies (b). Let  $x \in S$  and  $x^{-1} \in R(x)$ . Then

$$\begin{aligned} e(x^{-1}) = x^{-1}(x^{-1})^{-1} &\Rightarrow xe(x^{-1}) = xx^{-1}(x^{-1})^{-1} \\ &\Rightarrow xe(x^{-1}) = e(x)(x^{-1})^{-1} \\ &\Rightarrow xe(x) = e(x^{-1})(x^{-1})^{-1} \text{ (by (a))} \\ &\Rightarrow x = (x^{-1})^{-1} \end{aligned}$$

thus proving (b).

To prove that (b) implies (c), let  $x \in S, x^{-1} \in R(x)$  and  $(x^{-1})^{-1} \in R(x^{-1})$ . Then we have

$$\begin{aligned} x^{-1}(x^{-1})^{-1} = e(x^{-1}) &\Rightarrow x^{-1}(x^{-1})^{-1}x^{-1} = e(x^{-1})x^{-1} \\ &\Rightarrow x^{-1}(x^{-1})^{-1}x^{-1} = x^{-1} \\ &\Rightarrow x^{-1}xx^{-1} = x^{-1} \text{ (by (b))} \end{aligned}$$

and (c) follows.

To prove that (c) implies (d), let  $y, z \in R(x)$ . Then we have

$$y = yxy = y(xy) = ye(x) = y(xz) = (yx)z = e(x)z = (zx)z = z,$$

where the first and last equality follow from (c). It is proved that  $|R(x)| = 1$ .

Conversely, (d) implies (c). In fact, let  $x^{-1} \in R(x)$ . Then

$$x(x^{-1}xx^{-1}) = (xx^{-1})(xx^{-1}) = e(x)xx^{-1} = xx^{-1} = e(x)$$

and

$$(x^{-1}xx^{-1})x = x^{-1}xe(x) = x^{-1}x = e(x).$$

Thus  $x^{-1}xx^{-1} \in R(x)$  and hence, by (d),  $x^{-1}xx^{-1} = x^{-1}$ .

We prove now that (c) implies (a). In fact, for all  $x \in S$  and  $x^{-1} \in R(x)$ , we have  $e(x)x^{-1} = x^{-1}xx^{-1} = x^{-1}$  and  $x^{-1}e(x) = x^{-1}xx^{-1} = x^{-1}$ . Thus  $e(x) = e(x^{-1})$ . Thus the first four conditions are all equivalent.

Finally, it is obvious that (a) implies (e). Conversely, let  $x \in S$  and  $x \in R(x)$ . Then

$$\begin{aligned} e(x)e(x^{-1}) = e(x^{-1})e(x) &\Rightarrow e(x)e(x^{-1})x^{-1} = e(x^{-1})e(x)x^{-1} \\ &\Rightarrow e(x)x^{-1} = e(x^{-1})e(x)x^{-1} \\ &\Rightarrow e(x)x^{-1}x = e(x^{-1})e(x)x^{-1}x \\ &\Rightarrow e(x)e(x) = e(x^{-1})e(x)e(x) \\ &\Rightarrow e(x) = e(x^{-1})e(x). \end{aligned}$$

Now

$$e(x) = e(x^{-1})e(x) \Rightarrow e(x)x = e(x^{-1})e(x)x \Rightarrow x = e(x^{-1})x.$$

Similarly,

$$e(x) = e(x)e(x^{-1}) = e(x^{-1})e(x) \Rightarrow xe(x) = xe(x^{-1})e(x) \Rightarrow x = xe(x^{-1}).$$

It is proved that  $e(x) = e(x^{-1})$ . The lemma follows. ■

It is easy to prove, using Theorem 1.1, that each one of the equivalent conditions in the previous lemma hold in a Molaei's generalized group. However, we provide a direct proof for (a).

**Lemma 1.3** *Let  $x \in S$  and  $x \in R(x)$ . Then  $e(x) = e(x^{-1})$ .*

**Proof:** Let  $x \in S$ . Then

$$\begin{aligned} e(x)e(x^{-1}) &= (xx^{-1})((x^{-1})^{-1}x^{-1}) \\ &= x(x^{-1}(x^{-1})^{-1})x^{-1} \\ &= xe(x^{-1})x^{-1} \\ &= x(e(x^{-1})x^{-1}) \\ &= xx^{-1} \\ &= e(x). \end{aligned}$$

Similarly we prove that  $e(x^{-1})e(x) = e(x)$ . Thus it is proved that

$$e(x)e(x^{-1}) = e(x) = e(x^{-1})e(x). \quad (1)$$

On the other hand

$$e(x)e(x) = (xx^{-1})(xx^{-1}) = x(x^{-1}x)x^{-1} = xe(x)x^{-1} = (xe(x))x^{-1} = xx^{-1} = e(x). \quad (2)$$

Thus, by (M2) together with (1) and (2), it follows that  $e(x) = e(x^{-1})$ . The lemma follows. ■

## References

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*Acknowledgements:* The first author acknowledges with thanks the support of FCT, *POCTI/32440/MAT/2000*, and Fundação Calouste Gulbenkian. The second author was supported by a Mary Washington College Professional Activity Grant.