THE UNIVERSAL COVERING MAP IN O-MINIMAL EXPANSIONS OF GROUPS

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Abstract. In this paper we study locally definable manifolds and we prove: (i) the existence of universal locally definable covering maps; (ii) invariance results for locally definable covering maps, o-minimal fundamental groups and fundamental groupoids; (iii) monodromy equivalence for locally constant o-minimal sheaves; (iv) classification results for locally definable covering maps; (v) o-minimal Hurewicz and Seifert - van Kampen theorems.

1. Introduction

We fix an arbitrary o-minimal expansion $\mathcal{R} = \langle R, <, 0, +, \ldots \rangle$ of an ordered group. By "(locally) definable" we mean "(locally) definable in $\mathcal{R}$, possibly with parameters". We work in the category of locally definable manifolds with morphisms the locally definable continuous maps. Locally definable manifolds properly generalize the definable manifolds (see Section 2). For background on basic o-minimality we refer the reader to [9] for the definable setting and to [3] for the locally definable setting. For algebraic topology relevant to this paper, the reader should see [8], [22] and [23], for example.

The first main results of the paper are:

Theorem 1.1. *Let X be a definably connected Lindelöf locally definable manifold. Then the o-minimal fundamental group $\pi_1(X)$ of X is countable. In fact, if X is definable, then $\pi_1(X)$ is finitely generated.*

Theorem 1.2. *Let X be a definably connected locally definable manifold. Then there exists a universal locally definable covering map $u : U \to X$. Moreover, if X is Lindelöf (resp. paracompact), then U is also Lindelöf (resp. paracompact).*

Similar results were known before only in o-minimal expansions of real closed fields: in [4] Berarducci and Otero prove that the o-minimal fundamental group of a definable set is finitely generated; in [3] Baro and Otero prove the existence of o-minimal locally definable universal covers; in [6] Delfs and Knebusch prove the existence of locally semi-algebraic universal covers. Note also that in [13] the first two authors proved versions of the above theorems for definably connected locally definable groups. See also [20] and [21] for the special case of definable groups when

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$\mathcal{R}$ is linear. For background on locally definable groups we refer the reader to [10], [11], [13] and [25].

In this paper we also prove the following invariance results for both the universal locally definable covering map and the o-minimal fundamental group of a definably connected locally definable manifold:

Theorem 1.3. *Let $\mathcal{J}$ be an elementary extension of $\mathcal{R}$ or an o-minimal expansion of $\mathcal{R}$. Let X be a definably connected locally definable manifold. Then the following hold:*

(1) *A universal locally $\mathcal{J}$-definable covering map of X is $\mathcal{J}$-definably homeomorphic to a universal locally definable covering map of X.*
(2) *The o-minimal fundamental group of X in $\mathcal{J}$ is isomorphic to the o-minimal fundamental group of X in $\mathcal{R}$.*

Similarly, we have:

Theorem 1.4. *Suppose that $\mathcal{R}$ is an o-minimal expansion of the ordered group of real numbers. Let X be a definably connected locally definable manifold. Then the following hold:*

(1) *A topological universal covering map of X is topologically homeomorphic to the o-minimal universal locally definable covering map of X.*
(2) *The topological fundamental group of X is isomorphic to the o-minimal fundamental group of X.*

Analogues of these invariance results for o-minimal fundamental groups were proved before only for definable sets (resp. regular paracompact locally definable spaces) in o-minimal expansions of real closed fields in [4] and also [2] (resp. [3]). Versions of these invariance results for locally definable covering homomorphisms were proved before in [16].

The theorems above are generalizations to locally definable manifolds in arbitrary o-minimal expansions of ordered groups of the corresponding results by Baro and Otero ([3]) in o-minimal expansions of fields, which in turn are generalizations of similar results by Delfs and Knebusch ([6] and [7]) for locally semi-algebraic spaces. In the context of o-minimal expansions of fields or in the semi-algebraic context these results are consequences of o-minimal triangulation theorems [9] and [3] and the semi-algebraic triangulation theorem [7]. In our more general context, Theorems 1.1 and 1.2 are based on the following generalization of a result of Wilkie [27] on covering non-empty bounded open definable sets by finitely many open cells in o-minimal expansions of real closed fields: in a semi-bounded o-minimal expansion of an ordered group every non-empty open definable set is a finite union of open cells ([14]).

As we saw in Theorem 1.4 when $\mathcal{R}$ is an o-minimal expansion of the ordered group of real numbers, for definably connected locally definable manifolds, the theory developed in this paper coincides with the classical theory of topological covering maps ([22]). However, one should point out that, in an arbitrary o-minimal expansion $\mathcal{R}$ of an ordered group, the theory of topological covering maps is in some sense useless. In fact in that situation, if $\mathcal{R}$ is non archimedean, then all definably connected locally definable manifolds are, with their natural topology, totally disconnected spaces and so have no non trivial covering spaces. Our Theorem 1.2 shows that it is possible to find a suitable replacement of the theory of topological covering maps which in the archimedean case coincides with the classical theory and moreover it is preserved under elementary extensions (Theorem 1.3). As it is known developing these algebraic topology tools in the o-minimal context has proved to be very useful for example in the theory of definable groups ([17]).

In this paper we also obtain the monodromy equivalence for locally constant o-minimal sheaves (in an arbitrary o-minimal expansion $\mathcal{R}$ of an ordered group):

Theorem 1.5. *Let X be a locally definable manifold and let $\mathbf{J}$ be one of the categories: the category of sets; the category of G-torsors for a given discrete group G; the category of k-modules over a ring k. Then the monodromy functor*

$$\mu : \mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}}) \to \mathrm{Fct}(\Pi_1(X), \mathbf{J})$$

is an equivalence between the category of locally constant $\mathbf{J}$-sheaves on the o-minimal site X_{def} on X and the category of representations of the o-minimal fundamental groupoid $\Pi_1(X)$ of X in $\mathbf{J}$.

From Theorem 1.5 we obtain several classification results for locally definable covering maps. See Subsection 4.3. As a consequence of these classification results for locally definable covering maps we obtain o-minimal Hurewicz and Seifert - van Kampen theorems. Analogues of the o-minimal Hurewicz and Seifert - van Kampen theorems for definable sets in o-minimal expansions of fields were proved before in [17] and [4] respectively.

Structure of the paper.

In Section 2 we introduce preliminary notions and results about locally definable covering maps, o-minimal fundamental groups and o-minimal fundamental groupoids.

In Section 3 we prove the results on o-minimal fundamental groups and universal locally definable covering maps (Theorems 1.1, 1.2, 1.3 and 1.4). In Section 4 we prove the rest of the results of the paper namely: the monodromy equivalence for locally constant o-minimal sheaves, classification results for locally definable covering maps and o-minimal Hurewicz and Seifert - van Kampen theorems. In Section 5 we observe that all our results can be generalized to other categories of locally definable spaces in arbitrary o-minimal structures by pointing out exactly what is required in the proofs.

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2. Locally definable covering maps

This section contains general results and useful facts about locally definable covering maps. The results of the Subsection 2.1 hold in any o-minimal structure while those of Subsections 2.2 and 2.3 are related to o-minimal fundamental groups and fundamental groupoids and locally definable covering maps and so hold in o-minimal expansions of ordered groups. These results generalize the corresponding results for definable covering maps in o-minimal expansions of real closed fields that appear in [17].

2.1. **Locally definable covering maps.** Here we introduce some terminology and prove some preliminary results that will be useful later.

A *locally definable manifold (of dimension n)* is a triple $(S,(U_i,\theta_i)_{i\leq\kappa})$ where:

- $S=\bigcup_{i\leq\kappa}U_i$;
- each $\theta_i:U_i\to R^n$ is an injection such that $\theta_i(U_i)$ is an open definable subset of R^n;
- for all i,j, $\theta_i(U_i\cap U_j)$ is an open definable subset of $\theta_i(U_i)$ and the transition maps $\theta_{ij}:\theta_i(U_i\cap U_j)\to\theta_j(U_i\cap U_j):x\mapsto\theta_j(\theta_i^{-1}(x))$ are definable homeomorphisms.

We call the (U_i,θ_i)'s the *definable charts of* S. If $\kappa<\aleph_0$ then S is a *definable manifold*.

A locally definable manifold S has a topology such that each U_i is open and the θ_i's are homeomorphisms: a subset U of S is open in this topology if and only if for each i, $\theta_i(U\cap U_i)$ is an open definable subset of $\theta_i(U_i)$.

We say that a subset A of S is *definable* if and only if there is a finite $I_0\subseteq\kappa$ such that $A\subseteq\bigcup_{i\in I_0}U_i$ and for each $i\in I_0$, $\theta_i(A\cap U_i)$ is a definable subset of $\theta_i(U_i)$. A subset B of S is *locally definable* if and only if for each i, $B\cap U_i$ is a definable subset of S. We say that a locally definable manifold S is *definably connected* if it is not the disjoint union of two open and closed locally definable subsets. Note that in [6] and [3] "definably connected" is called "connected", but here prefer to make the distinction since often, e.g. when $\mathcal{R}$ is non-standard, definably connected locally definable manifolds are totally disconnected topological spaces.

Let $\mathcal{U}=\{U_\alpha\}_{\alpha\in I}$ be a cover of S by open locally definable subsets. We say that $\mathcal{U}$ is: (i) *admissible* if for each $i\leq\kappa$, the cover $\{U_\alpha\cap U_i\}_{\alpha\in I}$ of U_i admits a finite subcover; (ii) *locally finite* if for each $i\leq\kappa$, the set $\{\alpha\in I:U_\alpha\cap U_i\neq\emptyset\}$ is finite. If $\mathcal{V}=\{V_\beta\}_{\beta\in J}$ is another cover of S by open locally definable subsets, we say that $\mathcal{V}$ *refines* $\mathcal{U}$, denoted by $\mathcal{V}\leq\mathcal{U}$, if there is a map $\epsilon:J\to I$ such that $V_\beta\subseteq U_{\epsilon(\beta)}$ for all $\beta\in J$.

We say that a locally definable manifold S is *Lindelöf* if there exists an admissible cover of S by countably many open definable subsets.

The following observation is immediate:

Remark 2.1. A locally definable manifold S is Lindelöf if and only if every admissible cover of S by open locally definable subsets admits a refinement by an admissible cover of S by countably many open definable subsets. In particular, S is Lindelöf if and only if S has countably many definable charts.

The locally semi-algebraic analogue of Remark 2.1 is [7, Chapter I, Proposition 4.16]. Note also that by Remark 2.1 what is called "locally definable manifold" in the paper [16] corresponds exactly to Lindelöf locally definable manifolds.

We say that a locally definable manifold S is *paracompact* if there exists a locally finite (necessarily admissible) cover of S by open definable subsets.

The following observation is immediate:

Remark 2.2. A locally definable manifold S is paracompact if and only if every admissible cover of S by open locally definable subsets admits a refinement by a locally finite (necessarily admissible) cover of S by open definable subsets. In particular, S is paracompact if and only if S has a locally finite family of definable charts.

The locally semi-algebraic analogue of Remark 2.1 is [7, Chapter I, Proposition 4.5]. And the locally semi-algebraic analogue of the next easy remark is [7, Chapter I, Theorem 4.17]

Remark 2.3. A definably connected, paracompact locally definable manifold S is Lindelöf.

The above remarks are also mentioned in [3] where as in the locally semi-algebraic case ([6] and [7]) the notions of Lindelöf and paracompact play a more significant role then in this paper. We refer the reader to those papers for concrete examples of locally definable manifolds with or without these properties.

A map $f : X \to Y$ between locally definable manifolds with definable charts $(U_i, \theta_i)_{i \leq \kappa_X}$ and $(V_j, \delta_j)_{j \leq \kappa_Y}$ respectively is *a locally definable map* if for every finite $I \subseteq \kappa_X$ there is a finite $J \subseteq \kappa_Y$ such that:

- $f(\bigcup_{i \in I} U_i) \subseteq \bigcup_{j \in J} V_j$;
- the restriction $f_{|} : \bigcup_{i \in I} U_i \to \bigcup_{j \in J} V_j$ is a definable map between definable manifolds, i.e., for each $i \in I$ and every $j \in J$, $\delta_j \circ f \circ \theta_i^{-1} : \theta_i(U_i) \to \delta_j(V_j)$ is a definable map between definable sets.

Thus we have the category of locally definable manifolds with locally definable continuous maps.

Remark 2.4. Let $f : X \to Y$ be a locally definable continuous map, $D \subseteq Y$ a locally definable subset of Y and $\mathcal{V} = \{V_\beta\}_{\beta \in J}$ an admissible cover of Y by open locally definable subsets. Then

- $f^{-1}(D)$ is a locally definable subset of X.
- $f^{-1}\mathcal{V} = \{f^{-1}(V_\beta)\}_{\beta \in J}$ is an admissible cover of X by open locally definable subsets.

Indeed, if (U_i, θ_i) is a definable chart of X, then $f^{-1}(D) \cap U_i = (f_{|U_i})^{-1}(f_{|U_i}(U_i) \cap D)$ is definable since $f_{|U_i} : U_i \to f(U_i)$ is definable and $f_{|U_i}(U_i) \cap D$ is also definable; on the other hand the cover $\{f^{-1}(V_\beta) \cap U_i\}_{\beta \in J}$ of U_i admits a finite subcover since it is $\{(f_{|U_i})^{-1}(f_{|U_i}(U_i) \cap V_\beta)\}_{\beta \in J}$ and the cover $\{f_{|U_i}(U_i) \cap V_\beta)\}_{\beta \in J}$ of the definable subset $f_{|U_i}(U_i)$ admits a finite subcover.

Given a locally definable manifold S, a locally definable manifold X and an admissible cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of S by open definable subsets, we say that a continuous surjective locally definable map $p_X : X \to S$ is a *locally definable covering map trivial over* $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ if the following hold:

- $p_X^{-1}(U_\alpha) = \bigsqcup_{i \leq \lambda_\alpha} U_\alpha^i$ a disjoint union of open definable subsets of X;
- each $p_{X|U_\alpha^i} : U_\alpha^i \to U_\alpha$ is a definable homeomorphism.

A *locally definable covering map* $p_X : X \to S$ is a locally definable covering map trivial over some admissible cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of S by open definable subsets.

We say that two locally definable covering maps $p_X : X \to S$ and $p_Y : Y \to S$ are *locally definably homeomorphic* if there is a locally definable homeomorphism $F : X \to Y$ such that:

- $p_X = p_Y \circ F$.

Such $F : X \to Y$ is called a *locally definable covering homeomorphism.*

A locally definable covering map $p_X : X \to S$ is *trivial* if it is locally definably homeomorphic to a locally definable covering map $S \times M \to S : (s, m) \mapsto s$ for some set M.

Remark 2.5. If $p_X : X \to S$ is a locally definable covering map trivial over $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ with S definably connected, then there exists a fixed λ such that:

- $p_X^{-1}(U_\alpha) = \bigsqcup_{i \leq \lambda} U_\alpha^i$ (a disjoint union) of open definable subsets of X;
- each $p_{X|U_\alpha^i} : U_\alpha^i \to U_\alpha$ is a definable homeomorphism.

Remark 2.6. By Remark 2.1 what is called "locally definable covering map" in the paper [16] corresponds exactly to locally definable covering maps $p_X : X \to S$ between Lindelöf locally definable manifolds with S definably connected. These assumptions play no role in the proofs of that paper. Thus we will often use the results from [16] in the more general setting of this paper.

Let $D \subseteq R^m$ be a definable subset and Y a locally definable manifold with definable charts $(V_j, \delta_j)_{j \leq \kappa}$. We say that a map $f : D \to Y$ is a *definable map* if there is a finite $J \subseteq \kappa$ such that:

- $f(D) \subseteq \bigcup_{j \in J} V_j$;
- $f : D \to \bigcup_{j \in J} V_j$ is a definable map.

A definable map $f : D \to Y$ between a definable subset of R^m and a locally definable manifold Y is *continuous* if it is continuous when we put on D the induced topology from R^m.

Let $p_Y : Y \to T$ be a locally definable covering map, X be a locally definable manifold and let $f : X \to T$ be a locally definable map. A *lifting of* f is a continuous map $\widetilde{f} : X \to Y$ such that $p_Y \circ \widetilde{f} = f$. Note that a lifting of a continuous locally definable map need not be a locally definable map.

The proof of the following two lemmas can be easily be recovered adapting the results of [13], Section 2.

Lemma 2.7. *A locally definable covering map $p_X : X \to S$ is a continuous open surjection.*

Lemma 2.8. *Suppose that X is a locally definable manifold. Let $p_Y : Y \to T$ be a locally definable covering map and let $f, g : X \to Y$ be two continuous locally definable maps such that $p_Y \circ f = p_Y \circ g$. If X is definably connected and $f(x) = g(x)$ for some $x \in X$, then $f = g$.*

2.2. **Fundamental group and fundamental groupoid.** Here we recall the definitions of o-minimal fundamental groupoid and o-minimal fundamental group and prove their basic properties.

Let X be a locally definable manifold. A *path* $\alpha : [0, p] \to X$ is a continuous definable map. A path $\alpha : [0, p] \to X$ is *constant* if $\alpha(0) = \alpha(t)$ for all $t \in [0, p]$. A path $\alpha : [0, p] \to X$ is a *definable loop* if $\alpha(0) = \alpha(p)$. The *inverse* of a path $\alpha : [0, p] \to X$ is the path $\alpha^{-1} : [0, p] \to X$ given by $\alpha^{-1}(t) = \alpha(p - t)$ for all $t \in [0, p]$. A concatenation of two paths $\gamma : [0, p] \to X$ and $\delta : [0, q] \to X$ with

$\gamma(p) = \delta(0)$ is a path $\gamma \cdot \delta : [0, p+q] \to X$ with:

$$(\gamma \cdot \delta)(t) = \begin{cases} \gamma(t) & \text{if } t \in [0,p] \\ \\ \delta(t-p) & \text{if } t \in [p, p+q]. \end{cases}$$

We say that X is *definably path connected* if for every u, v in X there is a definable path $\alpha : [0, q] \to X$ such that $\alpha(0) = u$ and $\alpha(q) = v$.

Lemma 2.9. *Let X be a locally definable manifold. Then X is definably connected if and only if X is definably path connected. In fact, for any definably connected definable subset D of X there is a uniformly definable family of definable paths in D connecting a given fixed point in D to any other point in D.*

Proof. Suppose that X is definably connected with definable charts $(X_i, \theta_i)_{i \in I}$ each of which can be assumed without loss of generality to be definably connected. Then any definably connected definable subset D of X is contained in an open definably connected definable submanifold $\bigcup_{i \in I_0} X_i$ of X. Clearly, if each definable connected component of each $X_i \cap D$ ($i \in I_0$) is uniformly definable path connected, then so is D. Thus we have to show that for each $i \in I_0$, every definable connected definable subset C of X_i is uniformly definably path connected or equivalently, that every definable connected definable subset of $\theta_i(X_i) \subseteq R^{\dim X}$ is uniformly definably path connected. But this follows from by [9, Chapter 6, (3.2)] and its proof. The converse is immediate. □

Remark 2.10. *The previous proof is similar to* [13, Lemma 2.10] *but in* [13] *the argument uses group structure.*

Let X be a locally definable manifold. Given two definable continuous maps $f, g : Y \subseteq R^m \to X$, we say that a definable continuous map $F(t,s) : Y \times [0,q] \to X$ is a *definable homotopy between f and g* if $f = F_0$ and $g = F_q$, where $\forall s \in [0,q]$, $F_s := F(\cdot, s)$. In this situation we say that f and g are *definably homotopic*, denoted $f \sim g$.

Two definable paths $\gamma : [0,p] \to X$, $\delta : [0,q] \to X$, with $\gamma(0) = \delta(0)$ and $\gamma(p) = \delta(q)$, are called *definably homotopic* if there is some $t_0 \in [0, \min\{p,q\}]$, and a definable homotopy $F(t,s) : [0, \max\{p,q\}] \times [0,r] \to X$, for some $r > 0$ in R, between

$$\gamma_{|[0,t_0]} \cdot \mathbf{c} \cdot \gamma_{|[t_0,p]} \text{ and } \delta \text{ (if } p \leq q), \text{ or}$$

$$\delta_{|[0,t_0]} \cdot \mathbf{d} \cdot \delta_{|[t_0,q]} \text{ and } \gamma \text{ (if } q \leq p).$$

where $\mathbf{c}(t) = \gamma(t_0)$ and $\mathbf{d}(t) = \delta(t_0)$ are the constant definable paths with domain $[0, |p-q|]$.

Clearly, any two constant definable loops at the same point $c \in X$ are definably homotopic. We will thus write ϵ_c for the constant definable loop at c without specifying its domain.

Let X be a locally definable manifold and $x_0, x_1 \in X$. If $\mathbb{P}(X, x_0, x_1)$ denotes the set of all definable paths in X that start at x_0 and end at x_1, the restriction of $\sim$, the relation of being definably homotopic, to $\mathbb{P}(X, x_0, x_1) \times \mathbb{P}(X, x_0, x_1)$ is an equivalence relation on $\mathbb{P}(X, x_0, x_1)$.

If $\mathbb{L}(X, e_X)$ denotes the set of all definable loops that start and end at a fixed element e_X of X (i.e. $\mathbb{L}(X, e_X) = \mathbb{P}(X, e_X, e_X)$), the restriction of $\sim$ to $\mathbb{L}(X, e_X) \times$

$\mathbb{L}(X, e_X)$ is an equivalence relation on $\mathbb{L}(X, e_X)$. We define the *o-minimal fundamental group* $\pi_1(X, e_X)$ of X by

$$\pi_1(X, e_X) := \mathbb{L}(X, e_X)/\sim$$

and we set $[\gamma] :=$ the class of $\gamma \in \mathbb{L}(X, e_X)$. Note that $\pi_1(X, e_X)$ is indeed a group with group operation given by $[\gamma][\delta] = [\gamma \cdot \delta]$. Also this group depends on the topology on X.

If $f : X \to Y$ is a locally definable continuous map between two locally definable manifolds with $e_X \in X$ and $e_Y \in Y$ such that $f(e_X) = e_Y$, then we have an induced homomorphism $f_* : \pi_1(X, e_X) \to \pi_1(Y, e_Y) : [\sigma] \mapsto [f \circ \sigma]$ with the usual functorial properties.

We define the *o-minimal fundamental groupoid* $\Pi_1(X)$ of X to be the small category $\Pi_1(X)$ given by

$$\begin{aligned} &\mathrm{Ob}(\Pi_1(X)) = X, \\ &\mathrm{Hom}_{\Pi_1(X)}(x_0, x_1) = \mathbb{P}(X, x_0, x_1)/\sim . \end{aligned}$$

We set $[\gamma] :=$ the class of $\gamma \in \mathbb{P}(X, x_0, x_1)$. Note that $\Pi_1(X)$ is indeed a groupoid with operations $\mathrm{Hom}_{\Pi_1(X)}(x_0, x_1) \times \mathrm{Hom}_{\Pi_1(X)}(x_1, x_2) \to \mathrm{Hom}_{\Pi_1(X)}(x_0, x_2)$ given by $[\delta] \circ [\gamma] = [\gamma \cdot \delta]$.

Note that if $x \in X$, then $\mathbb{P}(X, x, x) = \mathbb{L}(X, x)$ and so

$$\pi_1(X, x) = \mathrm{Hom}_{\Pi_1(X)}(X, x, x).$$

If X is a locally definable manifold and $x \in X$, we define $\Pi_1(X, x)$ to be the category given by

$$\begin{aligned} &\mathrm{Ob}(\Pi_1(X, x)) = \{x\}, \\ &\mathrm{Hom}_{\Pi_1(X,x)}(x, x) = \pi_1(X, x). \end{aligned}$$

If $f : X \to Y$ is a locally definable continuous map between locally definable manifolds, then we have an induced functor $f_* : \Pi_1(X) \to \Pi_1(Y)$ which is a morphism of groupoids sending the object $x \in X$ to the object $f(x) \in Y$ and a morphism $[\gamma]$ of $\Pi_1(X)$ to the morphism $[f \circ \gamma]$ of $\Pi_1(Y)$.

Lemma 2.11. *Let X and Y be locally definable manifolds. Then*

1. *If X is definably connected then the natural functor $\Pi_1(X, x) \to \Pi_1(X)$ is an equivalence for every $x \in X$.*
2. *The natural functor $\Pi_1(X \times Y) \to \Pi_1(X) \times \Pi_1(Y)$ given by projection is an equivalence.*

Proof. (1) The functor $\Pi_1(X, x) \to \Pi_1(X)$ sends the object x of $\Pi_1(X, x)$ to the object x of $\Pi_1(X)$ and sends a morphism of $\Pi_1(X, x)$ represented by a definable loop at x to the morphim of $\Pi_1(X)$ represented by the same definable loop at x. By definition this morphism is fully faithfull. By Lemma 2.9, X is definably path connected, and so every object of $\Pi_1(X)$ is isomorphic to the object x. So the functor is also essentially surjective. Therefore, it is an equivalence.

(2) The functor $\Pi_1(X \times Y) \to \Pi_1(X) \times \Pi_1(Y)$ sends a morphism of $\Pi_1(X \times Y)$ represented by a definable path ρ in $X \times Y$ to the morphism of $\Pi_1(X) \times \Pi_1(Y)$ represented in each coordinate by the definable paths $q_1 \circ \rho$ in X and $q_2 \circ \rho$ in Y where q_1 and q_2 are the projections onto X and Y, respectively. This functor is an isomorphism with inverse given by the functor $\Pi_1(X) \times \Pi_1(Y) \to \Pi_1(X \times Y)$ that sends the object (x, y) of $\Pi_1(X) \times \Pi_1(Y)$ to the object (x, y) of $\Pi_1(X \times Y)$ and sends a morphism of $\Pi_1(X) \times \Pi_1(Y)$ represented by a pair of definable paths γ in

X and δ in Y to the morphism of $\Pi_1(X \times Y)$ represented by the definable path in $X \times Y$ with coordinates γ and δ. □

Corollary 2.12. *Let X and Y be locally definable manifolds with $e_X \in X$ and $e_Y \in Y$. Then*

1. *If X is definably connected then $\pi_1(X, e_X) \simeq \pi_1(X, x)$ for every $x \in X$.*
2. *$\pi_1(X, e_X) \times \pi_1(Y, e_Y) \simeq \pi_1(X \times Y, (e_X, e_Y))$.*

Notation: As usual for a definably connected locally definable manifold X if there is no need to mention a base point $e_X \in X$, then by Corollary 2.12 (1), we may denote $\pi_1(X, e_X)$ by $\pi_1(X)$.

2.3. **Locally definable covering maps and fundamental groups.** Below we introduce the crucial results relating the o-minimal fundamental groups and the locally definable covering maps. These results have analogues in the classical theory of topological covering maps and once we prove the locally definable analogue of the crucial lemma (Lemma 2.13) the remaining proofs are similar and so we omit them and refer reader to [22] or to [17], Section 2 for the definable case.

Lemma 2.13. *Suppose that $p_X : X \to S$ is a locally definable covering map. Then the following hold.*

1. *Let $\gamma : [0,p] \to X$ be a definable path in S and $x \in X$. If $p_X(x) = \gamma(0)$, then there is a unique definable path $\widetilde{\gamma} : [0,p] \to X$ in X, lifting γ, such that $\widetilde{\gamma}(0) = x$.*
2. *Suppose that $F : [0,p] \times [0,r] \to X$ is a definable homotopy between the definable paths γ and σ in S. Let $\widetilde{\gamma}$ be a definable path in X lifting γ. Then there is a unique definable lifting $\widetilde{F} : [0,p] \times [0,r] \to X$ of F, which is a definable homotopy between $\widetilde{\gamma}$ and $\widetilde{\sigma}$, where $\widetilde{\sigma}$ is a definable path in X lifting σ.*

Proof. The proof can be obtained adapting the results of [17], Section 2. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an admissible cover of S by open definable subsets over which $p_X : X \to S$ is trivial.

(1) Let $L \subseteq I$ be a finite subset such that $\gamma([0,p]) \subseteq \bigcup_{l \in L} U_l$. Then $[0,p] \subseteq \bigcup_{l \in L} \gamma^{-1}(U_l)$, with the $\gamma^{-1}(U_l)$'s open in $[0,p]$. Then, by [9, Chapter 6, (3.6)], for each $l \in L$ there is a $W_l \subset [0,p]$, open in $[0,p]$ such that $W_l \subset \overline{W_l} \subset \gamma^{-1}(U_l)$ and $[0,p] \subseteq \bigcup_{l \in L} W_l$. Therefore, there are $0 = s_0 < s_1 < \cdots < s_r = p$ such that for each $i = 0, \ldots, r-1$ we have $\gamma([s_i, s_{i+1}]) \subset U_{l(i)}$ (and $\gamma(s_{i+1}) \in U_{l(i)} \cap U_{l(i+1)}$). Lift $\gamma_1 = \gamma_{|[0,s_1]}$ to $\widetilde{\gamma}_1$, with $\widetilde{\gamma}_1(0) = x$, using the definable homeomorphism $p^{i_0}_{|U_{l(0)}} : U^{i_0}_{l(0)} \to U_{l(0)}$, where $U^{i_0}_{l(0)}$ is the definable connected component of $p^{-1}(U_{l(0)})$ in which x lays. Repeat the process for each $\gamma_{i+1} = \gamma_{|[s_i, s_{i+1}]}$ with $\widetilde{\gamma}_i(s_i)$ (intead of x). Patch the liftings together. Uniqueness follows (in each step) from Lemma 2.8.

(2) Let $L \subseteq I$ be a finite subset such that $F([0,p] \times [0,r]) \subseteq \bigcup_{l \in L} U_l$. Then $[0,p] \times [0,r] \subseteq \bigcup_{l \in L} F^{-1}(U_l)$, with the $F^{-1}(U_l)$'s open in $[0,p] \times [0,r]$. Then, by [9, Chapter 6, (3.6)], we have that for each $l \in L$ there is a $W_l \subset [0,p] \times [0,r]$, open in $[0,p] \times [0,r]$ such that $W_l \subset \overline{W_l} \subset F^{-1}(U_l)$ and $[0,p] \times [0,r] \subseteq \bigcup_{l \in L} W_l$. Now take a cell decomposition of R^2 compatible with the W_l's. This cell decomposition

induce a cell decomposition $(0 = t_0 < t_1 < \cdots < t_r = p)$ of $[0, p]$. For each two-dimensional cell C and each $s \in L$ such that $C \subset W_s$, we have $F(\overline{C}) \subset U_s$ and for any two-dimensional cells C_1 and C_2 in $[0,p] \times [0,r]$, and for each $s_1, s_2 \in L$ such that $C_1 \subset W_{s_1}$ and $C_2 \subset W_{s_2}$ we also have $F(\overline{C_1} \cap \overline{C_2}) \subset U_{s_1} \cap U_{s_2}$. Now we proceed as above lifting each $F_{|\overline{C}}$ using the relevant definable homeomorphism and then patching the liftings together (as in the classical case but working with the two–dimensional cells instead of rectangles); we start with the closure of the bottom two–dimensional cell above (t_0, t_1) and continue with the rest of the two-dimensional cells above (t_0, t_1), patching the liftings together; then we consider the next column of two-dimensional cells above (t_1, t_2) and we continue this way until we finish with the whole rectangle.

As above, uniqueness follows from Lemma 2.8. □

Notation: If $\gamma : [0, q] \to S$ is a definable path in S and $x \in X$, we denote by $x * \gamma$ the final point $\widetilde{\gamma}(q)$ of the lifting $\widetilde{\gamma}$ of γ with initial point $\widetilde{\gamma}(0) = x$.

As a consequence of Lemma 2.13 we have

Corollary 2.14. *Suppose that $p_X : X \to S$ is a locally definable covering map with $e_S \in S$ and $e_X \in X$ such that $p_X(e_X) = e_S$. Then the following hold.*

(1) *If σ is a definable loop in S that starts and ends at e_S, then $e_X = e_X * \sigma$ if and only if $[\sigma] \in p_{X*}(\pi_1(X, e_X))$.*
(2) *If σ and σ' are two definable paths in S from e_S to s, then $e_X * \sigma = e_X * \sigma'$ if and only if $[\sigma \cdot \sigma'^{-1}] \in p_{X*}(\pi_1(X, e_X))$.*

Here is an immediate consequence of Lemma 2.13 and Corollary 2.14:

Remark 2.15. Suppose that $p_X : X \to S$ is a locally definable covering map. Then, for each $s \in S$, there is a well defined right action

$$p_X^{-1}(s) \times \pi_1(S, s) \to p_X^{-1}(s) : (x, [\sigma]) \mapsto x * \sigma$$

such that the subgroup that acts trivially on a point $x \in p_X^{-1}(s)$ is $p_{X*}(\pi_1(X, x))$. If X is definably connected, then this action is transitive. So in this case, for a fixed $x \in p_X^{-1}(s)$ there is a canonical bijection

$$\pi_1(S, s)/p_{X*}(\pi_1(X, x)) \simeq p_X^{-1}(s).$$

If S is definably connected, then the action is transitive if and only if X is definably connected.

We obtain from Lemma 2.13 the following result:

Lemma 2.16. *Let $p : X \to S$ be a locally definable covering map. Then $p_* : \Pi_1(X) \to \Pi_1(S)$ is a faithful morphism and essentially surjective.*

Proof. Let $[\sigma], [\tau] \in \mathrm{Hom}_{\Pi_1(X)}(x_0, x_1)$ and suppose that we have $[p \circ \sigma] = [p \circ \tau] \in \mathrm{Hom}_{\Pi_1(S)}(p(x_0), p(x_1))$. By Lemma 2.13 a definable homotopy between $p \circ \sigma$ and $p \circ \tau$ lifts uniquely to a definable homotopy between σ (the unique lifting of $p \circ \sigma$ starting at x_0) and τ (the unique lifting of $p \circ \tau$ starting at x_0). So $[\sigma] = [\tau]$ as required. Since p is surjective, then p_* is essentially surjective. □

Corollary 2.17. *Let $p_X : X \to S$ be a locally definable covering map with $e_S \in S$ and $e_X \in X$ such that $p_X(e_X) = e_S$. Then $p_{X*} : \pi_1(X, e_X) \to \pi_1(S, e_S)$ is an injective homomorphism.*

We end this section with a necessary and sufficient condition for the existence of liftings of continuous locally definable maps.

Proposition 2.18. *Suppose $p_X : X \to S$ is a locally definable covering map with S definably connected. Let Y be a definably connected locally definable manifold and $f : Y \to S$ a continuous locally definable map. Let $e_S \in S, e_X \in X$ and $e_Y \in Y$ be such that $p_X(e_X) = f(e_Y) = e_S$. Then there is a continuous locally definable map $\widetilde{f} : Y \to X$ with $p_X \circ \widetilde{f} = f$ and $\widetilde{f}(e_Y) = e_X$ if and only if $f_*(\pi_1(Y, e_Y)) \subseteq p_{X*}(\pi_1(X, e_X))$. Such a lifting $\widetilde{f}$, when it exists, is unique.*

Proof. The necessity is clear from the functoriality of o-minimal fundamental groups, and the uniqueness follows from Lemma 2.8. The main point is to see that the lifting $\widetilde{f}$ is locally definable.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an admissible cover of S by open definably connected, definable subsets over which $p_X : X \to S$ is trivial. So for each $\alpha \in I$, $p_X^{-1}(U_\alpha) = \bigsqcup_{i \le \lambda_\alpha} U_\alpha^i$ and $p_{X|U_\alpha^i} : U_\alpha^i \to U_\alpha$ is a definable homeomorphism. For each $\alpha \in I$, let $\{V_\alpha^l : l \in L_\alpha\}$ be the definably connected components of $f^{-1}(U_\alpha)$. For all $\alpha \in I$, $l \in L_\alpha$, choose $y_\alpha^l \in V_\alpha^l$ such that if $e_Y \in V_\alpha^l$ then $e_Y = y_\alpha^l$, and, by Lemma 2.9, let η_α^l be a definable path in Y from e_Y to y_α^l. Since each V_α^l is definably connected, by Lemma 2.9 there is a uniformly definable family $\{\gamma_\alpha^l(w) : w \in V_\alpha^l\}$ of definable paths in V_α^l from y_α^l to w. For $w \in V_\alpha^l$, let $\delta_\alpha^l(w)$ be the definable path $\eta_\alpha^l \cdot \gamma_\alpha^l(w)$ from e_Y to w. Let $\sigma_\alpha^l(w) = f \circ \delta_\alpha^l(w)$. Then $\sigma_\alpha^l(w)$ is a definable path from e_S to $f(w)$. Set $\widetilde{f}(w) = e_X * \sigma_\alpha^l(w)$.

If $w \in V_\alpha^l \cap V_\beta^k$ then we have another definable path $\delta_\beta^k(w)$ from e_Y to w obtained from V_β^k, and $f \circ (\delta_\beta^k(w) \cdot (\delta_\alpha^l(w))^{-1}) = \sigma_\beta^k(w) \cdot (\sigma_\alpha^l(w))^{-1}$ is a definable path from e_S to e_S. By hypothesis, $[\sigma_\beta^k(w) \cdot (\sigma_\alpha^l(w))^{-1}] \in f_*(\pi_1(Y, e_Y)) \subseteq p_{X*}(\pi_1(X, e_X))$ and by Corollary 2.14 (2), $e_X * \sigma_\alpha^l(w) = e_X * \sigma_\beta^k(w)$ and so $\widetilde{f}$ is well defined. Note that the same argument shows that $\widetilde{f}$ does not depend on the choice of the points $y_\alpha^l \in V_\alpha^l$ or of the definable paths η_α^l. Furthermore, by construction, we clearly have

$$\widetilde{f}(e_Y) = e_X \ \text{ and } \ p_X \circ \widetilde{f} = f.$$

We now show that $\widetilde{f}$ is a locally definable continuous map. For this it is enough to show that each restriction $\widetilde{f}_{|V_\alpha^l}$ is a definable continuous map. But for $w \in V_\alpha^l$, we have $\widetilde{f}(w) = e_X * \sigma_\alpha^l(w)$ which is the endpoint of the lifting $\widetilde{\sigma_\alpha^l(w)}$ of $\sigma_\alpha^l(w)$ starting at e_X. Since $\sigma_\alpha^l(w) = (f \circ \eta_\alpha^l) \cdot (f \circ \gamma_\alpha^l(w))$, $\widetilde{f}(w)$ is the endpoint of the lifting $\widetilde{f \circ \gamma_\alpha^l(w)}$ of $f \circ \gamma_\alpha^l(w)$ starting at the endpoint $\widetilde{f \circ \eta_\alpha^l}(q_{\eta_\alpha^l})$ of the lifting $\widetilde{f \circ \eta_\alpha^l}$ of $f \circ \eta_\alpha^l$ starting at e_X. Note that since $f \circ \eta_\alpha^l(q_{\eta_\alpha^l}) = f(y_\alpha^l) \in U_\alpha$, there exists i such that $\widetilde{f \circ \eta_\alpha^l}(q_{\eta_\alpha^l}) = e_X * (f \circ \eta_\alpha^l) \in U_\alpha^i$. Furthermore, for each $w \in V_\alpha^l$, the lifting $\widetilde{f \circ \gamma_\alpha^l(w)}$ of $f \circ \gamma_\alpha^l(w)$ starting at $\widetilde{f \circ \eta_\alpha^l}(q_{\eta_\alpha^l})$ is contained in this U_α^i (since $f \circ \gamma_\alpha^l(w)$ is contained in U_α and by Lemma 2.13 the lifting $\widetilde{f \circ \gamma_\alpha^l(w)} : [0, q_{\gamma_\alpha^l(w)}] \to p_X^{-1}(U_\alpha)$, where $q_{\gamma_\alpha^l(w)}$ is the end point of the domain of $\gamma_\alpha^l(w)$, is continuous). Therefore, by uniqueness of such a lifting (Lemma 2.13),

$$\begin{aligned}\widetilde{f}_{|V_\alpha^l}(w) &= (p_{X|U_\alpha^i}^{-1} \circ (f \circ \gamma_\alpha^l(w)))(q_{\gamma_\alpha^l(w)}) \\ &= p_{X|U_\alpha^i}^{-1} \circ f(w)\end{aligned}$$

and the restriction $\widetilde{f}_{|V_\alpha^l}$ is a definable continuous map as required. □

Proposition 2.18 implies the following result

Corollary 2.19. *Suppose $p_X : X \to S$ and $p_Y : Y \to S$ are locally definable covering maps with X, Y and S definably connected. Let $e_S \in S, e_X \in X$ and $e_Y \in Y$ be such that $p_X(e_X) = p_Y(e_Y) = e_S$. If $p_{Y*}(\pi_1(Y, e_Y)) = p_{X*}(\pi_1(X, e_X))$, then there is a locally definable homeomorphism $\phi : Y \to X$ with $p_X \circ \phi = p_Y$ and $\phi(e_Y) = e_X$.*

As an immediate consequence of Corollary 2.19 we have

Corollary 2.20. *Suppose $p_X : X \to S$ is locally definable covering map with S definably connected and $\pi_1(S) = 1$. Then $p_X : X \to S$ is locally definably homeomorphic to a trivial locally definable covering map.*

3. O-minimal fundamental groups and universal covering maps

Here we prove one of the main results of the paper: (i) the existence of universal locally definable covering maps; (ii) invariance results for locally definable covering maps, o-minimal fundamental groups and o-minimal fundamental groupoids.

3.1. **The o-minimal fundamental group.** In this Subsection we will prove Theorem 1.1.

We start with the following fundamental result concerning the topology of locally definable manifolds, which is central to all our applications:

Proposition 3.1. *Let X be a locally definable manifold. Then there is an admissible cover $\{O_s\}_{s\in S}$ of X by open definably connected definable subsets such that:*

- *$\{O_s\}_{s\in S}$ refines the definable charts of X;*
- *for each $s \in S$, O_s is definably homeomorphic to an open cell in $R^{\dim X}$, in particular, the o-minimal fundamental group $\pi_1(O_s)$ is trivial.*

Proof. If $(X_i, \theta_i)_{i\in I}$ are the definable charts of X, then it is enough to show that each X_i has a finite cover $\{O_s\}_{s\in S_i}$ by open definably connected definable subsets each of which is definably homeomorphic to an open cell in $R^{\dim X}$. Equivalently it is enough to show that each $\theta_i(X_i)$ which is an open definable subset of $R^{\dim X}$ has a finite cover by open definably connected definable subsets each of which is definably homeomorphic to an open cell in $R^{\dim X}$. There are two cases to consider:

(i) $\mathcal{R}$ is an o-minimal expansion of a real closed field. Then we may replace $\theta_i(X_i)$ by a definably homeomorphic copy and assume that $\theta_i(X_i)$ is a bounded open definable subset of $R^{\dim X}$. In this situation, by [27], $\theta_i(X_i)$ is a finite union of open cells; Alternatively in this case one could also use the covers by proper sub-balls constructed in [5] or [12].

(ii) $\mathcal{R}$ is semi-bounded. Then by [14] and [1] we have that $\theta_i(X_i)$ is a finite union of open cells;

The proof of [13, Proposition 3.3] shows that open cells in $R^{\dim X}$ have trivial o-minimal fundamental groups. Therefore, the same is true for the o-minimal fundamental group of each O_s. □

By Remark 2.1 and the proof of Proposition 3.1 we have:

Remark 3.2. Let X be a Lindelöf locally definable manifold. Then there is a countable admissible cover $\{O_s\}_{s\in S}$ of X by open definably connected definable subsets such that:

- $\{O_s\}_{s\in S}$ refines the definable charts of X;
- for each $s \in S$, O_s is definably homeomorphic to an open cell in $R^{\dim X}$, in particular, $\pi_1(O_s) = 1$.

The proof of [13, Proposition 3.3] shows also that open cells in $\mathbb{R}^{\dim X}$ have trivial topological fundamental groups. Thus by Proposition 3.1 we have:

Remark 3.3. Suppose that $\mathcal{R}$ is an o-minimal expansion of the ordered group of real numbers. Let X be a locally definable manifold. Then there is an admissible cover $\{O_s\}_{s\in S}$ of X by open definably connected definable subsets such that:

- $\{O_s\}_{s\in S}$ refines the definable charts of X;
- for each $s \in S$, O_s is definably homeomorphic to an open cell in $\mathbb{R}^{\dim X}$, in particular, O_s is connected and the topological fundamental group $\pi_1^{\mathrm{top}}(O_s)$ is trivial.

Proof of Theorem 1.1: Let X be a definably connected Lindelöf locally definable manifold with $e_X \in X$.

Consider the countable admissible cover $\{O_s\}_{s\in S}$ of X by open definably connected, definably simply connected definable subsets given by Proposition 3.1 and Remark 3.2. For each pair of distinct elements $s, t \in S$ such that $O_s \cap O_t \neq \emptyset$ choose a point $a_{s,t} \in O_s \cap O_t$. For each pair $(a_{s,t}, a_{s',t'})$ of distinct points and $l \in \{s,t\} \cap \{s',t'\}$ let $\sigma^l_{s,t,s',t'}$ be a definable path in O_l from $a_{s,t}$ to $a_{s',t'}$. Also, for each $a_{s,t}$ such that $e_X \in O_s$, let $\sigma^s_{e_X,s,t}$ (respectively, $\sigma^s_{e_X,s,t}$) be a definable path in O_s from e_X to $a_{s,t}$ (respectively, from $a_{s,t}$ to e_X).

Let Σ be the countable collection of all definable paths $\sigma^l_{s,t,s',t'}$, $\sigma^s_{e_X,s,t}$ and $\sigma^s_{e_X,s,t}$ as above. Let K be the possibly infinite but countable simplicial complex of dimension one whose vertices are the end points of the definable paths in Σ and whose edges are the images of the definable paths in Σ. Clearly we have a homomorphism $\pi_1(|K|, e_X) \to \pi_1(X, e_X)$ which sends an edge loop in K into the definable loop it determines in X. This is well defined since if two edge loops are homotopic in $|K|$ then they are obviously definable homotopic in X. We now show that this homomorphism is surjective. Since the free group with generators set Σ is countable (it is a countable union of countable sets), $\pi_1(|K|, e_X)$ is a countable group and hence so is $\pi_1(X, e_X)$ as required.

Let $\gamma : [0,p] \to X$ be a definable loop in X at e_X. Then since $\{O_s\}_{s\in S}$ is an admissible cover of X there exists a finite subset $L \subseteq S$ such that $\gamma([0,p]) \subseteq \bigcup_{l\in L} O_l$. Then $[0,p] \subseteq \bigcup_{l\in L} \gamma^{-1}(O_l)$, with the $\gamma^{-1}(O_l)$'s open in $[0,p]$. Then, by [9, Chapter 6, (3.6)], for each $l \in L$ there is a $W_l \subset [0,p]$, open in $[0,p]$ such that $W_l \subset \overline{W_l} \subset \gamma^{-1}(O_l)$ and $[0,p] \subseteq \bigcup_{l\in L} W_l$. Therefore, there are $0 = s_0 < s_1 < \cdots < s_r = p$ such that for each $i = 0, \dots, r-1$ we have $\gamma([s_i, s_{i+1}]) \subset O_{l(i)}$ (and $\gamma(s_{i+1}) \in O_{l(i)} \cap O_{l(i+1)}$). Thus $\gamma = \gamma_0 \cdot \cdots \cdot \gamma_{r-1}$ where $\gamma_i = \gamma_{|[s_i,s_{i+1}]}$. For $i = 0, \dots, r-1$, let ϵ_i be a definable path in $O_{l(i)}$ from $a_{l(i),l(i+1)}$ to $\gamma_i(s_{i+1})$ and let δ_{i+1} be a definable path in $O_{l(i+1)}$ from $a_{l(i),l(i+1)}$ to $\gamma_{i+1}(s_{i+1})$. Let σ_0 be the definable path $\sigma^{l(0)}_{e_X,l(0),l(1)}$ in $O_{l(0)}$ and let $\sigma_{l(r)}$ be the definable path $\sigma^{l(r)}_{e_X,l(r-1),l(r)}$ in $O_{l(r)}$. Finally, for $i = 1, \dots, r-1$, let $\sigma_{l(i)}$ be the definable path $\sigma^{l(i)}_{l(i-1),l(i),l(i),l(i+1)}$

in $O_{l(i)}$. Since by Proposition 3.1, $\pi_1(O_{l(j)}) = 1$ for all $j = 0, \dots, r$, we have that σ_0 is definably homotopic to $\gamma_0 \cdot \epsilon_0^{-1}$, σ_r is definably homotopic to $\delta_r \cdot \gamma_r$ and, for each $i = 1, \dots, r-1$, σ_i is definably homotopic to $\delta_i \cdot \gamma_i \cdot \epsilon_i^{-1}$. Hence, γ is definably homotopic to $\sigma_0 \cdot \sigma_1 \cdot \dots \cdot \sigma_r$ as required.

Since $\pi_1(|K|, e_X)$ is countable, $\pi_1(X, e_X)$ is also countable. If X is definable, then K is finite simplicial complex of dimension one and as explained in [8, Chapter 3, Subsection 3.5.3], the fundamental group of a (finite) simplicial complex is finitely generated. Hence $\pi_1(X, e_X)$ is also finitely generated. □

3.2. **The universal locally definable covering map.** In this Subsection we will prove the existence of universal locally definable covering maps (Theorem 1.2).

Theorem 1.2 will be a consequence of the following stronger result:

Theorem 3.4. *Let X be a definably connected locally definable manifold with $e_X \in X$. For every subgroup $L \leq \pi_1(X), e_X$ there exists a locally definable covering map $v_L : V_L \to X$ with $e_{V_L} \in V_L$, V_L definably connected, $v_L(e_{V_L}) = e_X$ and $v_{L*}(\pi_1(V_L, e_{V_L})) = L$. Moreover, if X is Lindelöf (resp. paracompact), then V_L is also Lindelöf (resp. paracompact).*

Proof. This result was showed in [6, Theorem 5.11] in the semialgraic case and in [3, Fact 6.13] in o-minimal expansions of fields. By Proposition 3.1 one can do a similar proof which we include for the readers convinience.

Given two definable paths $\sigma : [0, q_\sigma] \to X$ and $\lambda : [0, q_\lambda] \to X$ in X, we put $\sigma \simeq \lambda$ if and only if $\sigma(0) = \lambda(0) = e_X$, $\sigma(q_\sigma) = \lambda(q_\lambda)$ and $[\sigma \cdot \lambda^{-1}] \in L \leq \pi_1(X, e_X)$. The relation $\simeq$ is an equivalence relation and we denote the equivalence class of σ under $\simeq$ by $\langle\sigma\rangle$.

Let $V := \{\langle\sigma\rangle : \sigma$ is a definable path in X such that $\sigma(0) = e_X\}$ and consider the well defined surjective map $v : V \to X : \langle\sigma\rangle \mapsto \sigma(q_\sigma)$. We will show that $v : V \to X$ is a locally definable covering map. Consider the admissible cover $\{O_s\}_{s \in S}$ of X by open definably connected, definably simply connected definable subsets given by Proposition 3.1. For each $s \in S$, we have $v^{-1}(O_s) = \{\langle\sigma\rangle : \sigma$ is a definable path in X such that $\sigma(0) = e_X$ and $\sigma(q_\sigma) \in O_s\}$. For each $s \in S$ fix a definable path $\sigma_s : [0, q_s] \to X$ such that $\sigma_s(0) = e_X$ and $\sigma_s(q_s) \in O_s$. Furthermore, assume also that if $e_X \in O_s$, then $\sigma_s = \epsilon_{e_X}$ (the trivial definable path at e_X).

Claim 3.5. *There is a well-defined bijection*

$$\phi_s : v^{-1}(O_s) \to O_s \times \pi_1(X, e_X)/L : \langle\lambda\rangle \mapsto (\lambda(q_\lambda), L[\lambda \cdot \eta \cdot \sigma_s^{-1}]),$$

where $\eta : [0, q_\eta] \to O_s$ is a definable path in O_s such that $\eta(0) = \lambda(q_\lambda)$ and $\eta(q_\eta) = \sigma_s(q_s)$.

Proof. Clearly, ϕ_s is well-defined, i.e. it does not depend on the choice of η since $\pi_1(O_s) = 1$ (Proposition 3.1) and for $\langle\lambda\rangle = \langle\lambda'\rangle$ we have $\lambda(q_\lambda) = \lambda(q_{\lambda'})$ and

$$\begin{aligned} L[\lambda \cdot \eta \cdot \sigma_s^{-1}] &= L[\lambda \cdot \lambda'^{-1} \cdot \lambda' \cdot \eta \cdot \sigma_s^{-1}] \\ &= L[\lambda \cdot \lambda'^{-1}][\lambda' \cdot \eta \cdot \sigma_s^{-1}] \\ &= L[\lambda' \cdot \eta \cdot \sigma_s^{-1}]. \end{aligned}$$

Also, for $o \in O_s$ and $L[\gamma] \in \pi_1(X, e_X)/L$ we have $\phi_s(\langle\lambda\rangle) = (o, L[\gamma])$ for $\lambda = \gamma \cdot \sigma_s \cdot \eta^{-1}$, where $\eta : [0, q_\eta] \to X$ is a definable path in O_s such that $\eta(0) = o$ and $\eta(q_\eta) = \sigma_s(q_s)$. Thus ϕ_s is surjective. On the other hand, suppose that

$\phi_s(\langle\lambda\rangle) = \phi_s(\langle\lambda'\rangle)$. Then $\lambda(q_\lambda) = \lambda'(q_{\lambda'})$ and $L[\lambda \cdot \eta \cdot \sigma_s^{-1}] = L[\lambda' \cdot \eta' \cdot \sigma_s^{-1}]$. Therefore $[\lambda' \cdot \eta' \cdot \sigma_s^{-1}][\lambda \cdot \eta \cdot \sigma_s^{-1}]^{-1} \in L$. But we also have

$$\begin{aligned}[\lambda' \cdot \eta' \cdot \sigma_s^{-1}][\lambda \cdot \eta \cdot \sigma_s^{-1}]^{-1} &= [\lambda' \cdot \eta' \cdot \sigma_s^{-1}][\sigma_s \cdot \eta^{-1} \cdot \lambda^{-1}] \\ &= [\lambda' \cdot \eta \cdot \sigma_s^{-1}][\sigma_s \cdot \eta^{-1} \cdot \lambda^{-1}] \\ &= [\lambda' \cdot \lambda^{-1}]\end{aligned}$$

(the fact $\pi_1(O_s) = 1$ (Proposition 3.1) implies that η and η' are definably homotopic and so $\lambda' \cdot \eta \cdot \sigma_s^{-1}$ is definably homotopic to $\lambda' \cdot \eta' \cdot \sigma_s^{-1}$). Thus we have $[\lambda \cdot \lambda'^{-1}] \in L$, $\langle\lambda\rangle = \langle\lambda'\rangle$ and ϕ_s is injective. □

For each $s \in S$ and $L[\gamma] \in \pi_1(X, e_X)/L$, set $O_s^{L[\gamma]} := \phi_s^{-1}(O_s \times \{L[\gamma]\})$. Then by Claim 3.5 and its proof we have, for each $s \in S$,

- $v^{-1}(O_s) = \bigsqcup_{L[\gamma]\in\pi_1(X,e_X)/L} O_s^{L[\gamma]}$;
- each $v_{|O_s^{L[\gamma]}} : O_s^{L[\gamma]} \to O_s$ is a bijection.

By [16, Lemma 2.1 (1)] (see Remark 2.6), it follows that there exists a locally definable manifold structure on V of dimension $n := \dim X$ such that $v : V \to X$ is a locally definable covering map trivial over $\mathcal{O} = \{O_s\}_{s\in S}$.

Note also that if $e_V := \langle\epsilon_{e_X}\rangle \in V$ (the equivalence class of the trivial definable loop at e_X), then $v(e_V) = e_X$.

Claim 3.6. *Let $\langle\lambda\rangle \in V$ with $\lambda : [0, q_\lambda] \to X$ a definable path in X such that $\lambda(0) = e_X$. Then the unique continuous definable lifting $\widetilde{\lambda} : [0, q_\lambda] \to V$ of λ starting at e_V satisfies $\widetilde{\lambda}(q) = \langle\lambda_{|[0,q]}\rangle$ for all $q \in [0, q_\lambda]$. In particular, V is definably path connected and $v_*(\pi_1(V, e_V)) = L \leq \pi_1(X, e_X)$.*

Proof. For $q \in [0, q_\lambda]$ let $\lambda_q : [0, q] \to X$ be the definable path in X given by $\lambda_q = \lambda_{|[0,q]}$. Note that $\lambda_0 = \epsilon_{e_X}$ and $\lambda_{q_\lambda} = \lambda$. Let $\widetilde{\lambda} : [0, q_\lambda] \to V$ be the unique continuous definable lifting of λ starting at e_V. So $\widetilde{\lambda}(0) = e_V$ and $v \circ \widetilde{\lambda} = \lambda$. We have to show that $\widetilde{\lambda}(q) = \langle\lambda_q\rangle$ for all $q \in [0, q_\lambda]$.

Since λ is a definable path and $\{O_s\}_{s\in S}$ is an admissible cover of X by open definable subsets, there exists points $0 = q_0 < q_1 < \cdots < q_k < q_{k+1} = q_\lambda$ such that for each $j = 0, \ldots, k$, we have $\lambda([q_j, q_{j+1}]) \subseteq O_{s(j)}$ for some $s(j) \in S$. Since $\widetilde{\lambda} : [0, q_\lambda] \to V$ is continuous, we have:

(1) If $\widetilde{\lambda}(q_j) \in O_s^{L[\gamma]}$ then $\widetilde{\lambda}(q) \in O_s^{L[\gamma]}$ for all $q \in [q_j, q_{j+1}]$.

On the other hand, we also have that:

(2) If $\langle\lambda_{q_j}\rangle \in O_s^{L[\delta]}$ then $\langle\lambda_q\rangle \in O_s^{L[\delta]}$ for all $q \in [q_j, q_{j+1}]$.

Indeed, for every $q \in [q_j, q_{j+1}]$ we have $L[\delta] = L[\lambda_{q_j} \cdot \eta_1 \cdot \sigma_s^{-1}] = L[\lambda_q \cdot \eta_2 \cdot \sigma_s^{-1}]$ since if $\delta : [0, q - q_j] \to X$ is the definable path given by $\delta(t) = \lambda(q_j + t)$, then $\lambda_q = \lambda_{q_j} \cdot \delta$ and $\delta \cdot \eta_2$ is definably homotopic to η_1 since $\pi_1(O_s) = 1$ (Proposition 3.1).

Now note that since $v(\widetilde{\lambda}(q)) = \lambda(q) = v(\langle\lambda_q\rangle)$, we have $\widetilde{\lambda}(q) = \langle\lambda_q\rangle$ if and only if whenever $\lambda(q) \in O_s$ then both $\widetilde{\lambda}(q)$ and $\langle\lambda_q\rangle$ belong to the same $O_s^{L[\gamma]}$. But since $e_V = \langle\epsilon_{e_X}\rangle \in O_{s(0)}^{L[\epsilon_{e_X}]}$ and $\widetilde{\lambda}(0) = e_V$ and $\lambda_0 = \epsilon_{e_X}$, by (1) and (2) above we obtain $\widetilde{\lambda}(q) = \langle\lambda_q\rangle$ for all $q \in [0, q_\lambda]$ as required. □

From Claim 3.6, it follows in particular that V is definably path connected and so by Lemma 2.9, V is definably connected. It remains to show that $v_*(\pi_1(V, e_V)) = L$. By Lemma 2.13, any definable loop δ in V at e_V is the unique lifting $\widetilde{\lambda}$ of a definable loop $\lambda = v \circ \delta$ in X at e_X. By Claim 3.6, $\langle \epsilon_{e_X} \rangle = e_V = \langle \lambda \rangle$. This implies that $[\lambda] \in L$ and so $v_*([\widetilde{\lambda}]) = [\lambda] \in L$. Conversely, if $[\lambda] \in L$, then $\langle \epsilon_{e_X} \rangle = e_V = \langle \lambda \rangle$ and by Claim 3.6, $[\widetilde{\lambda}] \in \pi_1(V, e_V)$ and $[\lambda] = v_*([\widetilde{\lambda}]) \in v_*(\pi_1(V, e_V))$.

By construction and Remarks 2.1 and 3.2 (resp. Remark 2.2), if X is Lindelöf (resp. paracompact), then V is also Lindelöf (resp. paracompact). □

By Lemma 2.8 and Theorem 3.4 we have:

Remark 3.7. Let $p : Y \to X$ be a locally definable covering map with $e_X \in X$, $e_Y \in Y$ and $p(e_Y) = e_X$. If X and Y are definably connected, then $p : Y \to X$ is locally definably isomorphic to a locally definable covering map $v_L : V_L \to X$ with $e_{V_L} \in V_L$ as constructed in Theorem 3.4.

Let X be a definably connected locally definable manifold with $e_X \in X$. A locally definable covering map $u : U \to X$ with U is definable connected is a called a *universal locally definable covering map* if:

- For every locally definable covering map $p : Y \to X$ with Y is definably connected, there exists a locally definable covering map $q : U \to Y$ such that $u = p \circ q$.
- $u : U \to X$ is unique up to locally definable covering homeomorphisms fixing the base points with the above universal property.

From Remark 3.7 we immediately obtain:

Remark 3.8. Let X be a definably connected locally definable manifold with $e_X \in X$. A locally definable covering map $u : U \to X$ with $e_U \in U$, U definable connected and $u(e_U) = e_X$ is a universal locally definable covering map if and only if $\pi_1(U, e_U) = 1$.

Proof of Theorem 1.2: Let X be a definably connected locally definable manifold. If we fix $e_X \in X$ and take $L = 1$ in Theorem 3.4 we get a locally definable covering map $u : U \to X$ with $e_U \in U$, U is definable connected, $u(e_U) = e_X$ and $u_*(\pi_1(U, e_U)) = 1$. Since by Lemma 2.17 the induced homomorphism is injective we have $\pi_1(U, e_U) \simeq u_*(\pi_1(U, e_U)) = 1$ and so the result follows from Remark 3.8. □

3.3. **Regular locally definable covering maps.** Let $p_X : X \to S$ be a locally definable covering map. The *group of locally definable covering homeomorphisms* is $\mathrm{Aut}_{p_X}(X/S) = \{\phi : X \to X : \phi$ is a locally definable homeomorphism such that $p_X = p_X \circ \phi\}$. (We often omit the subscript p_X if it is clear from the context). Note that if we put on $\mathrm{Aut}(X/S)$ the discrete topology, then we have a continuous locally definable action

$$\mathrm{Aut}(X/S) \times X \to X : x \mapsto \phi(x)$$

which induces an action on each fiber

$$\mathrm{Aut}(X/S) \times p_X^{-1}(s) \to p_X^{-1}(s)$$

where $s \in S$.

We say that a locally definable covering map $p_X : X \to S$ with X and S definably connected is *regular* if the action of $\mathrm{Aut}(X/S)$ on each fiber $p_X^{-1}(s)$ is transitive i.e. for any $x_1, x_2 \in p_X^{-1}(s)$ there is $\phi \in \mathrm{Aut}(X/S)$ such that $\phi(x_1) = x_2$. Since X is definably connected, by Lemma 2.8 such ϕ is unique. Thus, $p_X : X \to S$ is regular if and only if for each $x \in X$ the induces map

$$\mathrm{Aut}(X/S) \to X : \phi \mapsto \phi(x)$$

is a bijection.

As usual, given a subgroup L of a group G, we denote by $N_G(L) = \{g \in G : gL = Lg\}$ the normalizer of L in G.

We have the following useful characterization of regular locally definable covering maps.

Theorem 3.9. *Let X be a definably connected locally definable manifold with $e_X \in X$. If $p : Y \to X$ is a locally definable covering map with $e_Y \in Y$, Y definably connected and $p(e_Y) = e_X$ then there exists a canonical isomorphism*

$$N_{\pi_1(X,e_X)}(p_*(\pi_1(Y, e_Y)))/p_*(\pi_1(Y)) \to \mathrm{Aut}(Y/X).$$

Moreover, $p : Y \to X$ is regular if and only if $p_*(\pi_1(Y, e_Y)) \trianglelefteq \pi_1(X, e_X)$.

For the proof of this result we first require a couple of lemmas.

Lemma 3.10. *Let X be a definably connected locally definable manifold, $e_X \in X$, $L \leq \pi_1(X, e_X)$ and $v_L : V_L \to X$ the corresponding locally definable covering map with $e_{V_L} \in V_L$ and $v_{L*}(\pi_1(V_L, e_{V_L})) = L$. For each $[\delta] \in \pi_1(X, e_X)$ the map*

$$\phi_{[\delta]} : V_L \to V_L$$

given by $\phi_{[\delta]}(\langle\lambda\rangle) := \langle\delta\cdot\lambda\rangle$ is well defined if and only if $[\delta] \in N_{\pi_1(X,e_X)}(L)$. Furthermore, for every $[\delta] \in N_{\pi_1(X,e_X)}(L)$ the map $\phi_{[\delta]} : V_L \to V_L$ is a continuous locally definable map and we have:

- $v_L \circ \phi_{[\delta]} = v_L$;
- $\phi_{[\delta]}$ *is a locally definable homeomorphism with inverse* $\phi_{[\delta]^{-1}}$;
- $\phi_{[\delta_1][\delta_2]} = \phi_{[\delta_1]} \circ \phi_{[\delta_2]}$ *for every* $[\delta_1], [\delta_2] \in N_{\pi_1(X,e_X)}(L)$.

Proof. The map $\phi_{[\delta]} : V_L \to V_L$ is well defined if and only if $\langle\delta \cdot \lambda\rangle = \langle\delta' \cdot \lambda'\rangle$ whenever $\langle\lambda\rangle = \langle\lambda'\rangle$ and $[\delta] = [\delta']$. Suppose the map is well defined. Then for all $[\lambda] \in L$, we have $\langle\lambda\rangle = \langle\epsilon_{e_X}\rangle$ and so $\langle\delta \cdot \lambda\rangle = \langle\delta\rangle$ i.e., $[\delta][\lambda][\delta]^{-1} = [\delta \cdot \lambda \cdot \delta^{-1}] \in L$. Hence $[\delta]L[\delta]^{-1} \subseteq L$ i.e., $[\delta] \in N_{\pi_1(X,e_X)}(L)$. Conversely, if $[\delta] \in N_{\pi_1(X,e_X)}(L)$, then whenever $\langle\lambda\rangle = \langle\lambda'\rangle$ and $[\delta] = [\delta']$ we have

$$\begin{array}{lcl} [\lambda' \cdot \lambda^{-1}] \in L & \Rightarrow & [\delta][\lambda' \cdot \lambda^{-1}][\delta]^{-1} \in [\delta]L[\delta]^{-1} \subseteq L \\ & \Rightarrow & [\delta'][\lambda' \cdot \lambda^{-1}][\delta]^{-1} \in L \\ & \Rightarrow & [\delta' \cdot \lambda' \cdot (\delta \cdot \lambda)^{-1}] \in L \\ & \Rightarrow & \langle\delta \cdot \lambda\rangle = \langle\delta' \cdot \lambda'\rangle \end{array}$$

and $\phi_{[\delta]} : V_L \to V_L$ is well defined.

Now let $[\delta], [\delta_1], [\delta_2] \in N_{\pi_1(X,e_X)}(L)$. Then it is clear from the definition that:

- $v_L \circ \phi_{[\delta]} = v_L$;
- $\phi_{[\delta]}$ is a bijection with inverse $\phi_{[\delta^{-1}]}$;
- $\phi_{[\delta_1][\delta_2]} = \phi_{[\delta_1]} \circ \phi_{[\delta_2]}$.

It remains to show that $\phi_{[\delta]} : V_L \to V_L$ is a continuous locally definable map. For $s \in S$ and $L[\gamma] \in \pi_1(X, e_X)/L$ fix $\langle\lambda_0\rangle \in O_s^{L[\gamma]}$ (with the notation from the proof of Theorem 3.4). We claim first that for any other $\langle\lambda\rangle \in O_s^{L[\gamma]}$ we can assume that $\lambda = \lambda_0 \cdot \sigma$ for some $\sigma : [0, q_\sigma] \to O_s$ such that $\sigma(0) = v_L(\langle\lambda_0\rangle)$ and $\sigma(q_\sigma) = v_L(\langle\lambda\rangle)$. In fact, since $O_s^{L[\gamma]}$ is definably path connected, there is a definable path $\widetilde{\sigma} : [0, q_{\widetilde{\sigma}}] \to O_s^{L[\gamma]}$ such that $\widetilde{\sigma}(0) = \langle\lambda_0\rangle$ and $\widetilde{\sigma}(q_{\widetilde{\sigma}}) = \langle\lambda\rangle$, now take $\sigma = v_L \circ \widetilde{\sigma}$. Then the unique continuous definable lifting of $\lambda_0 \cdot \sigma$ starting at e_{V_L} is $\widetilde{\lambda_0} \cdot \widetilde{\sigma}$ and by Claim 3.6, its endpoint is $\langle\lambda_0 \cdot \sigma\rangle = \langle\lambda\rangle$.

Now observe that we have $L[\lambda_0 \cdot \eta_0 \cdot \sigma_s^{-1}] = L[\gamma] = L[\lambda \cdot \eta \cdot \sigma_s^{-1}]$ for any other $\langle\lambda\rangle \in O_s^{L[\gamma]}$. Moreover, if $\lambda = \lambda_0 \cdot \sigma$ as above, then we can take $\eta = \sigma^{-1} \cdot \eta_0$ and obtain

$$\begin{aligned} L[\delta \cdot \lambda \cdot \eta \cdot \sigma_s^{-1}] &= L[\delta \cdot (\lambda_0 \cdot \sigma) \cdot (\sigma^{-1} \cdot \eta_0) \cdot \sigma_s^{-1}] \\ &= L[\delta \cdot \lambda_0 \cdot \eta_0 \cdot \sigma_s^{-1}]. \end{aligned}$$

Thus, if $[\gamma'] := [\delta \cdot \lambda_0 \cdot \eta_0 \cdot \sigma_s^{-1}] \in \pi_1(X, e_X)$, then $\phi_{[\delta]|O_s^{L[\gamma]}} : O_s^{L[\gamma]} \to O_s^{L[\gamma']}$ is a bijection with inverse $\phi_{[\delta^{-1}]|O_s^{L[\gamma']}}$. Since $v_L \circ \phi_{[\delta]} = v_L$ we have $\phi_{[\delta]|O_s^{L[\gamma]}} = (v_{L|O_s^{L[\gamma']}})^{-1} \circ v_{L|O_s^{L[\gamma]}}$ and so $\phi_{[\delta]} : V_L \to V_L$ is a continuous locally definable map as required. □

Lemma 3.11. *Let X be a definably connected locally definable manifold, $e_X \in X$, $L \leq \pi_1(X, e_X)$ and $v_L : V_L \to X$ the corresponding locally definable covering map with $e_{V_L} \in V_L$ and $v_{L*}(\pi_1(V_L, e_{V_L})) = L$. There exists a canonical continuous locally definable action*

$$N_{\pi_1(X,e_X)}(L)/L \times V_L \to V_L$$

given by $L[\delta]\langle\lambda\rangle := \phi_{[\delta]}(\langle\lambda\rangle) = \langle\delta \cdot \lambda\rangle$, where on $N_{\pi_1(X,e_X)}(L)$ we put the discrete topology. Furthermore, there exists a canonical isomorphism

$$N_{\pi_1(X,e_X)}(L)/L \to \mathrm{Aut}(V_L/X)$$

and $v_L : V_L \to X$ is regular if and only if $L \trianglelefteq \pi_1(X, e_X)$.

Proof. First we show that the map $N_{\pi_1(X,e_X)}(L)/L \times V_L \to V_L$ is well defined. But

$$\begin{aligned} L[\delta] = L[\delta'] \wedge \langle\lambda\rangle = \langle\lambda'\rangle &\Rightarrow [\delta']L = L[\delta] \wedge [\lambda' \cdot \lambda^{-1}] \in L \\ &\Rightarrow [\delta']L[\delta]^{-1} = L \wedge [\lambda' \cdot \lambda^{-1}] \in L \\ &\Rightarrow [\delta' \cdot \lambda' \cdot \lambda^{-1} \cdot \delta^{-1}] \in L \\ &\Rightarrow [(\delta' \cdot \lambda') \cdot (\delta \cdot \lambda)^{-1}] \in L \\ &\Rightarrow \langle\delta' \cdot \lambda'\rangle = \langle\delta \cdot \lambda\rangle \\ &\Rightarrow L[\delta']\langle\lambda'\rangle = L[\delta]\langle\lambda\rangle. \end{aligned}$$

By Lemma 3.10 the map $N_{\pi_1(X,e_X)}(L)/L \times V_L \to V_L$ is a continuous locally definable map and so it remains to show that it is an action. But

- $$\begin{aligned} L[\delta_2](L[\delta_1]\langle\lambda\rangle) &= L[\delta_2]\langle\delta_1 \cdot \lambda\rangle \\ &= \langle\delta_2 \cdot (\delta_1 \cdot \lambda)\rangle \\ &= \langle(\delta_2 \cdot \delta_1) \cdot \lambda\rangle \\ &= L[\delta_2 \cdot \delta_1]\langle\lambda\rangle \\ &= (L[\delta_2] \cdot [\delta_1])\langle\lambda\rangle \\ &= (L[\delta_2] \cdot L[\delta_1])\langle\lambda\rangle \end{aligned}$$
  since $[\delta_2]L = L[\delta_2]$ because $[\delta_2] \in N_{\pi_1(X,e_X)}(L)$.
- $$\begin{aligned} L[\epsilon_{e_X}]\langle\lambda\rangle &= \langle\epsilon_{e_X} \cdot \lambda\rangle \\ &= \langle\lambda\rangle. \end{aligned}$$
- For a fixed $L[\delta] \in N_{\pi_1(X,e_X)}(L)/L$, the map $V_L \to V_L : \langle\lambda\rangle \mapsto L[\delta]\langle\lambda\rangle = \phi_{[\delta]}(\langle\lambda\rangle)$ is a bijection by Lemma 3.10.

By Lemma 3.10 we have a canonical homomorphism

$$N_{\pi_1(X,e_X)}(L) \to \mathrm{Aut}(V_L/X) : [\sigma] \mapsto \phi_{[\sigma]}$$

whose kernel is clearly L. So we must show that this homomorphism is surjective. Take $\phi \in \mathrm{Aut}(V_L/X)$. Then ϕ is determined by $\phi(e_{V_L})$ (by Lemma 2.8). (Recall that $e_{V_L} := \langle \epsilon_{e_X} \rangle$). Let $\langle \delta \rangle := \phi(e_{V_L}) = \phi(\langle \epsilon_{e_X} \rangle)$. Note that $\langle \delta \rangle \in v_L^{-1}(e_X)$ (because ϕ acts on $v_L^{-1}(e_X)$) and by Claim 3.6 $\langle \delta \rangle$ is the endpoint of the unique continuous definable lifting $\widetilde{\delta}$ of δ starting at e_{V_L}. So $[\delta] \in \pi_1(X, e_X)$. We have to show that $[\delta] \in N_{\pi_1(X,e_X)}(L)$ since that implies that $\phi_{[\delta]}$ is well defined, and since $\phi(\langle \epsilon_{e_X} \rangle) = \langle \delta \rangle = \phi_{[\delta]}(\langle \epsilon_{e_X} \rangle)$, we obtain from Lemma 2.8 that $\phi = \phi_{[\delta]}$. Take $[\lambda] \in L$. Then $\langle \epsilon_{e_X} \rangle = \langle \lambda \rangle$ and by Claim 3.6 the unique continuous definable lifting $\widetilde{\lambda}$ of λ starting at e_{V_L} is a definable loop at e_{V_L} (since $\langle \lambda \rangle$ is its endpoint). Since $v_L \circ \phi = v_L$, $\phi \circ \widetilde{\lambda}$ is the unique continuous definable lifting of λ starting at $\langle \delta \rangle$ and it is a definable loop at $\langle \delta \rangle$. Altogether this means that the unique continuous definable lifting of $\delta \cdot \lambda \cdot \delta^{-1}$ starting at e_{V_L} is $\widetilde{\delta} \cdot \phi \circ \widetilde{\lambda} \cdot \widetilde{\delta}^{-1}$ which is a definable loop at e_{V_L}. Hence, by Corollary 2.14 (1), $[\delta][\lambda][\delta]^{-1} = [\delta \cdot \lambda \cdot \delta^{-1}] \in L = v_{L*}(\pi_1(V_L, e_{V_L}))$ as required.

Suppose that $v_L : V_L \to X$ is regular. Let $[\delta] \in \pi_1(X, e_X)$. Then $\langle \delta \rangle \in v_L^{-1}(e_X)$ and there is a unique $\phi \in \mathrm{Aut}(V_L/X)$ such that $\langle \delta \rangle = \phi(\langle \epsilon_{e_X} \rangle)$. By the above $[\delta] \in N_{\pi_1(X,e_X)}(L)$ and so $\pi_1(X, e_X) = N_{\pi_1(X,e_X)}(L)$. Conversely, suppose that $L \trianglelefteq \pi_1(X, e_X)$. Let $\langle \delta \rangle \in v_L^{-1}(e_X)$. Then $[\delta] \in \pi_1(X, e_X) = N_{\pi_1(X,e_X)}(L)$, so $\phi_{[\delta]} \in \mathrm{Aut}(V_L/X)$ and $\langle \delta \rangle = \phi_{[\delta]}(\langle \epsilon_{e_X} \rangle)$. So the action of $\mathrm{Aut}(V_L/X)$ on $v_L^{-1}(e_X)$ is transitive and since e_X is arbitrary the same is true for the action on any other fiber. □

Proof of Theorem 3.9: This now follows at once from Remark 3.7 and Lemma 3.11. □

3.4. **The invariance results.** In this Subsection we prove the invariance results for the universal locally definable covering map, the o-minimal fundamental group and the o-minimal fundamental groupoid.

By Proposition 3.1, Corollary 2.20 together with [16, Corollary 2.2] (see Remark 2.6) we have:

Remark 3.12. Let $\mathcal{K}$ be a reduct of $\mathcal{R}$ which is still an o-minimal expansion of an ordered group or an elementary substructure of $\mathcal{R}$. Let X be a $\mathcal{K}$-definably connected locally $\mathcal{K}$-definable manifold defined without parameters and $p : Y \to X$ a locally definable covering map. Let also $\{O_s\}_{s \in S}$ be an admissible cover of X by open $\mathcal{K}$-definably connected $\mathcal{K}$-definable subsets defined without parameters given by Proposition 3.1. Then $p : Y \to X$ is locally definably homeomorphic to a locally $\mathcal{K}$-definable covering map trivial over $\{O_s\}_{s \in S}$.

By Proposition 3.1 , Corollary 2.20 together with [16, Corollary 2.3] (see Remark 2.6) we have:

Remark 3.13. Suppose that $\mathcal{R}$ is an o-minimal expansion of the ordered group of real numbers. Let X be a definably connected locally definable manifold defined without parameters and $p : Y \to X$ a topological covering map. Let also $\{O_s\}_{s \in S}$ be an admissible cover of X by open definably connected definable subsets defined without parameters given by Proposition 3.1. Then $p : Y \to X$ is topologically homeomorphic to a locally definable covering map trivial over $\{O_s\}_{s\in S}$.

We also have the following converse of both Remarks 3.12 and 3.13:

Remark 3.14. Let $\mathcal{K}$ be a reduct of $\mathcal{R}$ which is still an o-minimal expansion of an ordered group or an elementary substructure of $\mathcal{R}$. Suppose $p_X : X \to S$ is a locally $\mathcal{K}$-definable covering map defined without parameters with S and X both $\mathcal{K}$-definably connected. Then $p_X : X \to S$ is a locally definable covering map defined without parameters with S and X definably connected.

That $p_X : X \to S$ is a locally definable covering map defined without parameters is clear we just need to verify that S and X are also definably connected. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be an admissible cover of S by open $\mathcal{K}$-definably connected, $\mathcal{K}$-definable subsets defined without parameters over which $p_X : X \to S$ is trivial. So for each $\alpha \in I$, $p_X^{-1}(U_\alpha) = \bigsqcup_{i\leq\lambda} U_\alpha^i$ and $p_{X|U_\alpha^i} : U_\alpha^i \to U_\alpha$ is a $\mathcal{K}$-definable homeomorphism. Since, by cell decomposition, for $\mathcal{K}$-definable sets defined without parameters $\mathcal{K}$-definably connected is the same as definably connected, using the sets U_α's and U_α^i's the result follows.

Remark 3.15. Let $\mathcal{R}$ be an o-minimal expansion of the set of real numbers. Suppose $p_X : X \to S$ is a locally definable covering map defined with S and X both definably connected. Then $p_X : X \to S$ is a topological covering map with S and X connected.

That $p_X : X \to S$ is a topological covering map is clear we just need to verify that S and X are also connected. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be an admissible cover of S by open definably connected, definable subsets over which $p_X : X \to S$ is trivial. So for each $\alpha \in I$, $p_X^{-1}(U_\alpha) = \bigsqcup_{i\leq\lambda} U_\alpha^i$ and $p_{X|U_\alpha^i} : U_\alpha^i \to U_\alpha$ is a definable homeomorphism. Since, by cell decomposition, for definable sets definably connected is the same as connected, using the sets U_α's and U_α^i's the result follows.

Proof of Theorem 1.3: Suppose $\mathcal{J}$ is an elementary extension of $\mathcal{R}$ or an o-minimal expansion of $\mathcal{R}$. Let X be a definably connected locally definable manifold.

Fix $e_X \in X$. Clearly $X(J)$ is a $\mathcal{J}$-definable manifold. Consider the admissible cover $\{O_s\}_{s\in S}$ of X by open definably connected definable subsets given by Proposition 3.1. Using the $\mathcal{J}$-definably connectedness of the $O_s(J)$'s and the definable connectedness of X it follows that $X(J)$ is $\mathcal{J}$-definably connected. Therefore, by Theorem 1.2, $X(J)$ has a universal locally $\mathcal{J}$-definable covering map which is by Remark 3.12, up to locally $\mathcal{J}$-definably covering homeomorphism, of the form $w^J : W(J) \to X(J)$ where $w : W \to X$ is a locally definable covering map, with W definably connected, $e_W \in W$ and $w(e_W) = e_X$. In particular, $W(J)$ is also $\mathcal{J}$-definably connected and by Remark 3.8, we also have $\pi_1^{\mathcal{J}}(W(J), e_W) = 1$.

Let $u : U \to X$ be a universal locally definable map with U definably connected, $e_U \in U$ and $u(e_U) = e_X$. By Remark 3.8, $\pi_1(U, e_U) = 1$. Also there exists a locally definable covering map $q : U \to W$ such that

$$\begin{array}{ccc} U & \xrightarrow{q} & W \\ & \searrow^{u} & \downarrow{w} \\ & & X \end{array}$$

is a commutative diagram and $q(e_U) = e_W$. By Remark 3.14, $u^J : U(J) \to X(J)$ and $q^J : U(J) \to W(J)$ are also locally $\mathcal{J}$-definable covering maps. Since

$\pi_1^{\mathcal{J}}(U(J), e_U) \simeq (q^J)_*(\pi_1^{\mathcal{J}}(U(J), e_U)) \leq \pi_1^{\mathcal{J}}(W(J), e_W) = 1$ (Corollary 2.17), by Remark 3.8, $u^J : U(J) \to X(J)$ is a universal locally $\mathcal{J}$-definable covering map. Therefore, $u^J : U(J) \to X(J)$ and $w^J : W(J) \to X(J)$ are locally $\mathcal{J}$-definably homeomorphic as required.

We now show that the inclusion homomorphism

$$\mathrm{Aut}(U/X) \to \mathrm{Aut}^{\mathcal{J}}(U(J)/X(J)) : \varphi \mapsto \varphi^J$$

is an isomorphism. This homomorphism is clearly injective and it is surjective since the elements of $\mathrm{Aut}(U/X)$ (resp. of $\mathrm{Aut}^{\mathcal{J}}(U(J)/X(J))$) are determined by their value at $e_U \in u^{-1}(e_X) \subseteq U$ (resp. $e_U \in (u_X^J)^{-1}(e_X) \subseteq U(J)$) (Lemma 2.8), $(u^J)^{-1}(e_X) = (u^{-1}(e_X))(J) = u^{-1}(e_X)$ and $u : U \to X$ is regular (Theorem 3.9). By Theorem 3.9 we have $\pi_1(X, e_X) \simeq \mathrm{Aut}(U/X)$ and $\pi_1^{\mathcal{J}}(X(J), e_X) \simeq \mathrm{Aut}^{\mathcal{J}}(U(J)/X(J))$. Therefore, $\pi_1(X, e_X) \simeq \pi_1^{\mathcal{J}}(X(J), e_X)$. $\square$

Proof of Theorem 1.4: Suppose that $\mathcal{R}$ is an o-minimal expansion of the ordered group of real numbers. Let X a definably connected locally definable manifold.

Fix $e_X \in X$. Clearly X is a topological manifold. Consider the admissible cover $\{O_s\}_{s\in S}$ of X by open definably connected definable subsets given by Proposition 3.1. Using the connectedness of the O_s's and the definable connectedness of X it follows that X is connected. Applying Proposition 3.1 to an open definable neighborhood and using the definable connectedness of the corresponding O_s's and Lemma 2.9 it follows that X is locally path connected. We also have that X is semilocally simply connected (i.e. every point has a neighborhood, namely some O_s, such that every loop in the neighborhood is homotopic in X to a constant path). Therefore, by [22, Theorem 13.20], X has a topological universal covering map which is, by Remark 3.13, up to topological covering homeomorphism, a locally definable covering map $w : W \to X$ with W connected, $e_W \in W$ and $w(e_W) = e_X$.

Let $u : U \to X$ be a universal locally definable map with U connected, $e_U \in U$ and $u(e_U) = e_X$. By Remark 3.8, $\pi_1(U, e_U) = 1$. Also then there exists is a locally definable covering map $q : U \to W$ such that

$$\begin{array}{ccc} U & \xrightarrow{\;q\;} & W \\ & {\scriptstyle u}\searrow & \downarrow{\scriptstyle w} \\ & & X \end{array}$$

is a commutative diagram and $q(e_U) = e_W$. By Remark 3.15, $u : U \to X$ and $q : U \to W$ are also topological covering maps. Since $\pi_1^{\mathrm{top}}(U, e_U) \simeq q_*(\pi_1^{\mathrm{top}}(U, e_U)) \leq \pi_1^{\mathrm{top}}(W, e_W) = 1$ ([22, Lemma 13.1]), by the definition of topological universal covering map on [22, page 186], $u : U \to X$ is a topological universal covering map. Therefore, $u : U \to X$ and $w : W \to X$ are topologically homeomorphic by [22, Corollary 13.6] as required.

We now show that the inclusion homomorphism

$$\mathrm{Aut}(U/X) \to \mathrm{Aut}^{\mathrm{top}}(U/X) : \varphi \mapsto \varphi$$

is an isomorphism. This homomorphism is clearly injective and it is surjective since the elements of $\mathrm{Aut}(U/X)$ (resp. of $\mathrm{Aut}^{\mathrm{top}}(U/X)$) are determined by their value at $e_U \in u^{-1}(e_X) \subseteq U$ (Lemma 2.8) (resp. [22, Lemma 11.5]) and $u : U \to X$ is regular (Theorem 3.9). By Theorem 3.9 we have $\pi_1(X, e_X) \simeq \mathrm{Aut}(U/X)$ and by [22, Theorem 13.11] $\pi_1^{\mathrm{top}}(X, e_X) \simeq \mathrm{Aut}^{\mathrm{top}}(U/X)$. Therefore, $\pi_1(X, e_X) \simeq \pi_1^{\mathrm{top}}(X, e_X)$.

□

We have the following invariance results for the o-minimal fundamental groupoid, generalizing the one for the o-minimal fundamental group. First we have:

Theorem 3.16. *Let $\mathcal{J}$ be an elementary extension of $\mathcal{R}$ or an o-minimal expansion of $\mathcal{R}$. Let X be a locally definable manifold. Then the inclusion functor*

$$\Pi_1(X) \to \Pi_1^{\mathcal{J}}(X(J))$$

is an equivalence of categories.

Proof. (1) the inclusion is faithful. Let $x_0, x_1 \in X$ and consider $[\sigma], [\tau] \in \mathrm{Hom}_{\Pi_1(X)}(x_0, x_1)$ such that their image in $\mathrm{Hom}_{\Pi_1^{\mathcal{J}}(X(J))}(x_0, x_1)$ are equal, i.e. they are $\mathcal{J}$-definably homotopic. Let Y be a definably connected component of X such that $Y(J)$ which is $\mathcal{J}$-definably connected contains the image of this $\mathcal{J}$-definable homotopy. Then the image of $[\sigma \cdot \tau^{-1}]$ under the inclusion $\pi_1(Y, x_0) \to \pi_1^{\mathcal{J}}(Y(J), x_0)$ is trivial. By Theorem 1.3 (2), $\pi_1(Y, x_0) \to \pi_1^{\mathcal{J}}(Y(J), x_0)$ is an isomorphism and so $[\sigma \cdot \tau^{-1}]$ is trivial in $\pi_1(Y, x_0)$ and hence $[\sigma] = [\tau]$.

(2) the inclusion is full. Let $x_0, x_1 \in X$ and consider $[\delta] \in \mathrm{Hom}_{\Pi_1^{\mathcal{J}}(X(J))}(x_0, x_1)$ represented by a $\mathcal{J}$-definable path $\delta : [0, q_\delta] \to X(J)$. Let $\{U_\alpha\}_{\alpha \in I}$ be an admissible cover of X by open definably connected definable subsets, refining the definable charts of X and such that $\pi_1(U_\alpha) = 1$ for each $\alpha \in I$ (Proposition 3.1). Then $\{U_\alpha(J)\}_{\alpha \in I}$ is an admissible cover of $X(J)$ by open $\mathcal{J}$-definably connected $\mathcal{J}$-definable subsets, refining the $\mathcal{J}$-definable charts of X and such that $\pi_1^{\mathcal{J}}(U_\alpha(J)) = 1$ for each $\alpha \in I$ (by Theorem 1.3 (2)).

Let $L \subseteq I$ be a finite subset such that $\delta([0, q_\delta]) \subseteq \bigcup_{l \in L} U_l(J)$. Then $[0, q_\delta] \subseteq \bigcup_{l \in L} \delta^{-1}(U_l(J))$, with the $\delta^{-1}(U_l(J))$'s open in $[0, q_\delta]$. Then, by [9, Chapter 6, (3.6)], for each $l \in L$ there is a $W_l \subset [0, q_\delta]$, open in $[0, q_\delta]$ such that $W_l \subset \overline{W_l} \subset \delta^{-1}(U_l(J))$ and $[0, q_\delta] \subseteq \bigcup_{l \in L} W_l$. Therefore, there are $0 = s_0 < s_1 < \cdots < s_r = q_\delta$ such that for each $i = 0, \ldots, r-1$ we have $\delta([s_i, s_{i+1}]) \subset U_{l(i)}$ (and $\delta(s_{i+1}) \in U_{l(i)}(J) \cap U_{l(i+1)}(J)$). If we set $\delta_i = \delta_{|[s_i, s_{i+1}]}$, then $\delta = \delta_0 \cdot \cdots \cdot \delta_{r-1}$. Since $U_{l(i)}(J) \cap U_{l(i+1)}(J) \neq \emptyset$, we also have $U_{l(i)} \cap U_{l(i+1)} \neq \emptyset$. Choose elements $z_i \in U_{l(i)} \cap U_{l(i+1)}$ and definable paths σ_i in $U_{l(i)}$ connecting the elements $x_0, z_0, \ldots, z_{r-1}, x_1$. If $\sigma = \sigma_0 \cdot \cdots \cdot \sigma_{r-1}$, then $[\sigma] \in \mathrm{Hom}_{\Pi_1(X)}(x_0, x_1)$ and its image in $\mathrm{Hom}_{\Pi_1^{\mathcal{J}}(X(J))}(x_0, x_1)$ is $[\delta]$ since, for each i, δ_i is $\mathcal{J}$-definably homotopic to σ_i due to $\pi_1^{\mathcal{J}}(U_{l(i)}(J)) = 1$.

(3) the inclusion is essentially surjective. Let $x \in X(J)$ and let Y be a definably connected component of X such that $Y(J)$ which is $\mathcal{J}$-connected, contains x. Let $x_0 \in Y$. By Lemma 2.9, there is a $\mathcal{J}$-definable path $\delta : [0, q_\delta] \to Y(J) \subseteq X(J)$ with $\delta(0) = x_0$ and $\delta(q_\delta) = x$. Then $[\delta] \in \mathrm{Hom}_{\Pi_1^{\mathcal{J}}(X(J))}(x_0, x)$, $[\delta]^{-1} \in \mathrm{Hom}_{\Pi_1^{\mathcal{J}}(X(J))}(x, x_0)$ and x is isomorphic to x_0 in $\Pi_1^{\mathcal{J}}(X(J))$. □

Similarly we have:

Theorem 3.17. *Suppose that $\mathcal{R}$ is an o-minimal expansion of the ordered group of real numbers. Let X a locally definable manifold. Then the inclusion functor*

$$\Pi_1(X) \to \Pi_1^{\mathrm{top}}(X)$$

is an equivalence of categories.

4. Other applications

Here we prove the other main results of the paper, namely: the monodromy equivalence for locally constant o-minimal sheaves, classification results for locally definable covering maps and o-minimal Hurewicz and Seifert - van Kampen theorems.

4.1. **Locally definable coverings and locally constant sheaves.** Let X be a locally definable manifold. Then X is equipped with the o-minimal site $X_{\rm def}$ given by: (i) the category $\mathrm{Op}(X_{\rm def})$ of open definable subsets of X with morphisms being inclusions; (ii) the Grothendieck topology such that for $U \in \mathrm{Op}(X_{\rm def})$, a collection $\{U_j\}_{j\in J}$ of objects of $\mathrm{Op}(X_{\rm def})$ is an admissible cover of U if it admits a finite subcover.

Below we let $\mathbf{J}$ be one of these categories: the category **Set** of sets, the category G-**Tors** of G-torsors for a given discrete group G, the category $\mathrm{Mod}(k)$ of k-modules over a ring k. Recall that for G a discrete group, the category G-**Tors** of G-torsors is the category whose objects are sets M with a right action $M \times G \to M : (m, g) \mapsto m^g$ of G on M such that for each $m \in M$ the map $G \to M : g \mapsto m^g$ is a bijection and whose morphisms are maps $h : M \to N$ such that $h(m^g) = h(m)^g$.

Below, given a category $\mathbf{C}$, we denote by $\pi_0(\mathbf{C})$ the category of equivalence classes of objects of $\mathbf{C}$ under isomorphisms of $\mathbf{C}$. Later we also use the fact that, for a discrete abelian group G, $\pi_0(G\text{-}\mathbf{Tors})$ has an abelian group operation given, on representatives, by $M*N = (M\times N)/\Delta^*_G$ where $\Delta^*_G = \{(g,-g) : G\times G : g \in G\}$ acts on $M \times N$ by $(m,n)^{(g,-g)} = (m^g, n^{-g})$ and $M * N$ is the set of orbits. Since we can identify $\frac{G\times G}{\Delta^*_G}$ with G using addition (the sequence

$$0 \longrightarrow \Delta^*_G \hookrightarrow G \times G \xrightarrow{+} G \longrightarrow 0$$

is exact) and since G is abelian, we have a well defined action of $\frac{G\times G}{\Delta^*_G}$ on $M * N$ given by $[(x,y)]^{[(s,t)]} = [(x^s, y^t)]$ making $M * N$ into a G-torsor.

We denote by $\mathrm{Psh}_{\mathbf{J}}(X_{\rm def})$ the category of $\mathbf{J}$-pre-sheaves on the o-minimal site $X_{\rm def}$. By definitions, this is the category $\mathrm{Fct}(\mathrm{Op}(X_{\rm def})^{\rm op}, \mathbf{J})$ of contravariant functors

$$\begin{aligned} \mathcal{F} : \mathrm{Op}(X_{\rm def}) &\to \mathbf{J} \\ U &\mapsto \mathcal{F}(U) \\ (V \subset U) &\mapsto (\mathcal{F}(U) \to \mathcal{F}(V)) \\ &\qquad\qquad s \mapsto s_{|V} \end{aligned}$$

from $\mathrm{Op}(X_{\rm def})$ to $\mathbf{J}$ with morphisms being natural transformations of such functors. We denote by $\mathrm{Sh}_{\mathbf{J}}(X_{\rm def})$ the category of $\mathbf{J}$-sheaves on the o-minimal site $X_{\rm def}$, i.e., the full subcategory of $\mathrm{Psh}_{\mathbf{J}}(X_{\rm def})$ whose objects satisfy the following gluing conditions: for every $U \in \mathrm{Op}(X_{\rm def})$ and every admissible cover $\{U_j\}_{j\in J}$ of U we have:

- if $s, t \in \mathcal{F}(U)$ and $s_{|U_j} = t_{|U_j}$ for each j, then $s = t$;

- if $s_j \in \mathcal{F}(U_j)$ are such that $s_j = s_k$ on $U_j \cap U_k$ then they glue to $s \in \mathcal{F}(U)$ (i.e. $s_{|U_j} = s_j$).

We refer the reader to [15] and [19] for further details on the theory of o-minimal sheaves.

If $V \in \mathrm{Op}(X_{\mathrm{def}})$, a $\mathbf{J}$-sheaf $\mathcal{F}$ on V_{def} is constant if it is isomorphic to the $\mathbf{J}$-sheaf C_V on V_{def} associated to the $\mathbf{J}$-pre-sheaf sending every $W \in \mathrm{Op}(V_{\mathrm{def}})$ to a fixed $C \in \mathrm{Ob}\mathbf{J}$. We denote by $\mathrm{CSh}_{\mathbf{J}}(X_{\mathrm{def}})$ the category of constant $\mathbf{J}$-sheaves on the o-minimal site X_{def} on X. We denote by $\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}})$ the category of locally constant $\mathbf{J}$-sheaves on the o-minimal site X_{def} on X. By definition this means that if $\mathcal{F} \in \mathrm{Ob}\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}})$, then there exists an admissible cover $\{U_j\}_{j\in J}$ of X by open definable subsets such that the restriction $\mathcal{F}_{|U_j}$ is a constant $\mathbf{J}$-sheaf on $U_{j\,\mathrm{def}}$ for each $j \in J$.

Remark 4.1. Here we only consider the specific examples of $\mathbf{J}$ which are useful for the applications below. One could consider a category $\mathbf{J}$ admitting projective and inductive limits and satisfying the IPC property (see Definition 3.1.10 of [24] for more details). More generally, one could consider the constant stack associated to a category $\mathbf{J}$ as in [26].

Given S a locally definable manifold and $\mathbf{J}$ a category as above, we say that a locally definable covering map $p_X : X \to S$ trivial over $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ is a *locally definable* $\mathbf{J}$*-covering map trivial over* $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ if in addition the following hold:

- for every $U \in \mathrm{Op}(S_{\mathrm{def}})$, the set $\mathcal{P}_X(U) = \{f : U \to p_X^{-1}(U) : f$ a continuous locally definable map with $p_X \circ f = \mathrm{id}_U\}$ of continuous locally definable sections of $p_{X|p_X^{-1}(U)}$ is an object of $\mathbf{J}$.
- for every $V, U \in \mathrm{Op}(S_{\mathrm{def}})$ with $V \subseteq U$, the restriction map $\mathcal{P}_X(U) \to \mathcal{P}_X(V) : f \mapsto f_{|V}$ is a morphism of $\mathbf{J}$.

A *locally definable* $\mathbf{J}$*-covering map* $p_X : X \to S$ is a locally definable $\mathbf{J}$-covering map trivial over some admissible cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ of S by open definable subsets.

We say that two locally definable $\mathbf{J}$-covering maps $p_X : X \to S$ and $p_Y : Y \to S$ (trivial over $\mathcal{U}$) are *locally definably homeomorphic* if there is a locally definable homeomorphism $F : X \to Y$ such that:

- $p_X = p_Y \circ F$.
- The functor $F : \mathcal{P}_X \to \mathcal{P}_Y$ induced by composition by F is a morphism of $\mathrm{Sh}_{\mathbf{J}}(S_{\mathrm{def}})$.

Such $F : X \to Y$ is called a *locally definable* $\mathbf{J}$*-covering homeomorphism.*

A locally definable $\mathbf{J}$-covering map $p_X : X \to S$ is *trivial* if it is locally definably homeomorphic to a locally definable $\mathbf{J}$-covering map $S \times C \to S : (s, c) \mapsto s$ for some $C \in \mathrm{Ob}\mathbf{J}$.

We denote by $\mathbf{J}$-$\mathrm{Cov}_{\mathrm{ldef}}(X)$ the category of locally definable $\mathbf{J}$-covering maps and by $\mathbf{J}$-$\mathrm{Cov}_{0\,\mathrm{ldef}}(X)$ its full subcategory consisting of trivial locally definable $\mathbf{J}$-covering maps.

Example 4.2. (1) If we take $\mathbf{J}$ to be the category **Set** of sets, then of course we recover the previously defined notions.

(2) If G is a discrete group, then a locally definable G-**Tors**-covering map $p_X : X \to S$ trivial over $\mathcal{U}$ is exactly a locally definable G-covering map trivial over $\mathcal{U}$, i.e., a locally definable covering map $p_X : X \to S$ trivial over $\mathcal{U}$ with a continuous locally definable right action $X \times G \to X : (x, g) \mapsto x^g$ of G on X such that for each $s \in S$ there is an induced right action $p_X^{-1}(s) \times G \to p_X^{-1}(s)$ making the fiber $p_X^{-1}(s)$ a G-torsor. Also a locally definable G-**Tors**-covering homeomorphism $F : X \to Y$ between two locally definable G-**Tors**-covering maps $p_X : X \to S$ and $p_Y : Y \to S$ (trivial over $\mathcal{U}$) is exactly a locally definable G-covering homeomorphisms, i.e., such that:

- $p_X = p_Y \circ F$.
- For every $x \in X$ and $g \in G$, we have $F(x^g) = F(x)^g$.

Proposition 4.3. *Suppose that X is a locally definable manifold and $\mathbf{J}$ is a category as before. Then there is an equivalence*

$$\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}}) \to \mathbf{J}\text{-}\mathrm{Cov}_{\mathrm{ldef}}(X)$$

of categories which restricts to an equivalence

$$\mathrm{CSh}_{\mathbf{J}}(X_{\mathrm{def}}) \to \mathbf{J}\text{-}\mathrm{Cov}_{0\,\mathrm{ldef}}(X)$$

of subcategories.

Proof. Let $\mathcal{F}$ be an object of $\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}})$ and suppose that $\{U_j\}_{j \in J}$ is an admissible of X by open definable subsets such that for each $j \in J$, the restriction $\mathcal{F}_{|U_j}$ is isomorphic to a constant $\mathbf{J}$-sheaf. For each $j \in J$, if $\mathcal{F}_{|U_j} \to C_j$ is such an isomorphism in $\mathrm{LCSh}_{\mathbf{J}}(U_{j\,\mathrm{def}})$, we have for each $V \in \mathrm{Op}(U_{j\,\mathrm{def}})$ an induced isomorphism $\mathcal{F}_{|U_j}(V) \to C_j$ in $\mathbf{J}$ commuting with the restrictions

$$\begin{array}{ccc} \mathcal{F}_{|U_j}(V) & \longrightarrow & \mathcal{F}_{|U_j}(V') \\ & \searrow & \downarrow \\ & & C_j \end{array}$$

in $\mathbf{J}$ where $V' \subseteq V$ is in $\mathrm{Op}(U_{j,\,\mathrm{def}})$. Thus, if $x \in X$, since $x \in U_j$ for some $j \in J$, the stalk of $\mathcal{F}$ at x

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U)$$

with $U \in \mathrm{Op}(X_{\mathrm{def}})$, exists in $\mathbf{J}$ and we have a canonical surjective homomorphism

$$\mathcal{F}(U) \to \mathcal{F}_x : s \mapsto s_x$$

for every $U \in \mathrm{Op}(X_{\mathrm{def}})$ with $x \in U$.

Set $W_{\mathcal{F}} = \bigsqcup_{x \in X} \mathcal{F}_x$ and consider the obvious map $w_{\mathcal{F}} : W_{\mathcal{F}} \to X$ sending $s_x \in \mathcal{F}_x$ to x. For each $j \in J$ and $c \in C_j$ let $s^c \in \mathcal{F}_{|U_j}(U_j)$ be the corresponding section under the isomorphism $\mathcal{F}_{|U_j}(U_j) \to C_j$ in $\mathbf{J}$. Set $U_j^c = \{s_x^c : x \in U_j\}$. Then we have, for each $j \in J$,

- $w_{\mathcal{F}}^{-1}(U_j) = \bigsqcup_{c \in C_j} U_j^c$;
- each $w_{\mathcal{F}|U_j} : U_j^c \to U_j$ is a bijection.

By [16, Lemma 2.1 (1)] (see Remark 2.6), it follows that there exists a locally definable manifold structure on $W_{\mathcal{F}}$ of dimension $n := \dim X$ such that $w_{\mathcal{F}} : W_{\mathcal{F}} \to X$ is a locally definable covering map trivial over $\{U_j\}_{j \in J}$.

It is easy to see that for every $U \in \mathrm{Op}(X_{\mathrm{def}})$ we have $\{f : U \to w_{\mathcal{F}}^{-1}(U) : f$ is a continuous locally definable map such that $w_{\mathcal{F}} \circ f = \mathrm{id}_U\} = \mathcal{F}(U)$, an object of $\mathbf{J}$. Thus $w_{\mathcal{F}} : W_{\mathcal{F}} \to X$ is a locally definable $\mathbf{J}$-covering map trivial over $\{U_j\}_{j \in J}$. It is also standard to see that a morphism $\mathcal{F} \to \mathcal{G}$ in $\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}})$ determines a morphism

$$\begin{array}{ccc} W_{\mathcal{F}} & \longrightarrow & W_{\mathcal{G}} \\ & {\scriptstyle w_{\mathcal{F}}} \searrow & \downarrow {\scriptstyle w_{\mathcal{G}}} \\ & & X \end{array}$$

of locally definable $\mathbf{J}$-covering maps which is a locally definable homeomorphism if $\mathcal{F} \to \mathcal{G}$ is an isomorphism. Thus we have a well defined functor $\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}}) \to \mathbf{J}\text{-}\mathrm{Cov}_{\mathrm{ldef}}(X)$.

The inverse of the functor $\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}}) \to \mathbf{J}\text{-}\mathrm{Cov}_{\mathrm{ldef}}(X)$ just defined is the functor $\mathbf{J}\text{-}\mathrm{Cov}_{\mathrm{ldef}}(X) \to \mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}})$ sending $p : Y \to X$ to the $\mathbf{J}$-sheaf $\mathcal{P}_Y$ of continuos locally definable sections of $p : Y \to X$ and a morphism

$$\begin{array}{ccc} Y & \xrightarrow{\ r\ } & Z \\ & {\scriptstyle p} \searrow & \downarrow {\scriptstyle q} \\ & & X \end{array}$$

of locally definable $\mathbf{J}$-covering maps to the morphism $\mathcal{P}_Y \to \mathcal{Q}_Z$ in $\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}})$ induced by composition with r. It is standard to check that this is indeed a well defined functor which is the required inverse.

By construction, the isomorphism $\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}}) \to \mathbf{J}\text{-}\mathrm{Cov}_{\mathrm{ldef}}(X)$ restricts to an isomorphism $\mathrm{CSh}_{\mathbf{J}}(X_{\mathrm{def}}) \to \mathbf{J}\text{-}\mathrm{Cov}_{0\,\mathrm{ldef}}(X)$. □

It follows from Proposition 4.3 and the corresponding facts in $\mathbf{J}\text{-}\mathrm{Cov}_{\mathrm{ldef}}(X)$ that:

- We have an equivalence
$$\mathrm{CSh}_{\mathbf{J}}(X_{\mathrm{def}}) \to \mathbf{J}.$$
- An object $\mathcal{F}$ of $\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}})$ is determined by its stalks $\mathcal{F}_x$ $(x \in X)$ and a morphism $\phi : \mathcal{F} \to \mathcal{G}$ in $\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}})$ is determined by the induced homomorphism $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ $(x \in X)$ on stalks.
- If $\mathcal{F}$ is an object of $\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}})$ with X definably connected and $\pi_1(S) = 1$, then $\mathcal{F}$ is isomorphic to a constant $\mathbf{J}$-sheaf.

4.2. **Monodromy and representations of the fundamental groupoid.** Let X be a locally definable manifold and $\mathbf{J}$ a category as before. Consider the category

$$\mathrm{Fct}(\Pi_1(X), \mathbf{J})$$

of functors from the category $\Pi_1(X)$ to the category $\mathbf{J}$. The objects of this category are called *representation of the o-minimal fundamental groupoid* $\Pi_1(X)$ *in* $\mathbf{J}$. If X is definably connected with $e_X \in X$, then the objects of the category $\mathrm{Fct}(\Pi_1(X), \mathbf{J})$ are called *representation of the o-minimal fundamental group* $\pi_1(X, e_X)$ *in* $\mathbf{J}$.

By Lemma 2.11 (1) we have:

Remark 4.4. If X is definably connected with $e_X \in X$, then $\mathrm{Fct}(\Pi_1(X), \mathbf{J})$ is equivalent to the category whose objects are pairs (M, τ_M) with M an object of

$\mathbf{J}$ and $\tau_M \in \mathrm{Hom}(\pi_1(X, e_X)^{\mathrm{op}}, \mathrm{Aut}_{\mathbf{J}}(M))$ and whose morphisms $\tau_f : (M, \tau_M) \to (N, \tau_N)$ are morphisms $f : M \to N$ in $\mathbf{J}$ such that $\tau_N \circ f = f \circ \tau_M$.

The full subcategory of $\mathrm{Fct}(\Pi_1(X), \mathbf{J})$ consisting of trivial representations will be denoted by

$$\mathrm{Fct}_0(\Pi_1(X), \mathbf{J}).$$

A representation $\theta \in \mathrm{Fct}(\Pi_1(X), \mathbf{J})$ is trivial if it is isomorphic to a constant functor Δ_M which associates M to any object and id_M to any morphism. We have a natural equivalence

$$\mathbf{J} \to \mathrm{Fct}_0(\Pi_1(X), \mathbf{J})$$

given by the functor $M \mapsto \Delta_M$.

By Proposition 4.3 and Lemma 2.13 we have:

Lemma 4.5. *Suppose that X is a locally definable manifold and let $\mathcal{F}$ be an object of $\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}})$. Then the following hold.*

1. *If $\gamma : [0, p] \to X$ be a definable path in X, then the inverse image $\gamma^{-1}\mathcal{F}$ is an object of $\mathrm{CSh}_{\mathbf{J}}([0, p]_{\mathrm{def}})$.*
2. *If $h : [0, p] \times [0, r] \to X$ is a definable homotopy between the definable paths $\gamma, \sigma : [0, p] \to X$ in X, then the inverse image $h^{-1}\mathcal{F}$ is an object of $\mathrm{CSh}_{\mathbf{J}}(([0, p] \times [0, r])_{\mathrm{def}})$.*

Proof. Consider the locally definable $\mathbf{J}$-covering map $w_{\mathcal{F}} : W_{\mathcal{F}} \to X$ corresponding to $\mathcal{F}$ via Proposition 4.3.

(1) Consider a definable lifting $\widetilde{\gamma} : [0, p] \to W_{\mathcal{F}}$ of $\gamma : [0, p] \to X$ given by Lemma 2.13 (1). Since $(\gamma^{-1}\mathcal{F})_t = \mathcal{F}_{\gamma(t)}$ for each $t \in [0, p]$, the continuous (locally) definable map $\widetilde{\gamma} : [0, p] \to W_{\mathcal{F}}$ corresponds to a global section of $\gamma^{-1}\mathcal{F}$. Thus $\gamma^{-1}\mathcal{F}$ is constant.

(2) Consider a definable lifting $\widetilde{h} : [0, p] \times [0, r] \to W_{\mathcal{F}}$ of $h : [0, p] \times [0, r] \to X$ given by Lemma 2.13 (2). Since $(h^{-1}\mathcal{F})_{(t,s)} = \mathcal{F}_{h(t,s)}$ for each $(t, s) \in [0, p] \times [0, r]$, the continuous (locally) definable map $\widetilde{h} : [0, p] \times [0, r] \to W_{\mathcal{F}}$ corresponds to a global section of $h^{-1}\mathcal{F}$. Thus $h^{-1}\mathcal{F}$ is constant. □

Lemma 4.6. *Suppose that X is a locally definable manifold and let $\mathcal{F}$ be an object of $\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}})$. Then there exists a well defined functor*

$$\mu(\mathcal{F}) : \Pi_1(X) \to \mathbf{J}$$

sending $x \in X$ to $\mathcal{F}_x$ and sending $[\sigma] \in \mathrm{Hom}_{\Pi_1(X)}(x_0, x_1)$ to a canonical isomorphism $\mathcal{F}_{x_0} \to \mathcal{F}_{x_1}$ in $\mathbf{J}$.

Proof. Let $[\sigma] \in \mathrm{Hom}_{\Pi_1(X)}(x_0, x_1)$ where $\sigma : [0, p] \to X$ is a definable path in X from x_0 to x_1. By Lemma 4.5 (1), $\sigma^{-1}\mathcal{F}$ is constant and so we have isomorphisms

$$\mathcal{F}_{x_0} = (\sigma^{-1}\mathcal{F})_0 \xleftarrow{\sim} \sigma^{-1}\mathcal{F}([0, p]) \xrightarrow{\sim} (\sigma^{-1}F)_p = \mathcal{F}_{x_1}$$

defining the canonical isomorphism $\mathcal{F}_{x_0} \to \mathcal{F}_{x_1}$ in $\mathbf{J}$.

We need to show that this is well defined i.e., it does not depend on the representative of the definable homotopy class. Suppose that we have $[\sigma] = [\gamma]$ in $\mathrm{Hom}_{\Pi_1(X)}(x_0, x_1)$ where $\sigma, \gamma : [0, p] \to X$ are definable paths in X from x_0 to x_1. Let $h : [0, p] \times [0, r] \to X$ be a definable homotopy between σ and γ with

$h(t,0) = \sigma(t)$ and $h(t,r) = \gamma(t)$ for all $t \in [0,p]$. By Lemma 4.5 (2), $h^{-1}\mathcal{F}$ is constant and so we have isomorphisms

$$\begin{array}{ccccc}
\mathcal{F}_{x_0} = (\gamma^{-1}\mathcal{F})_0 & \xleftarrow[\sim]{} & \gamma^{-1}\mathcal{F}([0,p]) & \xrightarrow{\sim} & (\gamma^{-1}\mathcal{F})_p = \mathcal{F}_{x_1} \\
\uparrow{\scriptstyle =} & & \uparrow{\scriptstyle =} & & \uparrow{\scriptstyle =} \\
(h^{-1}\mathcal{F})_{(0,s)} & \xleftarrow[\sim]{} & h^{-1}\mathcal{F}([0,p]\times[0,r]) & \xrightarrow{\sim} & (h^{-1}\mathcal{F})_{(p,s)} \\
{\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow & & {\scriptstyle =}\downarrow \\
\mathcal{F}_{x_0} = (\sigma^{-1}\mathcal{F})_0 & \xleftarrow[\sim]{} & \sigma^{-1}\mathcal{F}([0,p]) & \xrightarrow{\sim} & (\sigma^{-1}\mathcal{F})_p = \mathcal{F}_{x_1}
\end{array}$$

for all $s \in [0,r]$, showing that the canonical isomorphism $\mathcal{F}_{x_0} \to \mathcal{F}_{x_1}$ just defined does not depend on the representative of the definable homotopy class. □

The functor of Lemma 4.6 induces the well defined *monodromy functor*

$$\mu : \mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}}) \to \mathrm{Fct}(\Pi_1(X), \mathbf{J})$$

sending an object $\mathcal{F}$ of $\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}})$ to the associated functor $\mu(\mathcal{F})$ and sending a morphism $\phi : \mathcal{F} \to \mathcal{G}$ in $\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}})$ to the induced natural transformation $\mu(\mathcal{F}) \to \mu(\mathcal{G})$ given by the commutative diagram

$$\begin{array}{ccc}
\mathcal{F}_{x_0} & \xrightarrow{\mu(\mathcal{F})([\gamma])} & \mathcal{F}_{x_1} \\
{\scriptstyle \phi_{x_0}}\downarrow & & \downarrow{\scriptstyle \phi_{x_1}} \\
\mathcal{G}_{x_0} & \xrightarrow{\mu(\mathcal{G})([\gamma])} & \mathcal{G}_{x_1}
\end{array}$$

in $\mathbf{J}$ for all $x_0, x_1 \in X$ and $[\gamma] \in \mathrm{Hom}_{\Pi_1(X)}(x_0, x_1)$.

Theorem 4.7. *Let X be a locally definable manifold and $\mathbf{J}$ a category as before. Then the monodromy functor*

$$\mu : \mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}}) \to \mathrm{Fct}(\Pi_1(X), \mathbf{J})$$

is an equivalence of categories.

Proof. (1) μ is faithful. Consider morphisms $\varphi, \psi : F \to G$ in $\mathrm{LCSh}_{\mathbf{J}}$ and suppose that $\mu(\varphi) = \mu(\psi)$. By definition, this means that $\varphi_x = \psi_x : F_x \to G_x$ for each $x \in X$. Hence $\varphi = \psi$.

(2) μ is full. Consider objects $F, G \in \mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}})$ and a morphism $u : \mu(F) \to \mu(G)$. This means that we have morphisms $u_x : F_x \to G_x$ in $\mathbf{J}$ for each $x \in X$. Let $\{U_\alpha\}_{\alpha \in I}$ be an admissible cover of X by open definable subsets such that both F and G are constant on $\{U_\alpha\}_{\alpha \in I}$. Without loss of generality we may assume that each U_α is definably connected. Let $x \in U_\alpha$. Then u_x extends uniquely to a morphism u_α. For $y \in U_\alpha$ we choose a path γ in U_α from x to y. Then $(u_\alpha)_y = \mu(G)(\gamma) \cdot (u_\alpha)_x \circ \mu(F)(\gamma^{-1})$. Moreover $(u_\alpha)_x = u_x$ and $u_y = \mu(G)(\gamma) \circ u_x \circ \mu(F)(\gamma^{-1})$ because u is a morphism of functors. Hence $(u_\alpha)_y = u_y$. So we have a morphism $u_\alpha : F_{|U_\alpha} \to G_{|U_\alpha}$ given by $(u_\alpha)_y = u_y$ for each $y \in U_\alpha$. Therefore, for $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$ we have $u_{\alpha|U_\alpha \cap U_\beta} = u_{\beta|U_\alpha \cap U_\beta}$. This proves that the $\{u_\alpha\}_{\alpha \in I}$ glue together in a morphism $v : F \to G$. Since, for each $\alpha \in I$, we have $v_{|U_\alpha} = u_\alpha$ and $(u_\alpha)_x = u_x : F_x \to G_x$ for each $x \in U_\alpha$, it follows that $\mu(v) = u$.

(3) μ is essentially surjective. Let $\{U_\alpha\}_{\alpha\in I}$ be an admissible cover of X by open definably connected definable subsets, refining the definable charts of X and such that $\pi_1(U_\alpha)=1$ for each $\alpha\in I$ (Proposition 3.1). Let $\alpha,\beta\in I$ be such that $U_\alpha\cap U_\beta\neq\emptyset$. The commutative diagram of inclusions

$$\begin{array}{ccc} U_\alpha & \xrightarrow{j_\alpha} & X \\ \uparrow{\scriptstyle i_{\alpha,\beta}} & & \uparrow{\scriptstyle j_\beta} \\ U_\alpha\cap U_\beta & \xrightarrow{i_{\beta,\alpha}} & U_\beta \end{array}$$

induces a commutative diagram of morphisms

$$\begin{array}{ccc} \Pi_1(U_\alpha) & \xrightarrow{j_{\alpha *}} & \Pi_1(X) \\ \uparrow{\scriptstyle i_{\alpha,\beta *}} & & \uparrow{\scriptstyle j_{\beta *}} \\ \Pi_1(U_\alpha\cap U_\beta) & \xrightarrow{i_{\beta,\alpha *}} & \Pi_1(U_\beta) \end{array}$$

which in turn induces a commutative diagram of morphims

$$\begin{array}{ccc} \mathrm{Fct}(\Pi_1(U_\alpha),\mathbf{J}) & \xleftarrow{\lambda_\alpha} & \mathrm{Fct}(\Pi_1(X),\mathbf{J}) \\ \downarrow{\scriptstyle \lambda_{\alpha,\beta}} & & \downarrow{\scriptstyle \lambda_\beta} \\ \mathrm{Fct}(\Pi_1(U_\alpha\cap U_\beta),\mathbf{J}) & \xleftarrow{\lambda_{\beta,\alpha}} & \mathrm{Fct}(\Pi_1(U_\beta),\mathbf{J}). \end{array}$$

Let A be an object of $\mathrm{Fct}(\Pi_1(X),\mathbf{J})$. Then for each $\alpha\in I$, there exists a costant sheaf F_α on U_α such that $\mu(F_\alpha)=\lambda_\alpha(A)$ since $\mu:\mathrm{CSh}_{\mathbf{J}}(U_{\alpha\mathrm{def}})\to\mathrm{Fct}(\Pi_1(U_\alpha),\mathbf{J})$ is an equivalence.

Let $\alpha,\beta\in I$ be such that $U_{\alpha\beta}:=U_\alpha\cap U_\beta\neq\emptyset$. We have $\mu(F_{\alpha|U_{\alpha\beta}})=\lambda_{\alpha,\beta}\circ\lambda_\alpha(A)$ and $\mu(F_{\beta|U_{\alpha\beta}})=\lambda_{\beta,\alpha}\circ\lambda_\beta(A)$. Since as seen above $\lambda_{\alpha,\beta}\circ\lambda_\alpha=\lambda_{\beta,\alpha}\circ\lambda_\beta$, we have $\mu(F_{\alpha|U_{\alpha\beta}})=\mu(F_{\beta|U_{\alpha\beta}})$. But $F_{\alpha|U_{\alpha\beta}}$ and $F_{\beta|U_{\alpha\beta}}$ are also objects of $\mathrm{CSh}_{\mathbf{J}}(U_{\alpha\beta\mathrm{def}})$. Hence, $\mu_0(F_{\alpha|U_{\alpha\beta}})=\mu_0(F_{\beta|U_{\alpha\beta}})$ and therefore, we have isomorphisms $\theta_{\alpha,\beta}:F_{\beta|U_{\alpha\beta}}\to F_{\alpha|U_{\alpha\beta}}$ such that $\theta_{\alpha,\alpha}=\mathrm{id}_{Y_\alpha}$ and $\theta_{\alpha,\beta}=\theta_{\alpha,\gamma}\circ\theta_{\gamma,\beta}$ on $U_\alpha\cap U_\beta\cap U_\gamma$. Then there exists $F\in\mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}})$ together with isomorphisms $\psi_\alpha:F_\alpha\simeq F_{|U_\alpha}$ such that $\theta_{\alpha,\beta}=\psi_\alpha^{-1}\circ\psi_\beta$ on $U_{\alpha\beta}$.

Now it remains to show that there exists an isomorphism $\psi:\mu(F)\to A$ in $\mathrm{Fct}(\Pi_1(X),\mathbf{J})$. If $x\in U_\alpha$, then $\lambda_\alpha A(x)=A(x)$ by definition of the functor λ_α. Since $\mu(F_\alpha)=\lambda_\alpha A$ we also have $F_{\alpha x}=\lambda_\alpha A(x)$. On the other hand, by construction of F we have an isomorphism $\psi_{\alpha x}:F_{\alpha x}\to F_x$. Thus we have an isomorphism $\psi(x):F_x\to A(x)$ that is, an isomorphism $\psi(x):\mu(F)(x)\to A(x)$. To conclude the proof we need to show that for $x_0,x_1\in X$ and $[\sigma]\in\mathrm{Hom}_{\Pi_1(X)}(x_0,x_1)$ (with $\sigma:[0,q_\sigma]\to X$ a path such that $\sigma(0)=x_0$ and $\sigma(q_\sigma)=x_1$), then we have a commutative diagram

$$\begin{array}{ccc} F_{x_0} & \xrightarrow{\psi(x_0)} & A(x_0) \\ \downarrow{\scriptstyle \mu(F)([\sigma])} & & \downarrow{\scriptstyle A([\sigma])} \\ F_{x_1} & \xrightarrow{\psi(x_1)} & A(x_1) \end{array}$$

in $\mathbf{J}$. Let $L\subseteq I$ be a finite subset such that $\sigma([0,q_\sigma])\subseteq\bigcup_{l\in L}U_l$. Then $[0,q_\sigma]\subseteq\bigcup_{l\in L}\sigma^{-1}(U_l)$, with the $\sigma^{-1}(U_l)$'s open in $[0,q_\sigma]$. Then, by [9, Chapter 6, (3.6)], for each $l \in L$ there is a $W_l \subset [0, q_\sigma]$, open in $[0, q_\sigma]$ such that $W_l \subset \overline{W_l} \subset \sigma^{-1}(U_l)$ and $[0, q_\sigma] \subseteq \bigcup_{l \in L} W_l$. Therefore, there are $0 = s_0 < s_1 < \cdots < s_r = q_\sigma$ such that for each $i = 0, \ldots, r-1$ we have $\sigma([s_i, s_{i+1}]) \subset U_{l(i)}$ (and $\sigma(s_{i+1}) \in U_{l(i)} \cap U_{l(i+1)}$). If we set $\sigma_i = \sigma_{|[s_i, s_{i+1}]}$, then $\sigma = \sigma_0 \cdot \cdots \cdot \sigma_{r-1}$. By the above constructions, the result holds for each σ_i and so it holds also for σ by composition. □

By the construction in Theorem 4.7 and Theorems 3.16 and 3.17 we have:

Corollary 4.8. *(1) Let $\mathcal{J}$ be an elementary extension of $\mathcal{R}$ or an o-minimal expansion of $\mathcal{R}$. Let X be a locally definable manifold. Then we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}}) & \xrightarrow{\mu} & \mathrm{Fct}(\Pi_1(X), \mathbf{J}) \\ \downarrow & & \uparrow \\ \mathrm{LCSh}_{\mathbf{J}}(X(J)_{\mathrm{def}}) & \xrightarrow{\mu} & \mathrm{Fct}(\Pi_1^{\mathcal{J}}(X(J)), \mathbf{J}) \end{array}$$

of equivalence of categories.

(2) Suppose that $\mathcal{R}$ is an o-minimal expansion of the ordered group of real numbers. Let X a locally definable manifold. Then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{LCSh}_{\mathbf{J}}(X_{\mathrm{def}}) & \xrightarrow{\mu} & \mathrm{Fct}(\Pi_1(X), \mathbf{J}) \\ \downarrow & & \uparrow \\ \mathrm{LCSh}_{\mathbf{J}}(X) & \xrightarrow{\mu} & \mathrm{Fct}(\Pi_1^{\mathrm{top}}(X), \mathbf{J}) \end{array}$$

of equivalences of categories where $\mathrm{LCSh}_{\mathbf{J}}(X)$ is the category of equivalence classes of locally constant $\mathbf{J}$-sheaves on the topological space X.

4.3. **Examples.** Here we deduce special cases of Theorem 4.7 which are more familiar.

Let X be a locally definable manifold. When $\mathbf{J} = \mathbf{Set}$ we denote by $\mathrm{Cov}_{\mathrm{ldef}}(X)$ the category of locally definable covering maps. If we denote by $\omega\mathbf{Set}$ the category of countable sets and by $\mathrm{f}\mathbf{Set}$ the category of finite sets, we denote by $\mathrm{Cov}_{\mathrm{ldef}_\omega}(X)$ and $\mathrm{Cov}_{\mathrm{def}}$ the full subcategories of the category $\mathrm{Cov}_{\mathrm{ldef}}(X)$ obtained by taking $\mathbf{J} = \omega\mathbf{Set}$ and $\mathbf{J} = \mathrm{f}\mathbf{Set}$ respectively (i.e. the fibers are countable and finite respectively).

Let X be a definably connected locally definable manifold with $e_X \in X$. Below we use the following notation: $\pi_1(X, e_X)$-$\mathbf{Set}$ is the category of $\pi_1(X, e_X)$-sets; $\pi_1(X, e_X)$-$\omega\mathbf{Set}$ is the full subcategory of countable $\pi_1(X, e_X)$-sets; $\pi_1(X, e_X)$-$\mathrm{f}\mathbf{Set}$ is the full subcategory of finite $\pi_1(X, e_X)$-sets.

Corollary 4.9. *Let X be a definably connected locally definable manifold with $e_X \in X$. There is a canonical equivalence*

$$\mathrm{Cov}_{\mathrm{ldef}}(X) \to \pi_1(X, e_X)\text{-}\mathbf{Set}$$

of categories. Moreover, if X is Lindelöf (resp. definable), then there is a canonical equivalence

$$\mathrm{Cov}_{\mathrm{ldef}_\omega}(X) \to \pi_1(X, e_X)\text{-}\omega\mathbf{Set}$$

(resp.

$$\mathrm{Cov}_{\mathrm{def}}(X, e) \to \pi_1(X, e_X)\text{-f}\mathbf{Set})$$

of categories.

Proof. By Theorem 4.7 and Proposition 4.3 we have that

$$\mu : \mathrm{Cov}_{\mathrm{ldef}}(X) \to \mathrm{Fct}(\Pi_1(X), \mathbf{Set})$$

is an equivalence of categories. Hence by Remark 4.4 we have a canonical equivalence

$$\mathrm{Cov}_{\mathrm{ldef}}(X) \to \pi_1(X, e_X)\text{-}\mathbf{Set}$$

of categories.

The other cases are similar. □

Let X be a locally definable manifold. When G be a discrete group and $\mathbf{J} = G\text{-}\mathbf{Tors}$ (see Example 4.2), we denote by $G\text{-Cov}_{\mathrm{ldef}}(X)$ the category whose objects are locally definable G-covering maps (i.e. locally definable covering maps with a G-action on the fibers). If G is a countable discrete group (resp. finite group) we denote by $G\text{-Cov}_{\mathrm{ldef}_\omega}(X)$ (resp. $G\text{-Cov}_{\mathrm{def}}$) the corresponding full subcategory of the category $G\text{-Cov}_{\mathrm{ldef}}(X)$.

Recall that, given a category $\mathbf{C}$, we denote by $\pi_0(\mathbf{C})$ the category of equivalence classes of object of $\mathbf{C}$ under isomorphisms of $\mathbf{C}$.

If we consider the categories of equivalence classes of locally definable G-coverings maps under locally definable G-covering homeomorphisms we obtain:

Corollary 4.10. *Let G be a discrete group and X a definably connected locally definable manifold with $e_X \in X$. Then there is a canonical bijection*

$$\pi_0(G\text{-Cov}_{\mathrm{ldef}}(X)) \to \mathrm{Hom}(\pi_1(X, e_X)^{\mathrm{op}}, G)/\textit{conjugacy}.$$

If X is Lindelöf (resp. is definable) and G countable (resp. is finite), then there is a canonical bijection

$$\pi_0(G\text{-Cov}_{\mathrm{ldef}_\omega}(X)) \to \mathrm{Hom}(\pi_1(X, e_X)^{\mathrm{op}}, G)/\textit{conjugacy}$$

(resp.

$$\pi_0(G\text{-Cov}_{\mathrm{def}}(X)) \to \mathrm{Hom}(\pi_1(X, e_X)^{\mathrm{op}}, G)/\textit{conjugacy}).$$

Proof. By Theorem 4.7 and Proposition 4.3, the monodromy functor

$$\mu_G : G\text{-Cov}_{\mathrm{ldef}}(X) \to \mathrm{Fct}(\Pi_1(X), G\text{-}\mathbf{Tors})$$

is an equivalence of categories. Hence, by Remark 4.4 we have

$$\begin{aligned} \pi_0(G\text{-Cov}_{\mathrm{ldef}}(X)) &\simeq \pi_0(\mathrm{Fct}(\Pi_1(X), G\text{-}\mathbf{Tors}))) \\ &\simeq \mathrm{Hom}(\pi_1(X, e_X)^{\mathrm{op}}, \mathrm{Aut}_{G\text{-}\mathbf{Tors}}(G))/\mathrm{conjugacy} \end{aligned}$$

and $\mathrm{Aut}_{G\text{-}\mathbf{Tors}}(G) \simeq G$.

The other cases are similar. □

If we consider the categories of equivalence classes of pointed locally definable G-covering maps, then we obtain:

Corollary 4.11. *Let G be a discrete group and X a definably connected locally definable manifold with $e_X \in X$. Then there is a one-to-one correspondence*

$$\pi_0(G\text{-}\mathrm{Cov}_{\mathrm{ldef}}(X, e_X)) \to \mathrm{Hom}(\pi_1(X, e_X)^{\mathrm{op}}, G).$$

If X is Lindelöf (resp. is definable) and G countable (resp. is finite), then there is a one-to-one correspondence

$$\pi_0(G\text{-}\mathrm{Cov}_{\mathrm{ldef}_\omega}(X, e_X)) \to \mathrm{Hom}(\pi_1(X, e_X)^{\mathrm{op}}, G)$$

(resp.

$$\pi_0(G\text{-}\mathrm{Cov}_{\mathrm{def}}(X, e_X)) \to \mathrm{Hom}(\pi_1(X, e_X)^{\mathrm{op}}, G)).$$

4.4. **O-minimal Hurewicz and Seifert-van Kampen theorems.** We start with the Seifert-van Kampen theorem:

Theorem 4.12 (**Seifert - van Kampen**). *Let X be a definably connected locally definable manifold with $e_X \in X$ and let $\mathcal{W} = \{W_\alpha\}_{\alpha \in I}$ be an admissible cover of X by open locally definable subsets. Suppose that for $\alpha, \beta \in I$:*

- *$e_X \in W_\alpha$ and W_α is definably connected;*
- *$W_\alpha \cap W_\beta \in \mathcal{W}$.*

Then

$$\varinjlim_{\alpha \in I} \pi_1(W_\alpha, e_X) \simeq \pi_1(X, e_X).$$

Proof. By Corollary 4.11, we have

$$\mathrm{Hom}(\pi_1(X, e_X)^{\mathrm{op}}, G) \simeq \pi_0(G\text{-}\mathrm{Cov}_{\mathrm{ldef}}(X)) \simeq \pi_0(\mathrm{LCSh}_{G\text{-}\mathbf{Tors}}(X_{\mathrm{def}}))$$

for any discrete group G. The same result holds with X replaced by W_α. The gluing properties of sheaves give the isomorphisms

$$\begin{aligned} \mathrm{Hom}(\pi_1(X, e_X)^{\mathrm{op}}, G) &\simeq \varprojlim_{\alpha} \mathrm{Hom}(\pi_1(W_\alpha, e_X)^{\mathrm{op}}, G) \\ &\simeq \mathrm{Hom}(\varinjlim_{\alpha} \pi_1(W_\alpha, e_X)^{\mathrm{op}}, G) \end{aligned}$$

for any discrete group G. Then the Yoneda's Lemma implies the result. □

We now proceed towards the proof of the o-minimal Hurewicz theorem. Let X be a locally definable manifold, $\mathcal{U} = \{U_i\}_{i \in I}$ an admissible cover of X by open definable subsets and $\mathcal{G}$ an object of $\mathrm{Sh}_{\mathbf{Gp}}(X_{\mathrm{def}})$. Below we write $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$. A Čech co-cycle for $\mathcal{U}$ with values in $\mathcal{G}$ is a family $(g_{ij})_{(i,j) \in I \times I}$ with $g_{ij} \in \mathcal{G}(U_{ij})$ such that

$$(g_{ij|U_{ijk}}) \cdot (g_{jk|U_{ijk}}) = g_{ik|U_{ijk}}, \text{ all, } i, j, k.$$

Two Čech co-cycles g and g' are cohomologous, denoted $g \sim g'$, if there is a family $(h_i)_{i \in I}$ with $h_i \in \mathcal{G}(U_i)$ such that

$$g'_{ij} = (h_{i|U_{ij}}) \cdot g_{ij} \cdot (h_{j|U_{ij}})^{-1}, \text{ all, } i, j.$$

This is an equivalence relation and the set of equivalence classes of Čech co-cycles is called the o-minimal Čech cohomology set with respect to the admissible cover $\mathcal{U}$ and is denoted by

$$\check{H}^1(X, \mathcal{U}; \mathcal{G}).$$

This is not in general a group, but it does have a distinguished element represented by the Čech co-cycle (g_{ij}) with $g_{ij} = 1$ for all i, j. It is a group if $\mathcal{G}$ is an object of the subcategory $\mathrm{Sh}_{\mathbf{Ab}}(X_{\mathrm{def}})$ and, it is called in this case the o-minimal Čech cohomology group with respect to the admissible cover $\mathcal{U}$.

Remark 4.13. If G is a discrete abelian group, then $\check{H}^1(X,\mathcal{U};G)$ is exactly the o-minimal Čech cohomology (abelian) group with respect to the admissible cover $\mathcal{U}$ in degree one of the o-minimal Čech cohomology theory defined in [18].

If $\mathcal{V}$ is another admissible cover of X by open definable subsets refining $\mathcal{U}$, then we have a canonical inclusion

$$\nu_{\mathcal{U}}^{\mathcal{V}} : \check{H}^1(X,\mathcal{U};\mathcal{G}) \to \check{H}^1(X,\mathcal{V};\mathcal{G})$$

induced by restrictions which is an injective homomorphism when $\mathcal{G}$ is an object of the subcategory $\mathrm{Sh}_{\mathbf{Ab}}(X_{\mathrm{def}})$.

We define the o-minimal Čech cohomology set (resp. group if $\mathcal{G}$ is an object of the subcategory $\mathrm{Sh}_{\mathbf{Ab}}(X_{\mathrm{def}})$)

$$\check{H}^1(X;\mathcal{G})$$

to be the disjoint union

$$\bigsqcup\{\check{H}^1(X,\mathcal{U};\mathcal{G}) : \mathcal{U} \text{ an admissible cover of } S \text{ by open definable subsets}\}$$

modulo the equivalence relation give by: an element from $\check{H}^1(X,\mathcal{U};\mathcal{G})$ is equivalent to an element from $\check{H}^1(X,\mathcal{V};\mathcal{G})$ if they have the same image under all the inclusion maps

$$\check{H}^1(X,\mathcal{U};\mathcal{G}) \xrightarrow{\nu_{\mathcal{U}}^{\mathcal{W}}} \check{H}^1(X,\mathcal{W};\mathcal{G}) \xleftarrow{\nu_{\mathcal{V}}^{\mathcal{W}}} \check{H}^1(X,\mathcal{V};\mathcal{G})$$

where $\mathcal{W}$ is an admissible cover of X by open definable subsets refining both $\mathcal{U}$ and $\mathcal{V}$. Clearly we then have canonical inclusions

$$\nu_{\mathcal{U}} : \check{H}^1(X,\mathcal{U};\mathcal{G}) \to \check{H}^1(X;\mathcal{G})$$

such that

$$\nu_{\mathcal{U}} = \nu_{\mathcal{V}} \circ \nu_{\mathcal{U}}^{\mathcal{V}}$$

whenever we have a refinement $\mathcal{V}$ of $\mathcal{U}$ by an admissible cover of X by open definable subsets.

When $\mathcal{G}$ is an object of the subcategory $\mathrm{Sh}_{\mathbf{Ab}}(X_{\mathrm{def}})$, then

$$\check{H}^1(X;\mathcal{G}) = \varinjlim_{\mathcal{U}} \check{H}^1(X,\mathcal{U};\mathcal{G})$$

where the direct limit is taken over admissible covers $\mathcal{U}$ of X by open definable subsets directed by refinement by admissible covers $\mathcal{U}$ of X by open definable subsets.

As above let X be a locally definable manifold and $\mathcal{G}$ an object of $\mathrm{Sh}_{\mathbf{Gp}}(X_{\mathrm{def}})$. We say that an object $\mathcal{S}$ of $\mathrm{Sh}(X_{\mathrm{def}})$ is a $\mathcal{G}$-torsor on X if $\mathcal{G}$ acts on $\mathcal{S}$ on the right and:

- there exists a admissible cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X by open definable subsets that splits $\mathcal{S}$ i.e., for all i, $\mathcal{S}(U_i) \neq \emptyset$;
- for every open definable subset U of X and $s \in \mathcal{S}(U)$, the map $\mathcal{G}_{|U} \to \mathcal{S}_{|U} : g \mapsto s^g$ is an isomorphism of sheaves.

A $\mathcal{G}$-torsor $\mathcal{S}$ on X is trivial if $\mathcal{S}(X) \neq \emptyset$, equivalently, if it is isomorphic to the $\mathcal{G}$-torsor $\mathcal{G}$ (with the action given by right multiplication).

Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an admissible cover of X by open definable subsets. We denote by $\mathcal{G}$-Tors$(X,\mathcal{U})$ the set of isomorphisms classes of $\mathcal{G}$-torsors on X split by $\mathcal{U} = \{U_i\}_{i\in I}$. If $\mathcal{G}$ is an object of the subcategory $\mathrm{Sh}_{\mathbf{Ab}}(X_{\mathrm{def}})$, then $\mathcal{G}$-Tors$(X,\mathcal{U})$ has an abelian group operation induced (on sections) by the abelian group operation on each $\pi_0(G\text{-}\mathbf{Tors})$ for G a discrete abelian group.

If $\mathcal{V}$ is another admissible cover of X by open definable subsets refining $\mathcal{U}$, then we have a canonical inclusion

$$i^{\mathcal{V}}_{\mathcal{U}} : \mathcal{G}\text{-Tors}(X,\mathcal{U}) \to \mathcal{G}\text{-Tors}(X,\mathcal{V})$$

induced by restrictions which is an injective homomorphism when $\mathcal{G}$ is an object of the subcategory $\mathrm{Sh}_{\mathbf{Ab}}(X_{\mathrm{def}})$.

We define the set (resp. group if $\mathcal{G}$ is an object of the subcategory $\mathrm{Sh}_{\mathbf{Ab}}(X_{\mathrm{def}})$)

$$\mathcal{G}\text{-Tors}(X)$$

to be the disjoint union

$$\bigsqcup \{\mathcal{G}\text{-Tors}(X,\mathcal{U}) : \mathcal{U} \text{ an admissible cover of } X \text{ by open definable subsets}\}$$

modulo the equivalence relation give by: an element from $\mathcal{G}$-Tors$(X,\mathcal{U})$ is equivalent to an element from $\mathcal{G}$-Tors$(X,\mathcal{V})$ if they have the same image under all the inclusion maps

$$\begin{array}{ccc} \mathcal{G}\text{-Tors}(X,\mathcal{U}) & & \mathcal{G}\text{-Tors}(X,\mathcal{V}) \\ & {\scriptstyle i^{\mathcal{W}}_{\mathcal{U}}} \searrow \quad \swarrow {\scriptstyle i^{\mathcal{W}}_{\mathcal{V}}} & \\ & \mathcal{G}\text{-Tors}(X,\mathcal{W}) & \end{array}$$

where $\mathcal{W}$ is an admissible cover of X by open definable subsets refining both $\mathcal{U}$ and $\mathcal{V}$. Clearly we then have canonical inclusions

$$i_{\mathcal{U}} : \mathcal{G}\text{-Tors}(X,\mathcal{U}) \to \mathcal{G}\text{-Tors}(X)$$

such that

$$i_{\mathcal{U}} = i_{\mathcal{V}} \circ i^{\mathcal{V}}_{\mathcal{U}}$$

whenever we have a refinement $\mathcal{V}$ of $\mathcal{U}$ by an admissible cover of X by open definable subsets.

Proposition 4.14. *Let X be a locally definable manifold, $\mathcal{G}$ an object of* $\mathrm{Sh}_{\mathbf{Gp}}(X_{\mathrm{def}})$ *and let $\mathcal{U} = \{U_i\}_{i\in I}$ be an admissible cover of X by open definable subsets. Then there exists a well defined bijection*

$$\mathcal{G}\text{-Tors}(X,\mathcal{U}) \to \check{H}^1(X,\mathcal{U};\mathcal{G})$$

which is an isomorphism when $\mathcal{G}$ is an object of the subcategory $\mathrm{Sh}_{\mathbf{Ab}}(X_{\mathrm{def}})$.

Proof. Let $\mathcal{S}$ be a $\mathcal{G}$-torsor on X split by $\mathcal{U} = \{U_i\}_{i\in I}$ and choose $s_i \in \mathcal{S}(U_i)$ for each i. Because of the second condition in the definition of $\mathcal{G}$-torsor, there are unique $g_{ij} \in \mathcal{G}(U_{ij})$, such that

$$(s_{i|U_{ij}})^{g_{ij}} = s_{j|U_{ij}}.$$

Then (g_{ij}) is a Čech co-cycle, because (omitting the restrictions signs)

$$s_i^{g_{ij}\cdot g_{jk}} = s_k = s_i^{g_{ik}}.$$

Moreover, replacing s_i with $s'_i = s_i^{h_i}$, $h_i \in \mathcal{G}(U_i)$ leads to a cohomologous co-cycle. Thus, $\mathcal{S}$ defines a class $c(\mathcal{S})$ in $\check{H}^1(X,\mathcal{U};\mathcal{G})$.

Let $\alpha : \mathcal{S} \to \mathcal{S}'$ be an isomorphism of $\mathcal{G}$-torsors on X split by $\mathcal{U} = \{U_i\}_{i\in I}$, and choose $s_i \in \mathcal{S}(U_i)$. Then $\alpha(s_i) \in \mathcal{S}'(U_i)$, and (omitting the restriction signs)

$$s_i^{g_{ij}} = s_j \Rightarrow \alpha(s_i)^{g_{ij}} = \alpha(s_j).$$

Thus the Čech co-cycle defined by the family $(\alpha(s_i))$ equals that defined by (s_i). Showing that $c(\mathcal{S})$ depends only on the isomorphism class of $\mathcal{S}$ and so the map $\mathcal{G}\text{-Tors}(X,\mathcal{U}) \to \check{H}^1(X,\mathcal{U};\mathcal{G})$ is well defined.

Suppose that $c(\mathcal{S}) = c(\mathcal{S}')$. Then we may choose sections $s_i \in \mathcal{S}(U_i)$ and $s'_i \in \mathcal{S}'(U_i)$ that define the same Čech co-cycle (g_{ij}). Let V be an open definable subset of X and set $V_i = U_i \cap V$, $V_{ij} = U_{ij} \cap V$ and $V_{ijk} = U_{ijk} \cap V$. Suppose that $t \in \mathcal{S}(V)$. Then

$$t_{|V_i} = (s_{i|V_i})^{g_i}$$

for a unique $g_i \in \mathcal{G}(V_i)$. From the equality $(t_{|V_i})_{|V_{ij}} = (t_{|V_j})_{|V_{ji}}$, we find that

$$(g_{i|V_{ij}}) = g_{ij} \cdot (g_{j|V_{ij}}) \qquad (*)$$

Since $\mathcal{S}$ is a sheaf, $t \mapsto (g_i)_{i\in I}$ is a bijection from $\mathcal{S}(V)$ onto the set of families $(g_i)_{i\in I}$, $g_i \in \mathcal{G}(V_i)$, satisfying (*). A similar statement holds for $\mathcal{S}'$, and so there is a canonical bijection $\mathcal{S}(V) \to \mathcal{S}'(V)$. The family of these bijections is an isomorphism $\mathcal{S} \to \mathcal{S}'$ of $\mathcal{G}$-torsors and so the map $\mathcal{G}\text{-Tors}(X,\mathcal{U}) \to \check{H}^1(X,\mathcal{U};\mathcal{G})$ is injective.

Let $(g_{ij})_{(i,j)\in I\times I}$ be a Čech co-cycle for $\mathcal{U} = \{U_i\}_{i\in I}$. For any V open definable subset of X set $V_i = U_i \cap V$, $V_{ij} = U_{ij} \cap V$ and $V_{ijk} = U_{ijk} \cap V$. Define $\mathcal{S}(V)$ to be the set of families $(g_i)_{i\in I}$, $g_i \in \mathcal{G}(V_i)$, such that

$$(g_{i|V_{ij}}) = g_{ij} \cdot (g_{j|V_{ij}}).$$

Showing that this defines a $\mathcal{G}$-torsor $\mathcal{S}$, and that $c(\mathcal{S})$ is represented by (g_{ij}) involves only routine checking and so the map $\mathcal{G}\text{-Tors}(X,\mathcal{U}) \to \check{H}^1(X,\mathcal{U};\mathcal{G})$ is surjective.

When $\mathcal{G}$ is an object of the subcategory $\mathrm{Sh}_{\mathbf{Ab}}(X_{\mathrm{def}})$ it is routine to show that the map $\mathcal{G}\text{-Tors}(X,\mathcal{U}) \to \check{H}^1(X,\mathcal{U};\mathcal{G})$ is a homomorphism. □

If $\mathcal{V}$ is another admissible cover of X by open definable subsets refining $\mathcal{U}$, then the bijections (resp. isomorphisms) of Proposition 4.14 commute with the canonical inclusions (resp. injective homomorphisms):

$$\begin{array}{ccc} \mathcal{G}\text{-Tors}(X,\mathcal{U}) & \longrightarrow & \check{H}^1(X,\mathcal{U};\mathcal{G}) \\ \downarrow & & \downarrow \\ \mathcal{G}\text{-Tors}(X,\mathcal{V}) & \longrightarrow & \check{H}^1(X,\mathcal{V};\mathcal{G}). \end{array}$$

Therefore, we have:

Corollary 4.15. *Let X be a locally definable manifold and let $\mathcal{G}$ an object of $\mathrm{Sh}_{\mathbf{Gp}}(X_{\mathrm{def}})$. Then there exists a well defined bijection*

$$\mathcal{G}\text{-Tors}(X) \to \check{H}^1(X;\mathcal{G})$$

which is an isomorphism when $\mathcal{G}$ is an object of the subcategory $\mathrm{Sh}_{\mathbf{Ab}}(X_{\mathrm{def}})$.

We are ready to prove the o-minimal Hurewicz theorem:

Theorem 4.16 (**Hurewicz theorem).** *Let G be a discrete group and X a definably connected locally definable manifold. Then there is a canonical bijection*

$$\mathrm{Hom}(\pi_1(X)^{\mathrm{op}}, G)/\mathit{conjugacy} \to \check{H}^1(X; G).$$

Moreover, if G is abelian, then we have a canonical isomorphism

$$\mathrm{Hom}(\pi_1(X)^{\mathrm{op}}, G) \simeq \check{H}^1(X; G).$$

Proof. If G is a discrete group, then we have canonical bijections

$$G\text{-Tors}(X) \simeq \pi_0(\mathrm{LCSh}_{G\text{-}\mathbf{Tors}}(X_{\mathrm{def}})) \simeq \pi_0(G\text{-}\mathrm{Cov}_{\mathrm{ldef}}(X))$$

by definitions and Proposition 4.3. Thus by the canonical bijections

$$\pi_0(G\text{-}\mathrm{Cov}_{\mathrm{ldef}}(X)) \simeq \mathrm{Hom}(\pi_1(X)^{\mathrm{op}}, G)/\mathrm{conjugacy}$$

and

$$\mathcal{G}\text{-Tors}(X) \simeq \check{H}^1(X; \mathcal{G})$$

(Corollaries 4.10 and 4.15), we have a canonical bijection

$$\mathrm{Hom}(\pi_1(X)^{\mathrm{op}}, G)/\mathrm{conjugacy} \to \check{H}^1(X; G).$$

If G is abelian, then both $\mathrm{Hom}(\pi_1(X)^{\mathrm{op}}, G)/\mathrm{conjugacy} = \mathrm{Hom}(\pi_1(X)^{\mathrm{op}}, G)$ and G-Tors(X) are abelian groups. Hence, $\pi_0(G\text{-}\mathrm{Cov}_{\mathrm{ldef}}(X)) \simeq \pi_0(\mathrm{LCSh}_{G\text{-}\mathbf{Tors}}(X_{\mathrm{def}}))$ $\simeq G$-Tors(X) also have canonical abelian group structures. It is routine to check that the canonical bijection

$$\pi_0(G\text{-}\mathrm{Cov}_{\mathrm{ldef}}(X)) \simeq \mathrm{Hom}(\pi_1(X)^{\mathrm{op}}, G)$$

(Corollary 4.10) induced by the monodromy functor (Theorem 4.7) is in fact a homomorphism. Therefore, by the canonical isomorphism

$$\mathcal{G}\text{-Tors}(X) \simeq \check{H}^1(X; \mathcal{G})$$

(Corollary 4.15), we have a canonical isomorphism

$$\mathrm{Hom}(\pi_1(X)^{\mathrm{op}}, G) \simeq \check{H}^1(X; G)$$

as required. □

5. Concluding remarks

Here we observe that all our results can be generalized to other categories of locally definable spaces in arbitrary o-minimal structures. To see this we point out exactly what is required in the proofs. So let $\mathcal{N}$ be an arbitrary o-minimal structure and let $\mathbf{A}$ be a full subcategory of the category of locally definable spaces in $\mathcal{N}$.

In $\mathcal{N}$ one can define and prove in exactly the same way all the basic concepts and properties about locally definable covering maps as in Subsection 2.1. So no special requirements are needed here, but in relation to the concepts and results of Subsection 2.2 we need to assume that the following holds in $\mathcal{N}$:

- (A1) It is possible to define good notions of definable paths and definable homotopies such that:
  - (a) every object of $\mathbf{A}$ which is definably connected is uniformly definably path connected;

(b) given a locally definable covering map $p_X : X \to S$ in $\mathbf{A}$ then: (i) every definable path γ in S has a unique lifting $\widetilde{\gamma}$ which is a definable path in X with a given base point; (ii) every definable homotopy F between definable paths γ and σ in S has a unique lifting $\widetilde{F}$ which is a definable homotopy between the definable paths $\widetilde{\gamma}$ and $\widetilde{\sigma}$ in X.

As the reader can easily verify these are indeed the only requirements needed to define and prove in exactly the same way all the basic concepts and properties of Subsection 2.2.

For the other results of the paper, we need on top of (A1) also the following requirement:

(A2) Every object of $\mathbf{A}$ has an admissible covers by definably simply connected, open definable subsets refining any admissible cover by open definable subsets.

With (A1) and (A2) one proves in exactly the same way the main result established in Subsection 3.1 (Theorem 1.1). For the proofs of the reminding results of Section 3, what we need, besides (A1), (A2) and the results previously obtained from these requirements, is: [16, Lemma 2.1 (1)] (in the proof of Theorem 3.4), [16, Corollary 2.2] (in Remark 3.12), [16, Corollary 2.3] (in Remark 3.13) and [9, Chapter 6, (3.6)] (in Theorem 3.16). Now the quoted results from [16] hold in arbitrary o-minimal structures (and for locally definable spaces as well). On the other hand, we used [9, Chapter 6, (3.6)] to notice that the domains of the definable paths are definably normal. But by [15, Remark 2.8, Proposition 2.12 and Theorem 2.13], in arbitrary o-minimal structures, every one dimensional definable space is definably normal. So if we assume as we should that the domains of the definable paths given by (A1) are one dimensional definable spaces, then the use of [9, Chapter 6, (3.6)] can be replaced by this last more general observation. In conclusion we saw that with (A1) and (A2) one proves in exactly the same way the main result of Section 3. Moreover, for the same reasons one sees that the same is true regarding all the results of Section 4.

The fact that (A1) and (A2) are the only requirements needed to develop this kind of theory is somewhat not surprising. Indeed in topology, where we have good notions of paths and homotopies with the lifting of paths and homotopies property, all one needs is existence of such nice open covers as in (A2). As we saw here, in o-minimal expansions of ordered groups, (A1) holds even in the category of locally definable spaces (but it not known to hold in arbitrary o-minimal structures) and (A2) holds in the category of locally definable manifolds. In the semi-algebraic case ([7]) and in the case of o-minimal expansions of fields ([3]) (A2) holds in the category of locally definable spaces by the semi-algebraic (resp. o-minimal) triangulation theorem. To obtain (A2) in o-minimal expansions of ordered groups for the category of locally definable spaces all that is needed is the following:

Conjecture. Every definable set is a finite union of relatively open definable subsets which are definably simply connected.

Finally observe that in our context, the role that (A1) (b) and (A2) play is similar to the role the analogue properties play in topology. However, (A2) is often used in combination with the results from [16] mentioned above to get local definability.

Also (A1) (a) is required essentially only once and to get local definability (see Proposition 2.18), the other places where it is used, it is used to replace definably connected by definably path connected.

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