Chapter 1

White Noise Analysis: An Introduction

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The starting point of White Noise Analysis[11] and[2,14–16,20,21,34,39] is a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and the corresponding norm $|\cdot|$, and a nuclear triple

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}',$$

where $\mathcal{N}$ is a nuclear space densely and continuously embedded in $\mathcal{H}$. Of course, in a general framework, *a priori* there are several different possible nuclear spaces. However, in concrete applications, the application will suggest the use of particular nuclear triples. For example, in the study of intersection local times of d-dimensional Brownian motions it is natural to consider the space $\mathcal{H} = L^2(\mathbb{R},\mathbb{R}^d) =: L^2_d(\mathbb{R})$ of all vector valued square integrable functions with respect to the Lebesgue measure on $\mathbb{R}$ and the Schwartz space $\mathcal{N} = S(\mathbb{R},\mathbb{R}^d) =: S_d(\mathbb{R})$ of vector valued test functions, while in the treatment of Feynman integrals the spaces $L^2(\mathbb{R}) := L^2(\mathbb{R},\mathbb{R})$, $S(\mathbb{R}) := S(\mathbb{R},\mathbb{R})$ are the natural ones.

Since nuclear triples are the basis of the whole White Noise Analysis, we start by briefly recalling the main background of the theory of nuclear spaces, due to A. Grothendieck.[7] For simplicity, instead of general nuclear spaces, cf. e.g.,[40,42,45,50] we just consider nuclear Fréchet spaces, which are the only ones needed in this book. For more details and the proofs see e.g.[2,3,9,14].

1. Nuclear Triples

As before, let $\mathcal{H}$ be a real separable Hilbert space. We consider a family of real separable Hilbert spaces $\mathcal{H}_p$, $p \in \mathbb{N}$, with Hilbertian norm $|\cdot|_p$ such that

$$\mathcal{H} \supset \mathcal{H}_1 \supset \ldots \supset \mathcal{H}_p \supset \ldots$$

so that the corresponding system of norms is ordered:

$$|\cdot| \leq |\cdot|_1 \leq \ldots \leq |\cdot|_p \leq \ldots.$$

In addition, we assume that the intersection of the Hilbert spaces $\mathcal{H}_p$, denoted by

$$\mathcal{N} := \bigcap_{p\in\mathbb{N}} \mathcal{H}_p, \tag{1}$$

is dense in each space $\mathcal{H}_p$, $p \in \mathbb{N}$.

Definition 1. The linear space $\mathcal{N}$ is called nuclear whenever for every $p \in \mathbb{N}$ there is a $q > p$ such that the canonical embedding $\mathcal{H}_q \hookrightarrow \mathcal{H}_p$ is a Hilbert-Schmidt operator.

From now on we shall assume that all spaces (1) are nuclear and fix on $\mathcal{N}$ the *projective limit topology*, that is, the coarsest topology on $\mathcal{N}$ with respect to which all canonical embeddings $\mathcal{N} \hookrightarrow \mathcal{H}_p$, $p \in \mathbb{N}$, are continuous. Or, in an equivalent way, a sequence $(\xi_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{N}$ converges to $\xi \in \mathcal{N}$ if and only if $(\xi_n)_{n\in\mathbb{N}}$ converges to ξ in every Hilbert space $\mathcal{H}_p$, $p \in \mathbb{N}$. It turns out that a nuclear space $\mathcal{N}$ endowed with the projective limit topology is a complete metrizable locally convex space, meaning that it is a Fréchet space. In order to mention explicitly this topology fixed on $\mathcal{N}$, we shall use the notation

$$\mathcal{N} = \underset{p\in\mathbb{N}}{\mathrm{prlim}} \mathcal{H}_p$$

and call such a topological space a *projective limit* or a *countable limit of the family* $(\mathcal{H}_p)_{p\in\mathbb{N}}$.

For each $p \in \mathbb{N}$, let now $\mathcal{H}_{-p}$ be the Hilbertian dual space of $\mathcal{H}_p$ with respect to $\mathcal{H}$ with the corresponding Hilbertian norm $|\cdot|_{-p}$. By the general duality theory cf. e.g.,[9] we have

$$\mathcal{N}' = \bigcup_{p\in\mathbb{N}} \mathcal{H}_{-p},$$

where $\mathcal{N}'$ is the dual space of $\mathcal{N}$ with respect to $\mathcal{H}$. Unless stated otherwise, we shall consider $\mathcal{N}'$ endowed with the inductive limit topology, that is, the finest topology on $\mathcal{N}'$ with respect to which all embeddings $\mathcal{H}_{-p} \hookrightarrow \mathcal{N}'$ are continuous. As a topological space, we shall denote it by

$$\mathcal{N}' = \underset{p\in\mathbb{N}}{\mathrm{indlim}} \mathcal{H}_{-p}$$

and call it an *inductive limit* of the family $(\mathcal{H}_{-p})_{p\in\mathbb{N}}$.

In this way, using the Riesz representation theorem to identify $\mathcal{H}$ with its dual space $\mathcal{H}'$, we have defined a so-called *nuclear* or *Gelfand triple*:

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'.$$

By construction, it turns out that the bilinear dual pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{N}'$ and $\mathcal{N}$ is defined as an extension of the inner product on $\mathcal{H}$:

$$\langle h, \xi \rangle = (h, \xi), \quad h \in \mathcal{H}, \xi \in \mathcal{N}.$$

Example 1. (i) The Schwartz space $S(\mathbb{R})$ of rapidly decreasing C^∞-functions on $\mathbb{R}$ endowed with its usual topology given by the system of seminorms

$$\sup_{u\in\mathbb{R}} \left| u^n \frac{d^m \xi}{du^m}(u) \right|, \quad \xi \in S(\mathbb{R}), m, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$$

is a first example of a nuclear space. Indeed, given the Hamiltonian of the quantum harmonic oscillator, that is, the self-adjoint operator on $L^2(\mathbb{R})$ defined on $S(\mathbb{R})$ by

$$(H\xi)(u) := -\frac{d^2\xi}{du^2}(u) + (u^2 + 1)\xi(u), \quad u \in \mathbb{R},$$

we can define a system of norms $|\cdot|_p$ by setting

$$|\xi|_p := |H^p \xi|, \quad \xi \in S(\mathbb{R}), p \in \mathbb{N},$$

where the last norm is the one on $L^2(\mathbb{R})$. It turns out (cf. e.g.,[12,43,47]) that this system of norms is equivalent to the initial system of seminorms, and thus both systems lead to equivalent topologies on $S(\mathbb{R})$. In addition, the completion of $S(\mathbb{R})$ with respect to each norm $|\cdot|_p$ yields a family of Hilbert spaces $\mathcal{H}_p$ and

$$S(\mathbb{R}) = \operatorname*{prlim}_{p\in\mathbb{N}} \mathcal{H}_p,$$

see e.g.,[14]. Therefore, for the dual space $S'(\mathbb{R})$ of $S(\mathbb{R})$ (with respect to $L^2(\mathbb{R})$) of Schwartz tempered distributions we have

$$S'(\mathbb{R}) = \operatorname*{indlim}_{p\in\mathbb{N}} \mathcal{H}_{-p}.$$

(ii) The previous example extends to the space $S_d(\mathbb{R})$ of vector valued Schwartz test functions for the operator H defined on $S_d(\mathbb{R})$ by

$$(H\boldsymbol{\xi})(u) := ((H\boldsymbol{\xi})_1(u), \ldots, (H\boldsymbol{\xi})_d(u)), \quad \boldsymbol{\xi} = (\xi_1, \ldots, \xi_d), \xi_i \in S(\mathbb{R}) \quad (2)$$

with

$$(H\boldsymbol{\xi})_i(u) := -\frac{d^2\xi_i}{du^2}(u) + (u^2+1)\xi_i(u) = (H\xi_i)(u), \quad i = 1, \ldots, d, u \in \mathbb{R}.$$

This leads to the following system of increasing Hilbertian norms $|\cdot|_p$, $p \in \mathbb{N}$,

$$|\boldsymbol{\xi}|_p^2 := \sum_{i=1}^d |\xi_i|_p^2 = \sum_{i=1}^d |H^p \xi_i|^2, \ \boldsymbol{\xi} = (\xi_1, \ldots, \xi_d), \xi_i \in S(\mathbb{R}), i = 1, \ldots, d, \tag{3}$$

where the last sum in (3) is the square of the $L^2_d(\mathbb{R})$-norm of (2), and to the corresponding Hilbert spaces $\mathcal{H}_p$ defined by completion of $S_d(\mathbb{R})$ with respect to the norms (3). As in (i), we have

$$S_d(\mathbb{R}) = \underset{p\in\mathbb{N}}{\mathrm{prlim}}\,\mathcal{H}_p, \quad S'_d(\mathbb{R}) = \underset{p\in\mathbb{N}}{\mathrm{indlim}}\,\mathcal{H}_{-p},$$

being $S'_d(\mathbb{R})$ the space of vector valued Schwartz tempered distributions.

(iii) Example (i) also extends to the Schwartz space $S(\mathbb{R}^d, \mathbb{R})$ of smooth functions on $\mathbb{R}^d$, $d \geq 2$, of rapid decrease (shortly $S(\mathbb{R}^d)$) and to its dual space $S'(\mathbb{R}^d)$ of Schwartz tempered distributions. In this case, the usual topology on $S(\mathbb{R}^d)$ is given by the family of seminorms indexed by multi-indices $(\alpha_1, \ldots, \alpha_d)$, $(\beta_1, \ldots, \beta_d)$ in $\mathbb{N}_0^d$,

$$\sup_{\mathbf{u}=(u_1,\ldots,u_d)\in\mathbb{R}^d} \left| u_1^{\alpha_1} \ldots u_d^{\alpha_d} \left(\partial_1^{\beta_1} \ldots \partial_d^{\beta_d} \xi \right)(\mathbf{u}) \right|, \quad \xi \in S(\mathbb{R}^d),$$

where ∂_i, $i = 1, \ldots, d$, is the partial derivative on $\mathbb{R}^d$ with respect to the i-th coordinate. Given the Hamiltonian of the quantum harmonic oscillator, that is, the self-adjoint operator on $L^2(\mathbb{R}^d, \mathbb{R}) =: L^2(\mathbb{R}^d)$ defined on $S(\mathbb{R}^d)$ by

$$(H\xi)(\mathbf{u}) := -(\Delta\xi)(\mathbf{u}) + (|\mathbf{u}|^2 + 1)\xi(\mathbf{u}), \quad \mathbf{u} \in \mathbb{R}^d,$$

being Δ the Laplacian on $\mathbb{R}^d$, we define a system of norms $|\cdot|_p$ on $S(\mathbb{R}^d)$ by

$$|\xi|_p := |H^p \xi|, \quad \xi \in S(\mathbb{R}^d), p \in \mathbb{N},$$

where the last norm is the one on $L^2(\mathbb{R}^d)$. As in Example (i), it turns out cf. e.g.,[12,43,47] that such a system is equivalent to the above system of seminorms, leading then to equivalent topologies on $S(\mathbb{R}^d)$. In addition, cf. e.g.,[14] we have

$$S(\mathbb{R}^d) = \underset{p\in\mathbb{N}}{\mathrm{prlim}}\,\mathcal{H}_p,$$

where each $\mathcal{H}_p$, $p \in \mathbb{N}$, is the Hilbert space obtained by completion of $S(\mathbb{R}^d)$ with respect to the norm $|\cdot|_p$. Thus

$$S'(\mathbb{R}^d) = \underset{p\in\mathbb{N}}{\mathrm{indlim}}\,\mathcal{H}_{-p}.$$

2. Gaussian Space

Given a nuclear triple $\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'$, let $\mathcal{C}_\sigma(\mathcal{N}')$ be the σ-algebra on $\mathcal{N}'$ generated by the *cylinder sets*

$$\{x \in \mathcal{N}' : (\langle x, \varphi_1\rangle, \ldots, \langle x, \varphi_n\rangle) \in B, \varphi_1, \ldots, \varphi_n \in \mathcal{N}, B \in \mathcal{B}(\mathbb{R}^n), n \in \mathbb{N}\},$$

where $\mathcal{B}(\mathbb{R}^n)$, $n \in \mathbb{N}$, is the Borel σ-algebra on $\mathbb{R}^n$.

Theorem 1. ***(The Minlos Theorem*[37]*)*** *Let C be a complex-valued function on $\mathcal{N}$ fulfilling the following three properties:*

(i) $C(0) = 1$,
(ii) *C is continuous on $\mathcal{N}$,*
(iii) *C is positive definite, i.e.,*

$$\sum_{i,j=1}^{n} C(\xi_i - \xi_j) z_i \overline{z_j} \geq 0, \quad \xi_1, \ldots, \xi_n \in \mathcal{N}, z_1, \ldots, z_n \in \mathbb{C}, n \in \mathbb{N}.$$

Then, there is a unique probability measure μ_C on $(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'))$ which characteristic function is equal to C, that is, for all $\xi \in \mathcal{N}$

$$\int_{\mathcal{N}'} \exp\left(i\langle x, \xi\rangle\right)\, d\mu_C(x) = C(\xi). \tag{4}$$

For a presentation of the Minlos theorem, including support properties of the probability measure given by this theorem see.[10]

Remark 1. The analogous statement of the Minlos theorem for the nuclear space $\mathcal{N}$ replaced by the finite dimensional space $\mathbb{R}^d$ is the well-known Bochner theorem. Because of this, in the literature Theorem 1 is quite often called the Bochner-Minlos theorem as well.

Consider now the following particular positive definite continuous function defined on $\mathcal{N}$ by

$$C(\xi) = \exp\left(-\frac{1}{2}|\xi|^2\right), \quad \xi \in \mathcal{N}. \tag{5}$$

Then, by the Minlos theorem, we are given a (Gaussian) measure μ on $(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'))$ defined by (4) and (5).

Definition 2. We call the probability space $(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'), \mu)$ the Gaussian space associated with $\mathcal{N}$ and $\mathcal{H}$.

In particular, if $\mathcal{N} = S(\mathbb{R}^d)$ with the topology described in Example 1, the space $(S'(\mathbb{R}^d), \mathcal{C}_\sigma(S'(\mathbb{R}^d)), \mu)$ is called white noise with d-dimensional time parameter. If $d = 1$, we simply call it white noise.

Definition 3. For short we set

$$(L^2) := L^2(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'), \mu)$$

for the complex L^2 space.

In applications of White Noise Analysis, the space (L^2) plays an essential role. In order to distinguish clearly the inner product $(\cdot,\cdot)$ and the Hilbertian norm $|\cdot|$ on the real space $\mathcal{H}$ from those defined on the complex space (L^2), we shall denote the inner product on (L^2) by $(\!(\cdot,\cdot)\!)$ and the corresponding norm by $\|\cdot\|$. Furthermore, we shall assume that $(\!(\cdot,\cdot)\!)$ is linear in the first factor and antilinear in the second one, that is,

$$(\!(F_1, F_2)\!) := \int_{\mathcal{N}'} F_1(x)\bar{F}_2(x)\, d\mu(x), \quad F_1, F_2 \in (L^2),$$

where $\bar{F}_2$ is the complex conjugate function of F_2.

From the definition of the Gaussian measure μ given by (4) and (5), it follows straightforwardly that for every $\xi \in \mathcal{N}$, $\langle\cdot,\xi\rangle$ is a normally distributed random variable with variance $|\xi|^2$. Thus, for all $\xi \in \mathcal{N}$, $\xi \neq 0$,

$$\|\langle\cdot,\xi\rangle\|^2 = \int_{\mathcal{N}'} \langle x,\xi\rangle^2\, d\mu(x) = \frac{1}{\sqrt{2\pi|\xi|^2}} \int_{-\infty}^{+\infty} u^2 \exp\left(-\frac{u^2}{2|\xi|^2}\right) du = |\xi|^2.$$

Moreover, again by (4) and (5), the real process X defined on $\mathcal{N}' \times \mathcal{N}$ by $X_\xi(x) = \langle x,\xi\rangle$ is centered Gaussian with covariance

$$\int_{\mathcal{N}'} \langle x,\xi_1\rangle\langle x,\xi_2\rangle\, d\mu(x) = \frac{1}{2}\left(\|\langle\cdot,\xi_1+\xi_2\rangle\|^2 - \|\langle\cdot,\xi_1\rangle\|^2 - \|\langle\cdot,\xi_2\rangle\|^2\right) = (\xi_1,\xi_2).$$

As we have mentioned above, in this book we shall mostly choose $\mathcal{N}$ to be the Schwartz space $S(\mathbb{R}^d)$, $S_d(\mathbb{R})$, or $S(\mathbb{R})$ of test functions and $\mathcal{H}$ to be $L^2(\mathbb{R}^d)$, $L^2_d(\mathbb{R})$, or $L^2(\mathbb{R})$, respectively. In all these cases, $\mathcal{N}$ is dense in $\mathcal{H}$. This is an assumption fixed on general $\mathcal{N}$ and $\mathcal{H}$ from the very beginning. Therefore, the above considerations allow an extension of the mapping

$$\mathcal{N} \ni \xi \mapsto \langle\cdot,\xi\rangle \in (L^2)$$

to a bounded linear operator

$$\mathcal{H} \ni f \mapsto \langle\cdot,f\rangle \in (L^2)$$

defined at each $f \in \mathcal{H}$ by

$$\langle\cdot,f\rangle := (L^2) - \lim_n \langle\cdot,\xi_n\rangle,$$

where $(\xi_n)_{n\in\mathbb{N}}$ is any sequence in $\mathcal{N}$ converging to f in $\mathcal{H}$. Moreover, $\|\langle\cdot,f\rangle\| = |f|$ for all $f \in \mathcal{H}$.

Proposition 1 ([14]). *The process X defined on $\mathcal{N}' \times \mathcal{H}$ by $X_f(x) = \langle x, f\rangle$ is centered Gaussian with covariance*

$$(\!(\langle \cdot, f\rangle, \langle \cdot, g\rangle)\!) = \int_{\mathcal{N}'} \langle x, f\rangle \langle x, g\rangle \, d\mu(x) = (f, g), \quad f, g \in \mathcal{H}.$$

In particular, for every $f \in \mathcal{H}$, $\langle \cdot, f\rangle$ is normally distributed with variance $|f|^2$. Thus, from its characteristic function we have

$$\int_{\mathcal{N}'} \exp\left(i\langle x, f\rangle\right) \, d\mu(x) = \exp\left(-\frac{1}{2}|f|^2\right), \tag{6}$$

which extends (4) and (5) to $f \in \mathcal{H}$.

More generally, for every $n \in \mathbb{N}_0$ and every $f \in \mathcal{H}$, $f \neq 0$, we can derive from the characteristic function (6),

$$\int_{\mathcal{N}'} \langle x, f\rangle^{2n} \, d\mu(x) = \frac{1}{\sqrt{2\pi |f|^2}} \int_{-\infty}^{+\infty} u^{2n} \exp\left(-\frac{u^2}{2|f|^2}\right) \, du = \frac{(2n)!}{n! 2^n} |f|^{2n}$$

$$\int_{\mathcal{N}'} \langle x, f\rangle^{2n+1} \, d\mu(x) = 0$$

and, by the polarization identity,

$$\int_{\mathcal{N}'} \langle x, f_1\rangle \ldots \langle x, f_n\rangle \, d\mu(x)$$
$$= \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n-k} \sum_{i_1 < \ldots < i_k} \int_{\mathcal{N}'} \langle x, f_{i_1} + \ldots + f_{i_k}\rangle^n \, d\mu(x),$$

for every $f_1, \ldots, f_n \in \mathcal{H}$, $n \in \mathbb{N}$.

Example 2. Coming back to the white noise space $(S'(\mathbb{R}), \mathcal{C}_\sigma(S'(\mathbb{R})), \mu)$, the previous proposition allows us to consider the Gaussian centered process X with independent increments,

$$X_{1\!\!1_{[0,t)}}(x) = \langle x, 1\!\!1_{[0,t)}\rangle, \quad t \geq 0,$$

being $1\!\!1_B$ the *indicator function* of a Borel set $B \subseteq \mathbb{R}$. This process has covariance

$$(\!(\langle \cdot, 1\!\!1_{[0,t)}\rangle, \langle \cdot, 1\!\!1_{[0,s)}\rangle)\!) = (1\!\!1_{[0,t)}, 1\!\!1_{[0,s)}) = s \wedge t,$$

and thus X is a one-dimensional Brownian motion starting at the origin at time zero. We shall denote this Brownian motion by B and $X_{1\!\!1_{[0,t)}}$ by B_t or $B(t, \cdot)$. Informally, note that

$$B_t(x) = \langle x, 1\!\!1_{[0,t)}\rangle = \int_0^t x(s) \, ds,$$

which suggests considering $x(t)$ as the time derivative of the Brownian motion. Of course, this time derivative does not exist in a pointwise sense. However, it exists as a distribution. From now on, we shall denote $x(t)$ by ω_t or $\omega(t)$ and call it **white noise**. As an aside, let us mention that this example is the connecting point for another direction inside infinite dimensional analysis, the well-known Malliavin Calculus.[36] For a clear explanation about the relation between both infinite dimensional analyses see e.g.[16,38].

Within the more general setting of the Gaussian space

$$(S'_d(\mathbb{R}), \mathcal{C}_\sigma(S'_d(\mathbb{R})), \mu),\ d > 1,$$

we can then introduce a d-dimensional Brownian motion $\mathbf{B}$ starting at the origin at time zero by

$$\mathbf{B}_t(\omega_1, \ldots, \omega_d) := \left(\langle \omega_1, 1\!\!1_{[0,t)}\rangle, \ldots, \langle \omega_d, 1\!\!1_{[0,t)}\rangle\right),\ (\omega_1, \ldots, \omega_d) \in S'_d(\mathbb{R}), t \geq 0.$$

3. Itô-Segal-Wiener Isomorphism

We verify from equalities above Example 2 that the important monomials of the type

$$\begin{aligned}
&\langle \cdot, f\rangle^n = \langle \cdot^{\otimes n}, f^{\otimes n}\rangle, \\
&\langle \cdot, f_1\rangle \ldots \langle \cdot, f_n\rangle = \langle \cdot^{\otimes n}, f_1 \otimes \ldots \otimes f_n\rangle = \langle \cdot^{\otimes n}, f_1 \widehat{\otimes} \ldots \widehat{\otimes} f_n\rangle,
\end{aligned}$$

do not verify an orthogonal relation. This fact is a reason for introducing the orthogonalized so-called Wick-ordered polynomials, a class of functions closely related to the orthogonal Hermite polynomials.

For each $x \in \mathcal{N}'$, let $: x^{\otimes n} :\in \mathcal{N}'^{\widehat{\otimes} n}$, $n \in \mathbb{N}_0$ (Appendix A.1.3) be the so-called *Wick power of order* n, inductively defined by

$$\begin{aligned}
&: x^0 := 1, \\
&: x^1 := x, \\
&: x^{\otimes n} :=: x^{\otimes(n-1)} : \widehat{\otimes} x - (n-1) : x^{\otimes(n-2)} : \widehat{\otimes} \mathrm{Tr}, \quad n \geq 2,
\end{aligned}$$

where $\mathrm{Tr} \in \mathcal{N}'^{\widehat{\otimes} 2}$ is given by

$$\langle \mathrm{Tr}, \xi_1 \otimes \xi_2\rangle = \langle \xi_1, \xi_2\rangle, \quad \xi_1, \xi_2 \in \mathcal{N}.$$

Thus, by induction, for all $x \in \mathcal{N}'$ and all $\xi \in \mathcal{N}$ we have

$$\langle : x^{\otimes n} :, \xi^{\otimes n}\rangle = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} \frac{(2k)!}{k! 2^k} (-\langle \xi, \xi\rangle)^k \langle x, \xi\rangle^{n-2k}, \tag{7}$$

where the right-hand side is the so-called Hermite polynomial in $\langle x,\xi\rangle$ of order n and parameter $\sqrt{\langle\xi,\xi\rangle}=|\xi|$. We recall that given a constant $\sigma>0$, the *n-th Hermite polynomial in $u\in\mathbb{R}$ with parameter σ* is defined by

$$\begin{aligned} :u^n:_{\sigma^2} &:= (-\sigma)^n \exp\left(\frac{u^2}{2\sigma^2}\right)\frac{d^n}{du^n}\exp\left(-\frac{u^2}{2\sigma^2}\right)\\ &= \left(\frac{\sigma}{\sqrt{2}}\right)^n H_n\left(\frac{u}{\sqrt{2}\sigma}\right), \end{aligned}$$

being H_n the *Hermite polynomial of order n*,

$$H_n(u) := (-1)^n \exp\left(u^2\right)\frac{d^n}{du^n}\exp\left(-u^2\right) = 2^n :u^n:_{\frac{1}{2}}, \quad u\in\mathbb{R}, n\in\mathbb{N}_0.$$

That is,

$$H_n(u) = 2^n \sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}\frac{(2k)!}{k!2^k}\left(-\frac{1}{2}\right)^k u^{n-2k}, \quad u\in\mathbb{R}, n\in\mathbb{N}_0.$$

Hence, for each $n\in\mathbb{N}_0$ and every $\xi\in\mathcal{N}$, $\xi\neq 0$, we have

$$\langle :x^{\otimes n}:,\xi^{\otimes n}\rangle =: \langle x,\xi\rangle^n :_{\langle\xi,\xi\rangle} = \left(\frac{|\xi|}{\sqrt{2}}\right)^n H_n\left(\frac{\langle x,\xi\rangle}{\sqrt{2}|\xi|}\right),$$

in accordance with (7). Of course, by the polarization identity, (7) also holds for $\xi\in\mathcal{N}_{\mathbb{C}} := \{\xi_1+i\xi_2 : \xi_1,\xi_2\in\mathcal{N}\}$ with

$$\langle x,\xi_1+i\xi_2\rangle := \langle x,\xi_1\rangle + i\langle x,\xi_2\rangle, \quad x\in\mathcal{N}', \xi_1,\xi_2\in\mathcal{N},$$

meaning that for $f\in\mathcal{H}$ or, more generally, for $f\in\mathcal{H}_{\mathbb{C}}$,

$$\langle f,\xi_1+i\xi_2\rangle = (f,\xi_1)+i(f,\xi_2), \quad \xi_1,\xi_2\in\mathcal{N}.$$

Proposition 2. *For all $\varphi^{(n)}\in\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}n}$ and all $\phi^{(m)}\in\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}m}$ the following orthogonal relation holds:*

$$\langle\!\langle \langle :x^{\otimes n}:,\varphi^{(n)}\rangle, \langle :x^{\otimes m}:,\phi^{(m)}\rangle\rangle\!\rangle = \delta_{n,m} n! (\varphi^{(n)},\phi^{(n)}). \tag{8}$$

Proof. (Sketch) Since elements in $\mathcal{N}^{\widehat{\otimes}n}$, $n\in\mathbb{N}_0$, are linear combinations of elements of the form $\xi^{\otimes n}$ with $\xi\in\mathcal{N}$, it is sufficient to prove (8) for $\varphi^{(n)}$, $\phi^{(m)}$ of the form $\varphi^{(n)}=\xi_1^{\otimes n}$, $\phi^{(m)}=\xi_2^{\otimes m}$, $\xi_1,\xi_2\in\mathcal{N}$. In this case, the proof follows from the orthogonality relation between Hermite polynomials,

$$\int_{-\infty}^{+\infty} H_n(u)H_m(u)\exp\left(-u^2\right)\,du = \delta_{n,m}\sqrt{\pi}2^n n!,$$

cf. e.g.,[2,14,39]. As before, the general case can then be derived from the real case by means of polarization identity. □

Conversely, since each monomial $u \mapsto u^n$, $n \in \mathbb{N}_0$ can be written as linear combination of Hermite polynomials in $u \in \mathbb{R}$ with any given parameter $\sigma > 0$,

$$u^n = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} \frac{(2k)!}{k!2^k} \sigma^{2k} : u^{n-2k} :_{\sigma^2}, \quad u \in \mathbb{R},$$

then, by the polarization identity, each monomial $\langle \cdot^{\otimes n}, \xi^{\otimes n} \rangle$, $\xi \in \mathcal{N}_{\mathbb{C}}$, can be written as

$$\begin{aligned}\langle x^{\otimes n}, \xi^{\otimes n} \rangle = \langle x, \xi \rangle^n &= \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} \frac{(2k)!}{k!2^k} \langle \xi, \xi \rangle^k : \langle x, \xi \rangle^{n-2k} :_{\langle \xi, \xi \rangle} \\ &= \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} \frac{(2k)!}{k!2^k} \langle \xi, \xi \rangle^k \langle : x^{\otimes(n-2k)} :, \xi^{\otimes(n-2k)} \rangle, \ x \in \mathcal{N}'.\end{aligned}$$

Therefore, the linear space of the so-called *smooth Wick-ordered polynomials*,

$$\mathcal{P}(\mathcal{N}') := \left\{ \Phi : \Phi(x) = \sum_{n=0}^{N} \langle : x^{\otimes n} :, \varphi^{(n)} \rangle, \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}, x \in \mathcal{N}', N \in \mathbb{N}_0 \right\}$$

coincides with the linear space

$$\left\{ \Phi : \Phi(x) = \sum_{n=0}^{N} \langle x^{\otimes n}, \varphi^{(n)} \rangle, \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}, x \in \mathcal{N}', N \in \mathbb{N}_0 \right\}.$$

In terms of (L^2) properties, it turns out that $\mathcal{P}(\mathcal{N}')$ is dense in (L^2).[48] As a consequence, for any $F \in (L^2)$ there is a sequence $\left(f^{(n)}\right)_{n \in \mathbb{N}_0}$ in the Fock space $\mathrm{Exp}(\mathcal{H}_{\mathbb{C}})$ (Appendix A.1.2) such that

$$F = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} :, f^{(n)} \rangle \tag{9}$$

and, moreover, by the orthogonality property (Proposition 2),

$$\|F\|^2 = \sum_{n=0}^{\infty} n! \left| f^{(n)} \right|^2 = \left\| \left(f^{(n)} \right)_{n \in \mathbb{N}_0} \right\|^2_{\mathrm{Exp}(\mathcal{H}_{\mathbb{C}})}.$$

And vice versa, any series of the form (9) with $\left(f^{(n)}\right)_{n \in \mathbb{N}_0} \in \mathrm{Exp}(\mathcal{H}_{\mathbb{C}})$ defines a function in (L^2). In other words, the expansion (9) yields a unitary isomorphism between the space (L^2) and the symmetric Fock space $\mathrm{Exp}(\mathcal{H}_{\mathbb{C}})$.

Definition 4. We call this unitary isomorphism the Itô-Segal-Wiener isomorphism. The expansion (9) with $\left(f^{(n)}\right)_{n \in \mathbb{N}_0} \in \mathrm{Exp}(\mathcal{H}_{\mathbb{C}})$ is called the Itô-Segal-Wiener chaos decomposition or simply the chaos decomposition of $F \in (L^2)$ and $f^{(n)}$, $n \in \mathbb{N}_0$, the kernels of F.

Remark 2. According to Section 2 and the considerations done just before Proposition 2, equality (7) can be extended to $f \in \mathcal{H}_{\mathbb{C}}$:

$$\langle : x^{\otimes n} :, f^{\otimes n} \rangle =: \langle x, f\rangle^n :_{\langle f,f\rangle} = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} \frac{(2k)!}{k! 2^k} (-\langle f, f\rangle)^k \langle x, f\rangle^{n-2k}, \ x \in \mathcal{N}'.$$

This yields an alternative approach to introduce the Itô-Segal-Wiener isomorphism. Let I be the set of all sequences $\alpha := (\alpha_n)_{n\in\mathbb{N}}$ such that all terms vanish except finitely many ones. For each $\alpha \in I$ set

$$\alpha! = \prod_{n=1}^{\infty} \alpha_n!.$$

Given an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}_{\mathbb{C}}$, it turns out that the family of functions H_α in (L^2) defined by

$$H_\alpha(x) := \prod_{n=1}^{\infty} : \langle x, e_n\rangle^{\alpha_n} :_{\langle e_n, e_n\rangle}, \quad x \in \mathcal{N}', \alpha \in I$$

is an orthogonal basis of (L^2) such that $(\!(H_\alpha, H_\beta)\!) = \delta_{\alpha,\beta}\alpha!$ for all $\alpha, \beta \in I$. Moreover, the space spanned by this family is dense in (L^2), leading then to the Itô-Segal-Wiener isomorphism cf. e.g.[2,14].

4. *S*- and *T*-transform

Among the (L^2) functions, we now consider in particular the class of functions with chaos decomposition of the form

$$\sum_{n=0}^{\infty} \frac{1}{n!} \langle : \cdot^{\otimes n} :, f^{\otimes n}\rangle, \quad f \in \mathcal{H}_{\mathbb{C}}. \tag{10}$$

Observe that their image under the Itô-Segal-Wiener isomorphism is equal to the exponential vectors $e(f) \in \mathrm{Exp}(\mathcal{H}_{\mathbb{C}})$ (Appendix A.1.2). Therefore,

$$\left\| \sum_{n=0}^{\infty} \frac{1}{n!} \langle : \cdot^{\otimes n} :, f^{\otimes n}\rangle \right\| = \exp\left(\frac{|f|^2}{2}\right)$$

and, more generally,

$$\left(\!\!\left(\sum_{n=0}^{\infty} \frac{1}{n!} \langle : \cdot^{\otimes n} :, f^{\otimes n}\rangle, \sum_{n=0}^{\infty} \frac{1}{n!} \langle : \cdot^{\otimes n} :, g^{\otimes n}\rangle \right)\!\!\right) = e^{(f,g)}, \quad f, g \in \mathcal{H}_{\mathbb{C}}.$$

We shall call (10) the *Wick or normalized exponential corresponding to* f and denote it by $: e^{\langle \cdot, f\rangle} :$.

As a first step towards an explicit form for the Wick exponentials (10), we observe that from the definition of the Hermite polynomials $: u^n :_{\sigma^2}$, $u \in \mathbb{R}$, we have

$$: u^n :_{\sigma^2} = \frac{d^n}{d\lambda^n} \exp\left(\lambda u - \frac{1}{2}\sigma^2\lambda^2\right)\Bigg|_{\lambda=0}.$$

Thus, for all $\xi \in \mathcal{N}_{\mathbb{C}}$,

$$: e^{\langle x,\xi\rangle} := \sum_{n=0}^{\infty} \frac{1}{n!} : \langle x,\xi\rangle^n :_{\langle \xi,\xi\rangle} = \exp\left(\langle x,\xi\rangle - \frac{1}{2}\langle \xi,\xi\rangle\right), \quad x \in \mathcal{N}'. \tag{11}$$

Definition 5. The S-transform of $F \in (L^2)$ is the mapping defined on $\mathcal{N}_{\mathbb{C}}$ by

$$(SF)(\xi) = \int_{\mathcal{N}'} : e^{\langle x,\xi\rangle} : F(x)\, d\mu(x), \quad \xi \in \mathcal{N}_{\mathbb{C}}.$$

Since the S-transform of a function F in (L^2) is defined by

$$(SF)(\xi) = \int_{\mathcal{N}'} : e^{\langle x,\xi\rangle} : F(x)\, d\mu(x) = \left(\left(: e^{\langle \cdot,\xi\rangle} :, \bar{F}\right)\right), \quad \xi \in \mathcal{N}_{\mathbb{C}},$$

being $\bar{F}$ the complex conjugate function of F, then in terms of chaos decomposition

$$F = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} :, f^{(n)} \rangle, \tag{12}$$

it follows from Proposition 2 that

$$(SF)(\xi) = \sum_{n=0}^{\infty} n! \left(\frac{\xi^{\otimes n}}{n!}, \overline{f^{(n)}}\right) = \sum_{n=0}^{\infty} \left(\xi^{\otimes n}, \overline{f^{(n)}}\right), \quad \xi \in \mathcal{N}_{\mathbb{C}}. \tag{13}$$

In particular, for $F =: e^{\langle \cdot,f\rangle} :$, $f \in \mathcal{H}_{\mathbb{C}}$,

$$\left(S : e^{\langle \cdot,f\rangle} :\right)(\xi) = e^{(\xi,\bar{f})}, \quad \xi \in \mathcal{N}_{\mathbb{C}}.$$

Remark 3. Equality (13) is of considerable practical importance. Whenever we can compute the S-transform of a $F \in (L^2)$, its expansion as in (13) immediately gives us the kernel functions of its Itô-Segal-Wiener decomposition (12).

The next result states another characterization of the S-transform, which is closely related to the Radon-Nikodym derivative of the translation of the Gaussian measure μ,

$$\frac{d\mu(\cdot - \xi)(x)}{d\mu(x)} =: e^{\langle x,\xi\rangle} :, \quad x \in \mathcal{N}', \xi \in \mathcal{N}.$$

See e.g.[2,12,39].

Proposition 3. *Let $F \in (L^2)$. Then, for all $\xi \in \mathcal{N}$,*

$$(SF)(\xi) = \int_{\mathcal{N}'} F(x+\xi)\, d\mu(x).$$

As a mapping, it is clear that the S-transform is linear on (L^2). Moreover, it is injective. In fact, since $\mathcal{N}$ is dense in $\mathcal{H}$, it follows from Proposition 5 in Appendix A.1.2 and the Itô-Segal-Wiener isomorphism that the space spanned by the set of Wick exponentials $: e^{\langle \cdot, \xi \rangle} :$, $\xi \in \mathcal{N}_{\mathbb{C}}$, is dense in (L^2). Therefore, if $SF = 0$, we have

$$0 = (SF)(\xi) = \left(\!\left(: e^{\langle \cdot, \xi \rangle} :, \bar{F}\right)\!\right), \quad \forall\, \xi \in \mathcal{N}_{\mathbb{C}},$$

which implies $F = 0$. As a particular application of the injective property of the S-transform we can now extend the explicit form (11) to $\mathcal{H}_{\mathbb{C}}$. For this purpose, we first observe that

$$\int_{\mathcal{N}'} \exp(\langle x, f\rangle)\, d\mu(x) = \exp\left(\frac{\langle f, f\rangle}{2}\right), \quad f \in \mathcal{H}_{\mathbb{C}}. \tag{14}$$

See e.g.[39] Thus $\exp(\langle \cdot, f\rangle - \frac{1}{2}\langle f, f\rangle) \in (L^2)$ and, yet by (14), for all $\xi \in \mathcal{N}_{\mathbb{C}}$ we have

$$\begin{aligned}
&S\left(\exp\left(\langle \cdot, f\rangle - \frac{1}{2}\langle f, f\rangle\right)\right)(\xi)\\
&= \exp\left(-\frac{1}{2}(\langle \xi, \xi\rangle + \langle f, f\rangle)\right) \int_{\mathcal{N}'} \exp\left(\langle x, \xi + f\rangle\right)\, d\mu(x)\\
&= e^{(\xi, \bar{f})} = \left(S : e^{\langle \cdot, f\rangle} :\right)(\xi).
\end{aligned}$$

Hence, by the injective property of the S-transform, for all $f \in \mathcal{H}_{\mathbb{C}}$ we find

$$: e^{\langle \cdot, f\rangle} := \exp\left(\langle \cdot, f\rangle - \frac{1}{2}\langle f, f\rangle\right).$$

Besides the aforementioned properties, it turns out that the S-transform is indeed a unitary isomorphism onto the so-called Bargmann-Segal space,[26] of holomorphic functions on $\mathcal{H}_{\mathbb{C}}$.

Another transformation, important as well in applications is the so-called T-transform.

Definition 6. The T-transform of $F \in (L^2)$ is the mapping defined on $\mathcal{N}_{\mathbb{C}}$ by

$$(TF)(\xi) = \int_{\mathcal{N}'} \exp\left(i\langle x, \xi\rangle\right) F(x)\, d\mu(x), \quad \xi \in \mathcal{N}_{\mathbb{C}}.$$

In other words,

$$(TF)(\xi) = (SF)(i\xi) \exp\left(-\frac{1}{2}\langle \xi, \xi\rangle\right), \quad F \in (L^2), \xi \in \mathcal{N}_{\mathbb{C}}.$$

Therefore, the T-transform has properties similar to the S-transform and all above expressions derived for the S-transform lead easily to corresponding expressions in terms of the T-transform.

5. Test and Generalized Functions

In order to define test and generalized functions of white noise, we shall again consider the space of smooth Wick-ordered polynomials (Section 3),

$$\mathcal{P}(\mathcal{N}') = \left\{ \Phi : \Phi(x) = \sum_{n=0}^{N} \langle : x^{\otimes n} :, \varphi^{(n)}\rangle, \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}, x \in \mathcal{N}', N \in \mathbb{N}_0 \right\}.$$

This space can be endowed with several different topologies, but there is a natural one such that $\mathcal{P}(\mathcal{N}')$ becomes a nuclear space.[2] With respect to this topology, a sequence $(\Phi_m)_{m\in\mathbb{N}}$ of Wick-ordered polynomials $\Phi_m = \sum_{n=0}^{N(\Phi_m)} \langle : \cdot^{\otimes n} :, \varphi_m^{(n)}\rangle$ converges to $\Phi = \sum_{n=0}^{N(\Phi)} \langle : \cdot^{\otimes n} :, \varphi^{(n)}\rangle \in \mathcal{P}(\mathcal{N}')$ if and only if the sequence $(N(\Phi_m))_{m\in\mathbb{N}}$ is bounded and the sequence $(\varphi_m^{(n)})_{m\in\mathbb{N}}$ converges to $\varphi^{(n)}$ in $\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}$ for all $n \in \mathbb{N}_0$. Here we have set $\varphi_m^{(n)} = 0$ for all $n > N(\Phi_m)$, $m \in \mathbb{N}$, and $\varphi^{(n)} = 0$ for all $n > N(\Phi)$. It turns out that the space $\mathcal{P}(\mathcal{N}')$ endowed with this topology is densely embedded in (L^2).[2,48]

Then we can consider the dual space $\mathcal{P}'(\mathcal{N}')$ of $\mathcal{P}(\mathcal{N}')$ with respect to (L^2) and in this way we have defined the triple

$$\mathcal{P}(\mathcal{N}') \subset (L^2) \subset \mathcal{P}'(\mathcal{N}').$$

The dual pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ between $\mathcal{P}'(\mathcal{N}')$ and $\mathcal{P}(\mathcal{N}')$ is defined as the bilinear extension of the (sesquilinear) inner product in (L^2), that is,

$$\begin{aligned}\langle\!\langle F, \Phi \rangle\!\rangle &= (\!(F, \bar{\Phi})\!) \\ &= \int_{\mathcal{N}'} F(x)\Phi(x)\, d\mu(x), \quad F \in (L^2), \Phi \in \mathcal{P}(\mathcal{N}').\end{aligned}$$

Remark 4. Since the function identically equal to 1 is a particular element of $\mathcal{P}(\mathcal{N}')$, we can use this equality to extend the concept of expectation to generalized functions:

$$\mathbb{E}(\Psi) := \langle\!\langle \Psi, 1 \rangle\!\rangle, \quad \Psi \in \mathcal{P}'(\mathcal{N}').$$

In order to define a space of test functions, observe that each kernel function $\varphi^{(n)}$, $n \in \mathbb{N}$ appearing in the chaos decomposition of a smooth Wick-ordered polynomial

$$\sum_{n=0}^{N} \langle : x^{\otimes n} :, \varphi^{(n)} \rangle$$

is in the space

$$\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n} = \operatorname*{prlim}_{p\in\mathbb{N}} \mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} n},$$

where $\mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} n}$, $p \in \mathbb{N}$, is the n-th symmetric tensor power of the complexified space $\mathcal{H}_{p,\mathbb{C}}$ of the Hilbert space $\mathcal{H}_p$ introduced in Section 1 (see Appendix A.1.3). Thus, $\varphi^{(n)} \in \mathcal{H}_{p,\mathbb{C}}^{\widehat{\otimes} n}$ for all $p \in \mathbb{N}$, which allows to define the family of Hilbertian norms $\|\cdot\|_{p,q,\beta}$, $p, q \in \mathbb{N}$, $\beta \in [0,1]$, on $\mathcal{P}(\mathcal{N}')$ by

$$\|\Phi\|_{p,q,\beta}^2 := \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nq} |\varphi^{(n)}|_p^2.$$

For each $p, q \in \mathbb{N}$ and each $\beta \in [0,1]$, let $(\mathcal{H}_p)_q^\beta$ be the Hilbert space obtained by completion of the space $\mathcal{P}(\mathcal{N}')$ with respect to the norm $\|\cdot\|_{p,q,\beta}$. That is,

$$(\mathcal{H}_p)_q^\beta = \left\{ \Phi = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} :, \varphi^{(n)} \rangle \in (L^2) : \|\Phi\|_{p,q,\beta} < \infty \right\}.$$

Then we can define a family of nuclear spaces continuously and densely embedded in (L^2) (cf. e.g.[17,30]) by

$$(\mathcal{N})^\beta := \operatorname{pr}\lim_{p,q\in\mathbb{N}} (\mathcal{H}_p)_q^\beta.$$

Therefore, by the general duality theory (Appendix A.1.3), the dual space $(\mathcal{N})^{-\beta}$ of $(\mathcal{N})^\beta$ with respect to (L^2) is given by

$$(\mathcal{N})^{-\beta} = \operatorname{ind}\lim_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})_{-q}^{-\beta},$$

where $(\mathcal{H}_{-p})_{-q}^{-\beta}$, $p, q \in \mathbb{N}$, $\beta \in [0,1]$, is the dual space of $(\mathcal{H}_p)_q^\beta$ with respect to (L^2). Moreover, since for all $\beta \in [0,1]$, $\mathcal{P}(\mathcal{N}') \subset (\mathcal{N})^\beta$, the spaces $(\mathcal{N})^{-\beta}$ may be regarded as subspaces of $\mathcal{P}'(\mathcal{N}')$ and hence we obtain the following extended chain of spaces:

$$\mathcal{P}(\mathcal{N}') \subset (\mathcal{N})^1 \subset (\mathcal{N})^\beta \subset (L^2) \subset (\mathcal{N})^{-\beta} \subset (\mathcal{N})^{-1} \subset \mathcal{P}'(\mathcal{N}'),\ \beta \in [0,1).$$

The space $(\mathcal{N})^{-1}$ is the so-called *Kondratiev space*.[19–21,30] For $\mathcal{N}$ being the Schwartz space $S(\mathbb{R}^d)$, $S_d(\mathbb{R})$, $d > 1$, or $S(\mathbb{R})$ with the Hilbertian norms $|\cdot|_p$ described in Example 1, the corresponding spaces $(\mathcal{N})^0$ and $(\mathcal{N})^{-0}$ are the so-called spaces of *Hida test functions* and *Hida distributions*, respectively.[2,11,14,18,22,23,25,31,32,41] Independently of the particular choice of the Schwartz space, we shall denote the spaces $(\mathcal{N})^0$ and $(\mathcal{N})^{-0}$ by $(\mathcal{S})$ and $(\mathcal{S})'$, respectively, and the Kondratiev space by $(\mathcal{S})^{-1}$.

The chaos decomposition provides a natural decomposition of the elements in $\mathcal{P}'(\mathcal{N}')$. In fact, it turns out[28,30] that for each $\psi^{(n)} \in \mathcal{N}'^{\widehat{\otimes} n}_{\mathbb{C}}$ there is a unique element in $\mathcal{P}'(\mathcal{N}')$, denoted informally by $\langle : x^{\otimes n} :, \psi^{(n)} \rangle$, acting on Wick polynomials $\Phi = \sum_{n=0}^N \langle : \cdot^{\otimes n} :, \varphi^{(n)} \rangle$ by

$$\langle\!\langle \langle : x^{\otimes n} :, \psi^{(n)} \rangle, \Phi \rangle\!\rangle = n! \langle \psi^{(n)}, \varphi^{(n)} \rangle.$$

Therefore, any element $\Psi \in \mathcal{P}'(\mathcal{N}')$ has a unique decomposition of the form

$$\Psi = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, \psi^{(n)} \rangle,$$

where the sum converges weakly in $\mathcal{P}'(\mathcal{N}')$, and we have

$$\langle\!\langle \Psi, \Phi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \langle \psi^{(n)}, \varphi^{(n)} \rangle \tag{15}$$

for all $\Phi = \sum_{n=0}^N \langle : \cdot^{\otimes n} :, \varphi^{(n)} \rangle \in \mathcal{P}(\mathcal{N}')$. For more details and the proofs see,[28,30].

This internal description of the space $\mathcal{P}'(\mathcal{N}')$ allows, in particular, to describe the distributions in each space $(\mathcal{N})^{-\beta} = \bigcup_{p,q \in \mathbb{N}} (\mathcal{H}_{-p})^{-\beta}_{-q}$, $\beta \in [0,1]$. In fact, it turns out from this construction that each Hilbert space $(\mathcal{H}_{-p})^{-\beta}_{-q}$, $p, q \in \mathbb{N}$, $\beta \in [0,1]$ consists in all $\Psi = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, \psi^{(n)} \rangle \in \mathcal{P}'(\mathcal{N}')$ such that

$$\|\Psi\|^2_{-p,-q,-\beta} := \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-nq} \left| \psi^{(n)} \right|^2_{-p} < \infty.$$

In[11] Hilbert spaces of smooth and generalized white noise functionals were introduced. For more details on this see Section 3.A of[14] .

6. Characterization Results

Both transformations introduced in Section 4, S- and T-transform, can be extended to $(\mathcal{N})^{-\beta}$, $\beta \in [0,1]$. This yields, in particular, characterization results for the Hida and Kondratiev distributions (Subsections 6.1 and 6.2 below) as well as for any distribution in $(\mathcal{N})^{-\beta}$, $\beta \in (0,1)$ cf. e.g.[34] For $\mathcal{N} = S(\mathbb{R})$ and $\beta \in (0,1)$ such results have been obtained in[26,27].

In order to extend Definitions 5 and 6 to distributions, we first observe that for a $\xi \in \mathcal{N}_{\mathbb{C}}$ we have

$$\| :e^{\langle \cdot,\xi\rangle}: \|^2_{p,q,\beta} = \sum_{n=0}^{\infty} (n!)^{1+\beta} 2^{nq} \left| \frac{\xi^{\otimes n}}{n!} \right|_p^2 = \sum_{n=0}^{\infty} (n!)^{\beta-1} 2^{nq} |\xi|_p^{2n}, \tag{16}$$

which is finite if and only if $\beta < 1$ or $2^q|\xi|_p^2 < 1$ if $\beta = 1$. That is, for all $\xi \in \mathcal{N}_{\mathbb{C}}$ we have

$$:e^{\langle \cdot,\xi\rangle}: \in (\mathcal{N})^{\beta}, \quad \forall\, \beta \in [0,1).$$

Thus, Definitions 5 and 6 can be directly extended to any $\Psi \in (\mathcal{N})^{-\beta}$, $\beta \in [0,1)$, by

$$(S\Psi)(\xi) := \langle\!\langle \Psi, :e^{\langle \cdot,\xi\rangle}: \rangle\!\rangle, \quad \xi \in \mathcal{N}_{\mathbb{C}}$$

and

$$(T\Psi)(\xi) := (S\Psi)(i\xi) \exp\left(-\frac{1}{2}\langle \xi,\xi\rangle\right), \quad \xi \in \mathcal{N}_{\mathbb{C}}, \tag{17}$$

respectively. Moreover, if $\Psi = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, \psi^{(n)}\rangle$, then

$$(S\Psi)(\xi) = \sum_{n=0}^{\infty} \langle \psi^{(n)}, \xi^{\otimes n}\rangle, \tag{18}$$

which extends equality (13) to distributions.

Although $:e^{\langle \cdot,\xi\rangle}: \notin (\mathcal{N})^1$ for $\xi \in \mathcal{N}_{\mathbb{C}} \setminus \{0\}$, computation (16) shows that $:e^{\langle \cdot,\xi\rangle}: \in (\mathcal{H}_p)^1_q$ whenever $2^q|\xi|_p^2 < 1$. This allows to define the S-transform of Kondratiev distributions as well. Let $\Psi \in (\mathcal{N})^{-1} = \bigcup_{p,q\in\mathbb{N}} (\mathcal{H}_{-p})^{-1}_{-q}$. Then, $\Psi \in (\mathcal{H}_{-p})^{-1}_{-q}$ for some $p, q \in \mathbb{N}$. So we define the S-transform of Ψ by

$$(S\Psi)(\xi) := \langle\!\langle \Psi, :e^{\langle \cdot,\xi\rangle}: \rangle\!\rangle,$$

for all $\xi \in \mathcal{N}_{\mathbb{C}}$ such that $2^q|\xi|_p^2 < 1$. Of course, for each such a function ξ the alternative description (18) still holds. In an analogous way, we define the T-transform of $\Psi \in (\mathcal{H}_{-p})^{-1}_{-q}$ by

$$(T\Psi)(\xi) := (S\Psi)(i\xi) \exp\left(-\frac{1}{2}\langle \xi,\xi\rangle\right),$$

for all $\xi \in \mathcal{N}_{\mathbb{C}}$ such that $2^q|\xi|_p^2 < 1$.

As we have mentioned above, the Hida and Kondratiev distributions can be characterized through their S- and T-transform. Since the definition of the T-transform of those distributions is based on the S-transform, we present these characterization results, as well as their corollaries, just in terms of the S-transform.

Remark 5. For the T-transform, analogous results hold by simply replacing the S by the T-transform.

6.1. *Hida Distributions*

We recall that in this case $\mathcal{N}$ can be any Schwartz space $S(\mathbb{R}^d)$, $S_d(\mathbb{R})$, $d \geq 1$. In order to cover all these possibilities, in this subsection we shall denote all possible Schwartz test function spaces simply by $\mathcal{S}$ and the corresponding dual space by $\mathcal{S}'$.

As a first step towards the characterization of Hida distributions through its S-transform, we need the following definition (Appendix A.1.4).

Definition 7. A function $F : \mathcal{S} \to \mathbb{C}$ is called a U-functional whenever
1. for every $\xi_1, \xi_2 \in \mathcal{S}$ the mapping $\mathbb{R} \ni \lambda \mapsto F(\lambda\xi_1 + \xi_2)$ has an entire extension to $\lambda \in \mathbb{C}$,
2. there are two constants $K_1, K_2 > 0$ such that

$$|F(z\xi)| \leq K_1 \exp\left(K_2 |z|^2 \|\xi\|^2\right), \quad \forall\, z \in \mathbb{C}, \xi \in \mathcal{S}$$

for some continuous norm $\|\cdot\|$ on $\mathcal{S}$.

We are now ready to state the aforementioned characterization result.

Theorem 2. *([18,41]) The S-transform defines a bijection between the space $(\mathcal{S})'$ and the space of U-functionals.*

As a consequence of Theorem 2 one may derive the next two statements. The first one concerns the convergence of sequences of Hida distributions and the second one the Bochner integration of families of Hida distributions. For more details and the proofs see[18,41] .

Corollary 1. *Let $(\Psi_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathcal{S})'$ such that*
1. for all $\xi \in \mathcal{S}$, $((S\Psi_n)(\xi))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$,
2. there are two constants $K_1, K_2 > 0$ such that for some continuous norm $\|\cdot\|$ on $\mathcal{S}$ we have

$$|(S\Psi_n)(z\xi)| \leq K_1 \exp\left(K_2|z|^2\|\xi\|^2\right), \quad \forall\, z \in \mathbb{C}, \xi \in \mathcal{S}, n \in \mathbb{N}.$$

Then $(\Psi_n)_{n\in\mathbb{N}}$ converges strongly in $(\mathcal{S})'$ to a unique Hida distribution.

Corollary 2. *Let $(\Lambda, \mathcal{B}, \nu)$ be a measure space and $\lambda \mapsto \Psi_\lambda$ be a mapping from Λ to $(\mathcal{S})'$. We assume that the S-transform of Ψ_λ fulfills the following two conditions:*
1. the mapping $\lambda \mapsto (S\Psi_\lambda)(\xi)$ is measurable for every $\xi \in \mathcal{S}$,
2. all $S\Psi_\lambda$ obey the bound

$$|(S\Psi_\lambda)(z\xi)| \leq C_1(\lambda) \exp\left(C_2(\lambda)|z|^2\|\xi\|^2\right), \quad z \in \mathbb{C}, \xi \in \mathcal{S},$$

for some continuous norm $\|\cdot\|$ *on* $\mathcal{S}$ *and for some* $C_1 \in L^1(\Lambda, \mathcal{B}, \nu)$, $C_2 \in L^\infty(\Lambda, \mathcal{B}, \nu)$.

Then the Bochner integral

$$\int_\Lambda \Psi_\lambda \, d\nu(\lambda)$$

exists in $(\mathcal{S})'$ *and*

$$S\left(\int_\Lambda \Psi_\lambda \, d\nu(\lambda)\right) = \int_\Lambda (S\Psi_\lambda) \, d\nu(\lambda).$$

Example 3. Given the one-dimensional Brownian motion $B_t = \langle \cdot, 1\!\!1_{[0,t)} \rangle$, $t \geq 0$ defined in Example 2 and the Dirac delta function $\delta_a \in S'(\mathbb{R})$ with mass at $a \in \mathbb{R}$, consider the informal composition

$$\delta_a(B_t) = \delta_0(B_t - a).$$

Based on an approximation procedure by Hida distributions (Corollary 1) and Corollary 2, a rigorous meaning of the Donsker's delta function $\delta_0(B_t - a)$ as the Bochner integral in $(\mathcal{S})'$

$$\delta_0(B_t - a) := \frac{1}{2\pi} \int_\mathbb{R} e^{iu(B_t - a)} \, du$$

has been given in.[35] Its S-transform is given (see e.g.[13,14,33,35]) by

$$(S\delta_0(B_t - a))(\xi) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}\left(a - \int_0^t \xi(u)\, du\right)^2\right), \quad \xi \in S(\mathbb{R}),$$

which is obviously a U-functional.

Among Hida distributions the positive ones have particular characteristics. We recall that a $\Psi \in (\mathcal{S})'$ is said to be *positive* whenever $\langle\!\langle \Psi, \Phi \rangle\!\rangle \geq 0$ for all $\Phi \in (\mathcal{S})$ μ-a.e. positive (being μ the Gaussian measure on $(\mathcal{S}', \mathcal{C}_\sigma(\mathcal{S}'))$). As shown independently in[24] and in[51] , we have the following result.

Theorem 3. *If* $\Psi \in (\mathcal{S})'$ *belongs to the cone* $(\mathcal{S})'_+$ *of positive Hida distributions, then there is a unique (positive) finite measure* ν_Ψ *on* $(\mathcal{S}', \mathcal{C}_\sigma(\mathcal{S}'))$ *such that*

$$\langle\!\langle \Psi, \Phi \rangle\!\rangle = \int_{\mathcal{S}'} \Phi(\omega) \, d\nu_\Psi(\omega)$$

for all $\Phi \in (\mathcal{S})$.

Example 4. Coming back to Example 3, in[49] the authors have proved that $\delta_0(B_t - a) \in (\mathcal{S})'_+$. On the other hand, Y. Yokoi has shown in[51] that $\Psi \in (\mathcal{S})'$ is positive if and only if $T\Psi$ is positive definite. By Example 3 and (17), it is clear that the latter condition holds for $\Psi = \delta_0(B_t - a)$. Thus, according to Theorem 3, the Donsker's delta function $\delta_0(B_t - a)$ defines a finite measure on $(S'(\mathbb{R}), \mathcal{C}_\sigma(S'(\mathbb{R})))$.

6.2. *Kondratiev Distributions*

Theorem 4. *([19]) Let $0 \in U \subset \mathcal{N}_{\mathbb{C}}$ be an open set and $F : U \to \mathbb{C}$ be a holomorphic function on U. Then there is a unique $\Psi \in (\mathcal{N})^{-1}$ such that $S\Psi = F$. Conversely, given a $\Psi \in (\mathcal{N})^{-1}$ the function $S\Psi$ is holomorphic on some open set in $\mathcal{N}_{\mathbb{C}}$ containing 0.*

The correspondence between F and Ψ is a bijection if one identifies holomorphic functions which coincide on some open neighborhood of 0 in $\mathcal{N}_{\mathbb{C}}$.

We shall do so. As a consequence, we can derive the next two statements. The first one concerns the convergence of sequences of Kondratiev distributions and the second one the Bochner integration of families of the same type of generalized functions.

Corollary 3 ([19]). *Let $(\Psi_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathcal{N})^{-1}$ such that there are $p, q \in \mathbb{N}$ so that*
1. all $S\Psi_n$ are holomorphic on the open neighborhood $U_{p,q} := \{\xi \in \mathcal{N}_{\mathbb{C}} : 2^q|\xi|_p^2 < 1\}$ of $0 \in \mathcal{N}_{\mathbb{C}}$,
2. there is a $C > 0$ such that $|S\Psi_n(\xi)| \leq C$ for all $\xi \in U_{p,q}$ and all $n \in \mathbb{N}$,
3. $(S\Psi_n(\xi))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for all $\xi \in U_{p,q}$.

Then $(\Psi_n)_{n\in\mathbb{N}}$ converges strongly in $(\mathcal{N})^{-1}$.

Corollary 4 ([19]). *Let $(\Lambda, \mathcal{B}, \nu)$ be a measure space and $\lambda \mapsto \Psi_\lambda$ be a mapping from Λ to $(\mathcal{N})^{-1}$. We assume that there is a $U_{p,q} = \{\xi \in \mathcal{N}_{\mathbb{C}} : 2^q|\xi|_p^2 < 1\}$, $p, q \in \mathbb{N}$, such that*
1. $S\Psi_\lambda$ is holomorphic on $U_{p,q}$ for every $\lambda \in \Lambda$,
2. the mapping $\lambda \mapsto (S\Psi_\lambda)(\xi)$ is measurable for every $\xi \in U_{p,q}$,
3. there is a $C \in L^1(\Lambda, \mathcal{B}, \nu)$ such that

$$|(S\Psi_\lambda)(\xi)| \leq C(\lambda), \quad \forall\, \xi \in U_{p,q},\, \nu - a.a.\, \lambda \in \Lambda.$$

Then there are $p', q' \in \mathbb{N}$, which only depend on p, q, such that

$$\int_\Lambda \Psi_\lambda \, d\nu(\lambda)$$

exists as a Bochner integral in $(\mathcal{H}_{-p'})^{-1}_{-q'}$. In particular, $S\left(\int_\Lambda \Phi_\lambda \, d\nu(\lambda)\right)$ is holomorphic on $U_{p',q'} = \{\xi \in \mathcal{N}_{\mathbb{C}} : 2^{q'}|\xi|_{p'}^2 < 1\}$ and

$$\left\langle\!\!\left\langle \int_\Lambda \Psi_\lambda \, d\nu(\lambda), \Phi \right\rangle\!\!\right\rangle = \int_\Lambda \langle\!\langle \Psi_\lambda, \Phi \rangle\!\rangle \, d\nu(\lambda), \quad \forall\, \Phi \in (\mathcal{N})^1.$$

Positive Kondratiev distributions are defined similarly to the Hida case. For such a particular class of generalized functions in $(\mathcal{N})^{-1}$ the following characterization result holds.

Theorem 5.[29] *If $\Psi \in (\mathcal{N})^{-1}$ belongs to the cone $(\mathcal{N})^{-1}_{+}$ of positive Kondratiev distributions, then there is a unique (positive) finite measure $\nu = \nu_\Psi$ on $(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'))$ such that for all $\Phi \in (\mathcal{N})^1$*

$$\langle\!\langle \Psi, \Phi \rangle\!\rangle = \int_{\mathcal{N}'} \Phi(x)\, d\nu(x) \tag{19}$$

and, moreover, there are $p \in \mathbb{N}$, $K, C > 0$ so that

$$\left| \int_{\mathcal{N}'} \langle x, \xi \rangle^n \, d\nu(x) \right| = K C^n n! |\xi|_p^n \tag{20}$$

for all $\xi \in \mathcal{N}$ and all $n \in \mathbb{N}_0$. Vice versa, any (positive) measure ν that obeys (20) *defines a positive distribution $\Psi \in (\mathcal{N})^{-1}_{+}$ by* (19).

7. Wick Product and $*$-Convolution

According to the Characterization Theorem 4, the S-transform on $(\mathcal{N})^{-1}$ is a bijection if we consider *germs* of holomorphic functions at zero, that is, if we identify holomorphic functions which coincide on some open neighborhood of 0 in $\mathcal{N}_{\mathbb{C}}$. Thus, we define $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ as the algebra of germs of holomorphic functions at zero. Algebraically, it is clear that $\mathrm{Hol}_0(\mathcal{N}_{\mathbb{C}})$ endowed with the pointwise multiplication of functions is an algebra. Therefore, by Theorem 4, for each pair $\Psi_1, \Psi_2 \in (\mathcal{N})^{-1}$ we can define the so-called *Wick product* $\Psi_1 \diamond \Psi_2 \in (\mathcal{N})^{-1}$ *of* Ψ_1 *and* Ψ_2 by

$$\Psi_1 \diamond \Psi_2 := S^{-1}\left((S\Psi_1)(S\Psi_2)\right).$$

It turns out that the space $(\mathcal{N})^{-1}$ endowed with the Wick product is a commutative algebra with unit element $: e^{\langle x, 0 \rangle} :\equiv 1$[14,15,19].

The Wick product can be described in terms of chaos decomposition as well. If $\Psi_i = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, \psi_i^{(n)} \rangle \in (\mathcal{N})^{-1}$, $i = 1, 2$, then

$$\Psi_1 \diamond \Psi_2 = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, \sum_{k=0}^{n} \psi_1^{(k)} \widehat{\otimes} \psi_2^{(n-k)} \right\rangle.$$

Clearly, we can also define

$$\Psi^{\diamond n} := S^{-1}((S\Psi)^n) = \Psi \diamond \ldots \diamond \Psi \ (n \text{ times}), \qquad \Psi \in (\mathcal{N})^{-1}$$

and thus finite linear combinations of the form $\sum_{n=0}^{N} a_n \Psi^{\diamond n}$ $(\Psi^{\diamond 0} := 1)$. Moreover, given a function $g : \mathbb{C} \to \mathbb{C}$ analytic on some neighborhood of $(S\Psi)(0) = \mathbb{E}(\Psi) \in \mathbb{C}$ (Remark 4), we can define $g^\diamond(\Psi) \in (\mathcal{N})^{-1}$ by $g^\diamond(\Psi) := S^{-1}(g(S\Psi))$[19,30]. In particular, we can define $\exp^\diamond \Psi \in (\mathcal{N})^{-1}$:

$$\exp^\diamond \Psi := S^{-1}(\exp(S\Psi)) = \sum_{n=0}^{\infty} \frac{1}{n!}\Psi^{\diamond n}.$$

In fact, if the power series representation of an analytic function g has the form

$$g(z) = \sum_{n=0}^{\infty} a_n(z - \mathbb{E}(\Psi))^n, \quad z \in \mathbb{C},$$

then the Wick series

$$\sum_{n=0}^{\infty} a_n(\Psi - \mathbb{E}(\Psi))^{\diamond n}$$

converges in $(\mathcal{N})^{-1}$ and, moreover,

$$g^\diamond(\Psi) = \sum_{n=0}^{\infty} a_n(\Psi - \mathbb{E}(\Psi))^{\diamond n}.$$

For more details concerning the Wick product see e.g.[14,15,19,30].

By Theorem 4 and Remark 5, the T-transform also yields the definition of an algebraic structure on $(\mathcal{N})^{-1}$. For each pair $\Psi_1, \Psi_2 \in (\mathcal{N})^{-1}$ we define the so-called *$*$-convolution* $\Psi_1 * \Psi_2 \in (\mathcal{N})^{-1}$ *of* Ψ_1 *and* Ψ_2 by

$$\Psi_1 * \Psi_2 := T^{-1}\left((T\Psi_1)(T\Psi_2)\right).$$

Due to the close relation between the T- and the S-transform on $(\mathcal{N})^{-1}$, all the above results quoted for the Wick product straightforwardly hold for the $*$-convolution. Moreover, through the so-called *Fourier transform on* $(\mathcal{N})^{-1}$,

$$\mathcal{F}\Psi := T^{-1}(S\Psi), \quad \Psi \in (\mathcal{N})^{-1},$$

both algebraic convolutions are related[19] by

$$\mathcal{F}(\Psi_1 \diamond \Psi_2) = (\mathcal{F}\Psi_1) * (\mathcal{F}\Psi_2), \quad \Psi_1, \Psi_2 \in (\mathcal{N})^{-1}.$$

For more details concerning this Fourier transform $\mathcal{F}$ see e.g.[14,33] and the references therein.

8. Annihilation, Creation and Second Quantization Operators

The Itô-Segal-Wiener isomorphism between the space (L^2) and the symmetric Fock space provides natural operators on (L^2) by carrying over standard Fock spaces operators, namely, the annihilation, creation and second quantizations operators cf. e.g.[2,44].

8.1. *Annihilation Operators*

Let $h \in \mathcal{H}$ be given. We recall that the so-called *annihilation operator* $a(h)$ is the operator acting on $f^{(n)} \in \mathrm{Exp}_n(\mathcal{H}_{\mathbb{C}})$, $n \in \mathbb{N}$, of the form (Appendix A.1.2)

$$f^{(n)} = f_1 \widehat{\otimes} \ldots \widehat{\otimes} f_n \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}, \quad f_i \in \mathcal{H}_{\mathbb{C}}, i = 1, ..., n \tag{21}$$

by

$$(a(h))f^{(n)} := \sum_{j=1}^{n} (h, f_j) f_1 \widehat{\otimes} \ldots \widehat{\otimes} f_{j-1} \widehat{\otimes} f_{j+1} \widehat{\otimes} \ldots \widehat{\otimes} f_n \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes}(n-1)}.$$

This definition can be linearly extended to the dense space in $\mathrm{Exp}_n(\mathcal{H}_{\mathbb{C}})$ spanned by elements of the form (21). Moreover, for such elements the following equality of norms holds (cf. e.g.[44]),

$$|(a(h))f^{(n)}|_{\mathrm{Exp}_{n-1}(\mathcal{H}_{\mathbb{C}})} \leq \sqrt{n}|h||f^{(n)}|_{\mathrm{Exp}_n(\mathcal{H}_{\mathbb{C}})}, \tag{22}$$

which allows to extend $a(h)$ to a bounded operator $a(h) : \mathrm{Exp}_n(\mathcal{H}_{\mathbb{C}}) \to \mathrm{Exp}_{n-1}(\mathcal{H}_{\mathbb{C}})$.

Therefore, in terms of the space (L^2), the Itô-Segal-Wiener isomorphism yields an operator, also denoted by $a(h)$, such that for all $x \in \mathcal{N}'$ and all $f_1, \ldots, f_n \in \mathcal{H}_{\mathbb{C}}$, $n \in \mathbb{N}$,

$$\begin{aligned}&(a(h))\langle : x^{\otimes n} :, f_1 \widehat{\otimes} \ldots \widehat{\otimes} f_n \rangle \\ &= \sum_{j=1}^{n} (h, f_j) \langle : x^{\otimes(n-1)} :, f_1 \widehat{\otimes} \ldots \widehat{\otimes} f_{j-1} \widehat{\otimes} f_{j+1} \widehat{\otimes} \ldots \widehat{\otimes} f_n \rangle.\end{aligned}$$

Due to (22), observe that for $x \in \mathcal{N}'$, $f_1, \ldots, f_n \in \mathcal{H}$, $n \in \mathbb{N}$, fixed, the linear functional on $\mathcal{H}$

$$\mathcal{H} \ni h \mapsto (a(h))P(x), \quad P(x) := \langle : x^{\otimes n} :, f_1 \widehat{\otimes} \ldots \widehat{\otimes} f_n \rangle$$

is bounded. Therefore, by the Riesz representation theorem, it is given by an inner product

$$(h, \nabla P(x)), \quad \forall\, h \in \mathcal{H}$$

for some $\nabla P(x) \in \mathcal{H}$. In particular, for $\mathcal{N} = S(\mathbb{R})$ and $\mathcal{H} = L^2(\mathbb{R})$, we have

$$(a(h))P(\omega) = (h, \nabla P(\omega)) = \int_{-\infty}^{+\infty} h(t) \partial_t P(\omega)\, dt, \quad h \in L^2(\mathbb{R}),$$

where ∂_t, $t \in \mathbb{R}$ is the Hida derivative introduced in[11] .

The reason for this name lies on the fact that the Hida derivative is indeed a (Gâteaux) derivative. For $\omega, \omega_0 \in S'(\mathbb{R})$ fixed, let F be a real or complex-valued function pointwisely defined on $S'(\mathbb{R})$. We say that F is *Gâteaux differentiable at ω in direction ω_0* if the function $\mathbb{R} \ni \lambda \mapsto F(\omega + \lambda\omega_0)$ is differentiable at $\lambda = 0$. In this case, we shall use the notation

$$D_{\omega_0} F(\omega) := \frac{d}{d\lambda} F(\omega + \lambda\omega_0)\Big|_{\lambda=0}. \tag{23}$$

In particular, for P defined as before we easily find

$$D_{\omega_0} P(\omega) = \sum_{j=1}^{n} \langle \omega_0, f_j \rangle \langle : \omega^{\otimes(n-1)} :, f_1 \widehat{\otimes} \ldots \widehat{\otimes} f_{j-1} \widehat{\otimes} f_{j+1} \widehat{\otimes} \ldots \widehat{\otimes} f_n \rangle.$$

This shows that $a(h)P(\omega)$, $h \in L^2(\mathbb{R})$ is, in particular, a Gâteaux derivative at the point $\omega \in S'(\mathbb{R})$ in direction $h \in L^2(\mathbb{R})$. Therefore, we can regard definition (23) as an extension of $a(h)$, $h \in L^2(\mathbb{R})$, to tempered distributions directions.

In particular, for the Dirac delta function $\delta_t \in S'(\mathbb{R})$, $t \in \mathbb{R}$, it turns out cf. e.g.[14] that

$$D_{\delta_t} P = \partial_t P,$$

where $\partial_t P$ is the Hida derivative of P. In fact,

$$\partial_{\cdot} \Phi(\omega) = \nabla \Phi(\omega)$$

defines a Fréchet derivative on $(\mathcal{S})$, see e.g.,[14] and for suitable positive $\Psi \in (\mathcal{S})'$

$$\varepsilon(\Phi) := \langle\!\langle \Psi, |\nabla \Phi|^2 \rangle\!\rangle$$

will give rise to (pre-)Dirichlet forms, see[14].

8.2. *Creation Operators*

In order to recall the definition of creation operators on the Fock space, we come back to the bounded operator $a(h) : \mathrm{Exp}_n(\mathcal{H}_\mathbb{C}) \to \mathrm{Exp}_{n-1}(\mathcal{H}_\mathbb{C})$, $h \in \mathcal{H}$, defined at the beginning of Subsection 8.1. Clearly, we can extend componentwisely $a(h)$ to the dense subspace of $\mathrm{Exp}(\mathcal{H}_\mathbb{C})$ consisting of all sequences $(f^{(n)})_{n\in\mathbb{N}_0}$ such that all terms vanish except finitely many ones. Therefore, the adjoint $a^*(h)$ of $a(h)$ is a well-defined operator on $\mathrm{Exp}(\mathcal{H}_\mathbb{C})$. A straightforward computation shows that its action on $f^{(n)} \in \mathrm{Exp}_n(\mathcal{H}_\mathbb{C})$, $n \in \mathbb{N}$, is given by

$$(a^*(h))f^{(n)} = h \widehat{\otimes} f^{(n)} \in \mathrm{Exp}_{n+1}(\mathcal{H}_\mathbb{C})$$

and, moreover,

$$|(a^*(h))f^{(n)}|_{\mathrm{Exp}_{n+1}(\mathcal{H}_\mathbb{C})} \leq \sqrt{n+1}|h||f^{(n)}|_{\mathrm{Exp}_n(\mathcal{H}_\mathbb{C})},$$

see e.g.[2,44]. Since $a(h)$ and $a^*(h)$ are densely defined, they are closable, and thus both operators can be extended to their closures. We shall denote both extended operators also by $a(h)$ and $a^*(h)$, respectively. The following equalities hold

$$[a(f), a(h)] = [a^*(f), a^*(h)] = 0, \quad [a(f), a^*(h)] \subseteq (f, h), \tag{24}$$

which are well-known as the canonical commutation relations. Here $[\cdot, \cdot]$ is the usual *commutator* between two operators, $[A, B] := AB - BA$.

Concerning $a^*(h)$, that is, the so-called *creation operator*, its image under the Itô-Segal-Wiener isomorphism leads as before to an operator on (L^2), also denoted by $a^*(h)$,

$$(a^*(h))\langle : x^{\otimes n} :, f^{(n)}\rangle = \langle : x^{\otimes(n+1)} :, h\widehat{\otimes} f^{(n)}\rangle, \quad f^{(n)} \in \mathcal{H}_\mathbb{C}^{\widehat{\otimes} n}, n \in \mathbb{N}.$$

Of course, by construction, $a^*(h)$ is the adjoint operator of $a(h)$ on (L^2) and relations corresponding to (24) hold.

In order to proceed towards distributions, let us consider $\mathcal{H} = L^2(\mathbb{R})$ and $\mathcal{N} = S(\mathbb{R})$. Concerning the corresponding space $(\mathcal{S})$ of Hida test functions, it turns out cf. e.g.[14] that each $\sum_{n=0}^\infty \langle : \cdot^{\otimes n} :, \varphi^{(n)}\rangle \in (\mathcal{S})$ has a continuous version. That is, each kernel $\varphi^{(n)}$, $n \in \mathbb{N}$, has a continuous version in $(S_\mathbb{C}(\mathbb{R}))^{\widehat{\otimes} n}$. For technical reasons, in what follows we shall always consider the continuous version of a Hida test function. Of course, it follows from the previous subsection that $D_{\omega_0}\langle : \omega^{\otimes n} :, \varphi^{(n)}\rangle$ exists for all $\omega_0 \in S'(\mathbb{R})$ and

$$D_{\omega_0}\langle : \omega^{\otimes n} :, \varphi^{(n)}\rangle = n\langle \omega_0 \widehat{\otimes} : \omega^{\otimes(n-1)} :, \varphi^{(n)}\rangle = n\left\langle : \omega^{\otimes(n-1)} :, \langle \omega_0, \varphi^{(n)}\rangle\right\rangle, \tag{25}$$

where $\langle \omega_0, \varphi^{(n)}\rangle$ means that ω_0 is evaluated on $\varphi^{(n)}$ in the first argument. Moreover, for a fixed $\omega_0 \in S'(\mathbb{R})$, D_{ω_0} is a continuous linear operator from $(\mathcal{S})$ into itself. This is a consequence of the fact that given a $\omega_0 \in \mathcal{H}_{-q}$, being $\mathcal{H}_{-q}$, $p \in \mathbb{N}$ the Hilbert spaces introduced in Example 1 (i) with the corresponding norm $|\cdot|_{-q}$, for every $\Phi \in (\mathcal{S})$ and every $p, r \in \mathbb{N}$ we find cf.,[14,39]

$$\|D_{\omega_0}\Phi\|_{p,r,0} \leq 2^{q-p}|\omega_0|_{-q}\|\Phi\|_{\max\{p,q\},r,0},$$

where the first and the last norms are the ones on $(\mathcal{H}_p)^0_r$, $p, r \in \mathbb{N}$, $\mathrm{prlim}_{p,r\in\mathbb{N}}(\mathcal{H}_p)^0_r = (\mathcal{S})$ (Section 5). Therefore, we can consider the adjoint operator $D^*_{\omega_0}$ of the Gâteaux derivative D_{ω_0}, $\omega_0 \in S'(\mathbb{R})$, which (strongly) continuously maps the space of Hida distributions $(\mathcal{S})'$ into itself. Due to (15) and (25), the action of $D^*_{\omega_0}$ on a $\Psi = \sum_{n=0}^{\infty}\langle :\omega^{\otimes n}:, \psi^{(n)}\rangle \in (\mathcal{S})'$, thus with symmetric tempered distribution kernels $\psi^{(n)}$, is given for all $\Phi = \sum_{n=0}^{\infty}\langle :\cdot^{\otimes n}:, \varphi^{(n)}\rangle \in (\mathcal{S})$ by

$$\langle\!\langle D^*_{\omega_0}\Psi, \Phi\rangle\!\rangle = \langle\!\langle \Psi, D_{\omega_0}\Phi\rangle\!\rangle = \sum_{n=0}^{\infty} n!(n+1)\left\langle \psi^{(n)}, \langle \omega_0, \varphi^{(n+1)}\rangle\right\rangle$$
$$= \sum_{n=0}^{\infty}(n+1)!\langle \omega_0\widehat{\otimes}\psi^{(n)}, \varphi^{(n+1)}\rangle.$$

That is,

$$D^*_{\omega_0}\Psi = \sum_{n=1}^{\infty}\langle :\omega^{\otimes n}:, \omega_0\widehat{\otimes}\psi^{(n-1)}\rangle. \tag{26}$$

It extends to tempered distributions the operator $a^*(h)$, $h \in L^2(\mathbb{R})$, above defined on (L^2). As we can expect, relations similar to (24) can be stated for D_{ω_0} and $D^*_{\omega_0}$, $\omega_0 \in S'(\mathbb{R})$, as well. From those, we give particular attention to the case $\omega_0 = \delta_t$, which leads to the *canonical commutation relations* used in quantum field theory. Informally, for $s, t \in \mathbb{R}$,

$$[\partial_s, \partial_t] = [\partial^*_s, \partial^*_t] = 0, \quad [\partial_s, \partial^*_t] = \delta(s-t),$$

with $\partial^*_t = D^*_{\delta_t}$, $t \in \mathbb{R}$. Furthermore, given the white noise $\omega(t)$ (Example 2), for all $\Phi \in (\mathcal{S})$ we find

$$\omega(t)\Phi = (\partial_t + \partial^*_t)\Phi,$$

where the multiplication appearing in the left-hand side is a Hida distribution. Indeed, since $(\mathcal{S})$ is closed under the pointwise multiplication and the multiplication of Hida test functions is a continuous bilinear mapping on $(\mathcal{S})$ cf. e.g.,[39] the multiplication $\omega(t)\Phi \in (\mathcal{S})'$ is well-defined by

$$\langle\!\langle \omega(t)\Phi, \Phi_0\rangle\!\rangle := \langle\!\langle \omega(t), \Phi\Phi_0\rangle\!\rangle, \quad \forall\, \Phi_0 \in (\mathcal{S}).$$

For more details and the proofs see e.g.[14,39].

Another application of this particular case concerns the definition of the so-called Hitsuda-Skorohod integral, related to the Skorohod and the Itô integrals, both well-known in stochastic analysis.

From now on, let $T \subset \mathbb{R}$ be a bounded interval with the Borel σ-algebra $\mathcal{B}$ and the Lebesgue measure. Since $\partial^*_t : (\mathcal{S})' \to (\mathcal{S})'$, given a mapping $\Psi : T \to (\mathcal{S})'$ defined a.e., we can then consider the mapping defined for a.a. $t \in T$ by $t \mapsto \partial_t^* \Psi(t)$. If, in addition, for every $\Phi \in (\mathcal{S})$, $\langle\!\langle \Psi(\cdot), \Phi \rangle\!\rangle \in L^1(T, \mathcal{B}, dt)$, then we have a well-defined *Pettis integral of* $\partial_t^* \Psi$,

$$\int_T \partial_t^* \Psi(t)\, dt \in (\mathcal{S})'. \tag{27}$$

That is, (27) is the unique element in $(\mathcal{S})'$ such that for all $\Phi \in (\mathcal{S})$

$$\left\langle\!\!\left\langle \int_T \partial_t^* \Psi(t)\, dt, \Phi \right\rangle\!\!\right\rangle = \int_T \langle\!\langle \partial_t^* \Psi(t), \Phi \rangle\!\rangle\, dt.$$

In particular, for $\Phi =: e^{\langle \cdot, \xi \rangle} :\in (\mathcal{S})$, $\xi \in S(\mathbb{R})$ (Section 6), it follows immediately from this definition that the S-transform of (27) is given by

$$S\left(\int_T \partial_t^* \Psi(t)\, dt\right)(\xi) = \int_T S\left(\partial_t^* \Psi(t)\right)(\xi)\, dt.$$

We call (27) the *Hitsuda-Skorohod integral of* Ψ.

In terms of chaos decomposition, observe that if $\Psi(t) = \sum_{n=0}^\infty \langle : \omega^{\otimes n} :, \psi^{(n)}(t) \rangle$ for a.a. $t \in T$, then, by (26),

$$\partial_t^* \Psi(t) = \sum_{n=1}^\infty \langle : \omega^{\otimes n} :, \delta_t \widehat{\otimes} \psi^{(n-1)}(t) \rangle = \omega(t) \diamond \sum_{n=0}^\infty \langle : \omega^{\otimes n} :, \psi^{(n)}(t) \rangle = \omega(t) \diamond \Psi(t),$$

where $\diamond$ is the Wick product introduced in Section 7. This yields another approach to introduce the Skorohod and the Itô integrals[15].

Now let $T = [0,1]$ and let $X : [0,1] \to (L^2)$ be a square integrable function with chaos decomposition

$$X(t) = \sum_{n=0}^\infty \langle : \omega^{\otimes n} :, f^{(n)}(t) \rangle \quad a.a.\ t \in [0,1]$$

such that for a.a. $t \in [0,1]$

$$\sum_{n=0}^\infty n n! |f^{(n)}(t)|^2 < \infty.$$

Then, it can be shown cf. e.g.[14] that the Hitsuda-Skorohod integral of X belongs to (L^2) and coincides with its Skorohod integral.

In order to recover the definition of the Itô integral, let $X : [0,1] \to (L^2)$ be a square integrable function which we assume to be adapted to the filtration generated by Brownian motion. In terms of chaos decomposition this means that if X is given as before, then for all $n \in \mathbb{N}$ and for a.a. $(u_1, \ldots, u_n) \in \mathbb{R}^n \setminus [0,t]^n$, $f^{(n)}(t; u_1, \ldots, u_n) = 0$, $t \in [0,1]$. Then, it turns out cf. e.g.[14] that the Hitsuda-Skorohod integral of X coincides with its Itô integral:

$$\int_0^1 \partial_t^* X(t)\, dt = \int_0^1 X(t)\, dB(t).$$

In accordance with the definition of Itô integral in stochastic analysis, here we should choose a continuous version of the Brownian motion defined in Example 2.

For more details and the proofs see e.g.[14,34].

8.3. *Second Quantization Operators*

Given a contraction operator B on $\mathcal{H}_{\mathbb{C}}$, we can define a contraction operator $\mathrm{Exp}B$ on the Fock space $\mathrm{Exp}(\mathcal{H}_{\mathbb{C}})$ defined on each space $\mathrm{Exp}_n(\mathcal{H}_{\mathbb{C}})$, $n \in \mathbb{N}$, by $B^{\otimes n}$, $\mathrm{Exp}B \restriction \mathrm{Exp}_0(\mathcal{H}_{\mathbb{C}}) := 1$. Therefore, for any coherent state $e(f)$, $f \in \mathcal{H}_{\mathbb{C}}$,

$$\mathrm{Exp}B\,(e(f)) = e(Bf).$$

In particular, given a positive self-adjoint operator A on $\mathcal{H}_{\mathbb{C}}$ and the contraction semigroup e^{-tA}, $t \geq 0$, we have defined a contraction semigroup $\mathrm{Exp}\left(e^{-tA}\right)$, $t \geq 0$, on $\mathrm{Exp}(\mathcal{H}_{\mathbb{C}})$. The generator of this semigroup is the so-called *second quantization operator corresponding to* A and we shall denote it by $d\mathrm{Exp}A$, i.e.,

$$\mathrm{Exp}\left(e^{-tA}\right) = \exp\left(-td\mathrm{Exp}A\right), \quad t \geq 0.$$

For more details see e.g.[2,43]. The action of $d\mathrm{Exp}A$ on coherent states $e(f)$ with f in the domain of A is given by

$$d\mathrm{Exp}A\,(e(f)) = \left((Af)\,\widehat{\otimes}\frac{f^{\otimes(n-1)}}{(n-1)!}\right)_{n\in\mathbb{N}_0}$$

with $(Af)\,\widehat{\otimes}\frac{f^{\otimes(n-1)}}{(n-1)!} = 0$ if $n = 0$, cf. e.g.[2].

The definition of second quantization operators leads, for instance, to the construction of countably Hilbert spaces and triples on Fock spaces and thus to the definition of new nuclear triples cf. e.g.[2,14,22,39]. In particular, for A being the Hamiltonian H of the quantum harmonic oscillator on $L^2(\mathbb{R})$ (Example 1),

$$(H\xi)(u) = -\frac{d^2\xi}{du^2}(u) + (u^2+1)\xi(u), \quad u \in \mathbb{R},$$

we rediscover the Gelfand triple

$$(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})'.$$

For more details concerning this constructive scheme and examples see e.g.[2,14,22].

A.1. Appendices

A.1.1. *Tensor Powers of Hilbert Spaces*

Instead of reproducing the abstract construction of tensor powers of general topological vector spaces like e.g. in[45,50], we follow closely the direct approach of[43] for Hilbert spaces.

Let $\mathcal{H}$ be a (real or complex) Hilbert space with inner product $(\cdot,\cdot)$. For each $n \in \mathbb{N}$, $n \geq 2$, and every $g_1, \ldots, g_n \in \mathcal{H}$, we consider the following n-linear form $g_1 \otimes \ldots \otimes g_n := \otimes_{i=1}^n g_i$ defined on $\mathcal{H}^n$ by

$$(g_1 \otimes \ldots \otimes g_n)(h_1, \ldots, h_n) := \prod_{i=1}^n (h_i, g_i), \quad h_1, \ldots, h_n \in \mathcal{H}.$$

We shall call the linear space spanned by such n-linear forms the *algebraic n-th tensor power of $\mathcal{H}$* and denote it by $\mathcal{H}^{\otimes_a n}$. In order to introduce a topological structure on $\mathcal{H}^{\otimes_a n}$, we define an inner product on $\mathcal{H}^{\otimes_a n}$, also denoted by $(\cdot,\cdot)$, acting on elements $\otimes_{i=1}^n g_{1i}, \otimes_{j=1}^n g_{2j} \in \mathcal{H}^{\otimes_a n}$ by

$$\left(\otimes_{i=1}^n g_{1i}, \otimes_{j=1}^n g_{2j}\right) := \prod_{k=1}^n (g_{1k}, g_{2k}). \tag{A.1}$$

Remark 6. It turns out (cf. e.g.[43]) that the value of (G_1, G_2), G_1, $G_2 \in \mathcal{H}^{\otimes_a n}$ is independent of the linear combinations used to express G_1 and G_2, and thus $(\cdot,\cdot)$ is well-defined.

Definition 8. The completion of $\mathcal{H}^{\otimes_a n}$ with respect to the norm induced by the inner product (A.1) is called the (*topological*) *n-th tensor power of $\mathcal{H}$*. We shall denoted it by $\mathcal{H}^{\otimes n}$.

We observe that if the Hilbert space $\mathcal{H}$ is separable and $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$, then $\mathcal{H}^{\otimes n}$ is also a separable Hilbert space and the set of elements of the form $e_{\mathbf{k}} := \otimes_{i=1}^n e_{k_i}$ indexed by $\mathbf{K} := (k_1, \ldots, k_n) \in \mathbb{N}^n$ is a Hilbertian basis of $\mathcal{H}^{\otimes n}$.

In particular, we can consider the Hilbert space $L^2(\mathbb{R})$ or its complexified space $L^2_{\mathbb{C}} := \{f_1 + i f_2 : f_1, f_2 \in L^2(\mathbb{R})\}$ (see Appendix A.1.2). In these cases the tensor powers $(L^2(\mathbb{R}))^{\otimes n}$, $(L^2_{\mathbb{C}}(\mathbb{R}))^{\otimes n}$ can be identified with $L^2(\mathbb{R}^n)$, $L^2_{\mathbb{C}}(\mathbb{R}^n)$, respectively.

Proposition 4. *The spaces $(L^2(\mathbb{R}))^{\otimes n}$ and $(L^2_{\mathbb{C}}(\mathbb{R}))^{\otimes n}$ are unitarily isomorphic to $L^2(\mathbb{R}^n)$ and $L^2_{\mathbb{C}}(\mathbb{R}^n)$, respectively.*

Proof. (Sketch) Given an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ of $L^2(\mathbb{R})$, consider the orthonormal basis $R_{\mathbf{K}}$, $\mathbf{K} := (k_1, \ldots, k_n) \in \mathbb{N}^n$, of $L^2(\mathbb{R}^n)$,

$$R_{\mathbf{K}}(x_1, \ldots, x_n) := e_{k_1}(x_1) \ldots e_{k_n}(x_n),$$

and the linear mapping R which maps the orthonormal basis $\{e_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^n}$ of $(L^2(\mathbb{R}))^{\otimes n}$ onto $\{R_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^n}$,

$$R : e_{k_1} \otimes \ldots \otimes e_{k_n} \mapsto R_{\mathbf{K}}, \quad \mathbf{K} = (k_1, \ldots, k_n).$$

Clearly the following equality of norms holds:

$$|R_{\mathbf{K}}|_{L^2(\mathbb{R}^n)} = |e_{k_1} \otimes \ldots \otimes e_{k_n}|_{(L^2(\mathbb{R}))^{\otimes n}}, \quad \forall\, \mathbf{K} = (k_1, \ldots, k_n) \in \mathbb{N}^n,$$

which leads to the existence of a unique extension of R to a unitary isomorphism of $(L^2(\mathbb{R}))^{\otimes n}$ onto $L^2(\mathbb{R}^n)$, also denoted by R. The assertion for the complex case follows by simply replacing each L^2-space by the corresponding $L^2_{\mathbb{C}}$ complexified space. □

Remark 7. In view of this proof, we shall identify each n-linear form $g_1 \otimes \ldots \otimes g_n$ with $R(g_1 \otimes \ldots \otimes g_n) \in L^2(\mathbb{R}^n)$, that is,

$$(g_1 \otimes \ldots \otimes g_n)(x_1, \ldots, x_n) := g_1(x_1) \ldots g_n(x_n),$$

and $g_1 \otimes \ldots \otimes g_n \in L^2(\mathbb{R}^n)$ will be called the *tensor product of* $g_1, \ldots, g_n$. The latter equality is adopted to define $g_1 \otimes \ldots \otimes g_n$ with $g_1, \ldots, g_n$ being elements of a generic space of functions.

A.1.2. *Fock Space*

The definition and main properties of the symmetric Fock spaces are described below. For more details see e.g.[8,43,46] .

Given a real separable Hilbert space $\mathcal{H}$, let $\mathcal{H}_{\mathbb{C}}$ be the complexified space of $\mathcal{H}$,

$$\mathcal{H}_{\mathbb{C}} := \{f + ig : f, g \in \mathcal{H}\}$$

with the inner product

$$(f_1 + ig_1, f_2 + ig_2) := (f_1, f_2) + (g_1, g_2) + i(g_1, f_2) - i(f_1, g_2)$$

(antilinear in the second factor) and the corresponding norm $|\cdot|$.

For each $n \in \mathbb{N}$ fixed and any $\iota \in S_n$ ($S_n :=$ the permutation group over $\{1, \ldots, n\}$), we consider the unitary isomorphism $U_{\iota,n}$ defined on the total set[a] of elements of the form $g_1 \otimes \ldots \otimes g_n \in \mathcal{H}_{\mathbb{C}}^{\otimes n}$, $g_i \in \mathcal{H}_{\mathbb{C}}$, $i = 1, \ldots, n$, by

$$U_{\iota,n}(g_1 \otimes \ldots \otimes g_n) := g_{\iota(1)} \otimes \ldots \otimes g_{\iota(n)}.$$

[a]A subset A of a Hilbert space is said to be *total* whenever the closure of the space spanned by A coincides with the whole space.

Then, given the family of unitary isomorphisms $U_{\iota,n}$ with $\iota \in S_n$, we define the operator P_n on $\mathcal{H}_{\mathbb{C}}^{\otimes n}$ by

$$P_n := \frac{1}{n!} \sum_{\iota \in S_n} U_{\iota,n}.$$

It is easy to check that $P_n \circ P_n = P_n$ and the adjoint operator of P_n coincides with P_n itself. That is, P_n is an orthogonal projection. We shall call the image of $\mathcal{H}_{\mathbb{C}}^{\otimes n}$ under P_n the *n-th symmetric tensor power of* $\mathcal{H}_{\mathbb{C}}$ and denote it by $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$. We shall denote each $P_n(g_1 \otimes \ldots \otimes g_n)$ by $g_1 \widehat{\otimes} \ldots \widehat{\otimes} g_n$.

Due to Proposition 4 and its proof, it is clear that in particular for $\mathcal{H}_{\mathbb{C}} = L^2_{\mathbb{C}}(\mathbb{R})$, the n-th symmetric tensor power $(L^2_{\mathbb{C}}(\mathbb{R}))^{\widehat{\otimes} n}$ is unitarily isomorphic to the subspace $\widehat{L}^2_{\mathbb{C}}(\mathbb{R}^n) \subset L^2_{\mathbb{C}}(\mathbb{R}^n)$ of all symmetric square integrable functions. For this reason, in accordance with Remark 7 we shall identify the space $(L^2_{\mathbb{C}}(\mathbb{R}))^{\widehat{\otimes} n}$ with the space $\widehat{L}^2_{\mathbb{C}}(\mathbb{R}^n)$.

Definition 9. The Bose or symmetric Fock space $\mathrm{Exp}\mathcal{H}_{\mathbb{C}}$ over $\mathcal{H}_{\mathbb{C}}$ is the Hilbert space defined by the Hilbertian direct sum

$$\mathrm{Exp}(\mathcal{H}_{\mathbb{C}}) := \bigoplus_{n=0}^{\infty} \mathrm{Exp}_n(\mathcal{H}_{\mathbb{C}}),$$

where $\mathrm{Exp}_0(\mathcal{H}_{\mathbb{C}}) := \mathbb{C}$ is the so-called vacuum subspace, and each $\mathrm{Exp}_n(\mathcal{H}_{\mathbb{C}})$, $n \in \mathbb{N}$, defined by the space $\mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$ endowed with the inner product $n!(\cdot,\cdot)_{\mathcal{H}_{\mathbb{C}}^{\otimes n}}$, is the so-called n-particle subspace.

In other words, a generic element $F \in \mathrm{Exp}(\mathcal{H}_{\mathbb{C}})$ is a sequence $F = (f^{(n)})_{n\in\mathbb{N}_0}$ with $f^{(n)} \in \mathcal{H}_{\mathbb{C}}^{\widehat{\otimes} n}$, $n \in \mathbb{N}$, and

$$\|F\|^2_{\mathrm{Exp}(\mathcal{H}_{\mathbb{C}})} := \sum_{n=0}^{\infty} n! |f^{(n)}|^2_{\mathcal{H}_{\mathbb{C}}^{\otimes n}} < \infty.$$

Among the elements in $\mathrm{Exp}(\mathcal{H}_{\mathbb{C}})$ we distinguish the so-called *exponential vectors* or *coherent states* $e(f) \in \mathrm{Exp}(\mathcal{H}_{\mathbb{C}})$ *corresponding to the one-particle vector* $f \in \mathcal{H}_{\mathbb{C}}$:

$$e(f) := \left(1, f, \frac{1}{2!} f^{\otimes 2}, \ldots, \frac{1}{n!} f^{\otimes n}, \ldots\right), \quad f^{\otimes n} := f \otimes \ldots \otimes f \ (n \text{ times}).$$

According to the definition of $\mathrm{Exp}(\mathcal{H}_{\mathbb{C}})$, we have

$$(e(f_1), e(f_2))_{\mathrm{Exp}(\mathcal{H}_{\mathbb{C}})} = \exp\left((f_1, f_2)\right), \quad f_1, f_2 \in \mathcal{H}_{\mathbb{C}},$$

and thus

$$\|e(f)\|^2_{\mathrm{Exp}(\mathcal{H}_{\mathbb{C}})} = \exp\left(|f|^2\right), \quad f \in \mathcal{H}_{\mathbb{C}}.$$

The next statement emphasizes the role of coherent states (see e.g.[2]).

Proposition 5. *Given a linear subspace $\mathcal{L} \subset \mathcal{H}_{\mathbb{C}}$, the family of coherent states $\{e(f) : f \in \mathcal{L}\}$ is total in* $\mathrm{Exp}(\mathcal{H}_{\mathbb{C}})$ *whenever $\mathcal{L}$ is dense in $\mathcal{H}_{\mathbb{C}}$.*

Remark 8. In view of Appendix A.1.3 below, let us mention that in an analogous way we can define symmetric tensor powers of real Hilbert spaces as well.

A.1.3. *Tensor Powers of Nuclear Spaces*

As in Section 1, let $\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'$ be a nuclear triple,

$$\mathcal{N} = \underset{p\in\mathbb{N}}{\mathrm{prlim}}\mathcal{H}_p, \quad \mathcal{N}' = \underset{p\in\mathbb{N}}{\mathrm{indlim}}\mathcal{H}_{-p}.$$

In order to define tensor powers $\mathcal{N}^{\otimes n}$ and symmetric tensor powers $\mathcal{N}^{\widehat{\otimes} n}$, $n \in \mathbb{N}$, $n \geq 2$, of the nuclear space $\mathcal{N}$, we consider the families of tensor powers of the Hilbert spaces $\mathcal{H}_p^{\otimes n}$ and $\mathcal{H}_p^{\widehat{\otimes} n}$, both indexed by $p \in \mathbb{N}$. Since there is no risk of confusion, we shall use the notation $|\cdot|_p$ also for the Hilbertian norm on $\mathcal{H}_p^{\otimes n}$. The *$n$-th tensor power $\mathcal{N}^{\otimes n}$ of $\mathcal{N}$* and the *n-th symmetric tensor power $\mathcal{N}^{\widehat{\otimes} n}$ of $\mathcal{N}$* are the nuclear Fréchet spaces defined by

$$\mathcal{N}^{\otimes n} := \underset{p\in\mathbb{N}}{\mathrm{pr\,lim}}\mathcal{H}_p^{\otimes n}, \quad \mathcal{N}^{\widehat{\otimes} n} := \underset{p\in\mathbb{N}}{\mathrm{pr\,lim}}\mathcal{H}_p^{\widehat{\otimes} n},$$

respectively.[2]

Moreover, if each $\mathcal{H}_{-p}^{\otimes n}$ (resp., $\mathcal{H}_{-p}^{\widehat{\otimes} n}$) is the dual space of $\mathcal{H}_p^{\otimes n}$ (resp., $\mathcal{H}_p^{\widehat{\otimes} n}$) with respect to $\mathcal{H}^{\otimes n}$ (resp., $\mathcal{H}^{\widehat{\otimes} n}$), then the dual space $\mathcal{N}'^{\otimes n}$ of $\mathcal{N}^{\otimes n}$ with respect to $\mathcal{H}^{\otimes n}$ and the dual space $\mathcal{N}'^{\widehat{\otimes} n}$ of $\mathcal{N}^{\widehat{\otimes} n}$ with respect to $\mathcal{H}^{\widehat{\otimes} n}$ can be written as

$$\mathcal{N}'^{\otimes n} = \underset{p\in\mathbb{N}}{\mathrm{indlim}}\mathcal{H}_{-p}^{\otimes n} \quad \text{and} \quad \mathcal{N}'^{\widehat{\otimes} n} = \underset{p\in\mathbb{N}}{\mathrm{indlim}}\mathcal{H}_{-p}^{\widehat{\otimes} n},$$

respectively. As before, in this work we shall also use the notation $|\cdot|_{-p}$ for the norm on $\mathcal{H}_{-p}^{\otimes n}$, $p \in \mathbb{N}$, and $\langle \cdot, \cdot \rangle$ for the dual pairing between $\mathcal{N}'^{\otimes n}$ and $\mathcal{N}^{\otimes n}$.

Thus we have defined the nuclear triples

$$\mathcal{N}^{\otimes n} \subset \mathcal{H}^{\otimes n} \subset \mathcal{N}'^{\otimes n} \quad \text{and} \quad \mathcal{N}^{\widehat{\otimes} n} \subset \mathcal{H}^{\widehat{\otimes} n} \subset \mathcal{N}'^{\widehat{\otimes} n}.$$

Remark 9. All the above results quoted still hold for complex spaces. In that case, we shall use the same notation as above.

A.1.4. *Holomorphy on Locally Convex Spaces*

In this part we generalize the notion of holomorphic or analytic functions in complex analysis to complex-valued functions defined on a locally convex topological vector space $\mathcal{E}$ over the complex field $\mathbb{C}$. For more details and the proofs see e.g.[1,4,5] .

A function $F : U \to \mathbb{C}$ defined on an open set $U \subset \mathcal{E}$ is called *G-holomorphic* or *Gâteaux-holomorphic* if for each $\xi_0 \in \mathcal{U}$ and each $\xi \in \mathcal{E}$ the complex-valued function

$$\mathbb{C} \ni z \mapsto F(\xi_0 + z\xi) \in \mathbb{C}$$

is analytic on some neighborhood of $0 \in \mathbb{C}$. Hence, given a G-holomorphic function $F : U \to \mathbb{C}$, it turns out by the general theory of complex analysis that for each $\xi_0 \in U$ we can write

$$F(\xi_0 + \xi) = \sum_{n=0}^{\infty} \frac{1}{n!} d^n F(\xi_0; \xi) \tag{A.2}$$

for all ξ in some open neighborhood of zero, where $d^n F(\xi_0; \xi)$ is the differential $d^n F(\xi_0)(\xi, \dots, \xi)$ of n-th order of F at the point ξ_0 along the direction $(\xi, \dots, \xi)$.

A G-holomorphic function $F : U \to \mathbb{C}$ is called *holomorphic* whenever for all $\xi_0 \in U$ there is an open neighborhood V of zero such that the series in (A.2) converges uniformly on V. It turns out (cf. e.g. [5, Lemma 2.8]) that a G-holomorphic function is holomorphic if and only if it is locally bounded.

A function F is said to be *holomorphic at a point* $\xi_0 \in \mathcal{E}$ if there is an open neighborhood $U \subset \mathcal{E}$ of ξ_0 such that $F : U \to \mathbb{C}$ is holomorphic.

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