

ON THE CONVERGENCE TO CRITICAL SCALING PROFILES IN SUBMONOLAYER DEPOSITION MODELS*

Dedicated to the memory of Jack Carr

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ABSTRACT. In this work we study the rate of convergence to similarity profiles in a mean field model for the deposition of a submonolayer of atoms in a crystal facet, when there is a critical minimal size $n \geq 2$ for the stability of the formed clusters. The work complements recently published related results by the same authors in which the rate of convergence was studied outside of a critical direction $x = \tau$ in the cluster size x vs. time τ plane. In this paper we consider a different similarity variable, $\xi := (x - \tau)/\sqrt{\tau}$, corresponding to an inner expansion of that critical direction, and prove the convergence of solutions to a similarity profile $\Phi_{2,n}(\xi)$ when $x, \tau \rightarrow +\infty$ with ξ fixed, as well as the rate at which the limit is approached.

1. INTRODUCTION

The deposition of a monolayer of atoms on top of a single crystal facet is a process of clear technological importance, and its theoretical understanding is a theme of current scientific interest (see, e.g., [15]).

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In the early stages of the deposition process the new monolayer is still far from being completed and only islands, or j -clusters, made up of a number j of adatoms exist, which, being far apart from each other on the crystal facet, do not interact among them.

One of the mathematical descriptions of this stage of the deposition process consists in a coagulation type differential equation [7, 8] obtained as follows: denote by $c_j = c_j(t)$ the concentration of j -clusters at time t on the crystal facet. The surface is hit at a constant rate $\alpha > 0$ by 1-clusters (also called *monomers*). Consider the non-interacting assumption above. Then, the only reactions of the clusters on the crystal surface are those in which a monomer takes part, namely $(1) + (j) \rightarrow (j+1)$, with rate coefficients $a_{1,j} \geq 0$, for $j \in \{1, 2, \dots\}$. Thus, the differential equations governing the dynamics of the population of clusters $\{c_j(t)\}$ are a Smoluchowski coagulation system with Becker-Döring like coagulation kernel (see, e.g., [5, 19]).

An additional assumption relevant to some applications is the existence of a critical cluster size n below which the clusters on the crystal facet are not stable and do not exist in any significant amount in the time scale of the coagulation reactions. Mathematically, this behaviour can be described as follows, [8]: clusters of size larger than 1 and smaller than or equal to $n-1$ cannot exist, and thus the smaller cluster that is not a monomer has size n and is formed when n monomers come together and react into an n -cluster. These “multiple collisions” can be thought of as follows [6]: if, as in Monte Carlo simulations, we consider the monomers sitting in the vertices of a lattice, and if we have $n-1$ monomers surrounding an empty site – as in the four nearest neighbours in a square lattice – which is suddenly hit by a monomer, we have one of these “multiple” collisions and the creation of an n -cluster in a single n body 1-cluster reaction.

Considering all coagulation rates to be time and cluster size independent ($a_{1,j} \equiv 1$, say) the mean-field model for the submonolayer deposition with a critical cluster size n is given by the following coagulation type ordinary differential equation system:

$$(1) \quad \begin{cases} \dot{c}_1 &= \alpha - nc_1^n - c_1 \sum_{j=n}^{\infty} c_j, \\ \dot{c}_n &= c_1^n - c_1 c_n, \\ \dot{c}_j &= c_1 c_{j-1} - c_1 c_j, \quad j \geq n+1. \end{cases}$$

Of course rate equations like (1), as all mathematical models, are an approximate description of what is happening in deposition processes [15], although they have proven very useful to relate measured cluster densities to atomistic parameters (see, e.g., [9] and references therein). More realistic rate equations, which include processes of cluster coalescence and fragmentation, reevaporation of monomers, etc., have also been considered [9]. Maybe the main limitation of (1) within the context of rate equation models is the assumption of constant rate coefficients. Typical rate coefficients depend on the effective cross section of the clusters for the coagulation reactions with adatoms, or their “capture numbers”, which measure the propensity of a cluster to coagulate with diffusing adatoms [10]. These quantities usually vary with the cluster size j although they do not depend strongly on j , and this supports the mathematical approximation of considering them constants

[10]. The study of more realistic cases seems to be at present outside the scope of rigorous mathematical analysis.

The study of the long time behaviour of solutions to coagulation type equations is a theme of current research interest. For the general Smoluchowski coagulation system (i.e., when coagulation reactions between all clusters $(j) + (k) \rightarrow (j+k)$ are allowed) a number of important papers concerning the existence of, and convergence to, self-similarity have been recently published (see, e.g.: [4, 13, 14, 18]).

For systems of Becker-Döring type (i.e., when only monomer-clusters reactions, but also including the reverse ones of fragmentation, $(1) + (j) \rightleftharpoons (1+j)$, are allowed) there is also an extensive bibliography dealing with many interesting and difficult problems posed by the long time behaviour of solutions, including the existence of dynamic phase transition [1], rates of convergence to equilibria in the subcritical regime [2, 3, 11, 16], or the scaling limit in the supercritical case [12, 17].

In the present paper we are concerned with the convergence of solutions of (1) to a scaling profile and with the rate at which this limit is approached.

Using the new time scale

$$(2) \quad \tau(t) := \int_{t_0}^t c_1(s) ds$$

it was proved in [8] that solutions of (1) in this time scale,

$$(3) \quad \tilde{c}_j(\tau) := c_j(t(\tau)),$$

satisfy the following scaling behaviour:

$$\lim_{\substack{j, \tau \rightarrow +\infty \\ \eta = j/\tau \text{ fixed} \\ \eta \neq 1}} \left(\frac{n\tau}{\alpha} \right)^{(n-1)/n} \tilde{c}_j(\tau) = \Phi_{1,n}(\eta) := \begin{cases} (1-\eta)^{-(n-1)/n}, & \text{if } 0 < \eta < 1 \\ 0, & \text{if } \eta > 1. \end{cases}$$

This extended the result in [7], valid in the absence of a critical cluster size (i.e., when $n = 2$). Exploiting techniques based on center manifold theory used in [7], we recently proved in [6] results about the rate of convergence to this scaling behaviour.

Along the direction $\eta := j/\tau = 1$ a different scaling variable is needed, which was identified in [7] as $\xi := (j - \tau)/\sqrt{\tau} \in \mathbb{R}$. Observing solutions of (1) along $\xi = \text{const.}$ corresponds to look at them with $\eta \rightarrow 1$ as $\tau \rightarrow \infty$ and so, in a certain sense, the new scaling variable ξ blows up to the whole real line the singular behaviour identified at the critical value $\eta = 1$.

In [7] the convergence to a similarity profile in this new similarity variable was proved for the case with no critical size clusters ($n = 2$). In this paper we extend that result to general $n \geq 2$ by proving that

$$\lim_{\substack{j, \tau \rightarrow +\infty \\ \xi = (j-\tau)/\sqrt{\tau} \text{ fixed} \\ \xi \in \mathbb{R}}} \frac{\sqrt{2\pi}}{\alpha} \left(\frac{\alpha}{n} \right)^{\frac{1}{n}} \tau^{\frac{n-1}{2n}} \tilde{c}_j(\tau) = \Phi_{2,n}(\xi)$$

where

$$\Phi_{2,n}(\xi) := e^{-\xi^2/2} \int_0^{+\infty} \exp\left(-\xi w^n - \frac{w^{2n}}{2}\right) dw.$$

The rate of convergence to similarity profiles in several types of coagulating systems has recently attracted some attention (see, e.g., [16, 18]). As stated above,

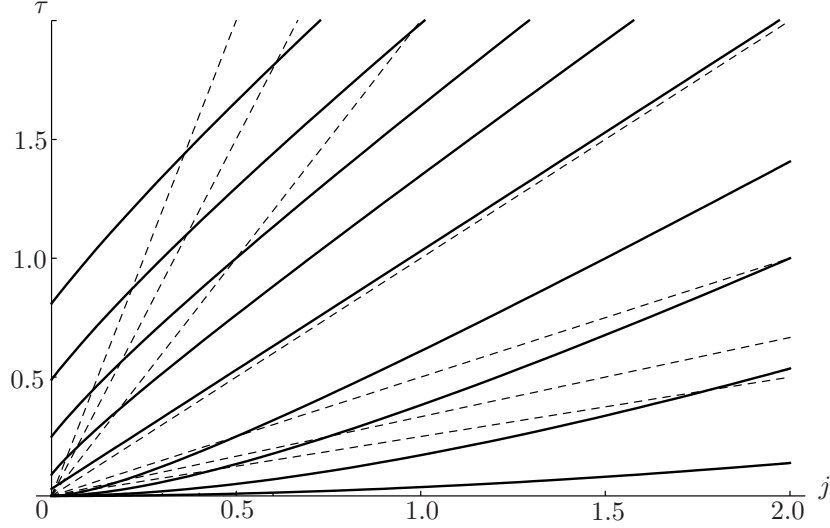


FIGURE 1. Lines with $\xi = \text{constant}$ (full), and with $\eta = \text{constant}$ (dashed) in the (j, τ) plane. The values used for these parameters are the following, in counterclockwise direction: $\xi = 5.0, 2.0, 1.0, 0.5, 0.0, -0.3, -0.5, -0.7, -0.9$, and $\eta = 1/4, 1/3, 1/2, 1, 2, 3, 4$.

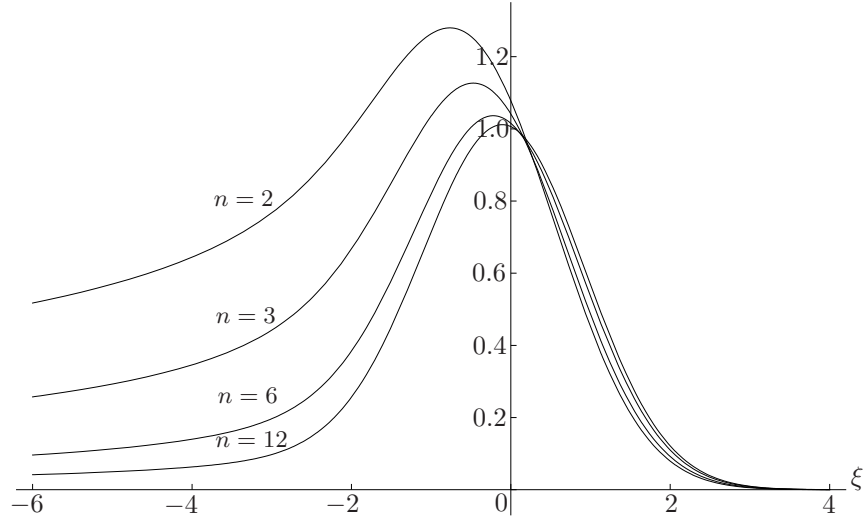


FIGURE 2. Graph of the similarity profile $\Phi_{2,n}(\xi)$ for different values of n .

for the deposition model (1) we recently studied in [6] the rate of convergence to the profile $\Phi_{1,n}$ in the similarity variable η . In the present paper we complement that work by studying the rate of convergence to $\Phi_{2,n}$ in the variable ξ , thus completing the study in [6, 7, 8].

Our main result is the following:

Theorem 1.1. *Let $n \geq 2$. For every solution (c_j) of (1) with initial condition (c_{j0}) consider the change of variables $(t, c_j) \mapsto (\tau, \tilde{c}_j)$ introduced in (2) and (3). If there exists $\rho_1, \rho_2 > 0, \mu > 1 - \frac{1}{n}$ such that for all $j \geq n$, $j^\mu c_j(0) \in [\rho_1, \rho_2]$ then, as $j, \tau \rightarrow +\infty$, with $\xi = \frac{j-\tau}{\sqrt{\tau}}$ fixed, the following holds:*

$$(4) \quad \left| \frac{\sqrt{2\pi}}{\alpha} \left(\frac{\alpha}{n} \right)^{\frac{1}{n}} \tau^{\frac{n-1}{2n}} \tilde{c}_j(\tau) - \Phi_{2,n}(\xi) \right| \leq \begin{cases} O(j^{-\frac{1}{2n} + \frac{1-\mu}{2}}) & \text{if } 1 - \frac{1}{n} < \mu < 1 \\ O(j^{-\frac{1}{2n}} \log j) & \text{if } \mu = 1 \\ O(j^{-\frac{1}{2n}}) & \text{if } \mu > 1. \end{cases}$$

As a consequence of the proof of this theorem, we obtain also the following result valid for an important case in applications: the deposition of monomers on top of a crystal facet having no j -cluster at the initial time $t = 0$, i.e., with initial conditions $c_j(0) = 0$ for all $j \geq n$:

Theorem 1.2. *With the notation used in Theorem 1.1 consider initial data with $c_{j0} = 0$ for all $j \geq n$. Then, as $j, \tau \rightarrow +\infty$, with $\xi = \frac{j-\tau}{\sqrt{\tau}}$ fixed, it holds:*

$$(5) \quad \left| \frac{\sqrt{2\pi}}{\alpha} \left(\frac{\alpha}{n} \right)^{\frac{1}{n}} \tau^{\frac{n-1}{2n}} \tilde{c}_j(\tau) - \Phi_{2,n}(\xi) \right| \leq C j^{-\frac{1}{2}} \log j.$$

2. PRELIMINARIES

We briefly present our approach to the study of (1), which follows the one used in [6, 7, 8] and consists of the exploration of the following two observations:

- (1) First note that the equation for c_1 depends only on c_1 and on the “bulk” quantity $y(t) := \sum_{j=n}^{\infty} c_j(t)$, which (formally) satisfies the differential equation $\dot{y} = c_1^n$. Thus, the definition of this bulk variable allow us to decouple the resulting infinite dimensional system into a closed two-dimensional system for the monomer–bulk variables (c_1, y) , from which we get all the needed information about the behaviour of c_1 .
- (2) Secondly, the remaining equations for c_j , with $j \geq n$, depend only on those same variables c_j , and on c_1 . However, the way they depend on c_1 is such that, by the change of time variable (2), the system is transformed into a linear lower triangular infinite system of ordinary differential equations for $\tilde{c}_j(\tau) = c_j(t(\tau))$, which can be recursively solved, in terms of \tilde{c}_1 , using the variation of constants formula.

In fact, writing the second and third equations of (1) in the form

$$(6) \quad \begin{cases} \dot{c}_n = c_1(c_1^{n-1} - c_n) \\ \dot{c}_j = c_1(c_{j-1} - c_j), \quad j \geq n+1, \end{cases}$$

and introducing the new time scale (2) along with scaled variables (3), system (6) reads

$$(7) \quad \begin{cases} \tilde{c}_n' = \tilde{c}_1^{n-1} - \tilde{c}_n \\ \tilde{c}_j' = \tilde{c}_{j-1} - \tilde{c}_j, \quad j \geq n+1, \end{cases}$$

where $(\cdot)' = d/d\tau$. Assuming \tilde{c}_1 is known, the equation for \tilde{c}_n in (7) can be readily solved by the variation of constants formula resulting in

$$\tilde{c}_n(\tau) = e^{-\tau}\tilde{c}_n(0) + \int_0^\tau (\tilde{c}_1(\tau-s))^{n-1}e^{-s}ds.$$

We can now plug this solution into the equation for \tilde{c}_j with $j = n+1$ and again use the variation of constants formula. Proceeding recursively for $j \geq n+1$ the following expression for $\tilde{c}_j(\tau)$ is obtained :

$$(8) \quad \tilde{c}_j(\tau) = \mathfrak{I}_1(j, \tau) + \mathfrak{I}_2(j, \tau),$$

where

$$(9) \quad \mathfrak{I}_1(j, \tau) := e^{-\tau} \sum_{k=n}^j \frac{\tau^{j-k}}{(j-k)!} c_k(0),$$

and

$$(10) \quad \mathfrak{I}_2(j, \tau) := \frac{1}{(j-n)!} \int_0^\tau (\tilde{c}_1(\tau-s))^{n-1} s^{j-n} e^{-s} ds.$$

The information about \tilde{c}_1 needed to study (10) is extracted from the two-dimensional system for (c_1, y) referred to in 1. above, which is

$$(11) \quad \begin{cases} \dot{x} = \alpha - nx^n - xy \\ \dot{y} = x^n, \end{cases}$$

where $x(t) := c_1(t)$ and $y(t) := \sum_{j=n}^\infty c_j(t)$. This was done in [6, 8] using asymptotic and center manifold methods. The result that we need below is the following:

Theorem 2.1. [6, Theorem 6] *With τ and $\tilde{c}_1(\tau)$ as before, the following holds:*

$$(12) \quad \left(\frac{n\tau}{\alpha}\right)^{(n-1)/n} (\tilde{c}_1(\tau))^{n-1} = 1 + (n-1) \left(1 - \frac{1}{n}\right) \frac{\log \tau}{\tau} + o\left(\frac{\log \tau}{\tau}\right), \text{ as } \tau \rightarrow \infty.$$

In the following two sections we prove Theorem 1.1 by studying separately the contributions of the two terms (9) and (10) to the limit (8). As a consequence of the estimates in section 4 a proof of Theorem 1.2 will also be obtained.

3. RATES OF CONVERGENCE TO $\Phi_{2,n}$: CONTRIBUTION OF \mathfrak{I}_1

In this section we study the limit

$$(13) \quad \lim_{\substack{j, \tau \rightarrow +\infty \\ \xi = \frac{j-\tau}{\sqrt{\tau}} \text{ fixed}}} \frac{\sqrt{2\pi}}{\alpha} \left(\frac{\alpha}{n}\right)^{\frac{1}{n}} \tau^{\frac{n-1}{2n}} \mathfrak{I}_1(j, \tau).$$

Part of the analysis was presented in [7] for the case $n = 2$ but the study of S_4 below is new and proves the conjecture left open in section 6.2 of [7]. Using the definition of ξ we can write $j = \tau + \xi\sqrt{\tau}$, which, solved for τ gives $\tau = j\Delta_j$ where the function $\Delta_j : \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$(14) \quad \Delta_j(\xi) := \left(\sqrt{1 + \frac{\xi^2}{4j}} + \text{sgn}(-\xi) \sqrt{\frac{\xi^2}{4j}} \right)^2.$$

A number of properties of this function can be easily established, in particular,

$$(15) \quad \Delta_j \begin{cases} < 1 & \text{if } \xi > 0 \\ = 1 & \text{if } \xi = 0 \\ > 1 & \text{if } \xi < 0, \end{cases}$$

and

$$(16) \quad \Delta_j \rightarrow 1 \text{ as } j \rightarrow \infty, \text{ for all } \xi.$$

Changing the summation variable to $\ell := j - k$, writing τ in terms of j as indicated, and not taking multiplication constants into account, the relevant limit to compute for (13) is the following:

$$(17) \quad \lim_{j \rightarrow \infty} (j\Delta_j)^{\frac{n-1}{2n}} e^{-j\Delta_j} \sum_{\ell=0}^{j-n} \frac{(j\Delta_j)^\ell}{\ell!} c_{j-\ell}(0).$$

In order to evaluate (17) we decompose the sum into a “small ℓ ” and a “large ℓ ” contribution. It is natural to consider a cut-off size separating the two sums that scales like τ as a function of j , which means that the cut-off value should behave as $\sim j - |\xi|\sqrt{j}$. So let us define

$$(18) \quad j^* := (j - n) - (1 + |\xi|)\sqrt{j - n},$$

and write the expression in (17) as

$$(j\Delta_j)^{\frac{n-1}{2n}} e^{-j\Delta_j} \left(\sum_{0 \leq \ell \leq j^*} \frac{(j\Delta_j)^\ell}{\ell!} c_{j-\ell}(0) + \sum_{j^* < \ell \leq j-n} \frac{(j\Delta_j)^\ell}{\ell!} c_{j-\ell}(0) \right) =: S_3(j) + S_4(j)$$

The “small ℓ ” sum, S_3 , that corresponds to the contribution of *large* cluster in the initial data (remember the change of variable $k \mapsto \ell$), can be estimated in the same way as the sum S_1 in the proof of Theorem 5.1 of [7], and was already done in section 6.2 of [7] with $n = 2$. The general n case is no different: assuming $c_k(0) \leq \rho/k^\mu$ we have

$$\begin{aligned} S_3(j) &\leq (j\Delta_j)^{\frac{n-1}{2n}} e^{-j\Delta_j} \rho \sum_{0 \leq \ell \leq j^*} \frac{(j\Delta_j)^\ell}{\ell! (j - j^*)^\mu} \\ &= j^{\frac{n-1}{2n} - \frac{\mu}{2}} \left(\frac{n}{\sqrt{j}} + (1 + |\xi|)\sqrt{1 - \frac{n}{j}} \right)^{-\mu} e^{-j\Delta_j} \sum_{0 \leq \ell \leq j^*} \frac{(j\Delta_j)^\ell}{\ell!} \\ (19) \quad &\leq C j^{\frac{n-1}{2n} - \frac{\mu}{2}} e^{-j\Delta_j} \rho \sum_{\ell=0}^{\infty} \frac{(j\Delta_j)^\ell}{\ell!} = C j^{\frac{n-1}{2n} - \frac{\mu}{2}}. \end{aligned}$$

The existence of a positive constant C is due to the fact that

$$\left(\frac{n}{\sqrt{j}} + (1 + |\xi|)\sqrt{1 - \frac{n}{j}} \right)^{-\mu} \longrightarrow (1 + |\xi|)^{-\mu} \in \mathbb{R}^+, \text{ as } j \rightarrow \infty.$$

We now consider the “large ℓ ” sum, S_4 , corresponding to the contribution of *small* clusters. Recalling Stirling’s expansion $\ell! = \sqrt{2\pi\ell}^{\frac{1}{2}} e^{-\ell} (1 + \frac{1}{12\ell} + O(\ell^{-2}))$,

we can write

$$\begin{aligned}
S_4(j) &\leq (j\Delta_j)^{\frac{n-1}{2n}} e^{-j\Delta_j} \rho \sum_{\ell=j^*+1}^{j-n} \frac{(j\Delta_j)^\ell}{\ell!(j-\ell)^\mu} \\
&= (j\Delta_j)^{\frac{n-1}{2n}} e^{-j\Delta_j} \frac{\rho}{\sqrt{2\pi}} \sum_{\ell=j^*+1}^{j-n} \frac{(j\Delta_j)^\ell}{\ell^{\ell+\frac{1}{2}} e^{-\ell} (1+O(\ell^{-1}))(j-\ell)^\mu} \\
(20) \quad &= \frac{\rho}{\sqrt{2\pi}} (1+O(j^{-1})) (j\Delta_j)^{\frac{n-1}{2n}} \sum_{\ell=j^*+1}^{j-n} \left(\frac{j\Delta_j}{\ell} e^{1-\frac{j\Delta_j}{\ell}} \right)^\ell \frac{1}{\ell^{1/2}(j-\ell)^\mu}.
\end{aligned}$$

Now, observing that the function $x \mapsto xe^{1-x}$, for positive x , is positive and has an absolute maximum at $x = 1$ with value equal to 1; recalling (16); and using the expression of j^* in (18) to conclude that, when $j^* + 1 \leq \ell \leq j - n$, we have $C_1 j^{-1/2} \leq \ell^{-1/2} \leq C_2 j^{-1/2}$, with $C_1, C_2 \rightarrow 1$ as $j \rightarrow \infty$, we can estimate (20) by

$$(21) \quad S_4(j) \leq \frac{\rho}{\sqrt{2\pi}} (1+O(j^{-1})) j^{\frac{n-1}{2n}-\frac{1}{2}} \sum_{\ell=j^*+1}^{j-n} \frac{1}{(j-\ell)^\mu}.$$

To estimate the sum in the right-hand side observe that

$$\sum_{\ell=j^*+1}^{j-n} \frac{1}{(j-\ell)^\mu} = \sum_{k=n}^{j-j^*-1} \frac{1}{k^\mu},$$

and the last sum is the lower Darboux sum of the integral of $x \mapsto x^{-\mu}$ between $n-1$ and $j-j^*-1$. Thus, denoting by C constants independent of j that can vary from case to case, we have: when $\mu \in (0, 1)$,

$$\begin{aligned}
\sum_{k=n}^{j-j^*-1} \frac{1}{k^\mu} &\leq \int_{n-1}^{j-j^*-1} \frac{1}{x^\mu} dx \\
&= \frac{1}{1-\mu} \left((j-j^*-1)^{1-\mu} - (n-1)^{1-\mu} \right) \\
&= \frac{1}{1-\mu} \left(((n-1) + (1+|\xi|)\sqrt{j-n})^{1-\mu} - (n-1)^{1-\mu} \right) \\
&\leq C j^{\frac{1-\mu}{2}},
\end{aligned}$$

when $\mu = 1$,

$$\sum_{k=n}^{j-j^*-1} \frac{1}{k} \leq \int_{n-1}^{j-j^*-1} \frac{1}{x} dx = \log \frac{j-j^*-1}{n-1} \leq C \log j,$$

and when $\mu > 1$,

$$\sum_{k=n}^{j-j^*-1} \frac{1}{k^\mu} \leq \int_{n-1}^{j-j^*-1} \frac{1}{x^\mu} dx = \frac{1}{\mu-1} \left((n-1)^{1-\mu} - (j-j^*-1)^{1-\mu} \right) \leq C.$$

Hence (21) can be estimated as

$$(21) \leq \begin{cases} C j^{-\frac{1}{2n} + \frac{1-\mu}{2}}, & \text{if } \mu \in (0, 1) \\ C j^{-\frac{1}{2n}} \log j, & \text{if } \mu = 1 \\ C j^{-\frac{1}{2n}}, & \text{if } \mu > 1. \end{cases}$$

Together with the estimate (19) we conclude that, for $\mu > 1 - \frac{1}{n}$, the limit (13) is equal to zero and the rate of convergence to the limit is the same as for S_4 above, namely,

$$(22) \quad \frac{\sqrt{2\pi}}{\alpha} \left(\frac{\alpha}{n}\right)^{\frac{1}{n}} \tau^{\frac{n-1}{2n}} \mathfrak{I}_1(j, \tau) \leq \begin{cases} Cj^{-\frac{1}{2n} + \frac{1-\mu}{2}}, & \text{if } \mu \in (1 - \frac{1}{n}, 1) \\ Cj^{-\frac{1}{2n}} \log j, & \text{if } \mu = 1 \\ Cj^{-\frac{1}{2n}}, & \text{if } \mu > 1. \end{cases}$$

4. RATE OF CONVERGENCE TO $\Phi_{2,n}$: CONTRIBUTION OF \mathfrak{I}_2

We now consider the limit

$$(23) \quad \lim_{\substack{j, \tau \rightarrow +\infty \\ \xi = \frac{j-\tau}{\sqrt{\tau}} \text{ fixed}}} \frac{\sqrt{2\pi}}{\alpha} \left(\frac{\alpha}{n}\right)^{\frac{1}{n}} \tau^{\frac{n-1}{2n}} \mathfrak{I}_2(j, \tau)$$

with $\mathfrak{I}_2(j, \tau)$ defined by (10). Analogously to what was done in [6, 7], instead of using (10) with $j \in \mathbb{N}$ we consider the function $\mathfrak{I}_2(x, \tau)$ defined for $(x, \tau) \in [n, \infty) \times [0, \infty)$ by the same expression as (10) and thus will consider instead the limit of

$$(24) \quad \varphi(x, \tau) = \frac{\sqrt{2\pi}}{\alpha} \left(\frac{\alpha}{n}\right)^{1/n} \tau^{\frac{n-1}{2n}} \int_0^\tau (\tilde{c}_1(\tau-s))^{n-1} s^{x-n} e^{-s} ds$$

when $x, \tau \rightarrow +\infty$ with $\xi = \frac{x-\tau}{\sqrt{\tau}}$ fixed. Hence, substituting $x = \tau + \xi\sqrt{\tau}$ in (24), the problem is reduced to the study of the limit of $\varphi(\tau + \xi\sqrt{\tau}, \tau)$ as $\tau \rightarrow +\infty$. Changing in (24) the integration variable $s \mapsto w$, where $\tau - s = w^n \sqrt{\tau}$, we can write

$$\begin{aligned} \varphi(\tau + \xi\sqrt{\tau}, \tau) &= \frac{\sqrt{2\pi}}{\alpha} \left(\frac{\alpha}{n}\right)^{1/n} \tau^{-\frac{1}{2n}} \times \\ &\times \int_0^{\tau^{1/2n}} (\tilde{c}_1(w^n \sqrt{\tau}))^{n-1} (\tau - w^n \sqrt{\tau})^{\tau + \xi\sqrt{\tau}} e^{-\tau - w^n \sqrt{\tau}} dw, \end{aligned}$$

and after a few algebraic manipulations we can rewrite this expression as

$$(25) \quad \varphi(\tau + \xi\sqrt{\tau}, \tau) = P(\xi, \tau) \mathcal{I}_n[1 + f_n, 1 + g](\xi, \tau),$$

where

$$P(\xi, \tau) := \frac{\sqrt{2\pi} e^{-\tau} \tau^{\tau + \xi\sqrt{\tau} - n + \frac{1}{2}}}{\Gamma(\tau + \xi\sqrt{\tau} - n + 1)},$$

$$(26) \quad f_n(\varsigma) := -1 + \left(\frac{n\varsigma}{\alpha}\right)^{(n-1)/n} (\tilde{c}_1(\varsigma))^{n-1},$$

$$g(w, \tau, \xi) := -1 + \left(1 - \frac{w^n}{\sqrt{\tau}}\right)^{\tau + \xi\sqrt{\tau} - n} \exp\left(\sqrt{\tau} w^{2n} + \xi w^n + \frac{w^{2n}}{2}\right),$$

and

$$\mathcal{I}_n[h_1, h_2](\xi, \tau) := \int_0^{\tau^{1/2n}} h_1(\sqrt{\tau} w^{2n}) h_2(w, \tau, \xi) e^{-\xi w^n - \frac{w^{2n}}{2}} dw.$$

Let us write (25) as

$$(27) \quad \varphi(\tau + \xi\sqrt{\tau}, \tau) = P(\xi, \tau) \sum_{j=1}^4 J_j(\xi, \tau)$$

where $J_1 = \mathcal{I}_n[1, 1]$, $J_2 = \mathcal{I}_n[f_n, 1]$, $J_3 = \mathcal{I}_n[1, g]$, and $J_4 = \mathcal{I}_n[f_n, g]$.

We shall now estimate the behaviour as $\tau \rightarrow +\infty$ of each of these terms P and J_j , paying attention not only to the limit but actually to the rate at which that limit is approached.

We start by estimating the prefactor P :

Lemma 4.1. $P(\xi, \tau) = e^{-\frac{\xi^2}{2}}(1 + O(\tau^{-1/2}))$, as $\tau \rightarrow +\infty$.

Proof. Using $x = \tau + \xi\sqrt{\tau}$, so that,

$$P = \frac{\sqrt{2\pi}e^{-\tau}\tau^{x-n+\frac{1}{2}}}{\Gamma(x-n+1)}.$$

Since by Stirling's expansion we have, as $x \rightarrow +\infty$,

$$\begin{aligned} \frac{1}{\Gamma(x-n+1)} &= \frac{1}{\Gamma(x)}(x-1)(x-2)\dots(x-(n-1)) \\ &= \frac{x^{n-1}}{\sqrt{2\pi}e^{-x}x^{x-\frac{1}{2}}} \frac{1+O(x^{-1})}{1+O(x^{-1})} \\ &= \frac{1}{\sqrt{2\pi}e^{-x}x^{x-n+\frac{1}{2}}}(1+O(x^{-1})), \end{aligned}$$

we obtain, as $\tau \rightarrow +\infty$,

$$\begin{aligned} P &= e^{x-\tau} \left(\frac{\tau}{x}\right)^{x-n+\frac{1}{2}} (1+O(x^{-1})) \\ &= e^{\xi\sqrt{\tau}} \left(\frac{\tau}{\tau+\xi\sqrt{\tau}}\right)^{\tau+\xi\sqrt{\tau}-n+\frac{1}{2}} (1+O(\tau^{-1/2})) \\ (28) \quad &= e^{\xi\sqrt{\tau}} \left(\frac{\tau}{\tau+\xi\sqrt{\tau}}\right)^{\tau+\xi\sqrt{\tau}} (1+O(\tau^{-1/2})). \end{aligned}$$

By further observing that,

$$\begin{aligned} e^{\xi\sqrt{\tau}} \left(\frac{\tau}{\tau+\xi\sqrt{\tau}}\right)^{\tau+\xi\sqrt{\tau}} &= \\ &= \exp \left[\xi\sqrt{\tau} + (\tau + \xi\sqrt{\tau}) \log \left(1 + \frac{-\xi\sqrt{\tau}}{\tau + \xi\sqrt{\tau}} \right) \right] \\ &= \exp \left[\xi\sqrt{\tau} + (\tau + \xi\sqrt{\tau}) \left(\frac{-\xi\sqrt{\tau}}{\tau + \xi\sqrt{\tau}} - \frac{1}{2} \left(\frac{-\xi\sqrt{\tau}}{\tau + \xi\sqrt{\tau}} \right)^2 + O(\tau^{-3/2}) \right) \right] \\ &= \exp \left[-\frac{1}{2}\xi^2 \frac{\tau}{\tau + \xi\sqrt{\tau}} + O(\tau^{-1/2}) \right] \\ &= \exp \left[-\frac{1}{2}\xi^2 (1 + O(\tau^{-1/2})) \right] \\ &= e^{-\frac{\xi^2}{2}}(1 + O(\tau^{-1/2})), \end{aligned}$$

as $\tau \rightarrow +\infty$, and using this in (28) the lemma is proved. \square

Lemma 4.2. $J_1(\xi, \tau) = \int_0^{+\infty} e^{-\xi w^n - \frac{w^{2n}}{2}} dw + O(e^{-\gamma\tau})$, as $\tau \rightarrow +\infty$, where $\gamma > 0$ only depends on n and ξ .

Proof. It is obvious that, as $\tau \rightarrow +\infty$,

$$J_1(\xi, \tau) = \int_0^{\tau^{1/2n}} e^{-\xi w^n - \frac{w^{2n}}{2}} dw \longrightarrow \int_0^{+\infty} e^{-\xi w^n - \frac{w^{2n}}{2}} dw.$$

The issue here is to estimate at what rate the limit is approached. Since the integrand is positive we have

$$(29) \quad \left| \int_0^{\tau^{1/2n}} e^{-\xi w^n - \frac{w^{2n}}{2}} dw - \int_0^{+\infty} e^{-\xi w^n - \frac{w^{2n}}{2}} dw \right| = \int_{\tau^{1/2n}}^{+\infty} e^{-\xi w^n - \frac{w^{2n}}{2}} dw.$$

Start by considering $\xi \geq 0$:

$$(30) \quad \begin{aligned} \int_{\tau^{1/2n}}^{+\infty} e^{-\xi w^n - \frac{w^{2n}}{2}} dw &< \int_{\tau^{1/2n}}^{+\infty} e^{-\frac{w^{2n}}{2}} dw = \frac{1}{2n} \int_{\tau}^{+\infty} z^{\frac{1}{2n}-1} e^{-z/2} dz \\ &< \frac{1}{2n} \int_{\tau}^{+\infty} e^{-z/2} dz = \frac{1}{n} e^{-\tau/2}. \end{aligned}$$

Now consider the case $\xi < 0$:

$$(31) \quad \begin{aligned} \int_{\tau^{1/2n}}^{+\infty} e^{-\xi w^n - \frac{w^{2n}}{2}} dw &= \int_{\tau^{1/2n}}^{+\infty} e^{|\xi| w^n - \frac{w^{2n}}{2}} dw = \int_{\tau^{1/2n}}^{+\infty} e^{-\frac{w^{2n}}{2} (1 - \frac{2|\xi|}{w^n})} dw \\ &< \frac{1}{2n\gamma(\xi)} e^{-\gamma(\xi)\tau}, \end{aligned}$$

where the last inequality is due to

$$1 - \frac{2|\xi|}{w^n} > 1 - \frac{2|\xi|}{\tau^{1/2}} > 1 - \frac{2|\xi|}{\tau_0^{1/2}} =: 2\gamma(\xi),$$

valid for all $\tau \in [\tau_0, +\infty)$, for every fixed $\tau_0 > 0$. Since $2\gamma(\xi) < 1$ we conclude that $\frac{1}{n} e^{-\tau/2} < \frac{1}{2n\gamma(\xi)} e^{-\gamma(\xi)\tau}$ and thus, from (30) and (31), we can estimate the right-hand side of (29) for all $\xi \in \mathbb{R}$ as follows:

$$(32) \quad \left| \int_0^{\tau^{1/2n}} e^{-\xi w^n - \frac{w^{2n}}{2}} dw - \int_0^{+\infty} e^{-\xi w^n - \frac{w^{2n}}{2}} dw \right| < \frac{1}{2n\gamma(\xi)} e^{-\gamma(\xi)\tau}.$$

(Observe that, due to (30), the convergence is uniform for $\xi \geq 0$.) \square

For the next lemma let us introduce the notation $\nu_0 := \sum_{k=n}^{\infty} c_k(0)$.

Lemma 4.3. *The following holds true for τ sufficiently large:*

$$(33) \quad \left| n\tau^{\frac{1}{2n}} J_2(\xi, \tau) + \nu_0 \left(\frac{n}{\alpha} \right)^{(n-1)/n} \right| \leq C\tau^{-\frac{1}{2} + \frac{1}{2n}} \log \tau, \quad \text{as } \tau \rightarrow \infty.$$

Proof. We estimate

$$J_2(\xi, \tau) = \int_0^{\tau^{1/2n}} f_n(w^n \sqrt{\tau}) e^{-\xi w^n - \frac{w^{2n}}{2}} dw.$$

By performing the change of variable $w = \frac{x^{1/n}}{\tau^{1/2n}}$, we obtain,

$$(34) \quad J_2(\xi, \tau) = \frac{1}{n\tau^{1/2n}} \int_0^{\tau} \frac{f_n(x)}{x^{1-1/n}} e^{-\xi \frac{x}{\sqrt{\tau}} - \frac{x^2}{2\tau}} dx = \frac{1}{n\tau^{1/2n}} (J_{2,1}(\xi, \tau) + J_{2,2}(\xi, \tau)),$$

where,

$$\begin{aligned} J_{2,1}(\xi, \tau) &= \int_0^\tau \frac{f_n(x)}{x^{1-1/n}} dx \\ J_{2,2}(\xi, \tau) &= \int_0^\tau \frac{f_n(x)}{x^{1-1/n}} (e^{-\xi \frac{x}{\sqrt{\tau}} - \frac{x^2}{2\tau}} - 1) dx. \end{aligned}$$

We start by estimating $J_{2,1}$. By (26),

$$\begin{aligned} J_{2,1}(\xi, \tau) &= \int_0^\tau \left[\left(\frac{n}{\alpha} \right)^{1-\frac{1}{n}} (\tilde{c}_1(x))^{n-1} - x^{\frac{1}{n}-1} \right] dx \\ (35) \quad &= \left(\frac{n}{\alpha} \right)^{1-\frac{1}{n}} \int_0^\tau (\tilde{c}_1(x))^{n-1} dx - n\tau^{\frac{1}{n}}. \end{aligned}$$

On the other hand, from (11) we have the equation,

$$\frac{d}{d\tau} \sum_{k=n}^{\infty} \tilde{c}_k(\tau) = (\tilde{c}_1(\tau))^{n-1},$$

from which we obtain, for each $\tau > 0$,

$$(36) \quad \int_0^\tau (\tilde{c}_1(x))^{n-1} dx = -\nu_0 + \sum_{k=n}^{\infty} \tilde{c}_k(\tau).$$

We next estimate the asymptotics of the sum in the right hand side of this equality. First, we observe that, from Theorem 2.1,

$$\tilde{c}_1(\tau) = \left(\frac{\alpha}{n\tau} \right)^{1/n} \left(1 + \left(1 - \frac{1}{n} \right) \frac{\log \tau}{\tau} + o\left(\frac{\log \tau}{\tau} \right) \right),$$

as $\tau \rightarrow +\infty$. Now note that, with the notation introduced previously for the bulk variable $\tilde{y}(\tau) = \sum_{k=n}^{\infty} \tilde{c}_k(\tau)$, the application of center manifold theory [6] allows us to conclude that the points $(\tilde{c}_1, \alpha - \tilde{c}_1 \tilde{y})$ lie on the graph of a function ϕ_n given in a neighborhood of the origin by equation (3.6) in [6], namely:

$$\phi_n(\tilde{c}_1) = n(\tilde{c}_1)^n - \frac{1}{\alpha}(\tilde{c}_1)^{n+2} + O((\tilde{c}_1)^{2n+2}) \quad \text{as } \tilde{c}_1 \rightarrow 0.$$

This entails that, as $\tau \rightarrow +\infty$,

$$\tilde{c}_1(\tau) \sum_{k=n}^{\infty} \tilde{c}_k(\tau) = \alpha - n(\tilde{c}_1(\tau))^n + \frac{1}{\alpha}(\tilde{c}_1(\tau))^{n+2} + O((\tilde{c}_1(\tau))^{2n+2}).$$

From this expression and the one for $\tilde{c}_1(\tau)$ above we deduce that, as $\tau \rightarrow \infty$,

$$\sum_{k=n}^{\infty} \tilde{c}_k(\tau) = \alpha \left(\frac{n}{\alpha} \right)^{1/n} \tau^{1/n} - \alpha \left(\frac{n}{\alpha} \right)^{1/n} \left(1 - \frac{1}{n} \right) \tau^{-1+1/n} \log \tau + O(\tau^{-1+1/n}).$$

Using this in (36) and substituting the result back into (35) we finally conclude that

$$\begin{aligned} J_{2,1}(\xi, \tau) &= -\nu_0 \left(\frac{n}{\alpha} \right)^{1-\frac{1}{n}} + \left(\frac{n}{\alpha} \right)^{1-\frac{1}{n}} \sum_{k=n}^{\infty} \tilde{c}_k(\tau) - n\tau^{\frac{1}{n}} \\ &= -\nu_0 \left(\frac{n}{\alpha} \right)^{(n-1)/n} - (n-1)\tau^{-1+\frac{1}{n}} \log \tau + O(\tau^{-1+1/n}). \end{aligned}$$

Now it remains to estimate

$$J_{2,2}(\xi, \tau) = \int_0^\tau \frac{f_n(x)}{x^{1-1/n}} \left(e^{-\xi \frac{x}{\sqrt{\tau}} - \frac{x^2}{2\tau}} - 1 \right) dx.$$

Let us start by observing that $1 - e^{-s} \leq \min\{1, s\}$. To apply this to the exponential in $J_{2,2}$ observe that $s_+ := -\xi + \sqrt{\xi^2 + 2}$ is the value of $s = \frac{x}{\sqrt{\tau}} > 0$ for which $\xi s + \frac{1}{2}s^2 = 1$. Thus, we can start by estimating $J_{2,2}$ as follows:

$$(37) \quad |J_{2,2}(\xi, \tau)| \leq \int_0^{\sqrt{\tau}s_+} \left| \frac{f_n(x)}{x^{1-1/n}} \right| \left(\left| \xi \right| \frac{x}{\sqrt{\tau}} + \frac{x^2}{2\tau} \right) dx + \int_{\sqrt{\tau}s_+}^\tau \left| \frac{f_n(x)}{x^{1-1/n}} \right| dx.$$

Now, since in the second integral in (37) we have $x > \sqrt{\tau}s_+$, if τ is sufficiently large we can substitute the asymptotic form of $f_n(x)$ obtained by using (12) in (26), in order to write

$$(38) \quad \begin{aligned} \int_{\sqrt{\tau}s_+}^\tau \left| \frac{f_n(x)}{x^{1-1/n}} \right| dx &= \int_{\sqrt{\tau}s_+}^\tau \left((n-1) \left(1 - \frac{1}{n}\right) \frac{\log x}{x^{2-1/n}} + o\left(\frac{\log x}{x^{2-1/n}}\right) \right) dx \\ &\leq C\tau^{-\frac{1}{2} + \frac{1}{2n}} \int_1^{\sqrt{\tau}/s_+} \frac{\log u}{u^{2-1/n}} du, \end{aligned}$$

where C is a constant depending on n and ξ (through s_+) and from now on can vary from line to line. Using the monotonicity of the logarithm to write $\log u < \log \sqrt{\tau}/s_+$ we can continue the estimate above as

$$(39) \quad \begin{aligned} &\leq C\tau^{-\frac{1}{2} + \frac{1}{2n}} \log \tau \int_1^{\sqrt{\tau}/s_+} \frac{1}{u^{2-1/n}} du \\ &\leq C\tau^{-\frac{1}{2} + \frac{1}{2n}} \log \tau. \end{aligned}$$

Now let us consider the first integral in (37). It is convenient to write it as an integral between fixed limits:

$$(40) \quad \begin{aligned} &\int_0^{\sqrt{\tau}s_+} \left| \frac{f_n(x)}{x^{1-1/n}} \right| \left(\left| \xi \right| \frac{x}{\sqrt{\tau}} + \frac{x^2}{2\tau} \right) dx \\ &= \int_0^1 \left| \frac{f_n(u\sqrt{\tau}s_+)}{(u\sqrt{\tau}s_+)^{1-1/n}} \right| \left(\left| \xi \right| s_+ u + \frac{1}{2} s_+^2 u^2 \right) \sqrt{\tau} s_+ du. \end{aligned}$$

In order to apply the asymptotic form of f_n , as in (38) we need to decompose the integral in (40) into a sum of two integrals such that in one we can apply the result about the asymptotic behaviour of f_n and the other can be adequately controlled.

Fix $\tilde{\varepsilon}, \delta > 0$. In (40) write $\int_0^1 = \int_0^{\delta/\tau^{-\tilde{\varepsilon}+1/2}} + \int_{\delta/\tau^{-\tilde{\varepsilon}+1/2}}^1$. Then we have

$$(41) \quad \begin{aligned} &\int_0^{\delta/\tau^{-\tilde{\varepsilon}+1/2}} \left| \frac{f_n(u\sqrt{\tau}s_+)}{(u\sqrt{\tau}s_+)^{1-1/n}} \right| \left(\left| \xi \right| s_+ u + \frac{1}{2} s_+^2 u^2 \right) \sqrt{\tau} s_+ du \\ &= (\sqrt{\tau}s_+)^{\frac{1}{n}} \int_0^{\delta/\tau^{-\tilde{\varepsilon}+1/2}} |f_n(u\sqrt{\tau}s_+)| \left(\left| \xi \right| s_+ + \frac{1}{2} s_+^2 u \right) u^{\frac{1}{n}} du \\ &\leq C\tau^{\frac{1}{2n}} \int_0^{\delta/\tau^{-\tilde{\varepsilon}+1/2}} \left(\left| \xi \right| s_+ u^{\frac{1}{n}} + \frac{1}{2} s_+^2 u^{1+\frac{1}{n}} \right) du \end{aligned}$$

$$(42) \quad \leq C\tau^{-\frac{1}{2} + \varepsilon},$$

with $\varepsilon = (1 + \frac{1}{n})\tilde{\varepsilon}$, where (41) is obtained by using the boundedness of f_n , due to its definition (26), and (42) is valid since we have τ with $\frac{\delta}{\tau - \tilde{\varepsilon} + 1/2} < 1$, otherwise the decomposition into two integrals would not make sense. Furthermore, again denoting by the same letter C constants that can be different, and again using (26) and (12) we have

$$\begin{aligned}
 & \int_{\delta/\tau - \tilde{\varepsilon} + 1/2}^1 \left| \frac{f_n(u\sqrt{\tau}s_+)}{(u\sqrt{\tau}s_+)^{1-1/n}} \right| \left(|\xi|s_+u + \frac{1}{2}s_+^2u^2 \right) \sqrt{\tau}s_+ du \\
 & \leq C \int_{\delta/\tau - \tilde{\varepsilon} + 1/2}^1 \left(\frac{|\log(u\sqrt{\tau}s_+)|}{(u\sqrt{\tau}s_+)^{2-\frac{1}{n}}} \right) \left(|\xi|s_+u + \frac{1}{2}s_+^2u^2 \right) \sqrt{\tau}s_+ du \\
 (43) \quad & \leq C\tau^{-\frac{1}{2} + \frac{1}{2n}} \int_{\delta/\tau - \tilde{\varepsilon} + 1/2}^1 \frac{|\log u|}{u^{2-\frac{1}{n}}} \left(|\xi|s_+u + \frac{1}{2}s_+^2u^2 \right) du
 \end{aligned}$$

$$\begin{aligned}
 (44) \quad & + C\tau^{-\frac{1}{2} + \frac{1}{2n}} \log \tau \int_{\delta/\tau - \tilde{\varepsilon} + 1/2}^1 \frac{1}{u^{2-\frac{1}{n}}} \left(|\xi|s_+u + \frac{1}{2}s_+^2u^2 \right) du.
 \end{aligned}$$

Since both integrals

$$\int_0^1 \frac{|\log u|}{u^{2-\frac{1}{n}}} \left(|\xi|s_+u + \frac{1}{2}s_+^2u^2 \right) du \quad \text{and} \quad \int_0^1 \frac{1}{u^{2-\frac{1}{n}}} \left(|\xi|s_+u + \frac{1}{2}s_+^2u^2 \right) du$$

are convergent, we conclude from (43) and (44) that for τ sufficiently large we have

$$(45) \quad \int_{\delta/\tau - \tilde{\varepsilon} + 1/2}^1 \left| \frac{f_n(u\sqrt{\tau}s_+)}{(u\sqrt{\tau}s_+)^{1-1/n}} \right| \left(|\xi|s_+u + \frac{1}{2}s_+^2u^2 \right) \sqrt{\tau}s_+ du \leq C\tau^{-\frac{1}{2} + \frac{1}{2n}} \log \tau.$$

Finally, from (39), (42), and (45) we conclude that, for τ sufficiently large, we have

$$|J_{2,2}(\xi, \tau)| \leq C\tau^{-\frac{1}{2} + \frac{1}{2n}} \log \tau,$$

and thus we conclude the estimate of the behaviour of J_2 proceeding from (34). \square

Lemma 4.4. $J_3(\xi, \tau) = O(\tau^{-1/2})$, as $\tau \rightarrow +\infty$.

Proof. We perform the change of variable $w \mapsto x = w^n$, and call $\theta = \sqrt{\tau}$, so that,

$$(46) \quad J_3(\xi, \tau) = \int_0^\theta \frac{\hat{g}(x, \theta, \xi)}{nx^{1-1/n}} e^{-\xi x - \frac{x^2}{2}} dx,$$

where,

$$\begin{aligned}
 \hat{g}(x, \theta, \xi) &= -1 + \left(1 - \frac{x}{\theta}\right)^{\theta^2 + \xi\theta - n} \exp\left(x\theta + \xi x + \frac{x^2}{2}\right) \\
 &= -1 + \exp\left((\theta^2 + \xi\theta - n) \log\left(1 - \frac{x}{\theta}\right) + x\theta + \xi x + \frac{x^2}{2}\right).
 \end{aligned}$$

It is straightforward to obtain the following three MacLaurin expansions, convergent when $0 \leq \frac{x}{\theta} < 1$:

$$\begin{aligned}\theta^2 \log \left(1 - \frac{x}{\theta}\right) + x\theta + \frac{x^2}{2} &= -\frac{x^3}{\theta} \sum_{k=0}^{\infty} \frac{1}{k+3} \left(\frac{x}{\theta}\right)^k, \\ \xi\theta \log \left(1 - \frac{x}{\theta}\right) + \xi x &= -\frac{\xi x^2}{\theta} \sum_{k=0}^{\infty} \frac{1}{k+2} \left(\frac{x}{\theta}\right)^k, \\ -n \log \left(1 - \frac{x}{\theta}\right) &= \frac{nx}{\theta} \sum_{k=0}^{\infty} \frac{1}{k+1} \left(\frac{x}{\theta}\right)^k.\end{aligned}$$

Thus, by collecting equal powers of x/θ in the above series, we obtain, for $0 \leq x < \theta$,

$$(47) \quad (\theta^2 + \xi\theta - n) \log \left(1 - \frac{x}{\theta}\right) + x\theta + \xi x + \frac{x^2}{2} = -\frac{x}{\theta} \sum_{k=0}^{\infty} p_k(x, \xi) \left(\frac{x}{\theta}\right)^k,$$

where

$$p_k(x, \xi) := \frac{x^2}{k+3} + \frac{\xi x}{k+2} - \frac{n}{k+1}.$$

For $k = 1, 2, \dots$, let $x_k(\xi)$ be the largest of the two zeros of $p_k(\cdot, \xi)$, that is,

$$x_k(\xi) = -\frac{k+3}{2(k+2)}\xi + \sqrt{\frac{(k+3)^2}{4(k+2)^2}\xi^2 + \frac{k+3}{k+1}n},$$

so that, for $x \geq x_k(\xi)$, we have, $p_k(x, \xi) \geq 0$. By observing that, as $k \rightarrow \infty$, $x_k(\xi) \rightarrow -\frac{\xi}{2} + \sqrt{\frac{\xi^2}{4} + n}$, we can conclude that $x_*(\xi) := \sup x_k(\xi) < +\infty$, and hence,

$$x \geq x_*(\xi) \quad \Rightarrow \quad \forall k \in \mathbb{N}, p_k(x, \xi) \geq 0.$$

Therefore, regardless of the sign of ξ ,

$$x_*(\xi) \leq x < \theta \quad \Rightarrow \quad (\theta^2 + \xi\theta - n) \log \left(1 - \frac{x}{\theta}\right) + x\theta + \xi x + \frac{x^2}{2} \leq 0,$$

which in turn implies that,

$$(48) \quad x_*(\xi) \leq x < \theta \quad \Rightarrow \quad |\hat{g}(x, \theta, \xi)| \leq 1.$$

Now we deal with the exponential term in (46). If $\xi \geq 0$, then obviously, for $x \geq 0$, $e^{-\xi x - \frac{x^2}{2}} \leq e^{-\frac{x^2}{2}}$. If $\xi < 0$, fix any $\omega \in]0, 1[$, and define $x_{**} := \frac{2|\xi|}{1-\omega}$. It is straightforward to check that,

$$(49) \quad x \geq x_{**} \quad \Rightarrow \quad e^{-\xi x - \frac{x^2}{2}} \leq e^{-\omega \frac{x^2}{2}}.$$

If $\xi \geq 0$ we take $x_{**} = 0$ and $\omega = 1$. Now, pick some $\eta \in]0, 1[$, and consider the splitting $J_3 = J_{3,1} + J_{3,2}$, where,

$$J_{3,1} := \int_0^{\eta\theta} \frac{\hat{g}(x, \theta, \xi)}{nx^{1-1/n}} e^{-\xi x - \frac{x^2}{2}} dx \quad \text{and} \quad J_{3,2} := \int_{\eta\theta}^{\theta} \frac{\hat{g}(x, \theta, \xi)}{nx^{1-1/n}} e^{-\xi x - \frac{x^2}{2}} dx.$$

Consider that $\theta \geq \frac{1}{\eta} \max(x_*, x_{**})$. Due to (48) and (49),

$$|J_{3,2}| \leq \int_{\eta\theta}^{\theta} \frac{1}{nx^{1-1/n}} e^{-\omega \frac{x^2}{2}} dx \leq (1 - \eta^{1/n}) \theta^{1/n} e^{-\frac{\omega \eta^2}{2} \theta^2},$$

and thus, if $\omega_* \in]0, \frac{\omega n^2}{2}[$, we have

$$(50) \quad |J_{3,2}| = o(e^{-\omega_* \theta^2}), \quad \text{as } \theta \rightarrow +\infty.$$

Now we estimate $|J_{3,1}|$. Let q be the left-hand side of (47), so that, $|\hat{g}| = |1 - e^q|$. Suppose first that $q \leq 0$. Then, $|\hat{g}| \leq -q$, which implies, by taking (47) in account,

$$|\hat{g}(x, \theta, \xi)| \leq \frac{x}{\theta} \left(\frac{x^2}{3} + \frac{|\xi|x}{2} \right) \frac{1}{1 - \frac{x}{\theta}} \leq C_1(x + x^2) \frac{x}{\theta},$$

for all $0 \leq x \leq \eta\theta$, for some positive constant C_1 only depending on ξ and η .

Now, suppose that $q > 0$. We have to find lower estimates for each polynomial $p_k(\cdot, \xi)$. If $\xi \geq 0$, then the minimum of $p_k(x, \cdot, \xi)$ for $x \in [0, \theta]$ is $p_k(0, \xi) = -\frac{n}{k+1}$. Then,

$$q \leq \frac{x}{\theta} \sum_{k=0}^{\infty} \frac{n}{k+1} \left(\frac{x}{\theta} \right)^k \leq C_2 \frac{x}{\theta},$$

for all $0 \leq x \leq \eta\theta$, and for some positive constant C_2 only depending on n and η . If $\xi < 0$, then the minimum of $p_k(x, \xi)$ for $x \in [0, \theta]$ is $m_k := p_k\left(\frac{k+3}{2k+4}|\xi|, \xi\right) = -\frac{k+3}{k(k+2)^2}\xi^2 - \frac{1}{k+1}n$. Then,

$$q \leq \frac{x}{\theta} \sum_{k=0}^{\infty} |m_k| \left(\frac{x}{\theta} \right)^k \leq C_3 \frac{x}{\theta},$$

for all $0 \leq x \leq \eta\theta$, for some positive constant C_3 , only depending on n, ξ and η .

For both cases (i) and (ii) we thus obtain, with $C = C_j$, $j = 2, 3$, accordingly,

$$|\hat{g}(x, \theta, \xi)| = e^q - 1 \leq \exp\left(\frac{Cx}{\theta}\right) - 1 = \frac{Cx}{\theta} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left(\frac{Cx}{\theta}\right)^k \leq C_4 \frac{x}{\theta},$$

for all $0 \leq x \leq \eta\theta$, for some positive constant C_4 , only depending on n, ξ and η .

Taking in account all the previous estimates, we conclude that, regardless the sign of ξ , there is a positive constant C_5 only depending on n, ξ and η such that, for all $0 \leq x \leq \eta\theta$,

$$|\hat{g}(x, \theta, \xi)| \leq \frac{C_5}{\theta}(x + x^2 + x^3).$$

Therefore,

$$(51) \quad \begin{aligned} |J_{3,1}| &\leq \frac{C_5}{n\theta} \int_0^{\eta\theta} (x^{1/n} + x^{1+1/n} + x^{2+1/n}) e^{-\xi x - \frac{x^2}{2}} dx \\ &\leq \frac{C_5}{n\theta} \int_0^{+\infty} (x^{1/n} + x^{1+1/n} + x^{2+1/n}) e^{-\xi x - \frac{x^2}{2}} dx \\ &\leq \frac{M}{\theta}, \end{aligned}$$

where $M > 0$ only depends on ξ, n and η . By taking together (50) and (51), we obtain $|J_3| = O(\theta^{-1})$, as $\theta \rightarrow +\infty$, and considering the definition of θ , the lemma is proved. \square

Lemma 4.5. $J_4(\xi, \tau) = O(\tau^{-1/2})$, as $\tau \rightarrow +\infty$.

Proof. By the continuity of f_n in $[0, +\infty[$ and Theorem 2.1, we know that f_n is bounded so that there is $M > 0$, which is independent of w and τ such that, for all $w \geq 0$ and $\tau \geq 0$, $|f_n(w^n \sqrt{\tau})g(w, \tau, \xi)| \leq M|g(w, \tau, \xi)|$, and therefore we can take the estimates from the proof of Lemma 4.4 to obtain the result. \square

Theorem 4.6. *With the notations and hypothesis introduced above in this paper we have for τ sufficiently large that*

$$\left| \tau^{1/2n} (\varphi(\tau + \xi \sqrt{\tau}, \tau) - \Phi_{2,n}(\xi)) + e^{-\xi^2/2} \left(\frac{\alpha}{n} \right)^{1/n} \frac{\nu_0}{\alpha} \right| \leq C \tau^{-\frac{1}{2} + \frac{1}{2n}} \log \tau.$$

Proof. The result follows directly from the lemmata 4.2–4.5. \square

If there are no clusters initially present on the crystal facet, $c_j(0) = 0$ for $j \geq n$, and thus $\nu_0 = 0$. In this case, since we know that $\tau = j\Delta_j$ with $\Delta_j = 1 + o(1)$ as $j \rightarrow \infty$ (see section 3), we immediately conclude that Theorem 4.6 reduces to Theorem 1.2.

5. CONCLUSION

We start this section by proving Theorem 1.1.

Proof of Theorem 1.1. In order to prove Theorem 1.1 it is enough to put together the result given in Theorem 4.6 with the estimate (22) established in section 3 for the contribution of the initial condition. Under the assumption of Theorem 1.1 that the initial condition $c_j(0)$ decays like $j^{-\mu}$, we conclude that the slowest decaying term arising from φ decays like $j^{-\frac{1}{2n}}$, while by (22) we see that the contribution coming from the initial data is in all cases never faster than this decay. Thus, the contribution coming from the initial condition is the dominant one for the decay rate, which proves Theorem 1.1. \square

We end by observing that Theorem 1.1 implies that when the tail of the initial condition decays sufficiently slow at infinity, with a rate $\mu \in (1 - \frac{1}{n}, 1)$, the rate of convergence to the similarity profile of the solution of the deposition model (1) will depend on μ . This lack of forgetfulness of the rate of convergence to the scaling profiles regarding some features of the initial condition was also observed recently in [6] for this deposition model along noncritical directions $\eta \neq 1$. So, although the similarity profile itself lacks any information about the initial condition, the rate at which it is approached by the solutions of (1) can still convey some information. A final word about the lower bound $1 - \frac{1}{n}$ on the allowed values of μ . This lower bound is clearly needed given the technique we use in the estimates of the contribution of the initial condition in section 3. A careful look at those estimates suggests that, if the initial condition is given exactly by $c_j(0) = \rho j^{-\mu}$, the right-hand side of (19) is likely to be the leading order contribution to S_3 (at least for some values of ξ) and thus, if μ is not bigger than $1 - \frac{1}{n}$, the limit (13) would not be equal to zero, which means that the solution to (1) would not approach the similarity profile $\Phi_{2,n}$. It will be interesting to explore in more detail these issues in the future and to investigate what occurs in this case.

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