

# Semigroups of Transformations Preserving an Equivalence Relation and a Cross-Section

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## Abstract

For a set  $X$ , an equivalence relation  $\rho$  on  $X$ , and a cross-section  $R$  of the partition  $X/\rho$  induced by  $\rho$ , consider the semigroup  $T(X, \rho, R)$  consisting of all mappings  $a$  from  $X$  to  $X$  such that  $a$  preserves both  $\rho$  (if  $(x, y) \in \rho$  then  $(xa, ya) \in \rho$ ) and  $R$  (if  $r \in R$  then  $ra \in R$ ). The semigroup  $T(X, \rho, R)$  is the centralizer of the idempotent transformation with kernel  $\rho$  and image  $R$ . We determine the structure of  $T(X, \rho, R)$  in terms of Green's relations, describe the regular elements of  $T(X, \rho, R)$ , and determine the following classes of the semigroups  $T(X, \rho, R)$ : regular, abundant, inverse, and completely regular.

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## 1 Introduction

Let  $X$  be an arbitrary nonempty set. The semigroup  $T(X)$  of full transformations on  $X$  consists of the mappings from  $X$  to  $X$  with composition as the semigroup operation.

Let  $\rho$  be an equivalence relation on  $X$  and let  $R$  be a cross-section of the partition  $X/\rho$  induced by  $\rho$ . Consider the following subset of  $T(X)$ :

$$T(X, \rho, R) = \{a \in T(X) : Ra \subseteq R \text{ and } (x, y) \in \rho \Rightarrow (xa, ya) \in \rho\}.$$

Clearly  $T(X, \rho, R)$  is a subsemigroup of  $T(X)$ . The family of semigroups  $T(X, \rho, R)$  includes the semigroup  $T(X)$  ( $T(X) = T(X, \Delta, X)$  where  $\Delta = \{(x, x) : x \in X\}$ ) and the semigroup  $PT(X')$  of partial transformations on  $X'$  where  $X'$  is  $X$  with one element removed (if  $X' = X - \{r\}$  then  $PT(X')$  is isomorphic to  $T(X, X \times X, \{r\})$ ).

Another way of describing the semigroups  $T(X, \rho, R)$  is through the notion of the centralizer. Let  $S$  be a semigroup and  $a \in S$ . The *centralizer*  $C(a)$  of  $a$  is defined as

$$C(a) = \{b \in S : ab = ba\}.$$

It is clear that  $C(a)$  is a subsemigroup of  $S$ .

The full transformation semigroup  $T(X)$  is the centralizer of the identity mapping  $id_X$  on  $X$ :  $T(X) = C(id_X)$ . More generally, the semigroups  $T(X, \rho, R)$  are the centralizers of the idempotent transformations:  $T(X, \rho, R)$  is the centralizer of the idempotent in  $T(X)$  with kernel  $\rho$  and image  $R$  [1].

Centralizers in  $T(X)$  for a finite set  $X$  have been studied by Higgins [4], Liskovec and Feĭnberg [10], [11], and Weaver [14]. The second author has studied centralizers in the semigroup  $PT(X)$  of partial transformations on a finite set  $X$  [6], [7], [8]. In [1], the authors determined the automorphism group of  $T(X, \rho, R)$ .

In this paper, we study the structure and regularity of the semigroups  $T(X, \rho, R)$  for an arbitrary set  $X$ . In Section 2, we determine Green's relations in  $T(X, \rho, R)$ . In particular, we find that, in general, the relations  $\mathcal{D}$  and  $\mathcal{J}$  are not the same in  $T(X, \rho, R)$ , and that the  $\mathcal{J}$ -classes of  $T(X, \rho, R)$  do not form a chain. We characterize the relations  $\rho$  for which  $\mathcal{D} = \mathcal{J}$  and the relations  $\rho$  for which the partially ordered set of  $\mathcal{J}$ -classes is a chain. In Section 3, we describe the regular elements of  $T(X, \rho, R)$  and characterize the relations  $\rho$  for which  $T(X, \rho, R)$  is a regular semigroup. In Section 4, we show that abundant semigroups  $T(X, \rho, R)$  are precisely those that are regular. Finally, in Section 5, we determine that  $T(X, \rho, R)$  is never an inverse semigroup (if  $|X| \geq 3$ ) or a completely regular semigroup (if  $|X| \geq 4$ ).

## 2 Green's Relations in $T(X, \rho, R)$

For  $a \in T(X)$ , we denote the kernel of  $a$  (the equivalence relation  $\{(x, y) \in X \times X : xa = ya\}$ ) by  $\text{Ker}(a)$  and the image of  $a$  by  $\nabla a$ . For  $Y \subseteq X$ ,  $Ya$  will denote the image of  $Y$  under  $a$ , that is,  $Ya = \{xa : x \in Y\}$ . As customary in transformation semigroup theory, we write transformations on the right (that is,  $xa$  instead of  $a(x)$ ).

Let  $\rho$  be an equivalence relation on  $X$  and  $R$  a cross-section of  $X/\rho$ . If  $x \in X$  then there is exactly one  $r \in R$  such that  $x\rho r$ , which will be denoted by  $r_x$ . Of course, for  $s \in R$ , we have  $r_s = s$ .

For the remainder of the paper,  $\rho$  will denote an equivalence relation on  $X$  and  $R$  will denote a cross-section of  $X/\rho$ .

If  $S$  is a semigroup and  $a, b \in S$ , we say that  $a \mathcal{R} b$  if  $aS^1 = bS^1$ ,  $a \mathcal{L} b$  if  $S^1a = S^1b$ , and  $a \mathcal{J} b$  if  $S^1aS^1 = S^1bS^1$ , where  $S^1$  is the semigroup  $S$  with an identity adjoined, if necessary. We define  $\mathcal{H}$  as the intersection of  $\mathcal{L}$  and  $\mathcal{R}$ , and  $\mathcal{D}$  as the join of  $\mathcal{L}$  and  $\mathcal{R}$ , that is, the smallest equivalence relation on  $S$  containing both  $\mathcal{L}$  and  $\mathcal{R}$ . These five equivalence relations on  $S$  are known as *Green's relations* [5, p. 45]. The relations  $\mathcal{L}$  and  $\mathcal{R}$  commute [5, Proposition 2.1.3], and consequently  $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ . If  $\mathcal{T}$  is one of Green's relations and  $a \in S$ , we denote the equivalence class of  $a$  with respect to  $\mathcal{T}$  by  $T_a$ . Since  $\mathcal{R}$ ,  $\mathcal{L}$ , and  $\mathcal{J}$  are defined in terms of principal ideals in  $S$ , which are partially ordered by inclusion, we have the induced partial orders in the sets of the equivalence classes of  $\mathcal{R}$ ,  $\mathcal{L}$ , and  $\mathcal{J}$ :  $R_a \leq R_b$  if  $aS^1 \subseteq bS^1$ ,  $L_a \leq L_b$  if  $S^1a \subseteq S^1b$ , and  $J_a \leq J_b$  if  $S^1aS^1 \subseteq S^1bS^1$ .

Green's relations in the semigroup  $T(X)$  are well known [5, Exercise 16, p. 63].

**Lemma 2.1** *If  $a, b \in T(X)$  then:*

- (1)  $a \mathcal{R} b \Leftrightarrow \text{Ker}(a) = \text{Ker}(b)$ .

- (2)  $a \mathcal{L} b \Leftrightarrow \nabla a = \nabla b$ .
- (3)  $a \mathcal{D} b \Leftrightarrow |\nabla a| = |\nabla b|$ .
- (4)  $\mathcal{D} = \mathcal{J}$ .

Our aim in this section is to describe Green's relations in the semigroups  $T(X, \rho, R)$ .

## 2.1 Relations $\mathcal{R}$ and $\mathcal{L}$

The relation  $\mathcal{R}$  in  $T(X, \rho, R)$  is simply the restriction of the relation  $\mathcal{R}$  in  $T(X)$  to  $T(X, \rho, R) \times T(X, \rho, R)$ . This result will follow from the following lemma.

**Lemma 2.2** *Let  $a, b \in T(X, \rho, R)$ . Then  $R_a \leq R_b$  if and only if  $\text{Ker}(b) \subseteq \text{Ker}(a)$ .*

**Proof:** Suppose  $R_a \leq R_b$ . Then there is  $c \in T(X, \rho, R)$  such that  $a = bc$ , and so for all  $x, y \in X$ ,  $xb = yb$  implies  $xa = (xb)c = (yb)c = ya$ . Thus  $\text{Ker}(b) \subseteq \text{Ker}(a)$ .

Conversely, suppose  $\text{Ker}(b) \subseteq \text{Ker}(a)$ . We shall construct  $c \in T(X, \rho, R)$  such that  $a = bc$ . Consider an equivalence class  $r\rho$  where  $r \in R$ . If  $r \notin \nabla b$ , define  $yc = y$  for every  $y \in r\rho$ . Suppose  $r \in \nabla b$ . Then, since  $b \in T(X, \rho, R)$ ,  $r = tb$  for some  $t \in R$ . Since  $a \in T(X, \rho, R)$ ,  $ta = p$  for some  $p \in R$ . Let  $y \in r\rho$ . If  $y = xb \in \nabla b$ , define  $yc = xa$ ; if  $y \notin \nabla b$ , define  $yc = p$ . Note that  $c$  is well defined since for all  $x, x' \in X$ , if  $xb = x'b$  then  $xa = x'a$  (since  $\text{Ker}(b) \subseteq \text{Ker}(a)$ ) and so  $(xb)c = xa = x'a = (x'b)c$ . It is clear by the construction of  $c$  that  $bc = a$ . It remains to show that  $c \in T(X, \rho, R)$ .

If  $r \notin \nabla b$  then  $rc = r \in R$  and  $(r\rho)c = r\rho$ . Suppose  $r \in \nabla b$ . By the definition of  $c$ ,  $rc = (tb)c = ta = p \in R$ . Next we show that  $(r\rho)c \subseteq p\rho$ . Let  $y \in r\rho$ . If  $y \notin \nabla b$  then  $yc = p \in p\rho$ , and so  $(r\rho)c \subseteq p\rho$ , in this case. Let  $y = xb \in \nabla b$ . Then  $x \in q\rho$  for some  $q \in R$ . Since  $x \in q\rho$  and  $xb \in r\rho$ ,  $qb = r$ . Since  $\text{Ker}(b) \subseteq \text{Ker}(a)$ ,  $tb = qb (= r)$  implies  $ta = qa$ . Thus  $qa = ta = p$ , and so  $(q\rho)a \subseteq p\rho$ . Hence  $yc = xa \in p\rho$ . It follows that  $c \in T(X, \rho, R)$ , and so  $R_a \leq R_b$ . ■

**Theorem 2.3** *Let  $a, b \in T(X, \rho, R)$ . Then  $a \mathcal{R} b$  if and only if  $\text{Ker}(a) = \text{Ker}(b)$ .*

**Proof:** It follows immediately from Lemma 2.2. ■

Let  $\mathcal{A}$  and  $\mathcal{B}$  be families of sets. We write  $\mathcal{A} \hookrightarrow \mathcal{B}$  if for every set  $C \in \mathcal{A}$  there is a set  $D \in \mathcal{B}$  such that  $C \subseteq D$ . If  $\mathcal{A} \hookrightarrow \mathcal{B}$  and  $\mathcal{B} \hookrightarrow \mathcal{A}$ , we write  $\mathcal{A} \leftrightarrow \mathcal{B}$ .

Our characterization of the relation  $\mathcal{L}$  in  $T(X, \rho, R)$  will follow from the following lemma. For  $a \in T(X, \rho, R)$ , we denote by  $\blacktriangledown a$  the family  $\{(r\rho)a : r \in R\}$ .

**Lemma 2.4** *Let  $a, b \in T(X, \rho, R)$ . Then  $L_a \leq L_b$  if and only if  $\blacktriangledown a \hookrightarrow \blacktriangledown b$ .*

**Proof:** Suppose  $L_a \leq L_b$ . Then there is  $c \in T(X, \rho, R)$  such that  $a = cb$ . Let  $A \in \blacktriangledown a$ . Then  $A = (r\rho)a = ((r\rho)c)b$  for some  $r \in R$ . Since  $c \in T(X, \rho, R)$ ,  $(r\rho)c \subseteq t\rho$  for some  $t \in R$ . Thus  $A \subseteq (t\rho)b \in \blacktriangledown b$ , and so  $\blacktriangledown a \hookrightarrow \blacktriangledown b$ .

Conversely, suppose  $\blacktriangledown a \hookrightarrow \blacktriangledown b$ . To construct  $c \in T(X, \rho, R)$  such that  $a = cb$ , consider  $r\rho$  ( $r \in R$ ). Since  $\blacktriangledown a \hookrightarrow \blacktriangledown b$  and  $b \in T(X, \rho, R)$ ,  $(r\rho)a \subseteq (t\rho)b \subseteq p\rho$  for some  $t, p \in R$ . Thus, for every  $x \in r\rho$ , we can select  $y_x \in t\rho$  such that  $xa = y_x b$  (if  $x = r$ , we may assume that  $y_x = t$  since  $ra = tb = p$ ) and define  $xc = y_x$ . By the construction of  $c$ ,  $a = cb$  and  $c \in T(X, \rho, R)$  (since  $(r\rho)c \subseteq t\rho$  and  $rc = t$ ). Thus  $L_a \leq L_b$ . ■

**Theorem 2.5** *Let  $a, b \in T(X, \rho, R)$ . Then  $a \mathcal{L} b$  if and only if  $\nabla a \leftrightarrow \nabla b$ .*

**Proof:** It follows immediately from Lemma 2.4. ■

For  $a \in T(X, \rho, R)$ , denote by  $m(\nabla a)$  the family of all sets maximal in  $\nabla a$  (with respect to inclusion). Suppose  $X$  is finite. Then for every  $A \in \nabla a$ , there is  $A' \in m(\nabla a)$  such that  $A \subseteq A'$ . (This is not necessarily true if  $X$  is infinite.) It easily follows that for all  $a, b \in T(X, \rho, R)$ ,  $\nabla a \leftrightarrow \nabla b$  if and only if  $m(\nabla a) = m(\nabla b)$ . Thus in the finite case,  $a \mathcal{L} b$  if and only if  $m(\nabla a) = m(\nabla b)$  [7].

## 2.2 Relations $\mathcal{D}$ and $\mathcal{J}$

Let  $f : Y \rightarrow Z$  be a function from a set  $Y$  to a set  $Z$ . For a family  $\mathcal{A}$  of subsets of  $Y$ ,  $f(\mathcal{A})$  denotes the family  $\{f(A) : A \in \mathcal{A}\}$  of subsets of  $Z$ . The following theorem characterizes Green's  $\mathcal{D}$ -relation in  $T(X, \rho, R)$ .

**Theorem 2.6** *Let  $a, b \in T(X, \rho, R)$ . Then  $a \mathcal{D} b$  if and only if there is a bijection  $f : \nabla a \rightarrow \nabla b$  such that  $f(R \cap \nabla a) \subseteq R$  and  $f(\nabla a) \leftrightarrow \nabla b$ .*

**Proof:** Suppose  $a \mathcal{D} b$ . Since  $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$ , there is  $c \in T(X, \rho, R)$  such that  $a \mathcal{R} c$  and  $c \mathcal{L} b$ . Then, by Theorem 2.3 and Theorem 2.5,  $\text{Ker}(a) = \text{Ker}(c)$  and  $\nabla c \leftrightarrow \nabla b$ .

Next we shall construct a bijection  $f : \nabla a \rightarrow \nabla c$  such that  $f(R \cap \nabla a) \subseteq R$  and  $f(\nabla a) = \nabla c$ . For every  $xa \in \nabla a$ , define  $f(xa) = xc$ . For all  $xa, x'a \in \nabla a$ ,  $f(xa) = f(x'a) \Leftrightarrow xc = x'c \Leftrightarrow xa = x'a$  (since  $\text{Ker}(a) = \text{Ker}(c)$ ). Thus  $f$  is well defined and one-to-one. It is obviously onto since for every  $xc \in \nabla c$ ,  $xc = f(xa)$ . Suppose  $r \in R \cap \nabla a$ . Then there is  $t \in R$  such that  $r = ta$ , and so  $f(r) = f(ta) = tc \in R$ . Thus  $f(R \cap \nabla a) \subseteq R$ . For every  $(r\rho)a \in \nabla a$  ( $r \in R$ ),  $f((r\rho)a) = \{f(xa) : x \in r\rho\} = \{xc : x \in r\rho\} = (r\rho)c$ . It follows that  $f(\nabla a) = \nabla c$ , and so  $f(\nabla a) \leftrightarrow \nabla b$ .

Conversely, suppose there is a bijection  $f : \nabla a \rightarrow \nabla b$  such that  $f(R \cap \nabla a) \subseteq R$  and  $f(\nabla a) \leftrightarrow \nabla b$ . Define  $c \in T(X)$  by  $xc = f(xa)$ . Let  $r \in R$ . Then  $ra \in R$  and so  $rc = f(ra) \in R$ . Thus  $Rc \subseteq R$ . Moreover,  $(r\rho)c = f((r\rho)a) \in f(\nabla a)$ . Since  $f(\nabla a) \leftrightarrow \nabla b$ , there is  $B \in \nabla b$  such that  $(r\rho)c \subseteq B$ . Since  $B \in \nabla b$ ,  $B \subseteq t\rho$  for some  $t \in R$ . Thus  $(r\rho)c \subseteq t\rho$ . It follows that  $c \in T(X, \rho, R)$ . For all  $x, x' \in X$ ,  $xc = x'c \Leftrightarrow f(xa) = f(x'a) \Leftrightarrow xa = x'a$  (since  $f$  is one-to-one). Thus  $\text{Ker}(a) = \text{Ker}(c)$ . Since for every  $r \in R$ ,  $(r\rho)c = f((r\rho)a)$ , we have  $\nabla c = f(\nabla a)$ . Thus  $\nabla c \leftrightarrow \nabla b$ . Hence, by Theorem 2.3 and Theorem 2.5,  $a \mathcal{R} c$  and  $c \mathcal{L} b$ , which gives  $a \mathcal{D} b$ . ■

Suppose  $X$  is finite and let  $f : \nabla a \rightarrow \nabla b$  be as in the statement of Theorem 2.6. Since  $f$  is a bijection,  $f(m(\nabla a)) = m(f(\nabla a))$ . Thus, by the argument after Theorem 2.5,  $f(\nabla a) \leftrightarrow \nabla b$  if and only if  $f(m(\nabla a)) = m(\nabla b)$ . Hence in the finite case,  $a \mathcal{D} b$  if and only if there is a bijection  $f : \nabla a \rightarrow \nabla b$  such that  $f(R \cap \nabla a) \subseteq R$  and  $f(m(\nabla a)) = m(\nabla b)$  [7].

In the semigroup  $T(X)$ ,  $\mathcal{D} = \mathcal{J}$  (Lemma 2.1). In general, this result is not true for  $T(X, \rho, R)$ . A characterization of the  $\mathcal{J}$ -relation in  $T(X, \rho, R)$  is provided by Theorem 2.8. We start with the theorem that determines the partial order of the  $\mathcal{J}$ -classes in  $T(X, \rho, R)$ .

**Theorem 2.7** *Let  $a, b \in T(X, \rho, R)$ . Then  $J_a \leq J_b$  if and only if there is a function  $g : \nabla b \rightarrow \nabla a$  such that  $g(R \cap \nabla b) \subseteq R$  and  $\nabla a \hookrightarrow g(\nabla b) \hookrightarrow X/\rho$ .*

**Proof:** Suppose  $J_a \leq J_b$ . Then  $a = cbd$  for some  $c, d \in T(X, \rho, R)$ . Fix  $r_0 \in \nabla a$  and define  $g : \nabla b \rightarrow \nabla a$  by:

$$g(x) = \begin{cases} xd & \text{if } x \in \nabla(cb) \\ r_x d & \text{if } x \notin \nabla(cb) \text{ but } x\rho \cap \nabla(cb) \neq \emptyset \\ r_0 & \text{if } x\rho \cap \nabla(cb) = \emptyset. \end{cases}$$

Let  $x \in \nabla b$ . If  $x \in \nabla(cb)$  then  $xd \in \nabla a$  (since  $a = cbd$ ). If  $x \notin \nabla(cb)$  but  $x\rho \cap \nabla(cb) \neq \emptyset$  then  $r_x \in \nabla(cb)$ , and so  $r_x d \in \nabla a$ . Thus  $g$  indeed maps  $\nabla b$  to  $\nabla a$ .

We have  $g(R \cap \nabla b) \subseteq R$  since  $Rd \subseteq R$ . Let  $C \in \blacktriangledown b$ . Then  $C = (r\rho)b$  for some  $r \in R$ . By the definition of  $g$ , either  $g(C) = \{r_0\} \subseteq r_0\rho$  (if  $x\rho \cap \nabla(cb) = \emptyset$ ) or  $g(C) = (C \cap \nabla(cb))d \subseteq Cd = (r\rho)bd \subseteq q\rho$  for some  $q \in R$  (since  $b, d \in T(X, \rho, R)$ ). It follows that  $g(\blacktriangledown b) \hookrightarrow X/\rho$ .

Let  $A \in \blacktriangledown a$ . Then  $A = (p\rho)a$  for some  $p \in R$ . Since  $c \in T(X, \rho, R)$ ,  $(p\rho)c \subseteq r\rho$  for some  $r \in R$ . Let  $C = (r\rho)b$ . Then  $C \in \blacktriangledown b$  and  $A = (p\rho)a = (p\rho)(cbd) \subseteq (C \cap \nabla(cb))d = g(C) \in g(\blacktriangledown b)$ . Thus  $\blacktriangledown a \hookrightarrow g(\blacktriangledown b)$ .

Conversely, suppose there is a function  $g : \nabla b \rightarrow \nabla a$  such that  $g(R \cap \nabla b) \subseteq R$  and  $\blacktriangledown a \hookrightarrow g(\blacktriangledown b) \hookrightarrow X/\rho$ . Let  $A \in \blacktriangledown a$ . Then there is a unique  $q_A \in R$  such that  $A \subseteq q_A\rho$ . Since  $\blacktriangledown a \hookrightarrow g(\blacktriangledown b)$ , there is  $C_A \in \blacktriangledown b$  such that  $A \subseteq g(C_A)$ . Since  $C_A \in \blacktriangledown b$ , there is  $r_A \in R$  such that  $C_A = (r_A\rho)b$ . Let  $t_A = r_A b$ . Since  $b \in T(X, \rho, R)$ ,  $t_A \in R$  and  $C_A = (r_A\rho)b \subseteq t_A\rho$ . For every  $z \in A$ , select  $u_z^A \in C_A$  such that  $z = g(u_z^A)$  (since  $g(t_A) = q_A$ , we may assume that  $u_z^A = t_A$  if  $z = q_A$ ). Let  $C'_A = \{u_z^A : z \in A\}$  and note that  $C'_A \subseteq C_A$ . For every  $y \in C'_A$ , select  $w_y^A \in r_A\rho$  such that  $y = w_y^A b$  (since  $r_A b = t_A$ , we may assume that  $w_y^A = r_A$  if  $y = t_A$ ).

We shall construct  $c, d \in T(X, \rho, R)$  such that  $a = cbd$ . To construct  $c$ , let  $x \in X$ . Then there is a unique  $p \in R$  such that  $x \in p\rho$ . Let  $A = (p\rho)a$  and  $z = xa$ . Then  $A \in \blacktriangledown a$  and  $z \in A$ . Let  $y = u_z^A$  and note that  $y \in C'_A$ . Define  $xc = w_y^A$ . By the definition of  $c$ , we have  $(p\rho)c \subseteq r_A\rho$ . Since  $pa = q_A$  (from the paragraph above), we have  $pc = r_A$  (since  $u_z^A = t_A$  for  $z = q_A$  and  $w_y^A = r_A$  for  $y = t_A$ ). It follows that  $c \in T(X, \rho, R)$ .

To construct  $d$ , let  $y \in X$ . Then there is a unique  $t \in R$  such that  $y \in t\rho$ . We define  $yd$  as follows:

- (i) If  $y \in C'_A$  for some  $A \in \blacktriangledown a$ , define  $yd = g(y)$ .
- (ii) If  $y \notin C'_B$  for every  $B \in \blacktriangledown a$  but  $C_A \subseteq t\rho$  for some  $A \in \blacktriangledown a$ , define  $yd = q_A$ .
- (iii) If there is no  $A \in \blacktriangledown a$  such that  $C_A \subseteq t\rho$ , define  $yd = y$ .

If  $C_A, C_B \subseteq t\rho$  for some  $A, B \in \blacktriangledown a$  then  $q_A = g(t) = q_B$ , and so the definition of  $d$  in (ii) does not depend on the choice of  $A$ . Thus  $d$  is well defined.

Let  $t \in R$  and consider  $t\rho$ . Suppose  $C_A \subseteq t\rho$  for some  $A \in \blacktriangledown a$ . Then, by (i),  $td = g(t) = q_A$ . Let  $B \in \blacktriangledown a$  be such that  $C_B \subseteq t\rho$ . Then  $q_A = g(t) = q_B$ . It follows that  $B \subseteq q_A\rho$  and so, by (i) and (ii),  $(t\rho)d \subseteq q_A\rho$ . If there is no  $A \in \blacktriangledown a$  such that  $C_A \subseteq t\rho$  then, by (iii),  $td = t$  and  $(t\rho)d = t\rho$ . It follows that  $d \in T(X, \rho, R)$ .

Let  $x \in X$ . Then there is a unique  $p \in R$  such that  $x \in p\rho$ . Let  $A = (p\rho)a$  and  $z = xa$ . Then  $A \in \blacktriangledown a$  and  $z \in A$ . Let  $y = u_z^A$  and note that  $y \in C'_A$ . By the definition of  $c$  and  $d$ ,  $xc = w_y^A$  (recall that  $w_y^A$  was selected so that  $w_y^A b = y$ ) and  $yd = g(y) = g(u_z^A) = z = xa$ . Thus  $x(cbd) = w_y^A(bd) = yd = xa$ . Hence  $a = cbd$  and so  $J_a \leq J_b$ . ■

**Theorem 2.8** *Let  $a, b \in T(X, \rho, R)$ . Then  $a \mathcal{J} b$  if and only if there are functions  $f : \nabla a \rightarrow \nabla b$  and  $g : \nabla b \rightarrow \nabla a$  such that  $f(R \cap \nabla a) \subseteq R$ ,  $\nabla b \hookrightarrow f(\nabla a) \hookrightarrow X/\rho$ ,  $g(R \cap \nabla b) \subseteq R$ , and  $\nabla a \hookrightarrow g(\nabla b) \hookrightarrow X/\rho$ .*

**Proof:** It is immediate by Theorem 2.7. ■

Let  $a, b \in T(X, \rho, R)$  with  $a \mathcal{J} b$ , and let  $f$  and  $g$  be functions as in the statement of Theorem 2.8. We make the following observations.

- (1) The functions  $f$  and  $g$  are onto.
- (2)  $|\nabla a| = |\nabla b|$ .
- (3) For every  $r \in R$ , there are  $s, t \in R$  such that  $f(r\rho \cap \nabla a) \subseteq s\rho$  and  $g(r\rho \cap \nabla b) \subseteq t\rho$ .

To illustrate Theorems 2.6 and 2.8, consider  $T(X, \rho, R)$ , where  $X = \{0, 1, 2, 3, 4, \dots\}$ ,

$$X/\rho = \{\{0, 2, 4\}, \{6, 8\}, \{10, 12\}, \{14, 16\}, \dots, \{1, 3\}, \{5, 7\}, \{9, 11\}, \dots\},$$

and  $R = \{0, 6, 10, 14, \dots, 1, 5, 9, \dots\}$ . Let  $a, b \in T(X, \rho, R)$  be such that

$$\begin{aligned}\nabla a &= \{\{0, 2\}, \{0, 4\}, \{6, 8\}, \{10, 12\}, \{14, 16\}, \dots\}, \\ \nabla b &= \{\{1, 3\}, \{5, 7\}, \{9, 11\}, \{13, 15\}, \{17, 19\}, \dots\}.\end{aligned}$$

It is clear that such  $a$  and  $b$  can be defined. It is also clear that we can define  $f : \nabla a \rightarrow \nabla b$  and  $g : \nabla b \rightarrow \nabla a$  such that  $f(R \cap \nabla a) \subseteq R$ ,  $g(R \cap \nabla b) \subseteq R$ , and

$$\begin{aligned}f(\{0, 2\}) &= \{1, 3\}, \quad f(\{0, 4\}) = \{1, 3\}, \quad f(\{6, 8\}) = \{5, 7\}, \quad f(\{10, 12\}) = \{9, 11\}, \dots \\ g(\{1, 3\}) &= \{0, 2\}, \quad g(\{5, 7\}) = \{0, 4\}, \quad g(\{9, 11\}) = \{6, 8\}, \quad g(\{13, 15\}) = \{10, 12\}, \dots\end{aligned}$$

Then  $f$  and  $g$  satisfy the conditions given in Theorem 2.8 (in fact,  $\nabla b = f(\nabla a)$  and  $\nabla a = g(\nabla b)$ ), and so  $a \mathcal{J} b$ . However, there is no bijection  $f : \nabla a \rightarrow \nabla b$  such that  $f(R \cap \nabla a) \subseteq R$  and  $f(\nabla a) \hookrightarrow \nabla b$  since if such an  $f$  existed, there would have to be  $C \in \nabla b$  such that  $f(\{0, 2\}) \subseteq C$  and  $f(\{0, 4\}) \subseteq C$ , which is impossible because every  $C \in \nabla b$  has 2 elements. Thus, by Theorem 2.6,  $a$  and  $b$  are not in the same  $\mathcal{D}$ -class of  $T(X, \rho, R)$ .

The above example shows that, in general,  $\mathcal{D} \neq \mathcal{J}$  in  $T(X, \rho, R)$ . This is in contrast with the semigroups  $T(X)$  and  $PT(X)$  of, respectively, full and partial transformations on  $X$ , in which  $\mathcal{D} = \mathcal{J}$  [5, Exercises 16 and 17, p. 63]. We shall characterize the equivalence relations  $\rho$  on  $X$  for which  $\mathcal{D} = \mathcal{J}$  in  $T(X, \rho, R)$ .

**Lemma 2.9** *Let  $\rho$  be a relation such that exactly one  $\rho$ -class has size at least 2. Then for all  $a, b \in T(X, \rho, R)$ , if  $a \mathcal{J} b$  then  $a \mathcal{D} b$ .*

**Proof:** Let  $a, b \in T(X, \rho, R)$  be such that  $a \mathcal{J} b$ . Then, by Theorem 2.8, there are functions  $f : \nabla a \rightarrow \nabla b$  and  $g : \nabla b \rightarrow \nabla a$  such that  $f(R \cap \nabla a) \subseteq R$ ,  $\nabla b \hookrightarrow f(\nabla a) \hookrightarrow X/\rho$ ,  $g(R \cap \nabla b) \subseteq R$ , and  $\nabla a \hookrightarrow g(\nabla b) \hookrightarrow X/\rho$ . Let  $r\rho$  be the  $\rho$ -class of size at least 2 ( $r \in R$ ).

Suppose  $\nabla a \subseteq R$ . Then every element of  $\nabla a$  has size 1, and so, since  $\nabla b \hookrightarrow f(\nabla a)$ , every element of  $\nabla b$  also has size 1 and  $\nabla b \subseteq R$ . Since  $|\nabla a| = |\nabla b|$  (see observation (2) after Theorem 2.8), there is a bijection  $h : \nabla a \rightarrow \nabla b$ . Since  $\nabla a, \nabla b \subseteq R$ , we clearly

have  $h(R \cap \nabla a) \subseteq R$ . Since  $\blacktriangledown a = \{\{s\} : s \in \nabla a\}$  and  $\blacktriangledown b = \{\{t\} : t \in \nabla b\}$ , we have  $h(\blacktriangledown a) = \blacktriangledown b$ . Thus  $a \mathcal{D} b$  by Theorem 2.6.

Suppose  $\nabla a \not\subseteq R$ . This can only happen when  $|r\rho \cap \nabla a| \geq 2$  (since  $r\rho$  is the only  $\rho$ -class of size at least 2). We claim that  $|r\rho \cap \nabla a| = |r\rho \cap \nabla b|$  and  $|\nabla a - r\rho| = |\nabla b - r\rho|$ .

To see that this claim is true, note that since  $g(R \cap \nabla b) \subseteq R$ ,  $g$  is onto, and  $g$  preserves  $\rho$ -classes (see observations (1) and (3) after Theorem 2.8), we have that  $|r\rho \cap \nabla a| \geq 2$  implies  $r\rho \cap \nabla a = g(r\rho \cap \nabla b)$ . Thus  $|r\rho \cap \nabla a| \leq |r\rho \cap \nabla b|$ . By a similar argument,  $|r\rho \cap \nabla b| \leq |r\rho \cap \nabla a|$ . Now  $r\rho \cap \nabla a = g(r\rho \cap \nabla b)$  and  $\nabla a = g(\nabla b)$  imply  $\nabla a - r\rho = g(\nabla b - r\rho)$ . Thus  $|\nabla a - r\rho| \leq |\nabla b - r\rho|$ . By a similar argument,  $|\nabla b - r\rho| \leq |\nabla a - r\rho|$ .

The claim has been proved. Thus there is a bijection  $h : \nabla a \rightarrow \nabla b$  such that  $h(r) = r$ ,  $h(r\rho \cap \nabla a) = r\rho \cap \nabla b$ , and  $h(\nabla a - r\rho) = \nabla b - r\rho$ . Note that if  $s \in (R \cap \nabla a) - \{r\}$  then  $s \in \nabla a - r\rho$ , and so  $h(s) \in \nabla b - r\rho$ . Since  $\nabla b - r\rho \subseteq R$ , it follows that  $h(R \cap \nabla a) \subseteq R$ . Since  $\blacktriangledown a = \{\{r\rho \cap \nabla a\}\} \cup \{\{s\} : s \in \nabla a - \{r\}\}$  and  $\blacktriangledown b = \{\{r\rho \cap \nabla b\}\} \cup \{\{t\} : t \in \nabla b - \{r\}\}$ , we have  $h(\blacktriangledown a) = \blacktriangledown b$ . Thus  $a \mathcal{D} b$  by Theorem 2.6. ■

**Lemma 2.10** *Let  $\rho$  be a relation such that all  $\rho$ -classes are finite and only finitely many of them have size at least 2. Then for all  $a, b \in T(X, \rho, R)$ , if  $a \mathcal{J} b$  then  $a \mathcal{D} b$ .*

**Proof:** Let  $a, b \in T(X, \rho, R)$  be such that  $a \mathcal{J} b$ . Then, by Theorem 2.8, there are functions  $f : \nabla a \rightarrow \nabla b$  and  $g : \nabla b \rightarrow \nabla a$  such that  $f(R \cap \nabla a) \subseteq R$ ,  $\blacktriangledown b \hookrightarrow f(\blacktriangledown a) \hookrightarrow X/\rho$ ,  $g(R \cap \nabla b) \subseteq R$ , and  $\blacktriangledown a \hookrightarrow g(\blacktriangledown b) \hookrightarrow X/\rho$ .

Since only finitely many  $\rho$ -classes have size at least 2, there are finitely many  $\rho$ -classes  $r\rho$  such that  $|r\rho \cap \nabla a| \geq 2$ , say  $r_1\rho, \dots, r_k\rho$  ( $r_i \in R, k \geq 0$ ), and finitely many  $\rho$ -classes  $s\rho$  such that  $|s\rho \cap \nabla b| \geq 2$ , say  $s_1\rho, \dots, s_m\rho$  ( $s_j \in R, m \geq 0$ ).

Note that for every  $i \in \{1, \dots, k\}$  there is  $j \in \{1, \dots, m\}$  such that  $g(s_j\rho \cap \nabla b) \subseteq r_i\rho$ . (Indeed, otherwise, since  $g$  preserves  $\rho$ -classes, there would be an  $i$  such that  $r_i\rho \cap g(s_j\rho \cap \nabla b) = \emptyset$  for all  $j \in \{1, \dots, m\}$ . But then, since  $g(R \cap \nabla b) \subseteq R$ , we would have  $r_i\rho \cap g(\nabla b) \subseteq \{r_i\}$ , which would contradict the fact that  $g$  is onto.) Since  $r_1\rho, \dots, r_k\rho$  are pairwise disjoint, it follows that  $k \leq m$ . By a similar argument, we have  $m \leq k$ , and so  $k = m$ .

Suppose  $k = 0$ . Then  $\blacktriangledown a = \{\{r\} : r \in \nabla a\}$  and  $\blacktriangledown b = \{\{s\} : s \in \nabla b\}$ . Since  $|\nabla a| = |\nabla b|$ , there is a bijection  $h : \nabla a \rightarrow \nabla b$ . Note that  $h(R \cap \nabla a) \subseteq R$  (since  $\nabla a, \nabla b \subseteq R$ ) and  $h(\blacktriangledown a) = \blacktriangledown b$ . Thus  $a \mathcal{D} b$  by Theorem 2.6.

Suppose  $k \geq 1$ . Let  $\{t_1\rho, \dots, t_p\rho\}$  be the set of all  $\rho$ -classes of size at least 2 ( $t_i \in R$ ). (Note that each  $r_i\rho$  and each  $s_i\rho$  ( $i = 1, \dots, k$ ) is an element of this set.) Let

$$X_0 = t_1\rho \cup \dots \cup t_p\rho, \quad Y = r_1\rho \cup \dots \cup r_k\rho, \quad \text{and} \quad Z = s_1\rho \cup \dots \cup s_k\rho.$$

Note that  $X_0$  is finite,  $Y \cup Z \subseteq X_0$ ,  $f(Y \cap \nabla a) \subseteq Z$ , and  $g(Z \cap \nabla b) \subseteq Y$ . (Indeed, if, say,  $g(Z \cap \nabla b) \not\subseteq Y$  then for at least one  $i \in \{1, \dots, k\}$  there would be no  $j \in \{1, \dots, k\}$  such that  $g(s_j\rho \cap \nabla b) \subseteq r_i\rho$ , which would contradict the observation made in the third paragraph of the proof.)

Consider the semigroup  $T(X_0, \rho_0, R_0)$ , where  $X_0$  is the set defined above,  $\rho_0$  is the equivalence relation on  $X_0$  with the partition  $\{t_1\rho, \dots, t_p\rho\}$ , and  $R_0 = \{t_1, \dots, t_p\}$ . Define  $a_0, b_0 \in T(X_0)$  by:

$$xa_0 = \begin{cases} xa & \text{if } xa \in Y \\ r_1 & \text{otherwise} \end{cases} \quad \text{and} \quad xb_0 = \begin{cases} xb & \text{if } xb \in Z \\ s_1 & \text{otherwise} \end{cases}.$$

Then  $a_0, b_0 \in T(X_0, \rho_0, R_0)$ ,  $\nabla a_0 = Y \cap \nabla a$ , and  $\nabla b_0 = Z \cap \nabla b$ . Moreover, there are sets (possibly empty)  $R' \subseteq \{r_1, \dots, r_k\}$  and  $S' \subseteq \{s_1, \dots, s_k\}$  such that

$$\begin{aligned}\blacktriangledown a_0 &= \{A \in \blacktriangledown a : |A| \geq 2\} \cup \{\{r\} : r \in R'\} \text{ and} \\ \blacktriangledown b_0 &= \{B \in \blacktriangledown b : |B| \geq 2\} \cup \{\{s\} : s \in S'\}.\end{aligned}$$

Define  $f_0 : \nabla a_0 \rightarrow \nabla b_0$  and  $g_0 : \nabla b_0 \rightarrow \nabla a_0$  by:  $f_0 = f|(Y \cap \nabla a)$  and  $g_0 = g|(Z \cap \nabla b)$ . Then  $f_0(R_0 \cap \nabla a_0) \subseteq R_0$ ,  $\blacktriangledown b_0 \hookrightarrow f_0(\blacktriangledown a_0) \hookrightarrow X_0/\rho_0$ ,  $g_0(R_0 \cap \nabla b_0) \subseteq R_0$ , and  $\blacktriangledown a_0 \hookrightarrow g_0(\blacktriangledown b_0) \hookrightarrow X_0/\rho_0$ . Thus  $a_0 \mathcal{J} b_0$  in  $T(X_0, \rho_0, R_0)$  by Theorem 2.8. Hence  $a_0 \mathcal{D} b_0$  in  $T(X_0, \rho_0, R_0)$  (since  $T(X_0, \rho_0, R_0)$  is finite and  $\mathcal{D} = \mathcal{J}$  in any finite semigroup).

Thus, by Theorem 2.6, there is a bijection  $h_0 : \nabla a_0 \rightarrow \nabla b_0$  such that  $h_0(R_0 \cap \nabla a_0) \subseteq R_0$  and  $h_0(\blacktriangledown a_0) \leftrightarrow \blacktriangledown b_0$ . It follows that  $|Y \cap \nabla a| = |Z \cap \nabla b|$  (since  $Y \cap \nabla a = \nabla a_0$  and  $Z \cap \nabla b = \nabla b_0$ ). Thus, since  $|\nabla a| = |\nabla b|$  and  $Y \cap \nabla a$  is finite, we have  $|\nabla a - Y| = |\nabla b - Z|$ . Hence there is a bijection  $h_1 : \nabla a - Y \rightarrow \nabla b - Z$ . Define  $h : \nabla a \rightarrow \nabla b$  by:

$$h(x) = \begin{cases} h_0(x) & \text{if } x \in Y \cap \nabla a \\ h_1(x) & \text{if } x \in \nabla a - Y. \end{cases}$$

Since  $h_0 : Y \cap \nabla a \rightarrow Z \cap \nabla b$  and  $h_1 : \nabla a - Y \rightarrow \nabla b - Z$  are bijections, we have that  $h$  is a bijection. Note that

$$\begin{aligned}\blacktriangledown a &= \{A \in \blacktriangledown a : A \subseteq Y\} \cup \{\{r\} : r \in \nabla a - Y\} \text{ and} \\ \blacktriangledown b &= \{B \in \blacktriangledown b : B \subseteq Z\} \cup \{\{s\} : s \in \nabla b - Z\}.\end{aligned}$$

Let  $r \in R \cap \nabla a$ . If  $r \in Y \cap \nabla a$  then  $h(r) = h_0(r) \in R_0 \subseteq R$ . If  $r \in \nabla a - Y$  then  $h(r) = h_1(r) \in R$  (since  $h_1 : \nabla a - Y \rightarrow \nabla b - Z$  and  $\nabla b - Z \subseteq R$ ). Thus  $h(R \cap \nabla a) \subseteq R$ .

Now,  $h_0(\blacktriangledown a_0) \leftrightarrow \blacktriangledown b_0$  implies  $h(\{A \in \blacktriangledown a : A \subseteq Y\}) \leftrightarrow \{B \in \blacktriangledown b : B \subseteq Z\}$ . Indeed, let  $B \in \blacktriangledown b$  and  $B \subseteq Z$ . Then  $B \subseteq s_i \rho$  for some  $i \in \{1, \dots, k\}$ . Since  $|s_i \rho \cap \nabla b| \geq 2$ , there is  $B_0 \in \blacktriangledown b$  such that  $|B_0| \geq 2$  and  $B \subseteq B_0 \subseteq s_i \rho$ . Then  $B_0 \in \blacktriangledown b_0$ , and so  $B_0 \subseteq h_0(A)$  for some  $A \in \blacktriangledown a_0$ . Note that  $|A| \geq 2$ , and so  $A \in \blacktriangledown a$ ,  $A \subseteq Y$ , and  $h_0(A) = h(A)$ . Thus  $B \subseteq B_0 \subseteq h(A)$ , and so  $\{B \in \blacktriangledown b : B \subseteq Z\} \hookrightarrow h(\{A \in \blacktriangledown a : A \subseteq Y\})$ . By a similar argument, we have  $h(\{A \in \blacktriangledown a : A \subseteq Y\}) \hookrightarrow \{B \in \blacktriangledown b : B \subseteq Z\}$ .

Finally, since  $h_1(\{\{r\} : r \in \nabla a - Y\}) = \{\{s\} : s \in \nabla b - Z\}$ , it follows that  $h(\blacktriangledown a) \leftrightarrow \blacktriangledown b$ . Hence  $a \mathcal{D} b$  by Theorem 2.6. ■

With the previous two lemmas, we are ready to describe the equivalence relations  $\rho$  on  $X$  for which  $\mathcal{D} = \mathcal{J}$  in  $T(X, \rho, R)$ .

**Theorem 2.11** *In the semigroup  $T(X, \rho, R)$ ,  $\mathcal{D} = \mathcal{J}$  if and only if  $\rho$  satisfies one of the following conditions:*

- (1) *Exactly one  $\rho$ -class is infinite and all other  $\rho$ -classes (if any) have size 1; or*
- (2) *All  $\rho$ -classes are finite and only finitely many of them have size at least 2.*

**Proof:** If (1) or (2) holds then  $\mathcal{J} \subseteq \mathcal{D}$  by Lemma 2.9 and Lemma 2.10, and so  $\mathcal{D} = \mathcal{J}$  (since  $\mathcal{D} \subseteq \mathcal{J}$  in any semigroup). Conversely, suppose that neither (1) nor (2) holds. Then there are two possible cases to consider.

**Case 1.** There is an infinite  $\rho$ -class and another  $\rho$ -class of size at least 2.



Let  $r\rho$  be infinite and  $s\rho$  be of size at least 2, where  $r, s \in R$  and  $r \neq s$ . Select  $x_0 \in r\rho$  such that  $x_0 \neq r$ , and  $y_0 \in s\rho$  such that  $y_0 \neq s$ . Consider the mappings

$$a = \begin{pmatrix} x & y \\ x & r \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} z & x_0 & y_0 & w \\ z & r & x_0 & r \end{pmatrix},$$

where  $x$  is an arbitrary element of  $r\rho$ ,  $y$  is an arbitrary element of  $X - r\rho$ ,  $z$  is an arbitrary element of  $r\rho - \{x_0\}$ , and  $w$  is an arbitrary element of  $X - (r\rho \cup \{y_0\})$ . Then  $a, b \in T(X, \rho, R)$  with  $\nabla a = \nabla b = r\rho$ ,

$$\blacktriangledown a = \{r\rho, \{r\}\} \quad \text{and} \quad \blacktriangledown b = \{r\rho - \{x_0\}, \{r, x_0\}\} \text{ or } \{r\rho - \{x_0\}, \{r, x_0\}, \{r\}\}.$$

Define  $f : r\rho \rightarrow r\rho$  by  $f(x) = x$ . Since  $r\rho$  is infinite, there is  $g : r\rho \rightarrow r\rho$  such that  $g(r) = r$  and  $g(r\rho - \{x_0\}) = r\rho$ . Then  $f$  and  $g$  satisfy the conditions from the statement of Theorem 2.8, and so  $a \mathcal{J} b$ .

Let  $h : r\rho \rightarrow r\rho$  be a bijection. Then  $h(r\rho) = r\rho$ . Note that  $r\rho$  is not included in  $r\rho - \{x_0\}$  or  $\{r, x_0\}$  or  $\{r\}$ . Since  $r\rho \in \blacktriangledown a$ , it follows that it is not true that  $h(\blacktriangledown a) \hookrightarrow \blacktriangledown b$ . Thus  $a$  and  $b$  are not  $\mathcal{D}$ -related by Theorem 2.6, and so  $\mathcal{D} \neq \mathcal{J}$ .

**Case 2.** There are infinitely many  $\rho$ -classes of size at least 2.

Let  $r_1\rho = \{r_1, x_1, \dots\}$ ,  $r_2\rho = \{r_2, x_2, \dots\}, \dots$  be an infinite sequence of  $\rho$ -classes of size at least 2 ( $r_i \in R$ ,  $x_i \neq r_i$ ,  $i = 1, 2, \dots$ ). We may assume that there is a  $\rho$ -class not in the sequence. Consider the mappings

$$a = \begin{pmatrix} r_i & x_i & y_i & y \\ r_i & x_i & r_i & r_1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} r_i & x_1 & x_j & y_i & y \\ r_i & r_1 & x_j & r_i & r_1 \end{pmatrix},$$

where  $i \geq 1$ ,  $j \geq 2$ ,  $y_i$  is an arbitrary element of  $r_i\rho - \{r_i, x_i\}$ , and  $y$  is an arbitrary element of  $X - (r_1\rho \cup r_2\rho \cup \dots)$ . Then  $a, b \in T(X, \rho, R)$  with

$$\nabla a = \{r_1, x_1, r_2, x_2, r_3, x_3, \dots\}, \quad \nabla b = \{r_1, r_2, x_2, r_3, x_3, \dots\},$$

$$\blacktriangledown a = \{\{r_1\}, \{r_1, x_1\}, \{r_2, x_2\}, \{r_3, x_3\}, \dots\} \quad \text{and} \quad \blacktriangledown b = \{\{r_1\}, \{r_2, x_2\}, \{r_3, x_3\}, \dots\}.$$

Define  $f : \nabla a \rightarrow \nabla b$  and  $g : \nabla b \rightarrow \nabla a$  by:

$$f = \begin{pmatrix} r_1 & x_1 & r_j & x_j \\ r_1 & r_1 & r_j & x_j \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} r_1 & r_j & x_j \\ r_1 & r_{j-1} & x_{j-1} \end{pmatrix},$$

where  $j \geq 2$ . Then  $f(\blacktriangledown a), g(\blacktriangledown b) \hookrightarrow X/\rho$ ,  $f(\blacktriangledown a) = \blacktriangledown b$ , and  $g(\blacktriangledown b) = \blacktriangledown a$ . Thus  $a \mathcal{J} b$  by Theorem 2.8.

Let  $h : \nabla a \rightarrow \nabla b$  be such that  $h(\blacktriangledown a) \hookrightarrow \blacktriangledown b$ . Then, since  $\{r_1\} \in \blacktriangledown b$ , there is  $A \in \blacktriangledown a$  such that  $h(A) \subseteq \{r_1\}$ . If  $A = \{r_i, x_i\}$  for some  $i$  then  $h$  is not one-to-one. Suppose  $A = \{r_1\}$ . Then  $h(r_1) = r_1$ . Since  $\{r_1, x_1\} \in \blacktriangledown a$ , we have  $h(\{r_1, x_1\}) \subseteq B$  for some  $B \in \blacktriangledown b$ . Since  $h(r_1) = r_1$  and  $\{r_1\}$  is the only element of  $\blacktriangledown b$  containing  $r_1$ ,  $B$  must be  $\{r_1\}$ . Then  $h(x_1) = r_1$ , and so again  $h$  is not one-to-one. It follows from Theorem 2.6 that  $a$  and  $b$  are not  $\mathcal{D}$ -related, and so  $\mathcal{D} \neq \mathcal{J}$ . ■

Recall that the  $\mathcal{J}$ -classes of any semigroup  $S$  are partially ordered by the relation  $\leq$  defined by:  $J_a \leq J_b$  if  $S^1 a S^1 \subseteq S^1 b S^1$ . The poset  $(S/\mathcal{J}, \leq)$  of  $\mathcal{J}$ -classes of  $S$  is isomorphic to the poset  $\{S^1 a S^1 : a \in S\}, \subseteq)$  of principal ideals of  $S$ .

In the semigroup  $T(X)$  of full transformations on  $X$ , the poset of  $\mathcal{J}$ -classes is a chain [9, Proposition 4.1]. We find that this never happens in the semigroup  $T(X, \rho, R)$  except in the two extreme cases when  $\rho$  is the identity relation on  $X$  or the universal relation on  $X$ .

**Theorem 2.12** *The partially ordered set of  $\mathcal{J}$ -classes of  $T(X, \rho, R)$  is a chain if and only if  $\rho = \{(x, x) : x \in X\}$  or  $\rho = X \times X$ .*

**Proof:** Suppose  $\rho = \{(x, x) : x \in X\}$ . Then  $T(X, \rho, R) = T(X)$  and it is well known that the  $\mathcal{J}$ -classes of  $T(X)$  form a chain [9, Proposition 4.1].

Suppose  $\rho = X \times X$ . Then  $R$  is a one-element set, say  $R = \{r\}$ , and  $r\rho = X$ . Let  $a, b \in T(X, \rho, R)$ . Note that  $r \in \nabla a \cap \nabla b$ . Suppose  $|\nabla a| \leq |\nabla b|$ . Then there is a function  $g : \nabla b \rightarrow \nabla a$  such that  $g(r) = r$  and  $g(\nabla b) = \nabla a$ . We have  $g(R \cap \nabla b) = g(\{r\}) = \{r\}$  and  $\nabla a \hookrightarrow g(\nabla b) \hookrightarrow X/\rho$  (since  $\nabla a = \{\nabla a\}$ ,  $g(\nabla b) = g(\{r\}) = \{r\}$ , and  $X/\rho = \{X\}$ ). Thus  $J_a \leq J_b$  by Theorem 2.7. Similarly, if  $|\nabla b| \leq |\nabla a|$  then  $J_b \leq J_a$ . It follows that the  $\mathcal{J}$ -classes of  $T(X, \rho, R)$  form a chain.

Conversely, suppose that  $\rho \neq \{(x, x) : x \in X\}$  and  $\rho \neq X \times X$ . Then there are two distinct  $\rho$ -classes of which at least one has size at least 2, say  $r\rho = \{r, x, \dots\}$  and  $s\rho = \{s, \dots\}$  ( $r, s \in R$ ). Consider the mappings

$$a = \begin{pmatrix} x & y \\ x & r \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} z & w \\ r & s \end{pmatrix},$$

where  $y$  is an arbitrary element of  $X - \{x\}$ ,  $z$  is an arbitrary element of  $r\rho$ , and  $w$  is an arbitrary element of  $X - r\rho$ . Then  $a, b \in T(X, \rho, R)$  with

$$\nabla a = \{r, x\}, \quad \nabla b = \{r, s\}, \quad \nabla a = \{\{r, x\}, \{r\}\}, \quad \text{and} \quad \nabla b = \{\{r\}, \{s\}\}.$$

Thus there is no  $f : \nabla a \rightarrow \nabla b$  such that  $\nabla b \hookrightarrow f(\nabla a) \hookrightarrow X/\rho$  (since otherwise we would have  $f(\{r, x\}) = \{r\}$  and  $f(\{r\}) = \{s\}$  or  $f(\{r, x\}) = \{s\}$  and  $f(\{r\}) = \{r\}$ , which is impossible). Similarly, there is no  $g : \nabla b \rightarrow \nabla a$  such that  $\nabla a \hookrightarrow g(\nabla b)$  (since  $\{r, x\} \in \nabla a$  has size 2 and, since  $\nabla b = \{\{r\}, \{s\}\}$ , every element of  $g(\nabla b)$  would have size 1).

It follows from Theorem 2.7 that  $J_a \not\leq J_b$  and  $J_b \not\leq J_a$ . Hence the partially ordered set of  $\mathcal{J}$ -classes of  $T(X, \rho, R)$  is not a chain. ■

We recall that if  $\rho = X \times X$  and  $|X| \geq 2$  then the semigroup  $T(X, \rho, R)$  is isomorphic to the semigroup  $PT(X')$  of partial transformations on  $X'$ , where  $X'$  is the set  $X$  with one element removed. Thus, the poset of  $\mathcal{J}$ -classes of  $T(X, \rho, R)$  is a chain if and only if  $T(X, \rho, R) = T(X)$  or  $T(X, \rho, R)$  is isomorphic to  $PT(X')$ .

### 3 Regular $T(X, \rho, R)$

An element  $a$  of a semigroup  $S$  is called *regular* if  $a = axa$  for some  $x$  in  $S$ . If all elements of  $S$  are regular, we say that  $S$  is a *regular semigroup*. If a  $\mathcal{D}$ -class  $D$  in  $S$  contains a regular element then all elements in  $D$  are regular [5, Proposition 2.3.1], and we call  $D$  a *regular  $\mathcal{D}$ -class*. In a regular  $\mathcal{D}$ -class, every  $\mathcal{L}$ -class and every  $\mathcal{R}$ -class contains an idempotent [5, Proposition 2.3.2]. It follows that an element of  $S$  is regular if and only if it is  $\mathcal{L}$ -related ( $\mathcal{R}$ -related) to an idempotent in  $S$ .

Let  $A$  be a nonempty subset of  $X$ . An equivalence relation  $\rho$  on  $X$  induces a partition  $A/\rho$  of  $A$ :

$$A/\rho = \{x\rho \cap A : x \in X \text{ and } x\rho \cap A \neq \emptyset\}.$$

The semigroup  $T(X)$  is regular [5, Exercise 15, p. 63]. This is not true in general about  $T(X, \rho, R)$ . The next theorem characterizes the regular elements of  $T(X, \rho, R)$ .

**Theorem 3.1** *Let  $a \in T(X, \rho, R)$ . Then  $a$  is regular if and only if  $\nabla a/\rho \subseteq \blacktriangledown a$ .*

**Proof:** Suppose  $a$  is regular, that is,  $a = aba$  for some  $b \in T(X, \rho, R)$ . Let  $r\rho \cap \nabla a \in \nabla a/\rho$  ( $r \in R$ ). Since  $b \in T(X, \rho, R)$ , there is  $p \in R$  such that  $(r\rho)b \subseteq p\rho$  and  $rb = p$ . Since  $r\rho \cap \nabla a \neq \emptyset$ ,  $r = ta$  for some  $t \in R$ , and so  $r = ta = ((ta)b)a = (rb)a = pa$ . It follows that  $(p\rho)a \subseteq r\rho \cap \nabla a$ . For the reverse inclusion, if  $xa \in r\rho \cap \nabla a$  then  $(xa)b \in p\rho$  and so  $xa = ((xa)b)a \in (p\rho)a$ . It follows that  $r\rho \cap \nabla a = (p\rho)a \in \blacktriangledown a$ , and so  $\nabla a/\rho \subseteq \blacktriangledown a$ .

Conversely, suppose  $\nabla a/\rho \subseteq \blacktriangledown a$ . We shall construct  $b \in T(X, \rho, R)$  such that  $a = aba$ . Consider  $r\rho$  ( $r \in R$ ). If  $r\rho \cap \nabla a = \emptyset$ , define  $xb = x$  for every  $x \in r\rho$ . Suppose  $r\rho \cap \nabla a \neq \emptyset$ . Then  $r\rho \cap \nabla a \in \nabla a/\rho \subseteq \blacktriangledown a$  and so there is  $p \in R$  such that  $r\rho \cap \nabla a = (p\rho)a$ . Let  $x \in r\rho$ . If  $x \in \nabla a$  then  $x = wa$  for some  $w \in p\rho$  (if  $x = r$ , we may assume  $w = p$ ), and we define  $xb = w$ . If  $x \notin \nabla a$ , we define  $xb = p$ .

By the construction of  $b$ ,  $b \in T(X, \rho, R)$  and  $a = aba$ . Thus  $a$  is regular. ■

Using the fact that  $a \in T(X, \rho, R)$  is regular if and only if  $a\mathcal{R}e$  for some idempotent  $e \in T(X, \rho, R)$ , we can obtain another characterization of the regular elements of  $T(X, \rho, R)$ .

For  $a \in T(X, \rho, R)$ , let  $\rho_a = \rho \vee \text{Ker}(a)$ , that is,  $\rho_a$  is the smallest equivalence relation on  $X$  that contains both  $\rho$  and  $\text{Ker}(a)$ . Note that every  $\rho_a$ -class  $x\rho_a$  is a union of  $\rho$ -classes and a union of  $\text{Ker}(a)$ -classes.

**Lemma 3.2** *For every  $x \in X$ ,  $(x\rho_a)a \subseteq r\rho$  for some  $r \in R$ .*

**Proof:** Let  $x \in X$ . Then  $xa \in r\rho$  for some  $r \in R$ . We claim that  $(x\rho_a)a \subseteq r\rho$ . Let  $y \in x\rho_a$ . Since  $\rho_a = \rho \vee \text{Ker}(a)$ , there are  $z_1, z_2, \dots, z_{2n-1} \in X$  ( $n \geq 1$ ) such that

$$(x, z_1) \in \rho, (z_1, z_2) \in \text{Ker}(a), (z_2, z_3) \in \rho, \dots, (z_{2n-1}, y) \in \text{Ker}(a).$$

Since  $a \in T(X, \rho, R)$ ,  $(xa, z_1a) \in \rho$  and so  $z_1a \in r\rho$ . Thus, since  $z_1a = z_2a$ ,  $z_2a \in r\rho$ . It follows by induction on  $n$  that  $ya \in r\rho$ , and so  $(x\rho_a)a \subseteq r\rho$ . ■

We say that an element  $a \in T(X, \rho, R)$  is *normal* if for every  $\rho_a$ -class  $x\rho_a$  there is a  $\rho$ -class  $r\rho$  that intersects all  $\text{Ker}(a)$ -classes included in  $x\rho_a$ . (Note that such a  $\rho$ -class  $r\rho$  must be included in  $x\rho_a$ .)

**Lemma 3.3** *Let  $a \in T(X, \rho, R)$  with  $X/\rho_a = \{E_i : i \in I\}$ . Then  $a$  is normal if and only if for every  $i \in I$  there is  $x_i \in E_i$  such that for every  $y \in E_i$ ,  $(x_i, y) \in \rho \circ \text{Ker}(a)$ .*

**Proof:** Suppose  $a$  is normal and let  $i \in I$ . Then there is  $x_i \in E_i$  such that  $(x_i\rho) \cap K \neq \emptyset$  for every  $\text{Ker}(a)$ -class  $K$  included in  $E_i$ . Let  $y \in E_i$ . Then  $y \in K$  for some  $\text{Ker}(a)$ -class  $K$ . Let  $z \in (x_i\rho) \cap K$ . Then  $(x_i, z) \in \rho$  and  $(z, y) \in \text{Ker}(a)$ . Thus  $(x_i, y) \in \rho \circ \text{Ker}(a)$ .

Conversely, suppose that the given condition holds and let  $i \in I$ . Then there is  $x_i \in E_i$  such that  $(x_i, y) \in \rho \circ \text{Ker}(a)$  for every  $y \in E_i$ . Let  $K$  be a  $\text{Ker}(a)$ -class included in  $E_i$  and let  $y \in K$ . Then, since  $(x_i, y) \in \rho \circ \text{Ker}(a)$ , there is  $z \in x_i\rho$  such that  $(z, y) \in \text{Ker}(a)$ . Thus  $z \in (x_i\rho) \cap K$ . It follows that  $x_i\rho$  intersects all  $\text{Ker}(a)$ -classes included in  $E_i$ , and so  $a$  is normal. ■

**Lemma 3.4** *Let  $e \in T(X, \rho, R)$  be an idempotent. Then  $e$  is normal.*

**Proof:** Consider  $x\rho_e$  ( $x \in X$ ). By Lemma 3.2,  $(x\rho_e)e \subseteq r\rho$  for some  $r \in R$ . We claim that  $r\rho$  intersects all  $\text{Ker}(e)$ -classes included in  $x\rho_e$ . Let  $K$  be a  $\text{Ker}(e)$ -class included in  $x\rho_e$ . Then  $Ke = \{y\}$  for some  $y \in r\rho$ . Since  $e$  is an idempotent and  $y \in \nabla e$ ,  $y = ye$ . Thus  $y \in K$  and so  $r\rho \cap K \neq \emptyset$ . It follows that  $e$  is normal. ■

**Corollary 3.5** *Let  $a, e \in T(X, \rho, R)$  such that  $e$  is an idempotent and  $\text{Ker}(a) = \text{Ker}(e)$ . Then  $a$  is normal.*

**Proof:** Since  $\text{Ker}(a) = \text{Ker}(e)$ ,  $\rho_a = \rho_e$ . Thus the result follows from Lemma 3.4 and the definition of normal elements. ■

**Theorem 3.6** *Let  $a \in T(X, \rho, R)$  with  $X/\rho_a = \{E_i : i \in I\}$ . Then the following are equivalent:*

- (1)  $a$  is regular.
- (2)  $a$  is normal.
- (3)  $(\forall i \in I)(\exists x_i \in E_i)(\forall y \in E_i) (x_i, y) \in \rho \circ \text{Ker}(a)$ .

**Proof:** (2) is equivalent to (3) by Lemma 3.3. Suppose  $a$  is regular. Then  $a\mathcal{R}e$  for some idempotent  $e \in T(X, \rho, R)$ . By Theorem 2.3,  $\text{Ker}(a) = \text{Ker}(e)$ . Thus  $a$  is normal by Corollary 3.5. Hence (1) implies (2).

It remains to show that (2) implies (1). Suppose  $a$  is normal. Then for every  $i \in I$  there is  $x_i \in E_i$  such that  $(x_i\rho) \cap K \neq \emptyset$  for every  $\text{Ker}(a)$ -class  $K$  included in  $E_i$ . We shall construct an idempotent  $e \in T(X, \rho, R)$  such that  $\text{Ker}(e) = \text{Ker}(a)$ . Let  $K$  be a  $\text{Ker}(a)$ -class. Then there is a unique  $i \in I$  such that  $K \subseteq E_i$ . Select  $y_i \in x_i\rho \cap K$  in such a way that  $y_i = r_{x_i}$  if  $r_{x_i} \in x_i\rho \cap K$ . Define  $e \in T(X)$  by  $Ke = \{y_i\}$ .

It is clear that  $\text{Ker}(e) = \text{Ker}(a)$  and that  $e$  preserves  $\rho$  (since it maps all  $\rho$ -classes included in  $E_i$  to  $x_i\rho$ ). By Lemma 3.2, all elements of  $R$  contained in  $E_i$  are in the same  $\text{Ker}(a)$ -class. Thus  $e$  maps all such elements to  $r_{x_i}$  and so it preserves  $R$ . Hence  $e \in T(X, \rho, R)$ . By Theorem 2.3,  $a\mathcal{R}e$  and so  $a$  is regular. ■

Let  $\rho$  be an equivalence relation on  $X$ . We say that  $\rho$  is a  $T$ -relation if there is at most one  $\rho$ -class containing two or more elements. If there is  $n \geq 1$  such that each  $\rho$ -class has at most  $n$  elements, we say that  $\rho$  is  $n$ -bounded.

The following theorem characterizes the equivalence relations  $\rho$  on  $X$  for which the semigroup  $T(X, \rho, R)$  is regular.

**Theorem 3.7** *The semigroup  $T(X, \rho, R)$  is regular if and only if  $\rho$  is 2-bounded or a  $T$ -relation.*

**Proof:** Suppose  $\rho$  is neither 2-bounded nor a  $T$ -relation. Then there are  $r, s \in R$  such that  $r \neq s$  and

$$r\rho = \{r, x_1, x_2, \dots\} \quad \text{and} \quad s\rho = \{s, y_1, \dots\}.$$

Consider the mapping

$$a = \begin{pmatrix} r & x_1 & x_2 & s & y_1 & z \\ r & x_1 & x_1 & r & x_2 & r \end{pmatrix},$$

where  $z$  denotes an arbitrary element in  $X - \{r, x_1, x_2, s, y_1\}$ . Then  $a \in T(X, \rho, R)$  with  $\nabla a / \rho = \{\{r, x_1, x_2\}\}$  and either  $\nabla a = \{\{r, x_1\}, \{r, x_2\}\}$  or  $\nabla a = \{\{r, x_1\}, \{r, x_2\}, \{r\}\}$ . In either case,  $\nabla a / \rho$  is not included in  $\nabla a$ , which implies that  $a$  is not regular (by Theorem 3.1).

Conversely, suppose that  $\rho$  is 2-bounded or a  $T$ -relation and let  $a \in T(X, \rho, R)$ . We shall prove that  $\nabla a / \rho \subseteq \nabla a$ . Let  $r\rho \cap \nabla a \in \nabla a / \rho$ . Then there is  $p \in R$  such that  $r = pa$  and  $(p\rho)a \subseteq r\rho$ .

Suppose  $r\rho$  has at least 3 elements. Then  $\rho$  is not 2-bounded and so it must be a  $T$ -relation. Thus every  $\rho$ -class except  $r\rho$  has 1 element. Hence  $r\rho \cap \nabla a = (r\rho)a$  (if  $(r\rho)a \subseteq r\rho$ ) or  $r\rho \cap \nabla a = \{r\} = (p\rho)a$  (if  $(r\rho)a$  is not included in  $r\rho$ ). Suppose  $r\rho = \{r, x\}$  has 2 elements. If  $x \in \nabla a$  then  $x \in (s\rho)a$  for some  $s \in R$ , and so  $r\rho \cap \nabla a = \{r, x\} = (s\rho)a$ . If  $x \notin \nabla a$  then  $r\rho \cap \nabla a = \{r\} = (p\rho)a$ . Finally, if  $r\rho$  has 1 element then  $r\rho \cap \nabla a = \{r\} = (p\rho)a$ .

It follows that  $r\rho \cap \nabla a \in \nabla a$ , and so  $a$  is regular by Theorem 3.1. ■

There is an asymmetry between the relations  $\mathcal{R}$  and  $\mathcal{L}$  in  $T(X, \rho, R)$ : while the  $\mathcal{R}$ -relation is simply the restriction of the  $\mathcal{R}$ -relation in  $T(X)$  to  $T(X, \rho, R) \times T(X, \rho, R)$ , the corresponding result is not true in general for the  $\mathcal{L}$ -relation. The following theorem determines the semigroups  $T(X, \rho, R)$  in which the  $\mathcal{L}$ -relation is the restriction of the  $\mathcal{L}$ -relation in  $T(X)$ .

**Theorem 3.8** *The  $\mathcal{L}$ -relation in  $T(X, \rho, R)$  is the restriction of the  $\mathcal{L}$ -relation in  $T(X)$  to  $T(X, \rho, R) \times T(X, \rho, R)$  if and only if  $T(X, \rho, R)$  is regular.*

**Proof:** Suppose  $T(X, \rho, R)$  is not regular. Then, by Theorem 3.7, there are  $r, s \in R$  such that  $r \neq s$  and

$$r\rho = \{r, x_1, x_2, \dots\} \quad \text{and} \quad s\rho = \{s, y_1, \dots\}.$$

Consider the mappings

$$a = \begin{pmatrix} r & x_1 & x_2 & s & y_1 & z \\ r & x_1 & x_1 & r & x_2 & r \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} r & x_1 & x_2 & s & y_1 & z \\ r & x_1 & x_2 & r & r & r \end{pmatrix},$$

where  $z$  is an arbitrary element in  $X - \{r, x_1, x_2, s, y_1\}$ . Then  $a, b \in T(X, \rho, R)$  with  $\nabla a = \nabla b = \{r, x_1, x_2\}$ . Thus, by Lemma 2.1,  $a \mathcal{L} b$  in  $T(X)$ . However,  $\{r, x_1, x_2\} \in \nabla b$  and  $\{r, x_1, x_2\}$  is not included in any  $A \in \nabla a$  (since  $\nabla a = \{\{r, x_1\}, \{r, x_2\}\}$  or  $\nabla a = \{\{r, x_1\}, \{r, x_2\}, \{r\}\}$ ). Thus, by Theorem 2.5,  $a$  and  $b$  are not  $\mathcal{L}$ -related in  $T(X, \rho, R)$ .

The converse follows from a general result saying that if  $T$  is a regular subsemigroup of a semigroup  $S$  then the relations  $\mathcal{L}$  and  $\mathcal{R}$  in  $T$  are the restrictions of the relations  $\mathcal{L}$  and  $\mathcal{R}$ , respectively, in  $S$  to  $T \times T$  [5, Proposition 2.4.2]. ■

## 4 Abundant $T(X, \rho, R)$

Let  $S$  be a semigroup. We say that  $a, b \in S$  are  $\mathcal{L}^*$ -related if they are  $\mathcal{L}$ -related in a semigroup  $T$  such that  $S$  is a subsemigroup of  $T$ . We have the dual definition of the  $\mathcal{R}^*$ -relation on  $S$  [3]. The relations  $\mathcal{L}^*$  and  $\mathcal{R}^*$  are equivalence relations. They have been studied by J. Fountain [2], [3] and others. A semigroup  $S$  is called *abundant* if every  $\mathcal{L}^*$ -class and every  $\mathcal{R}^*$ -class of  $S$  contains an idempotent [3]. As stated in [3], where the concept was introduced, the word “abundant” comes from the fact that such semigroups have a plentiful supply of idempotents.

It is clear from the definition of  $\mathcal{L}^*$  and  $\mathcal{R}^*$  that  $\mathcal{L} \subseteq \mathcal{L}^*$  and  $\mathcal{R} \subseteq \mathcal{R}^*$  in any semigroup  $S$ . Since in a regular semigroup, every  $\mathcal{L}$ -class and every  $\mathcal{R}$ -class contains an idempotent, we have that regular semigroups are abundant. Of course, the converse is not true. For example, A. Umar [12] proved that the semigroup of non-bijective, order-decreasing transformations on the set  $X = \{1, \dots, n\}$  is abundant but not regular.

We first note that every  $\mathcal{L}^*$ -class of  $T(X, \rho, R)$  contains an idempotent.

**Proposition 4.1** *Let  $a \in T(X, \rho, R)$ . Then there is an idempotent  $e \in T(X, \rho, R)$  such that  $\nabla a = \nabla e$ .*

**Proof:** Select  $r_0 \in \nabla a \cap R$  and define  $e \in T(X)$  as follows:

$$xe = \begin{cases} x & \text{if } x \in \nabla a \\ r_x & \text{if } x \notin \nabla a \text{ but } x\rho \cap \nabla a \neq \emptyset \\ r_0 & \text{if } x\rho \cap \nabla a = \emptyset. \end{cases}$$

By the definition of  $e$  and the fact that  $a \in T(X, \rho, R)$ , we have that  $e \in T(X, \rho, R)$ ,  $e$  is an idempotent, and  $\nabla e = \nabla a$ . ■

The statement preceding Proposition 4.1 follows since, by Lemma 2.1, the elements  $a$  and  $e$  are  $\mathcal{L}$ -related in  $T(X)$ . The corresponding statement for  $\mathcal{R}^*$ -classes of  $T(X, \rho, R)$  is not true, and so not every semigroup  $T(X, \rho, R)$  is abundant. Similar results have been obtained by A. Umar [13] for the semigroup of order-decreasing transformations on an infinite totally ordered set  $X$ . In contrast with Umar [13], who showed that in the class he studied the abundant semigroups are not regular, we find that abundant semigroups  $T(X, \rho, R)$  are precisely those that are regular.

**Theorem 4.2** *A semigroup  $T(X, \rho, R)$  is abundant if and only if it is regular.*

**Proof:** If  $T(X, \rho, R)$  is regular then it is abundant (since every regular semigroup is abundant). Conversely, suppose that  $T(X, \rho, R)$  is abundant, and let  $a \in T(X, \rho, R)$ . Then there is an idempotent  $e \in T(X, \rho, R)$  such that  $a\mathcal{R}^*e$ . By [3, the dual of Corollary 1.2],  $ea = a$  and for all  $c, d \in T(X, \rho, R)$ ,  $ca = da$  implies  $ce = de$ . We claim that  $\text{Ker}(a) = \text{Ker}(e)$ .

The inclusion  $\text{Ker}(e) \subseteq \text{Ker}(a)$  follows immediately from  $ea = a$ . Suppose  $(x, y) \in \text{Ker}(a)$ , that is,  $xa = ya$ . Since  $(x, r_x) \in \rho$ ,  $(y, r_y) \in \rho$ , and  $a \in T(X, \rho, R)$ , we have  $(xa, r_xa) \in \rho$  and  $(ya, r_ya) \in \rho$ . Thus, since  $xa = ya$ , we have  $(r_xa, r_ya) \in \rho$ , which implies  $r_xa = r_ya$  (since  $r_xa, r_ya \in R$  and  $R$  is a cross-section of  $X/\rho$ ).

Define  $c, d \in T(X)$  by :  $(X - R)c = \{x\}$ ,  $Rc = \{r_x\}$ ,  $(X - R)d = \{y\}$ , and  $Rd = \{r_y\}$ . It is clear that  $c, d \in T(X, \rho, R)$ , and that there is  $z_0 \in X$  such that  $z_0c = x$  and  $z_0d = y$ . Let  $z \in X$ . If  $z \in X - R$  then  $z(ca) = xa = ya = z(da)$ . If  $z \in R$  then  $z(ca) = r_xa = r_ya = z(da)$ . Hence  $ca = da$ , which implies  $ce = de$ . In particular,  $z_0(ce) = z_0(de)$ , which implies  $xe = ye$  (since  $z_0c = x$  and  $z_0d = y$ ). Hence  $\text{Ker}(a) \subseteq \text{Ker}(e)$ , and so  $\text{Ker}(a) = \text{Ker}(e)$ .

Thus  $a$  is normal (by Corollary 3.5), and so  $a$  is regular (by Theorem 3.6). It follows that  $T(X, \rho, R)$  is a regular semigroup. ■

## 5 Inverse $T(X, \rho, R)$ and Completely Regular $T(X, \rho, R)$

An element  $a'$  in a semigroup  $S$  is called an *inverse* of  $a \in S$  if  $a = aa'a$  and  $a' = a'aa'$ . If every element of  $S$  has exactly one inverse then  $S$  is called an *inverse semigroup*. An alternative definition is that  $S$  is an inverse semigroup if it is regular and its idempotents commute [5, Theorem 5.1.1]. If every element of  $S$  is in some subgroup of  $S$  then  $S$  is called a *completely regular semigroup*. Of course, both inverse semigroups and completely regular semigroups are regular semigroups.

**Theorem 5.1** Suppose  $|X| \geq 3$ . Then  $T(X, \rho, R)$  is not an inverse semigroup.

**Proof:** We shall construct idempotents  $e, f \in T(X, \rho, R)$  such that  $ef \neq fe$ .

Suppose there are at least two  $\rho$ -classes, that is, there are  $r\rho$  and  $s\rho$  ( $r, s \in R$ ) such that  $r \neq s$ . Define  $e, f \in T(X)$  by:

$$e = \begin{pmatrix} y & z \\ r & z \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} y & z \\ s & z \end{pmatrix},$$

where  $y$  is an arbitrary element in  $r\rho \cup s\rho$  and  $z$  is an arbitrary element in  $X - (r\rho \cup s\rho)$ . Note that  $r(ef) = s$  and  $r(fe) = r$ .

Suppose there is only one  $\rho$ -class, say  $r\rho$ . Since  $|X| \geq 3$ ,  $r\rho = \{r, x_1, x_2, \dots\}$ . Define  $e, f \in T(X)$  by:

$$e = \begin{pmatrix} x_1 & y \\ x_2 & y \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} x_2 & z \\ x_1 & z \end{pmatrix},$$

where  $y$  is an arbitrary element in  $X - \{x_1\}$  and  $z$  is an arbitrary element in  $X - \{x_2\}$ . Note that  $x_1(ef) = x_1$  and  $x_1(fe) = x_2$ .

In both cases we have:  $e, f \in T(X, \rho, R)$ ,  $e, f$  are idempotents, and  $ef \neq fe$ . It follows that  $T(X, \rho, R)$  is not an inverse semigroup (since idempotents in an inverse semigroup commute). ■

When  $|X| = 2$ ,  $T(X, \rho, R)$  is an inverse semigroup if  $X/\rho = \{\{r, x\}\}$  and  $T(X, \rho, R)$  is not inverse if  $X/\rho = \{\{r\}, \{s\}\}$ .

**Theorem 5.2** Suppose  $|X| \geq 4$ . Then  $T(X, \rho, R)$  is not a completely regular semigroup.

**Proof:** We shall construct  $a \in T(X, \rho, R)$  such that  $\nabla a \neq \nabla a^2$ .

Suppose there is a  $\rho$ -class with at least three elements, say  $r\rho = \{r, x_1, x_2, \dots\}$  ( $r \in R$ ). Define  $a \in T(X)$  by:

$$a = \begin{pmatrix} x_1 & z \\ x_2 & r \end{pmatrix},$$

where  $z$  is an arbitrary element in  $X - \{x_1\}$ . Note that  $\nabla a = \{r, x_2\}$  and  $\nabla a^2 = \{r\}$ .

Suppose there are at least three  $\rho$ -classes, that is, there are  $r\rho$ ,  $s\rho$ , and  $t\rho$  with  $r, s, t \in R$  pairwise distinct. Define  $a \in T(X)$  by:

$$a = \begin{pmatrix} y & z \\ s & t \end{pmatrix},$$

where  $y$  is an arbitrary element in  $r\rho$  and  $z$  is an arbitrary element in  $X - r\rho$ . Note that  $\nabla a = \{s, t\}$  and  $\nabla a^2 = \{t\}$ .

Since  $|X| \geq 4$ , the only remaining case to consider is when there are exactly two  $\rho$ -classes with two elements each, say  $r\rho = \{r, x\}$  and  $s\rho = \{s, y\}$  ( $r, s \in R$ ). Define  $a \in T(X)$  by:

$$a = \begin{pmatrix} x & z \\ y & s \end{pmatrix},$$

where  $z$  is an arbitrary element in  $X - \{x\}$ . Note that  $\nabla a = \{s, y\}$  and  $\nabla a^2 = \{s\}$ .

In all cases we have:  $a \in T(X, \rho, R)$  and  $\nabla a \neq \nabla a^2$ . By Lemma 2.1,  $\nabla a \neq \nabla a^2$  implies that  $a$  and  $a^2$  are not  $\mathcal{H}$ -related in  $T(X)$  (not even  $\mathcal{L}$ -related in  $T(X)$ ), and so they are not  $\mathcal{H}$ -related in  $T(X, \rho, R)$ . It follows that  $T(X, \rho, R)$  is not completely regular (since for every element  $a$  in a completely regular semigroup,  $a$  and  $a^2$  are  $\mathcal{H}$ -related [5, Proposition 4.1.1]). ■

When  $|X| = 3$ ,  $T(X, \rho, R)$  is completely regular if  $X/\rho = \{\{r, x\}, \{s\}\}$ , and  $T(X, \rho, R)$  is not completely regular if  $X/\rho = \{\{r\}, \{s\}, \{t\}\}$  or  $\{\{r, x_1, x_2\}\}$ . When  $|X| = 2$ ,  $T(X, \rho, R)$  is completely regular.

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