

Dense Relations Are Determined by Their Endomorphism Monoids

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Abstract

We introduce the class of dense relations on a set X and prove that for any finitary or infinitary dense relation ρ on X , the relational system (X, ρ) is determined up to semi-isomorphism by the monoid $\text{End}(X, \rho)$ of endomorphisms of (X, ρ) . In the case of binary relations, a semi-isomorphism is an isomorphism or an anti-isomorphism.

2000 *Mathematics Subject Classification*: 20M20, 20M15.

1 Introduction

For a mathematical structure M , let $\text{End}(M)$ denote the endomorphism monoid of M . A general problem, which attracted a considerable attention, can be stated as follows: Let M_1, M_2 be two mathematical structures. Given that $\text{End}(M_1) \cong \text{End}(M_2)$, what can we say about the relation between the structures M_1 and M_2 themselves? For example, Schein [8] proved that if M_1 and M_2 are two partially ordered sets, semilattices, distributive lattices, or Boolean algebras, then $\text{End}(M_1) \cong \text{End}(M_2)$ if and only if M_1 and M_2 are isomorphic or anti-isomorphic. For other results of this kind, see [2], [4], and [5].

The aim of this note is to prove a similar result for the class of dense relations. These relations include partial orders, binary relations that are reflexive and symmetric, and generalized equivalence relations.

Let I be an arbitrary non-empty index set. An I -tuple of elements of a set X is a mapping $f : I \rightarrow X$. If I is finite with $|I| = n$, we shall assume that $I = \{1, 2, \dots, n\}$, denote $f : I \rightarrow X$ by $(1f, 2f, \dots, nf)$, and refer to f as an n -tuple. An I -relation ρ on X is any set of I -tuples of elements of X . If $|I| = n$, an I -relation ρ is an n -ary relation on X , that is, a set of n -tuples of elements of X . By a *relation* on X , we shall mean an I -relation on X for some index set I .

Let ρ be an I -relation on X . An *endomorphism* of a relational system (X, ρ) is a mapping $a : X \rightarrow X$ that preserves ρ , that is, $fa \in \rho$ for every $f \in \rho$, where $fa : I \rightarrow X$ is the composition of $f : I \rightarrow X$ and $a : X \rightarrow X$. (We compose from left to right, that is, $i(fa) = (if)a$ for $i \in I$.) If ρ is an n -ary relation, we can use the n -tuple notation. With that notation, we have that $a : X \rightarrow X$ is an endomorphism of (X, ρ) if and only if $(x_1a, \dots, x_na) \in \rho$ for every $(x_1, \dots, x_n) \in \rho$. We denote by $\text{End}(X, \rho)$ the monoid of endomorphisms of (X, ρ) .

Turning to the definition of a dense relation, we denote by ρ^* the set of all mappings $f : I \rightarrow X$ such that $\sigma f \notin \rho$ for every $\sigma \in S(I)$, where $S(I)$ is the symmetric group of I .

That is,

$$\rho^* = \{f : I \rightarrow X : (\forall \sigma \in S(I)) \sigma f \notin \rho\}.$$

A reflexive I -relation ρ on X is said to be *dense* if it satisfies the following two properties:

- (D₁) For every injective $f_1 \in \rho \cup \rho^*$ and every $f \in \rho$, there is $a \in \text{End}(X, \rho)$ such that $f_1 a = f$.
- (D₂) There is an injective f_1 in ρ .

Examples of dense relations (for details see [1]) are partial orders, binary relations that are reflexive and symmetric, generalized equivalence relations (see [6]), relations defined by families of sets intersecting in at most one element (see [7]).

Let ρ_1, ρ_2 be I -relations on X_1, X_2 , respectively. We say that a bijection $g : X_1 \rightarrow X_2$ is a *semi-isomorphism* of (X_1, ρ_1) to (X_2, ρ_2) if there is a permutation $\sigma \in S(I)$ such that for all $f : I \rightarrow X_1$,

$$f \in \rho_1 \Leftrightarrow \sigma f g \in \rho_2.$$

We say that relational systems (X_1, ρ_1) and (X_2, ρ_2) are *semi-isomorphic* if there is a semi-isomorphism from (X_1, ρ_1) to (X_2, ρ_2) .

Note that if $I = \{1, 2\}$ then the only elements of $S(I)$ are id_I (the identity permutation of I) and the transposition $(1\ 2)$. It follows that if ρ_1 and ρ_2 are binary relations then any semi-isomorphism $g : X_1 \rightarrow X_2$ is either an isomorphism $((x, y) \in \rho_1 \Leftrightarrow (xg, yg) \in \rho_2)$ or an anti-isomorphism $((x, y) \in \rho_1 \Leftrightarrow (yg, xg) \in \rho_2)$.

In the next section, we prove that if ρ is a dense I -relation on X then $\text{End}(X, \rho)$ determines ρ up to a semi-isomorphism.

2 Main Theorem and Its Application to Binary Relations

The following lemma belongs to the folklore (see [2], for example) and we include a proof just for the sake of completeness.

Lemma 2.1 *Let ρ_1, ρ_2 be reflexive I -relations on X_1, X_2 , respectively, and suppose that $\phi : \text{End}(X_1, \rho_1) \rightarrow \text{End}(X_2, \rho_2)$ is an isomorphism. Then there exists a bijection $g : X_1 \rightarrow X_2$ such that $a\phi = g^{-1}ag$ for every $a \in \text{End}(X_1, \rho_1)$.*

Proof: For a set X and $x \in X$, we denote by X_x the constant mapping from X to X defined by: $yX_x = x$ for every $y \in X$. Let S be a semigroup of mappings from X to X with $X_x \in S$ for some $x \in X$. It is easy to see that $\{X_x\}$ is a minimal left ideal of S . In fact, all minimal left ideals of S are of the form $\{X_y\}$ where $y \in X$. For suppose J is a minimal left ideal of S and let $a \in J$. Then $X_x a \in J$ and $X_x a = X_{xa}$. Since $\{X_{xa}\} \subseteq J$ and J is a minimal left ideal, it follows that $\{X_{xa}\} = J$. Denote by $\mathcal{M}(S)$ the set of minimal left ideals of S . We proved that $\mathcal{M}(S) = \{\{X_x\} : X_x \in S\}$.

Since $\text{End}(X_1, \rho_1)$ and $\text{End}(X_2, \rho_2)$ contain all constant mappings on X_1 and X_2 , respectively, we have $\mathcal{M}(\text{End}(X_1, \rho_1)) = \{\{(X_1)_x\} : x \in X_1\}$ and $\mathcal{M}(\text{End}(X_2, \rho_2)) = \{\{(X_2)_y\} : y \in X_2\}$. Since an isomorphism must map minimal left ideals to minimal left ideals, it follows that $\{(X_1)_x\phi : x \in X_1\} = \{(X_2)_y : y \in X_2\}$, so that $|X_1| = |X_2|$. Define $g : X_1 \rightarrow X_2$ by: $xg = y$ if $(X_1)_x\phi = (X_2)_y$. Then g is a bijection and $(X_1)_x\phi = (X_2)_{xg}$ for every $x \in X_1$.

Let $x \in X_1$ and $a \in \text{End}(X_1, \rho_1)$. We claim that $(xg)(a\phi) = (xa)g$. Indeed, $(xg)(X_2)_{xg}(a\phi) = (xg)((X_1)_x\phi)(a\phi) = (xg)((X_1)_{xa}\phi) = (xg)((X_1)_{xa}\phi) = (xg)(X_2)_{(xa)g}$. Therefore $(xg)(a\phi) = (xg)(X_2)_{xg}(a\phi) = (xg)(X_2)_{(xa)g} = (xa)g$. It follows that $g(a\phi) = ag$ and hence $a\phi = g^{-1}ag$. ■

Note that if g is as in the statement of Lemma 2.1 then for all $a \in \text{End}(X_1, \rho_1)$ and $b \in \text{End}(X_2, \rho_2)$, $g^{-1}ag \in \text{End}(X_2, \rho_2)$ and $gbg^{-1} \in \text{End}(X_1, \rho_1)$.

Lemma 2.2 *Let ρ_1, ρ_2 be dense I -relations on X_1, X_2 , respectively, and let $g : X_1 \rightarrow X_2$ be a bijection such that $g^{-1}ag \in \text{End}(X_2, \rho_2)$ for every $a \in \text{End}(X_1, \rho_1)$. Then for every $f \in \rho_1$ there is a permutation $\sigma \in S(I)$ such that $\sigma fg \in \rho_2$.*

Proof: Fix an injective $f_2 \in \rho_2$ (which exists by (D_2)) and let $f_1 = f_2g^{-1}$. Note that f_1 is injective and $f_2 = f_1g$. Let f be an arbitrary element of ρ_1 .

Suppose $f_1 \in \rho_1^*$. Then, by (D_1) , there is $a \in \text{End}(X_1, \rho_1)$ such that $f_1a = f$. Thus, since $g^{-1}ag \in \text{End}(X_2, \rho_2)$, we have

$$f_2 \in \rho_2 \Rightarrow f_2(g^{-1}ag) \in \rho_2 \Rightarrow f_1ag \in \rho_2 \Rightarrow fg \in \rho_2 \Rightarrow id_I fg \in \rho_2.$$

Suppose $f_1 \notin \rho_1^*$. Then $\sigma f_1 \in \rho_1$ for some $\sigma \in S(I)$. By (D_1) , there is $a \in \text{End}(X_1, \rho_1)$ such that $\sigma f_1a = f$. Then $f_1a = \sigma^{-1}f$ and we have

$$f_2 \in \rho_2 \Rightarrow f_2(g^{-1}ag) \in \rho_2 \Rightarrow f_1ag \in \rho_2 \Rightarrow \sigma^{-1}fg \in \rho_2.$$

The lemma follows. ■

Lemma 2.3 *With the hypothesis of Lemma 2.2, let $f_1 \in \rho_1$ be injective and let $\sigma \in S(I)$. If $\sigma f_1g \in \rho_2$ then $\sigma fg \in \rho_2$ for every $f \in \rho_1$.*

Proof: Suppose $\sigma f_1g \in \rho_2$ and let $f \in \rho_1$. By (D_1) , there is $a \in \text{End}(X_1, \rho_1)$ such that $f_1a = f$. Since $g^{-1}ag \in \text{End}(X_2, \rho_2)$, we have

$$\sigma f_1g \in \rho_2 \Rightarrow \sigma f_1g(g^{-1}ag) \in \rho_2 \Rightarrow \sigma f_1ag \in \rho_2 \Rightarrow \sigma fg \in \rho_2,$$

which concludes the proof. ■

We can now prove the main theorem of this note.

Theorem 2.4 *Let ρ_1, ρ_2 be dense I -relations on X_1, X_2 , respectively. Then the endomorphism monoids $\text{End}(X_1, \rho_1)$ and $\text{End}(X_2, \rho_2)$ are isomorphic if and only if the relational systems (X_1, ρ_1) and (X_2, ρ_2) are semi-isomorphic.*

Proof: Suppose $\phi : \text{End}(X_1, \rho_1) \rightarrow \text{End}(X_2, \rho_2)$ is an isomorphism. By Lemma 2.1, there is a bijection $g : X_1 \rightarrow X_2$ such that $a\phi = g^{-1}ag$ for every $a \in \text{End}(X_1, \rho_1)$. We claim that g is a semi-isomorphism from (X_1, ρ_1) to (X_2, ρ_2) . Since ρ_1 and ρ_2 are dense, there are $f_1 \in \rho_1$ and $f_2 \in \rho_2$ such that f_1 and f_2 are injective. By Lemma 2.2, there is $\sigma \in S(I)$ such that $\sigma f_1g \in \rho_2$. By Lemma 2.3, $\sigma fg \in \rho_2$ for every $f \in \rho_1$. Now, note that $f'_2 = \sigma f_1g$ is an injective element of ρ_2 and $\sigma^{-1}f'_2g^{-1} = f_1 \in \rho_1$. Thus, by

Lemma 2.3, $\sigma^{-1}f'g^{-1} \in \rho_1$ for every $f' \in \rho_2$. It follows that for every $f : I \rightarrow X_1$, $f \in \rho_1 \Leftrightarrow \sigma fg \in \rho_2$, and so g is a semi-isomorphism.

Conversely, suppose that $g : X_1 \rightarrow X_2$ is a semi-isomorphism, and let $\sigma \in S(I)$ be such that for every $f : I \rightarrow X_1$, $f \in \rho_1 \Leftrightarrow \sigma fg \in \rho_2$. Then for all $a \in \text{End}(X_1, \rho_1)$ and $f' : I \rightarrow X_2$,

$$f' \in \rho_2 \Rightarrow \sigma^{-1}f'g^{-1} \in \rho_1 \Rightarrow \sigma^{-1}f'g^{-1}a \in \rho_1 \Rightarrow \sigma\sigma^{-1}f'g^{-1}ag \in \rho_2 \Rightarrow f'g^{-1}ag \in \rho_2.$$

It follows that $\phi : \text{End}(X_1, \rho_1) \rightarrow \text{End}(X_2, \rho_2)$ defined by: $a\phi = g^{-1}ag$ is an isomorphism. ■

Applying Theorem 2.4 to binary relations, we obtain the following corollary.

Corollary 2.5 *Let ρ_1, ρ_2 be binary dense relations on X_1, X_2 , respectively. Then the endomorphism monoids $\text{End}(X_1, \rho_1)$ and $\text{End}(X_2, \rho_2)$ are isomorphic if and only if the relational systems (X_1, ρ_1) and (X_2, ρ_2) are isomorphic or anti-isomorphic.*

Problem Classify the n -ary dense relations, for $n \geq 2$.

References

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