KAWSAKI DYNAMICS IN THE CONTINUUM VIA GENERATING FUNCTIONALS EVOLUTION

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To the memory of A.G. Kostyuchenko.

Abstract. We construct the time evolution of Kawasaki dynamics for a spatial infinite particle system in terms of generating functionals. This is carried out by an Ovsjannikov-type result in a scale of Banach spaces, which leads to a local (in time) solution. An application of this approach to Vlasov-type scaling in terms of generating functionals is considered as well.

1. INTRODUCTION

Originally, Bogoliubov generating functionals (GF for short) were introduced by N. N. Bogoliubov in [2] to define correlation functions for statistical mechanics systems. Apart from this specific application, and many others, GF are, by themselves, a subject of interest in infinite dimensional analysis. This is partially due to the fact that to a probability measure $\mu$ defined on the space $\Gamma$ of locally finite configurations $\gamma \subset \mathbb{R}^d$ one may associate a GF

$$B_\mu(\theta) := \int_\Gamma d\mu(\gamma) \prod_{x \in \gamma} (1 + \theta(x)),$$

yielding an alternative method to study the stochastic dynamics of an infinite particle system in the continuum by exploiting the close relation between measures and GF [4, 9].

Existence and uniqueness results for the Kawasaki dynamics through GF arise naturally from Picard-type approximations and a method by A. G. Kostyuchenko and G. E. Shilov presented in [6, Appendix 2, A2.1] in a scale of Banach spaces (see e.g. [5, Theorem 2.5]). This method, originally presented for equations with coefficients time independent, has been extended to an abstract and general framework by T. Yamanaka in [12] and L. V. Ovsjannikov in [10] in the linear case, and many applications were exposed by F. Treves in [11]. As an aside, within an analytical framework outside of our setting, all these statements are very closely related to variants of the abstract Cauchy-Kovalevskaya theorem. However, all these abstract forms only yield a local solution, that is, a solution which is defined on a finite time interval. Moreover, starting with an initial condition from a certain Banach space, in general the solution evolves on larger Banach spaces.

As a particular application, this work concludes with the study of the Vlasov-type scaling proposed in [3] for general continuous particle systems and accomplished in [1] for the Kawasaki dynamics. The general scheme proposed in [3] for correlation functions yields a limiting hierarchy which possesses a chaos preservation property, namely, starting...
with a Poissonian (non-homogeneous) initial state this structural property is preserved during the time evolution. In Section 4 the same problem is formulated in terms of GF and its analysis is carried out by the general Ovsjannikov-type result in a scale of Banach spaces presented in [5, Theorem 4.3].

2. General framework

In this section we briefly recall the concepts and results of combinatorial harmonic analysis on configuration spaces and Bogoliubov generating functionals needed throughout this work (for a detailed explanation see [7, 9]).

2.1. Harmonic analysis on configuration spaces. Let \( \Gamma := \Gamma_{\mathbb{R}^d} \) be the configuration space over \( \mathbb{R}^d, \, d \in \mathbb{N} \),

\[
\Gamma := \{ \gamma \subset \mathbb{R}^d : |\gamma \cap \Lambda| < \infty \text{ for every compact } \Lambda \subset \mathbb{R}^d \},
\]

where \( |\cdot| \) denotes the cardinality of a set. We identify each \( \gamma \in \Gamma \) with the non-negative Radon measure \( \sum_{x \in \gamma} \delta_x \) on the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^d) \), where \( \delta_x \) is the Dirac measure with mass at \( x \), which allows to endow \( \Gamma \) with the vague topology and the corresponding Borel \( \sigma \)-algebra \( \mathcal{B}(\Gamma) \).

For any \( n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) let

\[
\Gamma^{(n)} := \{ \gamma \in \Gamma : |\gamma| = n \}, \quad n \in \mathbb{N}, \quad \Gamma^{(0)} := \{ \emptyset \}.
\]

Clearly, each \( \Gamma^{(n)} \), \( n \in \mathbb{N} \), can be identify with the symmetrization of the set \( \{ (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n : x_i \neq x_j \text{ if } i \neq j \} \), which induces a natural (metrizable) topology on \( \Gamma^{(n)} \) and the corresponding Borel \( \sigma \)-algebra \( \mathcal{B}(\Gamma^{(n)}) \). In particular, for the Lebesgue product measure \( (dx)^\otimes n \) fixed on \( (\mathbb{R}^d)^n \), this identification yields a measure \( m^{(n)} \) on \( (\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)})) \).

For \( n = 0 \) we set \( m^{(0)}(\{ \emptyset \}) := 1 \). This leads to the definition of the space of finite configurations

\[
\Gamma_0 := \bigcup_{n=0}^{\infty} \Gamma^{(n)}
\]

endowed with the topology of disjoint union of topological spaces and the corresponding Borel \( \sigma \)-algebra \( \mathcal{B}(\Gamma_0) \), and to the so-called Lebesgue-Poisson measure on \( (\Gamma_0, \mathcal{B}(\Gamma_0)) \),

\[
\lambda := \lambda_{dx} := \sum_{n=0}^{\infty} \frac{1}{n!} m^{(n)}.
\]

Let \( \mathcal{B}_c(\mathbb{R}^d) \) be the set of all bounded Borel sets in \( \mathbb{R}^d \) and, for each \( \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \), let \( \Gamma_{\Lambda} := \{ \eta \in \Gamma : \eta \subset \Lambda \} \). Evidently \( \Gamma_{\Lambda} = \bigcup_{n=0}^{\infty} \Gamma_{\Lambda}^{(n)} \), where \( \Gamma_{\Lambda}^{(n)} := \Gamma_{\Lambda} \cap \Gamma^{(n)} \), \( n \in \mathbb{N}_0 \).

Given a complex-valued \( \mathcal{B}(\Gamma_0) \)-measurable function \( G \) such that \( G|_{\Gamma_{\Lambda}} \equiv 0 \) for some \( \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \), the \( K \)-transform of \( G \) is a mapping \( KG : \Gamma \to \mathbb{C} \) defined at each \( \gamma \in \Gamma \) by

\[
(KG)(\gamma) := \sum_{\eta \subset \gamma, |\eta| < \infty} G(\eta).
\]

It has been shown in [7] that the \( K \)-transform is a linear and invertible mapping.

Let \( \mathcal{M}_{\text{fin}}(\Gamma) \) be the set of all probability measures \( \mu \) on \( (\Gamma, \mathcal{B}(\Gamma)) \) with finite local moments of all orders, i.e.,

\[
\int_{\Gamma} d\mu(\gamma) |\gamma \cap \Lambda|^n < \infty \text{ for all } n \in \mathbb{N} \text{ and all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d),
\]

and let \( B_{\text{fin}}(\Gamma_0) \) be the set of all complex-valued bounded \( \mathcal{B}(\Gamma_0) \)-measurable functions with bounded support, i.e., \( G|_{\Gamma_0 \setminus (\bigcup_{n=0}^{\infty} \Gamma_{\Lambda}^{(n)})} \equiv 0 \) for some \( N \in \mathbb{N}_0, \Lambda \in \mathcal{B}_c(\mathbb{R}^d) \). Given
a $\mu \in \mathcal{M}_1^1(\Gamma)$, the so-called correlation measure $\rho_\mu$ corresponding to $\mu$ is a measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ defined for all $G \in B_{ba}(\Gamma_0)$ by

$$
(2.3) \quad \int_{\Gamma_0} d\rho_\mu(\eta) G(\eta) = \int_{\Gamma} d\mu(\gamma) (KG)(\gamma).
$$

This definition implies, in particular, that $B_{ba}(\Gamma_0) \subset L^1(\Gamma_0, \rho_\mu)$. Furthermore, still by (2.3), on $B_{ba}(\Gamma_0)$ the inequality $\|KG\|_{L^1(\Gamma, \mu)} \leq \|G\|_{L^1(\Gamma_0, \rho_\mu)}$ holds, allowing an extension of the $K$-transform to a bounded operator $K : L^1(\Gamma_0, \rho_\mu) \to L^1(\Gamma, \mu)$ in such a way that equality (2.3) still holds for any $G \in L^1(\Gamma_0, \rho_\mu)$. For the extended operator the explicit form (2.2) still holds, now $\mu$-a.e. In particular, for coherent states $e_\lambda(f)$ of complex-valued $B(\mathbb{R}^d)$-measurable functions $f$, 

$$(2.4) \quad e_\lambda(f, \eta) := \prod_{x \in \gamma} f(x), \quad \eta \in \Gamma_0 \setminus \{\emptyset\}, \quad e_\lambda(f, \emptyset) := 1.$$ 

Additionally, if $f$ has compact support we have

$$(2.5) \quad (Ke_\lambda(f))(\gamma) = \prod_{x \in \gamma} (1 + f(x))$$

for all $\gamma \in \Gamma$, while for functions $f$ such that $e_\lambda(f) \in L^1(\Gamma_0, \rho_\mu)$ equality (2.5) holds, but only for $\mu$-a.e. $\gamma \in \Gamma$. Concerning the Lebesgue-Poisson measure (2.1), we observe that $e_\lambda(f) \in L^p(\Gamma_0, \lambda)$ whenever $f \in L^p := L^p(\mathbb{R}^d, dx)$ for some $p \geq 1$. In this case, $\|e_\lambda(f)\|_{L^p} = \exp(\|f\|_{L^p})$. In particular, for $p = 1$, in addition we have

$$
\int_{\Gamma_0} d\lambda(\eta) e_\lambda(f, \eta) = \exp \left( \int_{\mathbb{R}^d} dx f(x) \right)
$$

for all $f \in L^1$. For more details see [8].

### 2.2. Bogoliubov generating functionals

Given a probability measure $\mu$ on $(\Gamma, \mathcal{B}(\Gamma))$ the so-called Bogoliubov generating functional (GF for short) $B_\mu$ corresponding to $\mu$ is the functional defined at each $B(\mathbb{R}^d)$-measurable function $\theta$ by

$$
(2.6) \quad B_\mu(\theta) := \int_{\Gamma} d\mu(\gamma) \prod_{x \in \gamma} (1 + \theta(x)),
$$

provided the right-hand side exists. It is clear from (2.6) that the domain of a GF $B_\mu$ depends on the underlying measure $\mu$ and, conversely, the domain of $B_\mu$ reflects special properties over the measure $\mu$. Throughout this work we will consider GF defined on the whole complex $L^1$ space. This implies, in particular, that the underlying measure $\mu$ has finite local exponential moments, i.e.,

$$
\int_{\Gamma} d\mu(\gamma) e^{\alpha|\gamma \Lambda|} < \infty \quad \text{for all} \quad \alpha > 0 \quad \text{and all} \quad \Lambda \in B_\gamma(\mathbb{R}^d)
$$

and thus $\mu \in \mathcal{M}_1^1(\Gamma)$. According to the previous subsection, this implies that to such a measure $\mu$ one may associate the correlation measure $\rho_\mu$, which leads to a description of the functional $B_\mu$ in terms of either the measure $\rho_\mu$

$$
B_\mu(\theta) = \int_{\Gamma} d\mu(\gamma) (Ke_\lambda(\theta))(\gamma) = \int_{\Gamma_0} d\rho_\mu(\eta) e_\lambda(\theta, \eta),
$$

or the so-called correlation function $k_\mu := \frac{d\rho_\mu}{d\alpha}$ corresponding to the measure $\mu$, if $\rho_\mu$ is absolutely continuous with respect to the Lebesgue–Poisson measure $\lambda$

$$
(2.7) \quad B_\mu(\theta) = \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) k_\mu(\eta).
$$

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1Throughout this work all $L^p$-spaces, $p \geq 1$, consist of complex-valued functions.
Throughout this work we will assume, in addition, that GF are entire on the $L^1$ space [9], which is a natural environment, namely, to recover the notion of correlation function. For a generic entire functional $B$ on $L^1$, this assumption implies that $B$ has a representation in terms of its Taylor expansion

$$B(\theta_0 + z\theta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} d^n B(\theta_0; \theta, \ldots, \theta), \quad z \in \mathbb{C}, \quad \theta \in L^1,$$

being each differential $d^n B(\theta_0; \cdot)$, $n \in \mathbb{N}$, $\theta_0 \in L^1$ defined by a symmetric kernel

$$\delta^n B(\theta_0; \cdot) \in L^\infty(\mathbb{R}^d) := L^\infty((\mathbb{R}^d)^n, (dx)^\otimes n),$$

called the variational derivative of $n$-th order of $B$ at the point $\theta_0$. That is,

$$d^n B(\theta_0; \theta_1, \ldots, \theta_n) := \frac{\partial^n}{\partial z_1 \cdots \partial z_n} B\left(\theta_0 + \sum_{i=1}^{n} z_i \theta_i\right) \bigg|_{z_1=\cdots=z_n=0}$$

(2.8)

$$= : \int_{(\mathbb{R}^d)^n} dx_1 \cdots dx_n \delta^n B(\theta_0; x_1, \ldots, x_n) \prod_{i=1}^{n} \theta_i(x_i)$$

for all $\theta_1, \ldots, \theta_n \in L^1$. Moreover, the operator norm of the bounded $n$-linear functional $d^n B(\theta_0; \cdot)$ is equal to $\|\delta^n B(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^d)}$ and for all $r > 0$ one has

$$\|\delta B(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{r} \sup_{\|\theta\|_{L^1} \leq r} |B(\theta_0 + \theta')|$$

(2.9)

and, for $n \geq 2$,

$$\|\delta^n B(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq n! \left(\frac{e}{r}\right)^n \sup_{\|\theta\|_{L^1} \leq r} |B(\theta_0 + \theta')|.$$  

(2.10)

In particular, if $B$ is an entire GF $B_\mu$ on $L^1$ then, in terms of the underlying measure $\mu$, the entireness property of $B_\mu$ implies that the correlation measure $\rho_\mu$ is absolutely continuous with respect to the Lebesgue-Poisson measure $\lambda$ and the Radon-Nykodim derivative $k_\mu = \frac{d\rho_\mu}{d\lambda}$ is given by

$$k_\mu(\eta) = \delta[\eta] B_\mu(0; \eta) \quad \text{for } \lambda\text{-a.a. } \eta \in \Gamma_0.$$

In what follows, for each $\alpha > 0$, we consider the Banach space $\mathcal{E}_\alpha$ of all entire functionals $B$ on $L^1$ such that

$$\|B\|_{\alpha} := \sup_{\theta \in L^1} \left( |B(\theta)| e^{-\frac{\beta}{2}\|\theta\|_{L^1}} \right) < \infty,$$

see [9]. This class of Banach spaces has the particularity that, for each $\alpha_0 > 0$, the family $\{\mathcal{E}_\alpha : 0 < \alpha \leq \alpha_0\}$ is a scale of Banach spaces, that is,

$$\mathcal{E}_{\alpha'} \subseteq \mathcal{E}_{\alpha}, \quad \|\cdot\|_{\alpha'} \leq \|\cdot\|_{\alpha}$$

for any pair $\alpha', \alpha''$ such that $0 < \alpha' < \alpha'' \leq \alpha_0$.

3. The Kawasaki dynamics

The Kawasaki dynamics is an example of a hopping particle model where, in this case, particles randomly hop over the space $\mathbb{R}^d$ according to a rate depending on the interaction between particles. More precisely, let $a : \mathbb{R}^d \to [0, +\infty)$ be an even and integrable function and let $\phi : \mathbb{R}^d \to [0, +\infty]$ be a pair potential, that is, a $B(\mathbb{R}^d)$-measurable function such that $\phi(-x) = \phi(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$, which we will assume to be integrable. A particle located at a site $x$ in a given configuration $\gamma \in \Gamma$
hops to a site \( y \) according to a rate given by \( a(x - y) \exp(-E(y, \gamma)) \), where \( E(y, \gamma) \) is a relative energy of interaction between the site \( y \) and the configuration \( \gamma \) defined by
\[
E(y, \gamma) := \sum_{x \in \gamma} \phi(x - y) \in [0, +\infty].
\]
Informally, the behavior of such an infinite particle system is described by
\[
(LF)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy \ a(x - y) e^{-E(y, \gamma)} \left( F(\gamma \setminus \{x\} \cup \{y\}) - F(\gamma) \right).
\]
Given an infinite particle system, as the Kawasaki dynamics, its time evolution in terms of states is informally given by the so-called Fokker-Planck equation,
\[
\frac{d\mu_t}{dt} = L^* \mu_t, \quad \mu_t|_{t=0} = \mu_0,
\]
where \( L^* \) is the dual operator of \( L \). Technically, the use of definition (2.3) allows an alternative approach to the study of (3.2) through the corresponding correlation functions \( k_t := k_{\mu_t}, t \geq 0 \), provided they exist. This leads to the Cauchy problem
\[
\frac{\partial}{\partial t} k_t = L^*k_t, \quad k_t|_{t=0} = k_0,
\]
where \( k_0 \) is the correlation function corresponding to the initial distribution \( \mu_0 \) and \( L^* \) is the dual operator of \( \hat{L} := K^{-1}LK \) in the sense
\[
\int_{\Gamma_0} d\lambda(\eta) (\hat{L}G(\eta))k(\eta) = \int_{\Gamma_0} d\lambda(\eta) G(\eta)(\hat{L}^*k)(\eta).
\]
Through the representation (2.7), this gives us a way to express the dynamics also in terms of the GF \( B_t \) corresponding to \( \mu_t \), i.e., informally,
\[
\frac{\partial}{\partial t} B_t(\theta) = \int_{\Gamma_0} d\lambda(\eta) e_{\lambda}(\theta, \eta) \left( \frac{\partial}{\partial t} k_t(\eta) \right) = \int_{\Gamma_0} d\lambda(\eta) e_{\lambda}(\theta, \eta)(\hat{L}^*k_t)(\eta)
\]
\[
= \int_{\Gamma_0} d\lambda(\eta) (\hat{L}e_{\lambda}(\theta))(\eta)k_t(\eta) := (\hat{L}B_t)(\theta).
\]
This leads to the time evolution equation
\[
\frac{\partial B_t}{\partial t} = \hat{L}B_t,
\]
where, in the case of the Kawasaki dynamics, \( \hat{L} \) is given cf. [4] by
\[
(\hat{L}B)(\theta)
\]
\[
= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy a(x - y)e^{-\phi(x - y)}(\theta(y) - \theta(x))\delta B(\theta e^{-\phi(y - \cdot)} + e^{-\phi(y - \cdot)} - 1; x).
\]

**Theorem 3.1.** Given an \( \alpha_0 > 0 \), let \( B_0 \in \mathcal{E}_{\alpha_0} \). For each \( \alpha \in (0, \alpha_0) \) there is a \( T > 0 \) (which depends on \( \alpha, \alpha_0 \)) such that there is a unique solution \( B_t, t \in [0, T), \) to the initial value problem (3.4), (3.5), \( B_t|_{t=0} = B_0 \) in the space \( \mathcal{E}_\alpha \).

This theorem follows as a particular application of an abstract Ovsjannikov-type result in a scale of Banach spaces which can be found e.g. in [5, Theorem 2.5], and the following estimate of norms.

**Proposition 3.2.** Let \( 0 < \alpha < \alpha_0 \) be given. If \( B \in \mathcal{E}_{\alpha''} \) for some \( \alpha'' \in (\alpha, \alpha_0) \), then \( \hat{L}B \in \mathcal{E}_{\alpha'} \) for all \( \alpha \leq \alpha' < \alpha'' \), and we have
\[
\|\hat{L}B\|_{\alpha'} \leq 2e^{\frac{\alpha_1 \alpha_0}{\alpha}}\|a\|_{L^1} \alpha_0^{\frac{\alpha}{\alpha'' - \alpha'}}\|B\|_{\alpha''}.
\]
To prove this result as well as other forthcoming ones the next lemma shows to be useful.

**Lemma 3.3.** Let \( \varphi, \psi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) be such that, for a.a. \( y \in \mathbb{R}^d \), \( \varphi(y, \cdot) \in L^\infty := L^\infty(\mathbb{R}^d), \psi(y, \cdot) \in L^1 \) and \( \|\varphi(y, \cdot)\|_{L^\infty} \leq c_0, \|\psi(y, \cdot)\|_{L^1} \leq c_1 \) for some constants \( c_0, c_1 > 0 \) independent of \( y \). For each \( \alpha > 0 \) and all \( B \in \mathcal{E}_\alpha \) let

\[
(L_0B)(\theta) := \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy a(x-y)e^{-k\phi(x-y)}(\theta(y) - \theta(x))\delta B(\varphi(y, \cdot)\theta + \psi(y, \cdot); x),
\]

\( \theta \in L^1 \). Here \( a \) and \( \phi \) are defined as before and \( k \geq 0 \) is a constant. Then, for all \( \alpha' > 0 \) such that \( c_0\alpha' < \alpha \), we have \( L_0B \in \mathcal{E}_{\alpha'} \) and

\[
\|L_0B\|_{\alpha'} \leq 2e^{\frac{c_1}{\alpha}}\|a\|_{L^1} \frac{\alpha'}{\alpha - c_0\alpha'} \|B\|_{\alpha}.
\]

**Proof.** First we observe that from the considerations done in Subsection 2.2 it follows that \( L_0B \) is an entire functional on \( L^1 \) and, in addition, that for all \( r > 0, \theta \in L^1 \), and a.a. \( x, y \in \mathbb{R}^d \),

\[
|\delta B(\varphi(y, \cdot)\theta + \psi(y, \cdot); x)| \leq \|\delta B(\varphi(y, \cdot)\theta + \psi(y, \cdot); \cdot)\|_{L^\infty} \leq \frac{1}{r} \sup_{\|\theta\|_{L^1} \leq r} |B(\varphi(y, \cdot)\theta + \psi(y, \cdot) + \theta_0)|,
\]

where, for all \( \theta_0 \in L^1 \) such that \( \|\theta_0\|_{L^1} \leq r \),

\[
|B(\varphi(y, \cdot)\theta + \psi(y, \cdot) + \theta_0)| \leq \|B\|_{\alpha} e^{\frac{\|\varphi(y, \cdot)\|_{L^\infty} + \|\psi(y, \cdot)\|_{L^1}}{\alpha}} \leq \|B\|_{\alpha} e^{\frac{c_0\|\theta\|_{L^1} + c_1 + r}{\alpha}}.
\]

As a result, due to the positiveness of \( \phi \) and to the fact that \( a \) is an even function, for all \( \theta \in L^1 \) one has

\[
\|L_0B\|_{\alpha} \leq \frac{1}{r} e^{\frac{c_0\|\theta\|_{L^1} + c_1 + r}{\alpha}} \|B\|_{\alpha} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy a(x-y)e^{-k\phi(x-y)}|\theta(y) - \theta(x)| \leq \frac{2}{r} e^{\frac{c_1 + r}{\alpha}} \|a\|_{L^1} \|\theta\|_{L^1} e^{\frac{c_0\|\theta\|_{L^1}}{\alpha}} \|B\|_{\alpha}.
\]

Thus,

\[
\|L_0B\|_{\alpha'} = \sup_{\theta \in L^1} \left( e^{-\frac{1}{\alpha'}} \|\theta\|_{L^1} \right) \left( |L_0B(\theta)| \right) \leq \frac{2}{r} e^{\frac{c_1 + r}{\alpha}} \|a\|_{L^1} \|\theta\|_{L^1} \sup_{\theta \in L^1} \left( e^{-\left(\frac{1}{\alpha'} - \frac{c_0}{\alpha}\right)} \|\theta\|_{L^1} \right) \|B\|_{\alpha},
\]

where the supremum is finite provided \( \frac{1}{\alpha'} - \frac{c_0}{\alpha} > 0 \). In such a situation, the use of the inequality \( xe^{-mx} \leq \frac{1}{e^m} \), \( x \geq 0, m > 0 \) leads for each \( r > 0 \) to

\[
\|L_0B\|_{\alpha'} \leq \frac{2}{r} \|a\|_{L^1} \|B\|_{\alpha} e^{\frac{c_1 + r}{\alpha}} \frac{\alpha'}{\alpha - c_0\alpha'} \|B\|_{\alpha}.
\]

The required estimate of norms follows by minimizing the expression \( \frac{1}{2} e^{\frac{c_1 + r}{\alpha}} \) in the parameter \( r \), that is, \( r = \alpha \).

**Proof of Proposition 3.2.** In Lemma 3.3 replace \( \varphi \) by \( e^{-\phi} \) and \( \psi \) by \( e^{-\phi} - 1 \), and consider \( k = 1 \). Due to the positiveness and integrability properties of \( \phi \) one has \( e^{-\phi} \leq 1 \) and \( |e^{-\phi} - 1| = 1 - e^{-\phi} \leq \phi \in L^1 \), ensuring the conditions to apply Lemma 3.3.

**Remark 3.4.** Concerning the initial conditions considered in Theorem 3.1, observe that, in particular, \( B_0 \) can be an entire GF \( B_{\mu_0} \) on \( L^1 \) such that, for some constants \( \alpha_0, C > 0 \),

\[
|B_{\mu_0}(\theta)| \leq C \exp\left(\frac{\|\theta\|_{L^1}}{\alpha_0}\right)
\]

for all \( \theta \in L^1 \). In such a situation an additional analysis is need in order to guarantee that for each \( t \) the local solution \( B_t \) given by Theorem 3.1 is a GF (corresponding to some measure). For more details see e.g. [5, 9] and references therein.
4. Vlasov scaling

We proceed to investigate the Vlasov-type scaling proposed in [3] for generic continuous particle systems and accomplished in [1] for the Kawasaki dynamics. As explained in both references, we start with a rescaling of an initial correlation function \( k_0 \), denoted by \( k_0^{(c)} \), \( c > 0 \), which has a singularity with respect to \( c \) of the type \( k_0^{(c)}(\eta) \sim c^{-\lvert\eta\rvert}r_0(\eta) \), \( \eta \in \Gamma_0 \), being \( r_0 \) a function independent of \( c \). The aim is to construct a scaling of the operator \( L \) defined in (3.1), \( L_\varepsilon \), \( \varepsilon > 0 \), in such a way that the following two conditions are fulfilled. The first one is that under the scaling \( L \mapsto L_\varepsilon \) the solution \( k_t^{(c)} \), \( t \geq 0 \), to

\[
\frac{\partial}{\partial t} k_t^{(c)} = \hat{L}_\varepsilon k_t^{(c)}, \quad k_t^{(c)} \big|_{t=0} = k_0^{(c)}
\]

preserves the order of the singularity with respect to \( c \), that is, \( k_t^{(c)}(\eta) \sim c^{-\lvert\eta\rvert}r_t(\eta) \), \( \eta \in \Gamma_0 \). The second condition is that the dynamics \( r_0 \mapsto r_t \) preserves the Lebesgue-Poisson exponents, that is, if \( r_0 \) is of the form \( r_0 = e_\lambda(\rho_0) \), then each \( r_t \), \( t > 0 \), is of the same type, i.e., \( r_t = e_\lambda(\rho_t) \), where \( \rho_t \) is a solution to a non-linear equation (called a Vlasov-type equation).

The previous scheme was accomplished in [1] through the scale transformation \( \phi \mapsto \varepsilon \phi \) of the operator \( L \), that is,

\[
(\varepsilon L_\varepsilon F)(\gamma) := \sum_{y \in \gamma} \int_{\mathbb{R}^d} dy a(x - y)e^{-\varepsilon E(y, \gamma)} (F(\gamma \setminus \{x\} \cup \{y\}) - F(\gamma)).
\]

As shown in [3, Example 12], [1], the corresponding Vlasov-type equation is given by

\[
\frac{\partial}{\partial t} \rho_t(x) = (\rho_t \ast a)(x)e^{-(\rho_t \ast \phi)(x) - \rho_t(x)(a \ast e^{-(\rho_t \ast \phi)})(x)}, \quad x \in \mathbb{R}^d,
\]

where \( \ast \) denotes the usual convolution of functions. Existence of classical solutions \( 0 \leq \rho_t \leq L_\infty \) to (4.1) has been discussed in [1]. Therefore, it is natural to consider the same scaling, but in GF.

To proceed towards GF, we consider \( k_t^{(c)} \) defined as before and \( k_{t, \text{ren}}^{(c)}(\eta) := \varepsilon^{|\eta|} k_t^{(c)}(\eta) \). In terms of GF, these yield

\[
B_t^{(c)}(\theta) := \int_{\Gamma_0} d\lambda(\eta)e_\lambda(\theta, \eta)k_t^{(c)}(\eta)
\]

and

\[
B_{t, \text{ren}}^{(c)}(\theta) := \int_{\Gamma_0} d\lambda(\eta)e_\lambda(\theta, \eta)k_{t, \text{ren}}^{(c)}(\eta) = \int_{\Gamma_0} d\lambda(\eta)e_\lambda(\varepsilon \theta, \eta)k_t^{(c)}(\eta) = B_t^{(c)}(\varepsilon \theta),
\]

leading, as in (3.3), to the initial value problem

\[
\frac{\partial}{\partial t} B_{t, \text{ren}}^{(c)} = \hat{L}_{\text{c, ren}} B_{t, \text{ren}}^{(c)}, \quad B_{t, \text{ren}}^{(c)} \big|_{t=0} = B_0^{(c)}.
\]

**Proposition 4.1.** For all \( \varepsilon > 0 \) and all \( \theta \in L^1 \), we have

\[
(\hat{L}_{\text{c, ren}} B)(\theta) = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy a(x - y)e^{-(\varepsilon \phi(x - y))}(\theta(y) - \theta(x))
\]

\[
\times \delta B \left( \theta e^{-\varepsilon \phi(y - \cdot)} + \frac{e^{-\varepsilon \phi(y - \cdot)} - 1}{\varepsilon}; x \right).
\]

**Proof.** Since

\[
(\hat{L}_{\text{c, ren}} B)(\theta) = \int_{\Gamma_0} d\lambda(\eta) (\hat{L}_{\text{c, ren}} e_\lambda(\theta))(\eta) k(\eta),
\]
first we have to calculate \((\hat{L}_{\varepsilon,\text{ren}}e_{\lambda}(\theta))(\eta) := e^{-|\eta|/\varepsilon}\hat{L}_{\varepsilon}(e_{\lambda}(\varepsilon\theta, \eta)), \hat{L}_{\varepsilon} = K^{-1}L_{\varepsilon}K\) cf. [3].

Similar calculations done in [4, Subsection 4.2.1] show

\[
\int_{\Gamma_0} d\lambda(\eta) k(\eta) \sum_{x \in \eta} \int_{\mathbb{R}^d} dy \, a(x-y) e^{-\varepsilon\phi(x-y)} (\theta(y) - \theta(x))
\times e_{\lambda} \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}, \eta \setminus \{x\} \right),
\]

and thus, using the relation between variational derivatives derived in [9, Proposition 11], one finds

\[
(\hat{L}_{\varepsilon,\text{ren}}B)(\theta) = \int_{\Gamma_0} d\lambda(\eta) k(\eta) \sum_{x \in \eta} \int_{\mathbb{R}^d} dy \, a(x-y) e^{-\varepsilon\phi(x-y)} (\theta(y) - \theta(x))
\times e_{\lambda} \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}, \eta \setminus \{x\} \right) = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \, a(x-y) e^{-\varepsilon\phi(x-y)} (\theta(y) - \theta(x))
\times \int_{\Gamma_0} d\lambda(\eta) k(\eta \cup \{x\}) e_{\lambda} \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}, \eta \right) = \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \, a(x-y) e^{-\varepsilon\phi(x-y)} (\theta(y) - \theta(x))
\times \delta B \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; x \right). \quad \square
\]

**Proposition 4.2.** (i) If \(B \in \mathcal{E}_\alpha\) for some \(\alpha > 0\), then, for all \(\theta \in L^1\), \((\hat{L}_{\varepsilon,\text{ren}}B)(\theta)\) converges as \(\varepsilon\) tends to zero to

\[
(\hat{L}_{V}B)(\theta) := \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \, a(x-y) (\theta(y) - \theta(x))\delta B(\theta - \phi(y-\cdot); x).
\]

(ii) Let \(\alpha_0 > \alpha > 0\) be given. If \(B \in \mathcal{E}_{\alpha''}\) for some \(\alpha'' \in (\alpha, \alpha_0]\), then \(\{\hat{L}_{\varepsilon,\text{ren}}B, \hat{L}_V B\} \subset \mathcal{E}_{\alpha'}\) for all \(\alpha \leq \alpha' < \alpha''\), and we have

\[
\|\hat{L}_B \|_{\alpha'} \leq 2\|a\|_{L^1} \frac{\alpha_0}{(\alpha'' - \alpha')} e^{\|\phi\|_{\alpha''}/2^1} \|B\|_{\alpha''},
\]

where \(\hat{L}_B = \hat{L}_{\varepsilon,\text{ren}}\) or \(\hat{L}_B = \hat{L}_V\).

**Proof.** (i) To prove this result we first analyze the pointwise convergence of the variational derivative (4.3) appearing in \(\hat{L}_{\varepsilon,\text{ren}}\). For this purpose we will use the relation between variational derivatives derived in [9, Proposition 11], i.e.,

\[
\delta B(\theta_1 + \theta_2; x) = \int_{\Gamma_0} d\lambda(\eta) \delta^{[\eta]+1} B(\theta_1; \eta \cup \{x\}) e_{\lambda}(\theta_2, \eta), \quad a.a. x \in \mathbb{R}^d, \quad \theta_1, \theta_2 \in L^1,
\]

which allows to rewrite (4.3) as

\[
\delta B \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; x \right) = \int_{\Gamma_0} d\lambda(\eta) \delta^{[\eta]+1} B(\theta - \phi(y-\cdot); \eta \cup \{x\})
\times e_{\lambda} \left( \theta \left( e^{-\varepsilon\phi(y-\cdot)} - 1 \right) \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon} + \phi(y-\cdot), \eta \right)
\]
for a.a. \( x,y \in \mathbb{R}^d \). Concerning the function

\[
f_\varepsilon := f_\varepsilon(\theta, \phi, y) := \theta \left( e^{-(\varepsilon \phi(y-\cdot))} - 1 \right) + \frac{e^{-\varepsilon \phi(y-\cdot)} - 1}{\varepsilon} + \phi(y - \cdot),
\]

which appears in (4.4), for a.a. \( y \in \mathbb{R}^d \), one clearly has \( \lim_{\varepsilon \to 0} f_\varepsilon = 0 \) a.e. in \( \mathbb{R}^d \). By definition (2.4), the latter implies that \( e_\lambda(f_\varepsilon) \) converges \( \lambda \)-a.e. to \( e_\lambda(0) \). Moreover, for the whole integrand function in (4.4), estimates (2.9), (2.10) yield for any \( r > 0 \) and \( \lambda \)-a.a. \( \eta \in \Gamma_0 \)

\[
\left| \delta^{[|\eta|+1]} B(\theta - \phi(y - \cdot); \eta \cup \{x\}) e_\lambda(f_\varepsilon, \eta) \right|
\leq \left\| \delta^{[|\eta|+1]} B(\theta - \phi(y - \cdot); \cdot) \right\|_{L^\infty(\mathbb{R}^d(|\eta|+1))} e_\lambda(|f_\varepsilon|, \eta)
\leq (|\eta|+1)! \left( \frac{e}{r} \right)^{|\eta|+1} e_\lambda(|f_\varepsilon|, \eta) \sup_{\|\theta_0\|_{L^1} \leq r} |B(\theta - \phi(y - \cdot) + \theta_0)|
\leq (|\eta|+1)! \left( \frac{e}{r} \right)^{|\eta|+1} e_\lambda(|\theta| + 2|\phi(y - \cdot)|, \eta) e^{-\frac{|y-\phi(y-\cdot)|L^1+r}{\eta}} \|B\|_\alpha
\]

with

\[
\int_{\Gamma_0} d\lambda(\eta) (|\eta|+1)! \left( \frac{e}{r} \right)^{|\eta|+1} e_\lambda(|\theta| + 2|\phi(y - \cdot)|, \eta) = \sum_{n=0}^{\infty} (n+1) \left( \frac{e}{r} \right)^{n+1} (|\|\theta\|_{L^1} + 2\|\phi\|_{L^1})^n
\]

being finite for any \( r > \epsilon (\|\theta\|_{L^1} + 2\|\phi\|_{L^1}) \).

As a result, by an application of the Lebesgue dominated convergence theorem we have proved that, for a.a. \( x,y \in \mathbb{R}^d \), (4.4) converges as \( \varepsilon \) tends to zero to

\[
\int_{\Gamma_0} d\lambda(\eta) \delta^{[|\eta|+1]} B(\theta - \phi(y - \cdot); \eta \cup \{x\}) e_\lambda(0, \eta) = \delta B(\theta - \phi(y - \cdot); x).
\]

In addition, for the integrand function which appears in \( \hat{L}_{\varepsilon, \text{ren}} B(\theta) \) we have

\[
\left| a(x-y) e^{-\varepsilon \phi(x-y)} (\theta(y) - \theta(x)) \delta B \left( \theta e^{-\varepsilon \phi(y-\cdot)} + \frac{e^{-\varepsilon \phi(y-\cdot)} - 1}{\varepsilon}; x \right) \right|
\leq \frac{e}{\alpha} a(x-y) |\theta(y) - \theta(x)| \|B\|_\alpha \exp \left( \frac{1}{\alpha} \|\theta\|_{L^1} + \frac{1}{\alpha} \|\phi\|_{L^1} \right)
\]

for all \( \varepsilon > 0 \) and a.a. \( x,y \in \mathbb{R}^d \), leading through a second application of the Lebesgue dominated convergence theorem to the required limit.

(ii) In Lemma 3.3 replace \( \varphi \) by \( e^{-\varepsilon \phi} \), \( \psi \) by \( e^{-\varepsilon \phi - \frac{1}{\varepsilon}} \), and \( k \) by \( \varepsilon \). Arguments similar to prove Proposition 3.2 complete the proof for \( \hat{L}_{\varepsilon, \text{ren}} \). A similar proof holds for \( \hat{L}_V \). \( \square \)

Proposition 4.2 (ii) provides similar estimate of norms for \( \hat{L}_{\varepsilon, \text{ren}}, \varepsilon > 0 \), and the limiting mapping \( \hat{L}_V \). According to the Ovsjannikov-type result used to prove Theorem 3.1, this means that given any \( B_{0,V}, B_{0,\text{ren}} \in \mathcal{E}_\alpha, \varepsilon > 0 \), for each \( \alpha \in (0, \alpha_0) \) there is a \( T > 0 \) such that there is a unique solution \( B_{t,V}^{(\varepsilon)} : [0, T) \to \mathcal{E}_\alpha, \varepsilon > 0 \), to each initial value problem (4.2) and a unique solution \( B_{t,\text{ren}} : [0, T) \to \mathcal{E}_\alpha \) to the initial value problem

\[
(4.5) \quad \frac{\partial}{\partial t} B_{t,V} = \hat{L}_V B_{t,V}, \quad B_{t,V} \big|_{t=0} = B_{0,V}.
\]

In other words, independent of the initial value problem under consideration, the solutions obtained are defined on the same time-interval and with values in the same Banach space. For more details see e.g. Theorem 2.5 and its proof in [5]. Therefore, it is natural to analyze under which conditions the solutions to (4.2) converge to the solution to (4.5). This follows from a general result presented in [5] (Theorem 4.3). However, to proceed to an application of this general result one needs the following estimate of norms.
Proposition 4.3. Assume that $0 \leq \phi \in L^1 \cap L^\infty$ and let $\alpha_0 > \alpha > 0$ be given. Then, for all $B \in \mathcal{E}_\alpha$, $\alpha' \in (\alpha, \alpha_0]$, the following estimate holds:

$$\|\hat{L}_{\varepsilon,\text{ren}} B - \hat{L}_V B\|_{\alpha'} \leq 2\varepsilon\|\alpha\| L^\infty \frac{e\alpha_0}{\alpha} \| B \|_{\alpha'} e^{\frac{\|\phi\| L^1}{\alpha}} \left( 2\varepsilon \|\phi\| L^1 + \frac{\alpha_0}{\varepsilon} \right) \frac{1}{\alpha'' - \alpha'} + \frac{8\alpha_0^2}{(\alpha'' - \alpha^2)}$$

for all $\alpha'$ such that $\alpha \leq \alpha' < \alpha''$ and all $\varepsilon > 0$.

Proof. First we observe that

$$\left| \langle \hat{L}_{\varepsilon,\text{ren}} B \rangle (\theta) - \langle \hat{L}_V B \rangle (\theta) \right| \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} dy a(x-y) |\theta(y) - \theta(x)|$$

with

$$\left| e^{-\varepsilon \phi(x-y)} \delta B \left( \theta e^{-\varepsilon \phi(y-\cdot)} + \frac{e^{-\varepsilon \phi(y-\cdot)} - 1}{\varepsilon}; x \right) - \delta B \left( \theta - \phi(y-\cdot); x \right) \right|$$

(4.6)

In order to estimate (4.6), given any $\theta_0, \theta_1, \theta_2 \in L^1$, let us consider the function $C_{\theta_0, \theta_1, \theta_2}(t) = dB(\theta_0 + (1-t)\theta_2; \theta_0)$, $t \in [0, 1]$, where $dB$ is the first order differential of $B$, defined in (2.8). One has

$$\frac{\partial}{\partial t} C_{\theta_0, \theta_1, \theta_2}(t) = \frac{\partial}{\partial s} C_{\theta_0, \theta_1, \theta_2}(t+s) \bigg|_{s=0}$$

$$= \frac{\partial}{\partial s} dB(\theta_2 + t(\theta_1 - \theta_2) + s(\theta_1 - \theta_2); \theta_0) \bigg|_{s=0}$$

$$= \frac{\partial^2}{\partial s \partial s_2} B(\theta_2 + t(\theta_1 - \theta_2) + s_1(\theta_1 - \theta_2) + s_2\theta_0) \bigg|_{s_1=s_2=0}$$

$$= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy (\theta_1(x) - \theta_2(x)) \delta^2 B(\theta_2 + t(\theta_1 - \theta_2); x, y),$$

leading to

$$|dB(\theta_1; \theta_0) - dB(\theta_2; \theta_0)|$$

$$= |C_{\theta_0, \theta_1, \theta_2}(1) - C_{\theta_0, \theta_1, \theta_2}(0)|$$

$$\leq \max_{t \in [0, 1]} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy |\theta_1(x) - \theta_2(x)| |\theta_0(y)| \|\delta^2 B(\theta_2 + t(\theta_1 - \theta_2); x, y)||_{L^\infty(\mathbb{R}^d)},$$

where, through estimate (2.10) with $r = \alpha''$

$$\|\delta^2 B(\theta_2 + t(\theta_1 - \theta_2); \cdot)||_{L^\infty(\mathbb{R}^d)} \leq 2 e^{\frac{3}{\alpha''}} \|B\|_{\alpha''} \exp \left( \frac{\|\theta_2 + t(\theta_1 - \theta_2)||_{L^1}}{\alpha''} \right).$$

As a result,

$$|dB(\theta_1; \theta_0) - dB(\theta_2; \theta_0)|$$

$$\leq 2 e^{\frac{3}{\alpha''}} \|\theta_1 - \theta_2\|_{L^1} \|\theta_0\|_{L^1} \|B\|_{\alpha''} \max_{t \in [0, 1]} \left( \frac{t\|\theta_1\|_{L^1} + (1-t)\|\theta_2\|_{L^1}}{\alpha''} \right)$$
for all $\theta_0, \theta_1, \theta_2 \in L^1$. In particular, this shows that for all $\theta_0 \in L^1$,

\[
\left| dB \left( \theta e^{-\varepsilon \phi(y - \cdot)} + \frac{e^{-\varepsilon \phi(y - \cdot)} - 1}{\varepsilon}; \theta_0 \right) - dB \left( \theta - \phi(y - \cdot); \theta_0 \right) \right|
\leq 2\varepsilon \frac{e^3}{\alpha^2} \| \phi \|_{L^\infty} \| B \|_{\alpha^\nu} (\| \theta \|_{L^1} + \| \phi \|_{L^1}) \| \theta_0 \|_{L^1}
\times \max_{t \in [0,1]} \exp \left( \frac{1}{\alpha^\nu} (t (\| \theta \|_{L^1} + \| \phi \|_{L^1}) + (1 - t) (\| \theta \|_{L^1} + \| \phi \|_{L^1})) \right)
= 2\varepsilon \frac{e^3}{\alpha^2} \| \phi \|_{L^\infty} \| B \|_{\alpha^\nu} (\| \theta \|_{L^1} + \| \phi \|_{L^1}) \exp \left( \frac{1}{\alpha^\nu} (\| \theta \|_{L^1} + \| \phi \|_{L^1}) \right) \| \theta_0 \|_{L^1}.
\]

where we have used the inequalities

\[
\| \theta e^{-\varepsilon \phi(y - \cdot)} - \theta \|_{L^1} \leq \varepsilon \| \phi \|_{L^\infty} \| \theta \|_{L^1},
\| \frac{e^{-\varepsilon \phi(y - \cdot)} - 1}{\varepsilon} + \phi(y - \cdot) \|_{L^1} \leq \varepsilon \| \phi \|_{L^\infty} \| \phi \|_{L^1},
\| \theta e^{-\varepsilon \phi(y - \cdot)} + \frac{e^{-\varepsilon \phi(y - \cdot)} - 1}{\varepsilon} \|_{L^1} \leq \| \theta \|_{L^1} + \| \phi \|_{L^1}.
\]

In other words, we have shown that the norm of the bounded linear functional on $L^1$

\[
L^1 \ni \theta_0 \mapsto dB \left( \theta e^{-\varepsilon \phi(y - \cdot)} + \frac{e^{-\varepsilon \phi(y - \cdot)} - 1}{\varepsilon}; \theta_0 \right) - dB \left( \theta - \phi(y - \cdot); \theta_0 \right)
\]

is bounded by

\[
Q := 2\varepsilon \frac{e^3}{\alpha^2} \| \phi \|_{L^\infty} \| B \|_{\alpha^\nu} (\| \theta \|_{L^1} + \| \phi \|_{L^1}) \exp \left( \frac{1}{\alpha^\nu} (\| \theta \|_{L^1} + \| \phi \|_{L^1}) \right).
\]

Since this operator norm is given by

\[
\left\| \delta B \left( \theta e^{-\varepsilon \phi(y - \cdot)} + \frac{e^{-\varepsilon \phi(y - \cdot)} - 1}{\varepsilon}; \cdot \right) - \delta B \left( \theta - \phi(y - \cdot); \cdot \right) \right\|_{L^\infty}
\]

cf. Subsection 2.2, this means that

\[
\left\| \delta B \left( \theta e^{-\varepsilon \phi(y - \cdot)} + \frac{e^{-\varepsilon \phi(y - \cdot)} - 1}{\varepsilon}; \cdot \right) - \delta B \left( \theta - \phi(y - \cdot); \cdot \right) \right\|_{L^\infty} \leq Q.
\]

In this way we obtain

\[
\left| \left( \tilde{L}_{\varepsilon, ren} B \right)(\theta) - \left( \tilde{L}_V B \right)(\theta) \right|
\leq \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy \, a(x - y) \, | \theta(y) - \theta(x) |
\times \left\{ \left\| \delta B \left( \theta e^{-\varepsilon \phi(y - \cdot)} + \frac{e^{-\varepsilon \phi(y - \cdot)} - 1}{\varepsilon}; \cdot \right) - \delta B \left( \theta - \phi(y - \cdot); \cdot \right) \right\|_{L^\infty}
+ \varepsilon \| \phi \|_{L^\infty} \| \delta B \left( \theta - \phi(y - \cdot); \cdot \right) \|_{L^\infty} \right\}
\leq 2 \varepsilon \| \phi \|_{L^\infty} \| a \|_{L^1} \frac{e}{\alpha^\nu} \exp \left( \frac{1}{\alpha^\nu} (\| \theta \|_{L^1} + \| \phi \|_{L^1}) \right) \| \theta \|_{L^1}
\times \left\{ 2 \varepsilon^2 \left( \| \theta \|_{L^1} + \| \phi \|_{L^1} \right) + 1 \right\} \| B \|_{\alpha^\nu}.
\]
and thus
\[ \|L_{c,\text{ren}}B - \tilde{L}_V B\|_{\alpha'} \]
\[ \leq 2\varepsilon\|\phi\|_{L^\infty}\|d\|_{L^1} \frac{e^{\varepsilon\|\phi\|_{L^\infty}}}{\alpha'} \left\{ 2\varepsilon^2 \sup_{\theta \in L^1} \left( \|\theta\|_{L^1} \exp \left( \frac{1}{\alpha'} - \frac{1}{\alpha'} \right) \right) \right\} \]
\[ + \left( 2\varepsilon^2 \|\phi\|_{L^1} + 1 \right) \sup_{\theta \in L^1} \left( \|\theta\|_{L^1} \exp \left( \frac{1}{\alpha'} - \frac{1}{\alpha'} \right) \right) \|B\|_{\alpha''}, \]
and the proof follows using the inequalities \( xe^{-mx} \leq \frac{1}{me} \) and \( x^2e^{-mx} \leq \frac{4}{m^2e} \) for \( x \geq 0, m > 0. \)

We are now in conditions to state the following result.

**Theorem 4.4.** Given an \( 0 < \alpha < \alpha_0 \), let \( B^{(\varepsilon)}_{t,\text{ren}}, B_{t,V}, t \in [0,T) \), be the local solutions in \( E_\alpha \) to the initial value problems (4.2), (4.5) with \( B^{(\varepsilon)}_{0,\text{ren}}, B_{0,V} \in E_{\alpha_0} \). If \( 0 \leq \phi \in L^1 \cap L^\infty \) and \( \lim_{\varepsilon \to 0} \|B^{(\varepsilon)}_{0,\text{ren}} - B_{0,V}\|_{\alpha_0} = 0 \), then, for each \( t \in [0,T) \),

\[ \lim_{\varepsilon \to 0} \|B^{(\varepsilon)}_{t,\text{ren}} - B_{t,V}\|_{\alpha} = 0. \]

Moreover, if \( B_{0,V}(\theta) = \exp \left( \int_{R^d} dx \rho_0(x)\theta(x) \right) \), \( \theta \in L^1 \), for some function \( 0 \leq \rho_0 \in L^\infty \) such that \( \|\rho_0\|_{L^\infty} \leq \frac{1}{\alpha_0} \), then for each \( t \in [0,T) \),

\[ B_{t,V}(\theta) = \exp \left( \int_{R^d} dx \rho_0(x)\theta(x) \right), \quad \theta \in L^1, \]

where \( 0 \leq \rho \in L^\infty \) is a classical solution to the equation (4.1).

**Proof.** The first part follows directly from Proposition 4.3 and [5, Theorem 4.3], taking in [5, Theorem 4.3] \( p = 2 \) and

\[ N_\varepsilon = 2\varepsilon\|a\|_{L^1}\|\phi\|_{L^\infty} \frac{e^{\varepsilon\|\phi\|_{L^1}}}{\alpha'} \max \left\{ 2e\|\phi\|_{L^1} + \frac{\alpha_0}{\alpha}, 8\alpha_0^2 \right\}. \]

Concerning the last part, we begin by observing that it has been shown in [1, Subsection 4.2] that given a \( 0 \leq \rho_0 \in L^\infty \) such that \( \|\rho_0\|_{L^\infty} \leq \frac{1}{\alpha_0} \), there is a solution \( 0 \leq \rho \in L^\infty \) to (4.1) such that \( \|\rho\|_{L^\infty} \leq \frac{1}{\alpha_0} \). This implies that \( B_{t,V} \), given by (4.7), does not leave the initial Banach space \( E_{\alpha_0} \subset E_\alpha \). Then, by an argument of uniqueness, to prove the last assertion amounts to show that \( B_{t,V} \) solves equation (4.5). For this purpose we note that for any \( \theta, \theta_1 \in L^1 \) we have

\[ \frac{\partial}{\partial z_1} B_{t,V}(\theta + z_1\theta_1) \big|_{z_1 = 0} = B_{t,V}(\theta) \int_{R^d} dx \rho_0(x)\theta_1(x), \]

and thus \( \delta B_{t,V}(\theta; x) = B_{t,V}(\theta)\rho_1(x) \). Hence, for all \( \theta \in L^1 \),

\[ (\tilde{L}_V B_{t,V})(\theta) = B_{t,V}(\theta) \int_{R^d} dx \int_{R^d} dy \frac{a(x-y)\theta(y)}{\theta(y) - \theta(x)} \rho_1(x)e^{-(\rho_1+\phi)(y)} \]
\[ = B_{t,V}(\theta) \int_{R^d} dy \frac{a * \rho_1}{(a * \rho_1)}(y)e^{-(\rho_1+\phi)(y)} - \int_{R^d} dx \theta(x) \frac{a * e^{-(\rho_1+\phi)}(y)}{(x)\rho_1(x)}. \]

Since \( \rho_1 \) is a classical solution to (4.1), \( \rho_1 \) solves a weak form of equation (4.1), that is, the right-hand side of the latter equality is equal to

\[ B_{t,V}(\theta) \frac{d}{dt} \int_{R^d} dx \rho_1(x)\theta(x) = \frac{\partial}{\partial t} B_{t,V}(\theta). \]
REFERENCES


