

## A HYBRID METHOD FOR INVERSE SCATTERING FOR SOUND-SOFT OBSTACLES IN $\mathbb{R}^3$

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**ABSTRACT.** We present a hybrid method to numerically solve the inverse acoustic sound-soft obstacle scattering problem in  $\mathbb{R}^3$ , given the far-field pattern for one incident direction. This method combines ideas of both iterative and decomposition methods, inheriting advantages of each of them, such as getting good reconstructions and not needing a forward solver at each step. A related Newton method is presented to show convergence of the method and numerical results show its feasibility.

**1. Introduction.** Non-destructive obstacle detection through low-frequency wave propagation motivates several mathematical and numerical problems with applications such as radar, sonar or medical imaging. In particular, confining ourselves to time-harmonic acoustic scattering, we are interested in numerical methods to reconstruct inaccessible impenetrable scattering obstacles within a homogeneous background from the knowledge of the incident field and the corresponding scattered field at large distances (far-field pattern).

Given an open bounded obstacle  $D \subset \mathbb{R}^3$  with an unbounded and connected complement and given an incident field  $u^i$ , the direct scattering problem consists of finding the total field  $u = u^i + u^s$  as the sum of the known incident field  $u^i$  and the scattered field  $u^s$  such that both the Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D} \quad (1.1)$$

with wave number  $k > 0$  and the Dirichlet boundary condition

$$u = 0 \quad \text{on } \Gamma := \partial D \quad (1.2)$$

are satisfied. An obstacle satisfying the above boundary condition is called sound-soft.

To ensure well-posedness, at infinity one needs to impose the Sommerfeld radiation condition

$$\lim_{r \rightarrow \infty} r \left( \frac{\partial u^s}{\partial r} - i k u^s \right) = 0, \quad r = |x| \quad (1.3)$$

with the limit satisfied uniformly in all directions. Then it is known (e.g. [7, Ch.2]) that the solution  $u^s$  has an asymptotic behaviour of the form

$$u^s(x) = \frac{e^{ik|x|}}{|x|} \left( u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right) \right), \quad |x| \rightarrow \infty,$$

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where  $\hat{x} = x/|x|$ . The function  $u_\infty$  is called the far-field pattern and is defined on the unit sphere  $\Omega$ . By Rellich's lemma the scattered field  $u^s$  is uniquely determined by its far-field pattern.

The inverse problem that we are interested in is the following: Given an incident field  $u^i$  and the corresponding far-field pattern  $u_\infty$ , determine the position and shape of the obstacle  $D$ . This problem is non-linear in the sense that the scattered field depends non-linearly on the obstacle and it is also ill-posed in the sense that the determination of  $D$  does not depend continuously on the far-field pattern. In this way, every numerical method to solve the inverse problem must take these two difficulties into account.

Several methods have been used to numerically solve this inverse scattering problem and they can be classified roughly in three classes: iterative methods, decomposition methods and sampling methods. We will focus on regularized iterative methods and decomposition methods, since they are the "parents" of the hybrid method we will consider throughout this paper. For details on sampling methods we refer to [3, 5, 25]

*Newton iterative methods* appeared in the beginning of the 80's. These methods pose the inverse problem as an ill-posed operator equation and then solve it by regularized Newton iterations. For instance, for a single fixed incident field  $u^i$ , the solution to the direct scattering problem defines the operator

$$F : \gamma \mapsto u_\infty \quad (1.4)$$

that maps the boundary  $\gamma$  of some obstacle onto the far-field corresponding to scattering by that obstacle. In this sense, given the far-field pattern  $u_\infty$ , the inverse problem is equivalent to finding the solution to the nonlinear and ill-posed operator equation

$$F(\Gamma) = u_\infty \quad (1.5)$$

for the unknown boundary  $\Gamma$ . Regularized Newton iterations applied to (1.5) have been broadly studied. Their idea is to linearize (1.5) based on the Fréchet differentiability of the operator  $F$  (see [12, 24]), solve the linearized ill-posed equation through a regularization method and iterate this procedure. Due to the ill-posedness of  $F'$  regularization is required in each iteration step. As the Fréchet derivative  $F'$  is characterized in terms of the solution to a direct problem, the main drawback of this method is that it requires a forward solver to be used at each iteration step, which is costly in terms of computations. A reasonable initial guess is also needed to start the iterations. As for the theoretical background, the convergence proofs for these methods are not yet completely satisfactory, though there has been some progress in that matter (see [13, 14]).

As an alternative approach appearing on the second half of the 80's, *decomposition methods* take care of the ill-posedness and the nonlinearity of the inverse scattering problem separately (see Kirsch and Kress [17] or Potthast's point source method [23]). Their idea is the following: In a first ill-posed step the total field  $u$  is reconstructed from the given far-field pattern  $u_\infty$ . Then, in a second step, one tries to find the boundary  $\Gamma$  as the location where the boundary condition (1.2) is satisfied in a least squares sense or just by plotting  $|u|$ . This second step is clearly non-linear. Though these methods do not need the solution to the forward problem, the reconstructions obtained are not as accurate as those obtained by Newton's iterations. As for the theoretical background, these methods are usually compared with a minimization problem (see [7, sec.5.4]) but there is a gap between the theory and the implementation of the methods.

Recently methods that could recover both the obstacle and the impedance at its boundary where also developed, with representatives in each of the three classes mentioned (see [2, 20, 28, 29]).

In the meantime several other methods were suggested to solve the inverse scattering problem (see [3, 7, 8, 19, 27] for details on the state of the art), always trying to get good reconstructions with small computational cost and needing only few input data. In the next sections, the hybrid method will be suggested as a good compromise between these three aspects. This method is based in [18], where it was suggested to combine ideas of both iterative and decomposition methods in order to create an iterative method using as background idea analytic continuation of the total field. In this sense this new method is called hybrid. The same idea was applied to an inverse boundary value problem in potential theory [4] and to inverse scattering for sound-soft cracks [21] and for sound-hard obstacles [22]. Later this method was also extended to scattering for shape and impedance [29], recovering both the obstacle and the unknown impedance on its boundary. This new method does not need a forward solver and the accuracy of the reconstructions is as satisfactory as for Newton iterations, provided the initial guess is close enough to the exact boundary. In this paper we present a new approach for a convergence proof and we also describe the implementation of this method in  $\mathbb{R}^3$ .

This paper is organized as follows. In the next section we will briefly show the procedure to solve the direct scattering problem through an integral representation of the solution, in order to introduce some concepts that will be important later when solving the inverse problem. In section 3 we will present the hybrid method to numerically solve the inverse problem for the Dirichlet boundary condition in  $\mathbb{R}^3$ . In section 4 we will prove convergence of the hybrid method under some assumptions by relating it to a Newton method. Finally we will present the numerical implementation and some numerical results in section 5.

**2. The Direct Problem.** In this section we will show a procedure to solve the direct scattering problem via integral representation of the solution, following [7]. In this way we will introduce some concepts and ideas that are important later for the inverse problem. We are looking for a solution  $u^s$  to

$$\Delta u^s + k^2 u^s = 0, \quad x \in \mathbb{R}^3 \setminus \overline{D}, \quad (2.1)$$

$$u = 0, \quad x \in \Gamma, \quad (2.2)$$

$$\lim_{r \rightarrow \infty} r^{\frac{m-1}{2}} \left( \frac{\partial u^s}{\partial r} - iku^s \right) = 0, \quad (2.3)$$

where the total field  $u$  is the sum of the scattered field  $u^s$  and the given incident field  $u^i$ . The boundary  $\Gamma$  of the open domain  $D$  is considered to be  $C^2$ -smooth and connected. In this way, we represent the solution  $u^s$  by a combined single-and double-layer potential over  $\Gamma$

$$u^s(x) = \int_{\Gamma} \left( \frac{\partial \Phi(x, y)}{\partial \nu(y)} - i\eta \Phi(x, y) \right) \varphi(y) ds(y), \quad x \in \mathbb{R}^3 \setminus \Gamma$$

with density  $\varphi \in C(\Gamma)$ . By the properties of the layer potentials, a combined single-and double-layer representation for the solution of the exterior Dirichlet problem is always possible (see [7]). Note that by the properties of the fundamental solution

$$\Phi(x, y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}$$

the layer potentials satisfy the Helmholtz equation (2.1) in  $\mathbb{R}^3 \setminus \overline{D}$  and the Sommerfeld radiation condition (2.3).

By the jump relations and the boundary condition, the density  $\varphi$  is given by the unique solution to the well-posed integral equation (see [7])

$$\frac{\varphi}{2} + (K_\Gamma - i\eta S_\Gamma) \varphi = -u^i \quad \text{on } \Gamma,$$

where the single-layer operator over some closed  $C^2$ -smooth surface  $\gamma$  is given by

$$(S_\gamma \varphi)(x) = \int_\gamma \Phi(x, y) \varphi(y) ds(y), \quad x \in \gamma, \quad (2.4)$$

and the double-layer operator is given by

$$(K_\gamma \varphi)(x) = \int_\gamma \frac{\partial \Phi(x, y)}{\partial \nu(y)} \varphi(y) ds(y), \quad x \in \gamma. \quad (2.5)$$

Then, by the asymptotics of the layer potentials we would get that the far-field pattern of  $u^s$  would be given by

$$u_\infty = F_{\infty, \Gamma} \varphi \quad \text{on } \Omega,$$

where  $F_{\infty, \gamma} := (K_{\infty, \gamma} - i\eta S_{\infty, \gamma})$  and the single and double far-field layer operators are given respectively by

$$(S_{\infty, \gamma} \varphi)(\hat{x}) := \frac{1}{4\pi} \int_\gamma e^{-ik\hat{x} \cdot y} \varphi(y) ds(y), \quad \hat{x} \in \Omega \quad (2.6)$$

$$(K_{\infty, \gamma} \varphi)(\hat{x}) := \frac{1}{4\pi} \int_\gamma \frac{\partial e^{-ik\hat{x} \cdot y}}{\partial \nu(y)} \varphi(y) ds(y), \quad \hat{x} \in \Omega. \quad (2.7)$$

To numerically solve the problem we used a Galerkin method [11], whose numerical implementation is described in detail in [9].

**3. The Hybrid Method.** In this section we will consider the hybrid method for the numerical solution of the inverse obstacle scattering problem with Dirichlet boundary condition. This means that the solution  $u^s$  to the direct problem satisfies (2.1)–(2.3). Again the total field  $u$  is the sum of the given incident field  $u^i$  and the scattered field  $u^s$ . The goal is to recover the domain  $D$  from the knowledge of the far-field pattern  $u_\infty$  for one incident plane wave  $u^i(x) = e^{ikx \cdot d}$ ,  $|d| = 1$ .

At the  $n$ -iteration of the hybrid method, let  $\gamma_n$ , parameterized by  $z_n$ , be the current approximation to the boundary  $\Gamma$ . Having the first step in mind, one wants to reconstruct the scattered field  $u^s$ . Therefore, assuming that  $k^2$  is not an interior Dirichlet eigenvalue of the negative Laplacian, we can represent the scattered field  $u^s$  as a single layer potential over  $\gamma_n$  (see [6, thm. 3.30]), that is,

$$u^s(x) = \int_{\gamma_n} \Phi(x, y) \varphi(y) ds(y), \quad x \in \mathbb{R}^m \setminus \gamma_n, \quad (3.1)$$

with density  $\varphi \in C(\gamma_n)$ .

**Remark 1.** We choose a single-layer representation because it leads to a simpler numerical implementation. Indeed, for a combined single-and double-layer representation, a proper numerical treatment to compute the trace of the normal derivative is still open. The main drawback of this single layer representation is that the condition on the wave number  $k$  is needed and can not be guaranteed *a priori* for the successive approximations  $\gamma_n$ .

Actually one also needs that the scattered field  $u^s$  can be analytically extended up to  $\gamma_n$ . Due to analyticity of the single layer potential in  $\mathbb{R}^m \setminus \gamma_n$ , it is clear that if the previous assumption does not hold, then the previous representation would not make sense for the cases where  $\gamma_n \cap D \neq \emptyset$ . Therefore we will assume the following continuation principle.

**Assumption 3.1** (Analytic Continuation Principle). The solution  $u^s$  to the direct problem of scattering by  $D$  can be analytically extended as a solution to the Helmholtz equation in a neighbourhood of the boundary  $\Gamma$  of  $D$ .

**Remark 2.** If the boundary  $\Gamma$  is analytic, then assumption 3.1 holds (see [10]).

In this way, assuming that  $\gamma_n$  is sufficiently close to  $\Gamma$ , the representation (3.1) is valid. By the asymptotics of the single layer potential, the density  $\varphi$  must satisfy the far-field equation

$$S_{\gamma_n, \infty} \varphi = u_\infty \text{ in } \Omega \quad (3.2)$$

with the far-field operator  $S_{\gamma, \infty} : C(\gamma) \rightarrow C(\Omega)$  given by

$$S_{\gamma, \infty} \varphi = \frac{1}{4\pi} \int_{\gamma} e^{-ik\hat{x} \cdot y} \varphi(y) ds(y), \quad \hat{x} \in \Omega.$$

The previous operator is compact since it has a continuous kernel, so for numerical purposes (3.2) must be replaced by a regularized equation. Moreover, if the far-field pattern contains noise, then equation (3.2) might not even have a solution, since the right hand side may lie outside the range of the operator. In this way it only makes sense to speak about a regularized solution. In order to show that a regularization scheme is applicable one needs to show that the operator  $S_{\gamma_n, \infty}$  is injective. We will consider the operator  $S_{\gamma, \infty} : L^2(\gamma) \rightarrow L^2(\Omega)$  since we need an Hilbert space setting and refer to [7] for the proof.

**Theorem 3.2.** Assume that  $k^2$  is not an interior Dirichlet eigenvalue of the negative Laplacian with respect to the open bounded domain  $D_\gamma$  with boundary  $\gamma$ . Then the single far-field layer operator  $S_{\gamma, \infty} : L^2(\gamma) \rightarrow L^2(\Omega)$  is injective and has dense range.

Using the well-established Tikhonov regularization, we replace (3.2) by

$$(\alpha_n I + S_{\gamma_n, \infty}^* S_{\gamma_n, \infty}) \varphi^{(n)} = S_{\gamma_n, \infty}^* u_\infty \quad (3.3)$$

solving it with respect to  $\varphi^{(n)}$  for some regularization parameter  $\alpha_n > 0$  that decreases exponentially with  $n$ . The scattered field  $u^s$  can now be approximated by

$$u_n^s(x) := \int_{\gamma_n} \Phi(x, y) \varphi^{(n)} ds(y), \quad x \in \mathbb{R}^m \setminus \gamma_n. \quad (3.4)$$

and using the jump relations (e.g. [7]) we also get the approximations for  $u^s$  and its exterior normal derivative  $\partial u^s / \partial \nu$  on  $\gamma_n$  given by

$$u_n^s(x) = (S_{\gamma_n} \varphi^{(n)})(x), \quad x \in \gamma_n, \quad (3.5)$$

$$\frac{\partial u_n^s}{\partial \nu}(x) = -\frac{1}{2} \varphi^{(n)} + (K_{\gamma_n}^* \varphi^{(n)})(x), \quad x \in \gamma_n, \quad (3.6)$$

respectively. This settles the first step of the method.

For the second step, we define the operator  $G$  that, for a fixed analytic field  $u$ , maps the parameterization  $z$  of the contour  $\gamma$  to the exterior trace of the field  $u$  over  $\gamma$ , that is,

$$G : z \mapsto u \circ z.$$

If the field  $u$  is the total field, then in order to find the position of the boundary of the obstacle  $D$  as the location where the boundary condition is satisfied, we want to find the solution to

$$G(z) = 0.$$

In the spirit of a Newton method we now linearize the previous equation around the current approximation  $z_n$  and solve the linearized equation

$$G(z_n) + G'(z_n)h = 0 \quad \text{in } X \quad (3.7)$$

with respect to the shift  $h$ . In the next theorem we characterize the Fréchet derivative of  $G$ . We will consider closed surfaces  $\gamma$  given by

$$\gamma = \{z(t) : t \in X\}$$

where the parameterization domain is  $X := [0, \pi] \times [0, 2\pi]$  and the parameterization  $z : X \rightarrow \mathbb{R}^3$  is considered to be such that  $\gamma$  is  $C^2$ -smooth.

**Theorem 3.3.** *The operator  $G : C^2(X) \rightarrow C(X)$  is Fréchet differentiable and the Fréchet derivative is given by*

$$G'(z)h = (\text{grad } u \circ z) \cdot h.$$

*Proof.* By Taylor's formula, the Fréchet differentiability of  $G$  is a direct consequence of the analyticity of  $u$  and the  $C^2$ -smoothness of  $z$ . Moreover, from the Taylor formula for  $u$  one gets for each  $t \in X$  that

$$u(z(t) + h(t)) = u(z(t)) + \text{grad } u(z(t)) \cdot h(t) + O(|h(t)|^2),$$

as  $\|h\|_\infty \rightarrow 0$ . Therefore by definition of  $G$  we have

$$\|G(z+h) - G(z) - (\text{grad } u \circ z) \cdot h\|_\infty = O(\|h\|_\infty^2)$$

as  $\|h\|_\infty \rightarrow 0$  and by definition of the Fréchet derivative one has the result.  $\square$

With this characterization, equation (3.7) can be rewritten in the following way

$$(\text{grad } u \circ z_n) \cdot h = -u \circ z \quad \text{in } X.$$

In this way, at each iteration  $n$  we approximate  $u^s$  by  $u_n^s$  obtained in the first step of the iteration and solve

$$((\text{grad } u_n^s + \text{grad } u^i) \circ z_n) \cdot h = -(u_n^s + u^i) \circ z \quad \text{in } X \quad (3.8)$$

with respect to  $h$  in a least squares sense, obtaining the new approximation  $\gamma_{n+1}$  parameterized by  $z_{n+1} = z_n + h$ . Note that we use the jump relations (3.5) and (3.6) to compute the terms involved, namely using the decomposition

$$\text{grad } u|_{\gamma_n} = \nu \frac{\partial u}{\partial \nu} \Big|_{\gamma_n} + \nabla_t u, \quad (3.9)$$

where  $\nabla_t u$  represents the surface gradient of  $u$ . In the spirit of an iterative method, we now repeat the two steps until some stopping criterion is fulfilled. The details on the numerical implementation will be given in section 5.

**Remark 3.** In the initial Kirsch and Kress method [17], no iteration on the surface  $\gamma$  was made, since  $\gamma$  was not seen as an approximation to  $\Gamma$ . In fact, the surface  $\gamma$  had a role of an auxiliary surface to recover the total field  $u$ . Then one would obtain a linearized equation similar to (3.8) that was solved iteratively using always the fixed field obtained in the first step. In this way it was of crucial importance that the approximation (3.4) was defined over  $\Gamma$  and so one needed the assumption that  $\gamma$  is inside  $D$ . The way the hybrid method was constructed eliminates this need because both  $u$  and its derivatives are evaluated at each step only over the current approximation  $\gamma_n$ .

The hybrid method is closer to a decomposition method than to a Newton method. It does not need a forward solver and also separates the ill-posedness from the non-linearity of the problem into two steps in each iteration. The previous remark clarifies the changes made to a particular decomposition method - the Kirsch and Kress method [17] - in order to allow it to be iterated, reviving it with a more competitive version. Moreover, the hybrid method is more effective than the Kirsch and Kress method in the sense that it gets better reconstructions (as we will see in section 5), being able to compete with approximations obtained by the usual regularized Newton method. In this way, this method provides a good compromise between data required, computational costs and numerical accuracy.

**4. A Related Newton Method.** In this section we will establish a local convergence result for the hybrid method as an alternative approach to the usual comparison to a minimization problem as presented in [21, 29]. We will follow the same procedure for a convergence proof as in [26] with differences in the considered operators. In [26] the convergence analysis is done for the boundary data to far-field pattern operator (1.4), or in other words, for the usual Newton method applied to the operator that maps the boundary to the far-field pattern of the corresponding scattered wave. In this work we are interested in a different operator that is related to the hybrid method. We will define the hybrid method as a point-wise iterative scheme and derive its convergence under some assumptions. We assume the domain to be star-shaped, that is, the boundary is of the form

$$\Gamma = \{z_*(\theta, \phi) = r_*(\hat{x}(\theta, \phi)) \hat{x}(\theta, \phi) \mid r : \Omega \rightarrow \mathbb{R}^3, (\theta, \phi) \in X\}. \quad (4.1)$$

for a radial parameterization  $r_* \in C^2(\Omega)$  where

$$\hat{x}(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (4.2)$$

for  $(\theta, \phi) \in X = [0, \pi] \times [0, 2\pi]$ , but a proof can be carried out for a general-shaped and sufficiently smooth domain.

**4.1. Noise free data.** For a first analysis, we assume that the given far-field data  $u_\infty$  is noise free.

Considering star-shaped domains as in (4.1), let  $r_n$  be the radial parameterization to the current approximation  $\gamma_n$  and let  $r_*$  be the radial parameterization to the solution  $\Gamma$  to the inverse problem. The first step of the hybrid method consists of reconstructing the scattered field. To do so, we assume that the solution  $u_*$  to the forward problem can be analytically extended (possibly through the interior of  $D$ ) up to  $\gamma_n$ , as referred in the analytic continuation principle 3.1. Then it is possible to represent  $u^s$  as a combined layer potential over  $\gamma_n$

$$u^s(x) = \int_{\gamma_n} \left( \frac{\partial \Phi(x, y)}{\partial \nu(y)} - i\eta \Phi(x, y) \right) \varphi_*^{(n)}(y) ds(y), \quad x \in \mathbb{R}^m \setminus \gamma_n$$

with density  $\varphi_*^{(n)}$  being the unique solution to the far-field equation

$$F_{\gamma_n, \infty} \varphi_*^{(n)} = u_\infty \quad (4.3)$$

where again  $F_{\gamma, \infty} = (K_{\gamma, \infty} - i\eta S_{\gamma, \infty})$ .

**Remark 4.** We just used a combined single-and double-layer representation to get rid of the assumption that  $k^2$  is not an interior eigenvalue of the negative Laplacian of the successive approximations  $\gamma_n$ . With this last assumption, a single-layer representation could be carried out in exactly the same way.

Therefore, at each step  $n$  the reconstructed total field represented as a combined layer potential over  $\gamma_n$  coincides with the true solution  $u_*$  to the forward problem, as well as its trace and normal trace on the boundary, respectively, given by

$$u_* = u^i + L_{\gamma_n} \varphi_*^{(n)} \quad \text{on } \gamma_n \quad (4.4)$$

and

$$\frac{\partial u_*}{\partial \nu} = \frac{\partial u^i}{\partial \nu} + N_{\gamma_n} \varphi_*^{(n)} \quad \text{on } \gamma_n, \quad (4.5)$$

where the operators involved are defined as

$$\begin{aligned} L_\gamma &= (I/2 + K_\gamma - i\eta S_\gamma), \\ N_\gamma &= (i\eta I/2 + T_\gamma - i\eta K_\gamma^*). \end{aligned}$$

Using the hybrid method for the Dirichlet case, we end up at each step with the linearized equation

$$(u_* + \text{grad } u_* \cdot h)|_{\gamma_n} = 0,$$

that for star-shaped domains with some abuse of notation  $h(\hat{x}) = h \hat{x}$  reduces to

$$\left( u_* + \frac{\partial u_*}{\partial \hat{x}} h \right) \Big|_{\gamma_n} = 0.$$

We now assume that the radial derivatives do not vanish in a small closed neighbourhood  $U$  of  $\Gamma$ , that is,

$$\left| \frac{\partial u_*}{\partial \hat{x}} \Big|_\gamma \right| > 0 \quad (4.6)$$

for  $\gamma \in U$ . For continuous radial derivatives the condition (4.6) implies there exists an  $\epsilon > 0$  such that

$$\left| \frac{\partial u_*}{\partial \hat{x}} \Big|_\gamma \right| \geq 4\epsilon \quad (4.7)$$

for  $\gamma \in U$ .

**Remark 5.** One needs to justify that the previous assumption is not artificial and makes sense in practise. In [1, pp.360], it is shown that for scattering by a sound-soft sphere with radius  $a$  in  $\mathbb{R}^3$  considering plane wave incidence with direction  $d = (0, 0, -1)$ , the normal derivative of the corresponding total field  $u$  measured over the same sphere is given explicitly by

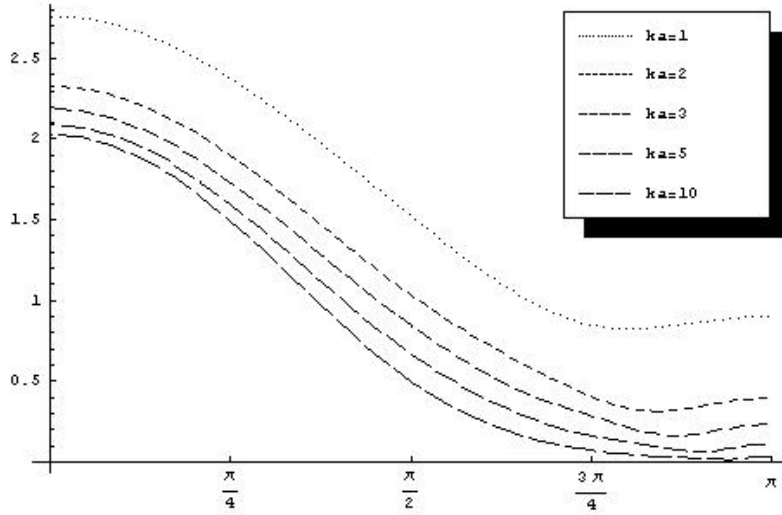
$$\frac{1}{k} \frac{\partial u}{\partial \hat{x}}(a \hat{x}(\theta, \phi)) = \frac{1}{k} \frac{\partial u}{\partial \nu}(a \hat{x}(\theta, \phi)) = -\frac{i}{(ka)^2} \sum_{n=0}^{\infty} (-i)^n (2n+1) \frac{P_n(\cos \theta)}{h_n^{(1)}(ka)}$$

for  $\hat{x}(\theta, \phi)$  given as in (4.2) and where  $h_n^{(1)}$  holds for a spherical Bessel function of the first kind and order  $n$  (e.g. [7, sec.2.4]). Plots of  $|1/k \partial u / \partial \hat{x}|$  (see [1, fig.10.2] or figure 1) for several values of  $ka$  show numerically that the radial derivative on the surface does not vanish. Therefore, by continuous dependence of the total field  $u$  on the scatterer, we conclude that for domains close to a sphere it makes sense to make the assumption (4.6). However, the value of  $|1/k \partial u / \partial \hat{x}|$  gets smaller in the shadow region as  $ka$  increases and assumption (4.7) might not hold for large obstacles or high frequency.

Then, the linearized equation can be written pointwise as

$$h = - \left( u_* / \frac{\partial u_*}{\partial \hat{x}} \right) \Big|_{\gamma_n}. \quad (4.8)$$



FIGURE 1. Plot of  $|1/k \partial u / \partial \hat{x}|$  in terms of  $\theta$ .

Note that the quantity on the left-hand side of the previous equation is real-valued, while the quantity on the right hand side might be complex-value. Therefore, the shift might be computed for numerical purposes by

$$h = -\operatorname{Re} \left( u_* / \frac{\partial u_*}{\partial \hat{x}} \right) \Big|_{\gamma_n} \quad (4.9)$$

In this case we have that (4.7) must be replaced by

$$\left| \operatorname{Re} \left( \frac{\partial u_*}{\partial \hat{x}} \Big|_{\gamma} \right) \right| > 4\epsilon, \quad (4.10)$$

which is still covered by a similar argument to remark 5 for low frequency.

**Remark 6.** Another possibility is to consider

$$h = -\frac{1}{2} \left( \operatorname{Re} \left( u_* / \frac{\partial u_*}{\partial \hat{x}} \right) + \operatorname{Im} \left( u_* / \frac{\partial u_*}{\partial \hat{x}} \right) \right) \Big|_{\gamma_n}. \quad (4.11)$$

This is a more natural way since one fits both the real and imaginary parts when trying to find the zeros. In this case we would have that (4.7) must be replaced by

$$\left| \operatorname{Re} \left( \frac{\partial u_*}{\partial \hat{x}} \Big|_{\gamma} \right) \right| > 4\epsilon, \quad \left| \operatorname{Im} \left( \frac{\partial u_*}{\partial \hat{x}} \Big|_{\gamma} \right) \right| > 4\epsilon. \quad (4.12)$$

In order to keep the notation shorter we will continue the analysis for (4.9). One should have in mind that all the estimates obtained also hold for this last case (4.11) just by using the property that the  $C$ -norm of a complex function is greater than or equal to the  $C$ -norm of its real or imaginary part.

Therefore the update is obtained by the iterative scheme

$$r_{n+1} = r_n - \operatorname{Re} \left( u_* / \frac{\partial u_*}{\partial \hat{x}} \right) \Big|_{\gamma_n}, \quad (4.13)$$

where we recall that  $u_*$  is the true solution to the forward problem with scatterer  $D$ . Note that given an analytic initial guess  $r_0$ , at each step the updated approximation  $r_{n+1}$  is still analytic, due to the analyticity of  $u_*$ .

By Taylor's Formula we have pointwisely that

$$\left| u_* \circ r_n - u_* \circ r_* - \frac{\partial u_*}{\partial \hat{x}} \circ r_n (r_n - r_*) \right| = O(|r_n - r_*|^2)$$

and by the boundary condition, as  $u_*|_{\Gamma} = 0$ , we get

$$\left\| u_*|_{\gamma_n} - \frac{\partial u_*}{\partial \hat{x}}|_{\gamma_n} (r_n - r_*) \right\|_{\infty} = O(\|r_n - r_*\|^2).$$

In this way we obtain

$$\begin{aligned} r_{n+1} - r_* &= r_n - \operatorname{Re} \left( u_* / \frac{\partial u_*}{\partial \hat{x}} \right) \Big|_{\gamma_n} - r_* \\ &= \operatorname{Re} \left[ \left( (r_n - r_*) \frac{\partial u_*}{\partial \hat{x}} \Big|_{\gamma_n} - u_*|_{\gamma_n} \right) / \left( \frac{\partial u_*}{\partial \hat{x}} \Big|_{\gamma_n} \right) \right]. \end{aligned}$$

which implies that

$$\|r_{n+1} - r_*\| \leq \frac{C}{4\epsilon} \|r_n - r_*\|^2$$

showing that (4.13) converges (superlinearly) to the solution in a small neighbourhood  $U$  of  $\Gamma$ .

**4.2. Noisy data.** If the far-field data  $u_{\infty}^{\delta} \in L^2(\Omega)$  has noise with magnitude  $\delta$ , that is,

$$\|u_{\infty}^{\delta} - u_{\infty}\|_{L^2(\Omega)} < \delta \quad (4.14)$$

then (4.3) in general has no solution. Therefore we look for a regularized solution to (4.3), that is,

$$\varphi_{\alpha}^{(n)} := R_{n,\alpha} u_{\infty}^{\delta} \quad (4.15)$$

where  $R_{n,\alpha}$  is a regularization scheme for  $F_{\gamma_n,\infty}^{-1}$ . As before we apply Tikhonov regularization, that is,

$$R_{n,\alpha} = (\alpha I + F_{\gamma_n,\infty}^* F_{\gamma_n,\infty})^{-1} F_{\gamma_n,\infty}^*. \quad (4.16)$$

Therefore the iterative scheme (4.13) must be replaced by

$$r_{n+1} = r_n - \operatorname{Re} \left( u_{n,\alpha} / \frac{\partial u_{n,\alpha}}{\partial \hat{x}} \right) \Big|_{\gamma_n} \quad (4.17)$$

where

$$u_{n,\alpha}|_{\gamma_n} = u^i|_{\gamma_n} + L_{\gamma_n} R_{n,\alpha} u_{\infty}^{\delta}, \quad (4.18)$$

$$\frac{\partial u_{n,\alpha}}{\partial \hat{x}} \Big|_{\gamma_n} = \frac{\partial u^i}{\partial \hat{x}} \Big|_{\gamma_n} + D_{\gamma_n} R_{n,\alpha} u_{\infty}^{\delta}. \quad (4.19)$$

and

$$D_{\gamma} : \varphi \mapsto \hat{x} \cdot (\nu N_{\gamma} \varphi + \nabla_{\tau} L_{\gamma} \varphi) \Big|_{\gamma}$$

maps  $\varphi_\alpha^{(n)}$  into  $\left(\frac{\partial u_{n,\alpha}^s}{\partial \hat{x}}\right)\Big|_{\gamma_n}$ . In  $\mathbb{R}^3$ , we consider the operators

$$F_{\gamma,\infty} : H^3(\gamma) \rightarrow L^2(\Omega), \quad (4.20)$$

$$L_\gamma : H^3(\gamma) \rightarrow H^3(\gamma), \quad (4.21)$$

$$N_\gamma : H^3(\gamma) \rightarrow H^2(\gamma), \quad (4.22)$$

$$D_\gamma : H^3(\gamma) \rightarrow H^2(\gamma), \quad (4.23)$$

that by Sobolev's imbedding theorems and the mapping properties of the layer potentials (see [16]) are bounded for a  $C^{4,\alpha}$ -smooth surface  $\gamma$  and in particular for analytic boundaries.

As the operator  $F_{\infty,\gamma_n}$  is injective and has dense range (see [7, 15]), in  $\mathbb{R}^3$  it makes sense to consider the regularization operator

$$R_{n,\alpha} : L^2(\gamma) \rightarrow H^3(\gamma)$$

as defined in (4.16). In this way, it is clear that the exterior trace  $u_{n,\alpha}|_{\gamma_n} \in H^3(\gamma_n)$  and also that the exterior normal trace  $(\partial u_{n,\alpha}/\partial \hat{x})|_{\gamma_n} \in H^2(\gamma_n)$ . Therefore we can consider (4.24) in a pointwise sense, since for a contour  $\gamma \in \mathbb{R}^3$  by the Sobolev imbedding theorems the  $H^2(\gamma)$ -norm is stronger than the  $C(\gamma)$ -norm. However, unlike in the no noise case, by (4.17) even if  $r_n$  is analytic the update  $r_{n+1}$  is just in  $H^2(\Omega)$ . In this way, one needs extra regularization so that the operators (4.20)–(4.23) remain bounded in the considered spaces for the next Newton step. Having this in mind, we redefine the iterative method as

$$\tilde{r}_{n+1} = r_n - \operatorname{Re} \left( u_{n,\alpha} / \frac{\partial u_{n,\alpha}}{\partial \hat{x}} \right) \Big|_{\gamma_n} \quad (4.24)$$

and  $r_{n+1} = Q(\tilde{r}_{n+1})$ , where  $Q : C(\Omega) \rightarrow C^{4,\alpha}(\Omega)$  maps  $\tilde{r}_{n+1}$  to a quasi-solution  $r_{n+1}$  with constraint  $C_0$ , that is,

$$\|\tilde{r}_{n+1} - r_{n+1}\|_{C(\Omega)} \leq \|\tilde{r}_{n+1} - r\|_{C(\Omega)}$$

for all  $r \in C^{4,\alpha}(\Omega)$  with  $\|r\|_{C^{4,\alpha}(\Omega)} \leq C_0$ . We will assume as *a priori* knowledge that  $\|r_*\|_{C^{4,\alpha}(\Omega)} \leq C_0$ .

**Remark 7.** In practise, one does not follow the procedure of finding a quasi-solution at each step. Instead, one finds the best approximation to  $\tilde{r}_{n+1}$  in some sufficiently smooth finite dimensional approximation space.

We are now in a position to head for the convergence result. We start by establishing the estimate

$$\begin{aligned} \|u_{n,\alpha} - u_*\|_{C(\gamma_n)} &= \|L_{\gamma_n} \varphi_\alpha^{(n)} - L_{\gamma_n} \varphi_*^{(n)}\|_{C(\gamma_n)} \\ &\leq \|L_{\gamma_n}\| \|\varphi_\alpha^{(n)} - \varphi_*^{(n)}\|_{C(\gamma_n)} \\ &\leq \|L_{\gamma_n}\| \|R_{n,\alpha}(u_\infty^\delta - u_\infty) + R_{n,\alpha} L_{\gamma_n}^\infty \varphi_*^{(n)} - \varphi_*^{(n)}\|_{C(\gamma_n)} \\ &\leq C \left( \|R_{n,\alpha}\| \delta + \|R_{n,\alpha} L_{\gamma_n}^\infty \varphi_*^{(n)} - \varphi_*^{(n)}\|_{C(\gamma_n)} \right). \end{aligned}$$

For Tikhonov regularization, picking  $\alpha(\delta) \rightarrow 0$  such that

$$\frac{\delta^2}{\alpha(\delta)} \rightarrow 0, \text{ for } \delta \rightarrow 0$$

we have that (see thm.4.13 in [7])

$$\|u_{n,\alpha} - u_*\|_{C(\gamma_n)} \leq \xi_1(\delta) \quad (4.25)$$

where  $\xi_1(\delta)$  is monotonously decreasing and  $\xi_1(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ . By a similar argument we can conclude that

$$\begin{aligned} \left\| \frac{\partial u_{n,\alpha}}{\partial \hat{x}} - \frac{\partial u_*}{\partial \hat{x}} \right\|_{C(\gamma_n)} &\leq \|D_{\gamma_n} \varphi_\alpha^{(n)} - D_{\gamma_n} \varphi_*^{(n)}\|_{C(\gamma_n)} \\ &\leq \tilde{C} \left( \|R_{n,\alpha}\| \delta + \|R_{n,\alpha} L_{\gamma_n}^\infty \varphi_*^{(n)} - \varphi_*^{(n)}\|_{C(\gamma_n)} \right) \end{aligned}$$

and again similarly we conclude that

$$\left\| \frac{\partial u_{n,\alpha}}{\partial \hat{x}} - \frac{\partial u_*}{\partial \hat{x}} \right\|_{C(\gamma_n)} \leq \xi_2(\delta) \quad (4.26)$$

such that  $\xi_2(\delta)$  is monotonously decreasing and  $\xi_2(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ . Note that if (4.7) holds, then for a sufficiently small closed neighbourhood  $U$  of  $\Gamma$  and a sufficiently small  $\delta$  we have

$$\left| \frac{\partial u_{n,\alpha}}{\partial \hat{x}} \Big|_\gamma \right| \geq 2\epsilon \quad (4.27)$$

for  $\gamma \in U$ .

As the method involves regularization, a stopping rule is of major importance to establish convergence of the method.

**Definition 4.1** (Stopping Rule). Given noisy data  $u_\infty^\delta$  fulfilling (4.14) we stop the iterative scheme (4.24) if two successive approximations satisfy

$$\|r_{n+1} - r_n\|_{C(\Omega)} \leq C_1(\delta) \quad (4.28)$$

where  $C_1(\delta) = \frac{4\xi_1(\delta)}{\epsilon - 2\xi_2(\delta)}$ .

Note that the stopping rule gets more strict as  $\delta$  decreases, since  $C_1(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ . When the stopping criterion is fulfilled, we establish by definition  $r_\delta := r_{n+1}$ .

**Theorem 4.2** (Convergence). *Let  $\Gamma$  be analytic and assume that (4.27) holds. Then the iterative scheme (4.24) with a regularization scheme satisfying (4.25) and (4.26) with a stopping rule (4.28) is locally convergent, in the sense that*

$$\|r_\delta - r_*\|_{C(\Omega)} \rightarrow 0, \quad \delta \rightarrow 0.$$

*Proof.* As compared to the noise free data, the proof must be changed in the following way. We start by noting that by definition of a quasi-solution, since the solution  $r_*$  is assumed to be analytic (and therefore  $r_* \in C^{4,\alpha}(\Omega)$ ) we have

$$\begin{aligned} \|r_{n+1} - r_*\|_{C(\Omega)} &\leq \|r_{n+1} - \tilde{r}_{n+1}\|_{C(\Omega)} + \|\tilde{r}_{n+1} - r_*\|_{C(\Omega)} \\ &\leq 2 \|\tilde{r}_{n+1} - r_*\|_{C(\Omega)}. \end{aligned}$$

We also have that

$$\begin{aligned} \tilde{r}_{n+1} - r_* &= r_n - \operatorname{Re} \left( u_{n,\alpha} / \frac{\partial u_{n,\alpha}}{\partial \hat{x}} \right) \Big|_{\gamma_n} - r_* \\ &= \operatorname{Re} \left[ \left( -u_{n,\alpha}|_{\gamma_n} + (r_n - r_*) \frac{\partial u_{n,\alpha}}{\partial \hat{x}} \Big|_{\gamma_n} \right) / \left( \frac{\partial u_{n,\alpha}}{\partial \hat{x}} \Big|_{\gamma_n} \right) \right] \\ &= \operatorname{Re} \left[ \left\{ -u_*|_{\gamma_n} - (r_* - r_n) \frac{\partial u_*}{\partial \hat{x}} \Big|_{\gamma_n} + (u_* - u_{n,\alpha})|_{\gamma_n} \right. \right. \\ &\quad \left. \left. + (r_n - r_*) \left( \frac{\partial u_{n,\alpha}}{\partial \hat{x}} - \frac{\partial u_*}{\partial \hat{x}} \right) \Big|_{\gamma_n} \right\} / \left( \frac{\partial u_{n,\alpha}}{\partial \hat{x}} \Big|_{\gamma_n} \right) \right] \end{aligned}$$

and therefore by Taylor's formula and by (4.25) and (4.26) we get

$$\begin{aligned} \|r_{n+1} - r_*\|_{C(\Omega)} &\leq 2 \|\tilde{r}_{n+1} - r_*\|_{C(\Omega)} \\ &\leq \frac{1}{\epsilon} \left( C \|r_n - r_*\|_{C(\Omega)}^2 + \xi_2(\delta) \|r_n - r_*\|_{C(\Omega)} + \xi_1(\delta) \right) \end{aligned}$$

where  $C$  is a constant depending on the solution  $u_*$ .

In order to estimate the error on the  $(n+1)$ -approximation in terms of the error on the  $n$ -approximation, we want to explore under which conditions we have

$$\frac{1}{\epsilon} \left( C \|r_n - r_*\|_{C(\Omega)}^2 + \xi_2(\delta) \|r_n - r_*\|_{C(\Omega)} + \xi_1(\delta) \right) < \frac{\|r_n - r_*\|_{C(\Omega)}}{2}.$$

Defining  $\beta := \epsilon/2 - \xi_2(\delta)$ , that is positive for a sufficiently small  $\delta$ , by the solution formula for quadratic equations we have

$$\frac{\beta - \sqrt{\beta^2 - 4C\xi_1(\delta)}}{2C} < \|r_n - r_*\|_{C(\Omega)} < \frac{\beta + \sqrt{\beta^2 - 4C\xi_1(\delta)}}{2C}.$$

Therefore, by Taylor's expansion of the square root function around  $\beta$ , for a sufficiently small fixed  $\delta$  one concludes that if

$$C_1(\delta) = \frac{4\xi_1(\delta)}{\epsilon - 2\xi_2(\delta)} < \|r_n - r_*\|_{C(\Omega)} < \frac{\epsilon - 2\xi_2(\delta)}{4C} \quad (4.29)$$

we get

$$\|r_{n+1} - r_*\|_{C(\Omega)} \leq \frac{\|r_n - r_*\|_{C(\Omega)}}{2}. \quad (4.30)$$

This shows that for a sufficiently small  $\delta$ , we have that (4.30) holds for a starting value in some neighbourhood of  $\Gamma$ , as long as  $r_n$  satisfies (4.29). In order to justify the choice for the stopping criterion, we note that under (4.29) we have that

$$\begin{aligned} \|r_{n+1} - r_n\|_{C(\Omega)} &= \|r_{n+1} - r_* - (r_n - r_*)\|_{C(\Omega)} \\ &\geq \|r_n - r_*\|_{C(\Omega)} - \|r_{n+1} - r_*\|_{C(\Omega)} \\ &\geq \frac{1}{2} \|r_n - r_*\|_{C(\Omega)} \end{aligned}$$

and therefore if the stopping criterion (4.28) is satisfied then  $r_n$  would be the last iteration to satisfy (4.29) since

$$\|r_{n+1} - r_*\|_{C(\Omega)} \leq \frac{\|r_n - r_*\|_{C(\Omega)}}{2} \leq \|r_{n+1} - r_n\|_{C(\Omega)} \leq C_1(\delta)$$

and further convergence could not be guaranteed. Therefore, by definition of  $r_\delta$  one gets

$$\|r_\delta - r_*\|_{C(\Omega)} \leq C_1(\delta) \rightarrow 0$$

as  $\delta \rightarrow 0$  and the proof is finished.  $\square$

This result proves convergence of the hybrid method for the Dirichlet case under certain assumptions, taking into account the iterative procedure and the linearization of the second step unlike the minimization problem approach in [21, 29]. However, some of the assumptions need further work. Assumption (4.6) still requires a rigorous theoretical proof, though its validity can be supported in some cases in low frequency, as for domains close to spheres (see remark 5). This approach also requires more smoothness on the boundary, needing to be analytic (or at least  $C^{4,\alpha}$ -smooth, which is a very artificial space). Another drawback of this approach is that its extension to the Neumann (and consequently to the Robin) case is not trivial, since the characterization of the Fréchet derivative of  $G$  would depend on  $h$  and its tangential derivatives (see [22, 29]). In this way we are not able to

write the shift  $h$  explicitly (as in (4.8) for the Dirichlet case) for a general  $h$  and therefore we do not get an iterative method of the form (4.13) equivalent to the hybrid method.

**5. Numerical Results.** Since there is no straightforward simple quadrature rule available that deals with the singularity of the fundamental solution in  $\mathbb{R}^3$ , the Nyström method loses some of its charm. We will base our approach on quadrature rules that are exact for spherical harmonics of order  $l$  given by

$$Y_l^j(\hat{x}(\theta, \phi)) = (-1)^{(j+|j|)/2} \sqrt{\frac{2j+1}{4\pi} \frac{(l-|j|)!}{(l+|j|)!}} P_l^{|j|}(\cos \theta) e^{ij\phi},$$

for  $l = 1, 2, \dots$  and  $j \leq |l|$ , where  $P_l^{|j|}$  are the associated Legendre polynomials and for  $\hat{x} \in \Omega$  we consider the parametric form (4.2). For domains homeomorphic to the unit sphere, spherical harmonics can be seen as a global approximation analog to the trigonometric polynomials for the two dimensional case, since linear combinations of these functions defined over the unit sphere are dense in  $L^2$  (e.g. [7]). We also note that the spherical harmonics of order less than or equal to  $N$  are the basis of the space  $\mathbb{P}_N$ , that stands for the restriction to  $\Omega$  of polynomials in  $\mathbb{R}^3$  of order less or equal to  $N$ .

To generate the far-field pattern synthetic data we followed [11], where a fully discrete spectral method for the direct scattering problem is presented for smooth domains homeomorphic to a sphere. This method has superalgebraic convergence for analytic surfaces and right-hand sides. Its numerical implementation is presented in detail in [9]. With this Galerkin method we generated 800 far-field pattern data points. We will use the quadrature rules in [9] to numerically approximate the integral operators that appear while solving the inverse problem. We restrict our analysis to star-shaped domains with  $C^2$  boundary of the form

$$\gamma = \{z(\hat{x}) := r(\hat{x})\hat{x} : \hat{x} \in \Omega\}$$

for a radial function  $r$  at least  $C^2$ -smooth over the unit sphere  $\Omega$ .

Let us consider that  $\gamma_n$  is our current approximation to the correct boundary  $\Gamma$ . Following the procedure in section 3, we start by considering a single-layer representation for the scattered field  $u^s$

$$u^s(x) = \int_{\gamma_n} \Phi(x, y) \varphi(y) ds(y)$$

which can be done under certain assumptions as already mentioned in that section. From the representation of the scattered field one arrives at a first kind far-field integral equation given by

$$S_{\gamma_n, \infty} \varphi = u_\infty \quad \text{on } \Omega,$$

where the single layer far-field operator

$$(S_{\gamma, \infty} \varphi)(\hat{x}) = \frac{1}{4\pi} \int_{\gamma} e^{-ik\hat{x} \cdot y} \varphi(y) ds(y), \quad \hat{x} \in \Omega,$$

with some abuse of notation can also be given by

$$(S_{\gamma, \infty} \psi)(\hat{x}) = \frac{1}{4\pi} \int_{\Omega} e^{-ik\hat{x} \cdot z(\hat{y})} \psi(\hat{y}) J_z(\hat{y}) ds(y), \quad \hat{x} \in \Omega,$$

for  $\psi(\hat{y}) = \varphi(z(\hat{y}))$ . As mentioned in section 3, as the operator  $S_{\gamma_n, \infty}$  is compact and the right-hand side  $u_\infty$  might be noisy, one uses Tikhonov regularization to solve the previous equation. In this way we solve

$$(\alpha_n I + S_{\gamma_n, \infty}^* S_{\gamma_n, \infty}) \varphi^{(n)} = S_{\gamma_n, \infty}^* u_\infty \quad \text{on } \Omega. \quad (5.1)$$

where again  $\psi^{(n)}(\hat{y}) = \varphi^{(n)}(z(\hat{y}))$  and the adjoint operator  $S_{\gamma_n, \infty}^*$  is given by

$$(S_{\gamma_n, \infty}^* \varphi)(y) = \frac{1}{4\pi} \int_{\Omega} e^{ik\hat{x} \cdot y} \varphi(\hat{x}) ds(\hat{x}), \quad y \in \gamma_n.$$

One now looks for an approximation  $\psi_N^{(n)} \in \mathbb{P}_{N-1}$  given by

$$\psi_N^{(n)}(\hat{x}) = \sum_{l=0}^{N-1} \sum_{j=-l}^l a_l^j Y_l^j(\hat{x}), \quad (5.2)$$

to the density  $\psi^{(n)}$ . By combining the one-dimensional Gauss and trapezoidal quadrature rules, one gets the  $m := 2N^2$ -point Gauss-trapezoidal rule (e.g. [7, 9]) over the unit sphere given by

$$Q_m(\Psi) = \frac{\pi}{N} \sum_{j=1}^N \sum_{k=0}^{2N-1} \alpha_j \Psi(\hat{x}_{jk}) \quad (5.3)$$

for  $\hat{x}_{jk} = \hat{x}(\theta_j, \phi_k)$ . In the direction  $\phi$  one uses the trapezoidal rule for periodic functions with equidistant points  $\phi_k = k\pi/N$ ,  $k = 0, \dots, 2N-1$ . For direction  $\theta$  one chooses the Gauss rule with integration points  $\theta_j = \arccos t_j$ ,  $j = 1, \dots, N$  and weights

$$\alpha_j = \frac{2(1-t_j^2)}{[NP_{N-1}(t_j)]^2}, \quad j = 1, \dots, N$$

where  $-1 < t_1 < t_2 < \dots < t_N < 1$  denote the  $N$  zeros of the Legendre polynomial  $P_N$  of order  $N$  in the interval  $(-1, 1)$ . The quadrature rule  $Q_m$  is exact for spherical polynomials of degree  $2(N-1)$ . As the integral kernel of  $S_{\gamma_n, \infty}$  is continuous, one uses this quadrature rule to discretize this operator, getting from (5.1) a  $2N^2 \times 2N^2$  linear system on the coefficients  $a_l^j$ ,  $l = 0, N-1$ ,  $j \leq |l|$  of  $\psi_N^{(n)}$ . Numerically this was done considering  $N = 8$ . In this way we reconstruct an approximation for the density  $\psi^{(n)}$ . Preparing the second step, by the jump relations one obtains approximations for the total field  $u$  over the contour  $\gamma_n$  given by

$$u_n(x) = u^i(x) + \int_{\Omega} \Phi(x, z(\hat{y})) \psi_N^{(n)}(\hat{y}) J_z(\hat{y}) ds(y), \quad x \in \gamma_n \quad (5.4)$$

and for its normal derivative over the contour  $\gamma_n$  given by

$$\begin{aligned} \frac{\partial u_n}{\partial \nu}(z(\hat{x})) &= \frac{\partial u^i}{\partial \nu}(z(\hat{x})) - \frac{\psi_N^{(n)}(\hat{x})}{2} \\ &+ \int_{\Omega} \frac{\partial \Phi(z(\hat{x}), z(\hat{y}))}{\partial \nu(z(\hat{x}))} \psi_N^{(n)}(\hat{y}) J_z(\hat{y}) ds(y), \quad \hat{x} \in \Omega. \end{aligned} \quad (5.5)$$

The first goal is to split the singular part of the kernels from the analytic one. We will be interested in an integral operator  $\mathcal{M}$  of the form

$$\mathcal{M}\varphi(x) = \frac{1}{4\pi} \int_{\Gamma} \left( \frac{1}{|x-y|} M_1(x, y) + M_2(x, y) \right) \varphi(y) ds(y)$$

where the kernels  $M_1$  and  $M_2$  are smooth. Note that both the single operator  $S$  and the corresponding normal derivative operator  $K^*$  are of the previous form. For instance, for

the single layer operator

$$\begin{aligned} S\varphi(x) &= \int_{\gamma} \Phi(x, y) \varphi(y) ds(y) \\ &= \frac{1}{4\pi} \int_{\gamma} \left( \frac{1}{|x - y|} M_1^S(x, y) + M_2^S(x, y) \right) \varphi(y) ds(y) \end{aligned}$$

we would have

$$\begin{aligned} M_1^S(x, y) &= \cos(k|x - y|) \\ M_2^S(x, y) &= i\mathcal{S}^s(x, y) \end{aligned}$$

while for

$$\begin{aligned} K^*\varphi(x) &= \int_{\gamma} \frac{\partial \Phi(x, y)}{\partial \nu(x)} \varphi(y) ds(y) \\ &= \frac{1}{4\pi} \int_{\gamma} \left( \frac{1}{|x - y|} M_1^{K^*}(x, y) + M_2^{K^*}(x, y) \right) \varphi(y) ds(y) \end{aligned}$$

we would have

$$\begin{aligned} M_1^{K^*}(x, y) &= \frac{(y - x) \cdot \nu(x)}{|x - y|^2} \cos(k|x - y|) + k \left( (y - x) \cdot \nu(x) \right) \mathcal{S}^s(x, y) \\ M_2^{K^*}(x, y) &= i \frac{(y - x) \cdot \nu(x)}{|x - y|^2} \left( \mathcal{S}^s(x, y) - k \cos(k|x - y|) \right), \end{aligned}$$

where

$$\mathcal{S}^s(x, y) = \begin{cases} \frac{\sin(k|x - y|)}{|x - y|}, & x \neq y \\ k, & x = y. \end{cases}$$

For  $\Gamma$  homeomorphic to the unit sphere  $\Omega$  we can parameterize the operator  $\mathcal{M}$  and obtain a parametric integral operator  $\mathcal{H}$  that can be decomposed as

$$\begin{aligned} \mathcal{H}\psi(\hat{y}) &= \frac{1}{4\pi} \int_{\Omega} \left( \frac{1}{|\hat{x} - \hat{y}|} H_1(\hat{x}, \hat{y}) + H_2(\hat{x}, \hat{y}) \right) \psi(\hat{y}) ds(\hat{y}) \\ &= \frac{1}{4\pi} (\mathcal{H}_1\psi(\hat{y}) + \mathcal{H}_2\psi(\hat{y})), \end{aligned} \tag{5.6}$$

where  $\psi(\hat{x}) = \varphi(z(\hat{x}))$ . Note that in our particular case we assume that  $\Gamma$  is star-shaped, that is,

$$\Gamma = \{z(\hat{x}) := r(\hat{x})\hat{x} : \hat{x} \in \Omega\}$$

with some  $r \in C^2(\Omega)$ . We then define the integral operator  $\mathcal{H}_1$  with singular kernel and the integral operator  $\mathcal{H}_2$  with smooth kernel as

$$\begin{aligned} \mathcal{H}_1\psi(\hat{y}) &= \int_{\Omega} \frac{1}{|\hat{x} - \hat{y}|} H_1(\hat{x}, \hat{y}) \psi(\hat{y}) ds(\hat{y}) \\ \mathcal{H}_2\psi(\hat{y}) &= \int_{\Omega} H_2(\hat{x}, \hat{y}) \psi(\hat{y}) ds(\hat{y}) \end{aligned}$$



with the parameterized kernels

$$\begin{aligned} H_1(\hat{x}, \hat{y}) &= M_1(z(\hat{x}), z(\hat{y})) \mathcal{R}(\hat{x}, \hat{y}) J_z(\hat{y}) \\ H_2(\hat{x}, \hat{y}) &= M_2(z(\hat{x}), z(\hat{y})) J_z(\hat{y}) \end{aligned} \quad (5.7)$$

for

$$\mathcal{R}(\hat{x}, \hat{y}) = \frac{|\hat{x} - \hat{y}|}{|z(\hat{x}) - z(\hat{y})|}$$

and where the Jacobian of the transformation is given by

$$J_z = r \sqrt{r^2 + |\nabla r|^2}$$

where  $\nabla r$  is the gradient of  $r$  over the unit sphere  $\Omega$ . We are of course interested in approximations to  $\mathcal{H}_1$  and  $\mathcal{H}_2$  in the cases where  $M$  is replaced by  $M^S$  or  $M^{K^*}$ .

Following the ideas of [9], to deal with the singular operator  $\mathcal{H}_1$  we introduce a change of coordinate system in  $\Omega$ , in order to take the singularity to the north pole. In this way we consider the orthogonal continuous transformation  $T_{\hat{x}}$  that maps  $\hat{x}$  to the north pole  $\hat{n} := (0, 0, 1)$  and we also define the transformation  $\mathcal{T}_{\hat{x}}$  such that

$$\mathcal{T}_{\hat{x}}\Psi(\hat{y}) = \Psi(T_{\hat{x}}^{-1}\hat{y}), \quad \Psi \in C(\Omega), \quad \hat{y} \in \Omega$$

and its bivariate analogue

$$\mathcal{T}_{\hat{x}}\Psi(\hat{y}_1, \hat{y}_2) = \Psi(T_{\hat{x}}^{-1}\hat{y}_1, T_{\hat{x}}^{-1}\hat{y}_2), \quad \Psi \in C(\Omega \times \Omega), \quad \hat{y}_1, \hat{y}_2 \in \Omega.$$

From the orthogonality of  $T_{\hat{x}}$  we have that

$$\mathcal{H}_1\psi(\hat{x}) = \int_{\Omega} \frac{1}{|\hat{n} - \hat{y}|} \mathcal{T}_{\hat{x}}H_1(\hat{n}, \hat{y}) \mathcal{T}_{\hat{x}}\psi(\hat{y}) ds(\hat{y}). \quad (5.8)$$

With the transformation  $\mathcal{T}_{\hat{x}}$  and using polar coordinates for computing  $\mathcal{H}_1$ , the singularity of the denominator quantity  $|\hat{n} - \hat{y}| = 2 \sin(\theta/2)$  is cancelled out by the surface element  $ds(\hat{y}) = \sin \theta d\theta d\phi$ , that corresponds to the polar coordinate transformation. Moreover, it turns out that the mapping  $(\theta, \phi) \mapsto \mathcal{T}_{\hat{x}}H_1(\hat{n}, \hat{y})$  is smooth (e.g. [11, lem. 4.6]). These two aspects indicate the crucial importance of the use of the rotated coordinate system in (5.8).

As this latter mapping  $(\theta, \phi) \mapsto \mathcal{T}_{\hat{x}}H_1(\hat{n}, \hat{y})$  is continuous it makes sense to take an interpolation of the integrand term  $\mathcal{T}_{\hat{x}}H_1(\hat{n}, \hat{y}) \mathcal{T}_{\hat{x}}\psi(\hat{y})$  in  $\mathcal{P}_{N'}$ , for some positive integer  $N'$ . Accordingly, using the  $m' = 2N'^2$ -point quadrature rule  $Q_{m'}$  defined in (5.3) with  $N$  replaced by  $N'$ , which is exact for spherical polynomials of degree  $2(N' - 1)$ , we could integrate this interpolator exactly and by the properties of the spherical harmonics and the addition theorem we would get an approximation  $\mathcal{H}_{1,N'}$  to  $\mathcal{H}_1$  (see [9]) given by

$$\mathcal{H}_{1,N'}(\hat{x}) = \sum_{j'=0}^{N'} \sum_{k'=0}^{2N'-1} \beta_{j'} \mathcal{T}_{\hat{x}}H_1(\hat{n}, \hat{x}'_{j'k'}) \mathcal{T}_{\hat{x}}\psi(\hat{x}'_{j'k'}) \quad (5.9)$$

where

$$\beta_{j'} = \frac{\pi \alpha'_{j'}}{N'} \sum_{n=0}^{N'-1} P_n(t'_{j'})$$

and the quadrature points  $\hat{x}'_{j'k'}$  and the weights  $\alpha'_{j'}$  correspond to the quadrature rule  $Q_{m'}$ , that is, the same procedure as for (5.3) with  $N$  replaced by  $N'$ .

As for the smooth part  $\mathcal{H}_2$  of (5.6), we get an approximation by the quadrature rule  $Q_{m'}$  in the rotated system

$$\mathcal{H}_{2,N'}(\hat{x}) = \frac{\pi}{N'} \sum_{j'=1}^N \sum_{k'=0}^{2N-1} \alpha'_{j'} \mathcal{T}_{\hat{x}} H_2(\hat{n}, \hat{x}'_{j'k'}) \mathcal{T}_{\hat{x}} \psi(\hat{x}'_{j'k'}).$$

which corresponds to interpolating the integrand function by spherical polynomials of order less than or equal to  $2(N' - 1)$  and integrating exactly. This procedure establishes approximations for the total field  $u$  and its normal derivative over  $\gamma_n$  by (5.4) and (5.5) respectively. We used  $N' = 9$  in the numerical examples.

To compute the tangential component of the gradient over  $\gamma_n$  one uses the analog to trigonometric differentiation in  $\mathbb{R}^3$ , that is, one interpolates  $u(z(\cdot))$  defined in the unit sphere by spherical harmonics and uses the tangential gradient of the interpolation as approximation to  $\nabla_\tau u$ . In this way, using the decomposition

$$\text{grad } u|_{\gamma_n} = \nu \left. \frac{\partial u}{\partial \nu} \right|_{\gamma_n} + \nabla_\tau u$$

accordingly with (3.8), one solves

$$\text{grad } u_n(z(\hat{x})) \cdot (h(\hat{x}) \hat{x}) = -u_n(z(\hat{x})), \quad \hat{x} \in \Omega$$

in a least squares sense, in order to find a shift  $h \in V_M$  where  $V_M$  is the space of linear combinations of real parts of spherical harmonics with order less or equal to  $M$  given by

$$V_M = \left\{ h \in \mathbb{P}_M : f(\hat{x}) = \sum_{l=0}^M \sum_{j=0}^l a_l^j \text{Re}(Y_l^j(\hat{x})), a_l^j \in \mathbb{R} \right\},$$

which is an approximation space of dimension  $(M+1) \times (M+2)/2$ . We used 128 points over  $\Omega$  for the least squares minimization and considered  $M = 8$ .

The two steps are then iterated while  $\|u_n\|_{L^2(\gamma_n)}$  is decreasing.

We have applied the method to several examples and the numerical reconstructions were quite satisfying. We considered the wave number  $k = 1$  and a plane incident wave with direction  $d = (0, 1, 0)$ . The incident direction is indicated in the pictures by an arrow. All the reconstructions were made with 2% noise in the maximum norm on the far-field data and using as initial guess a circle of radius  $4Y_0^0$ .

As a first example in figure 2 we present an acorn shaped obstacle with parameterization

$$r(\theta, \phi) = 0.6 + \sqrt{4.25 + 2 \cos 3\theta},$$

for  $\theta \in [0, \pi]$ ,  $\phi \in [0, 2\pi]$ .

In figure 3 we present a pinched ball obstacle parameterized by

$$r(\theta, \phi) = \sqrt{1.44 + .5 \cos 2\phi (\cos 2\theta - 1)}$$

for  $\theta \in [0, \pi]$ ,  $\phi \in [0, 2\pi]$ . The reconstruction is worst in the non-convex part as usually occurs in the methods with few incident waves.

To better illustrate this handicap we present in figure 4 a pinched-acorn obstacle with parameterization

$$r(\theta, \phi) = 0.6 * \sqrt{(1.44 + 0.5(\cos 2\theta - 1)(\cos 4\phi)) (4.25 + 2 \cos 3\phi)},$$

for  $\theta \in [0, \pi]$ ,  $\phi \in [0, 2\pi]$ . The reconstructions are nonetheless quite good.

In figure 5 we present an cushion shaped obstacle parameterized by

$$r(\theta, \phi) = \sqrt{0.8 + 0.5(\cos 2\phi - 1)(\cos 4\theta - 1)}$$

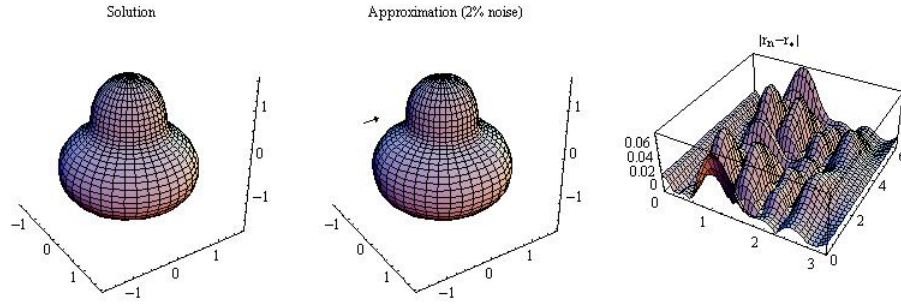


FIGURE 2. Acorn shaped obstacle on the left, reconstruction with 2% noise in the middle and error on the maximum norm on the parameterization space on the right.

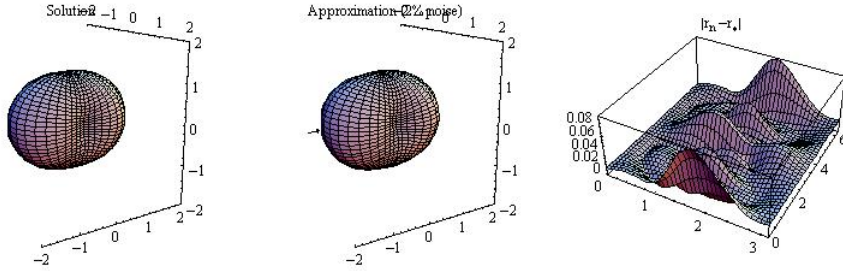


FIGURE 3. Pinched-ball shaped obstacle on the left, reconstruction with 2% noise in the middle and error on the maximum norm on the parameterization space on the right.

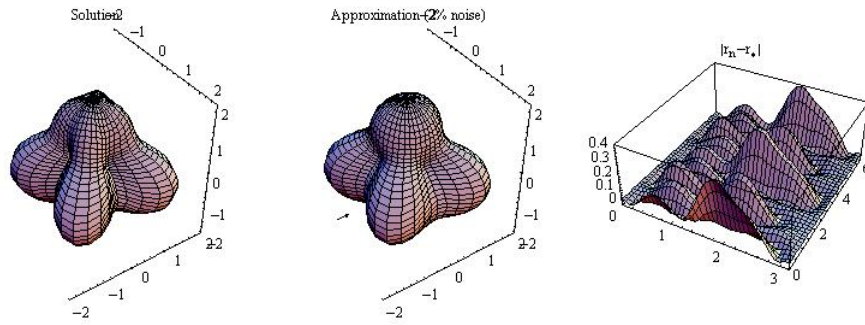


FIGURE 4. Pinched-acorn shaped obstacle on the left, reconstruction with 2% noise in the middle and error on the maximum norm on the parameterization space on the right.

for  $\theta \in [0, \pi]$ ,  $\phi \in [0, 2\pi]$ .

The numerical reconstructions show the feasibility of the method and its robust behaviour with noisy data.

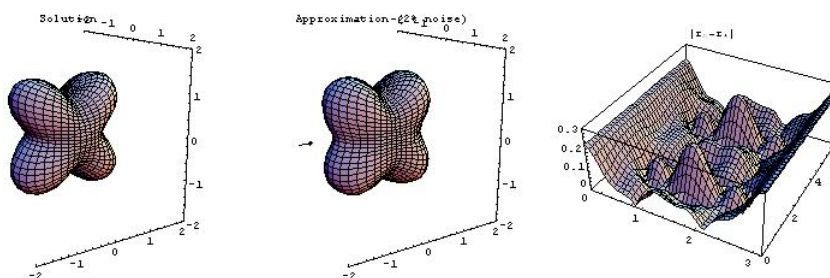


FIGURE 5. Cushion shaped obstacle on the left, reconstruction with 2% noise in the middle and error on the maximum norm on the parameterization space on the right.

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