UNIVERSIDADE ABERTA

Dynamical problems in coagulation equations

Rafael Silva Sasportes

Doutoramento em Matemática

“Esta tese não inclui as críticas e as sugestões feitas pelo júri”

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Prof. Doutor Fernando Pestana da Costa

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À Isabel

Ao Miguel, à Madalena e à Matilde

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Resumo

Neste trabalho são analisados alguns aspectos do comportamento asimptótico dos sistemas de um número infinito de equações diferenciais ordinárias que modelam a cinética de partículas de coagulação dados por

$$\begin{cases} \dot{c}_1 = \alpha t^\omega - c_1^2 - c_1 \displaystyle\sum_{j=1}^{\infty} c_j \\ \dot{c}_j = c_1 c_{j-1} - c_1 c_j, \quad j \geqslant 2, \end{cases}$$

onde $\alpha > 0$ e ω são constantes.

Abordamos dois aspectos particularmente importantes do comportamento dinâmico das soluções deste sistema. Primeiro, o comportamento pontual das soluções quando $t \to +\infty$ e o comportamento da quantidade total de agregados definido por $\sum_{j=1}^{\infty} c_j$. O segundo aspecto prende-se com a ocorrência de comportamentos auto-semelhantes.

No Capítulo 2 estudamos o caso $\omega > -1/2$, no Capítulo 4 o caso $\omega = -1/2$ e no no Capítulo 5 o caso $\omega > -1/2$ utilizando uma mudança de variáveis apropriada. No Capítulo 3 consideramos uma extensão dos resultados do Capítulo 2, para fontes de monómeros do tipo

$$J_1(t) = \alpha t^\omega (1 + \varepsilon(t)),$$

onde $\varepsilon(\cdot)$ é uma função contínua satisfazendo $\varepsilon(t) \to 0$ quando $t \to +\infty$.

Os casos $-1 < \omega < -1/2$ e $\omega < -1$ são tratados no Capítulo 5 utilizando uma abordagem diferente, assente numa análise das propriedades de monotonicidade

das soluções.

Os resultados obtidos permitem-nos mostrar a existência de uma função $\varsigma(t) \sim t^{\frac{\omega+2}{3}}$ e uma família de funções de escalamento $\Phi_{1,\omega}$ para $\omega > -\frac{1}{2}$ tais que

$$c_j(t) \sim \varsigma(t)^{-a}\Phi(j\varsigma(t)^{-b})$$

se verifica para $a = \frac{1-\omega}{2+\omega}$ e $b = 1$.

Resultados semelhantes são também obtidos no caso $\omega = -\frac{1}{2}$.

Para o caso $\omega < -\frac{1}{2}$ alguns resultados parcias, e evidência numérica, sugerem que isso não acontece.

Abstract

In this work we study some aspects of the long time behaviour of a system of an infinite number of ordinary differential equations modelling the kinetics of particle coagulation given by

$$\begin{cases} \dot{c}_1 = \alpha t^{\omega} - c_1^2 - c_1 \displaystyle\sum_{j=1}^{\infty} c_j \\ \dot{c}_j = c_1 c_{j-1} - c_1 c_j, \quad j \geqslant 2, \end{cases}$$

where $\alpha > 0$ and ω are constant parameters.

We will study two aspects of the dynamical behaviour of solutions to this system.

First, we establish the componentwise behaviour of the solution as $t \to +\infty$, as well as the behaviour of the total amount of clusters present (the *bulk*), defined by $\sum_{j=1}^{\infty} c_j$.

The other aspect we are interested in concerns the existence of self-similar solutions.

In Chapter 2 we will study the case $\omega > -1/2$, in Chapter 4 the case $\omega = -1/2$ and in Chapter 5 the case $\omega = -1$ using the ansatz provided by an appropriate change of variables.

In Chapter 3 we consider an extension of the results obtained in Chapter 2, to monomer input

$$J_1(t) = \alpha t^{\omega}(1 + \varepsilon(t)),$$

where $\varepsilon(\cdot)$ is a continuous function satisfying $\varepsilon(t) \to 0$ as $t \to +\infty$.

The cases $-1 < \omega < -1/2$ and $\omega < -1$ will be treated in Chapter 5 using a different approach, based on the analysis of the monotonicity behaviour of solutions.

The results obtained allow us to establish the existence of a function $\varsigma(t) \sim t^{\frac{\omega+2}{3}}$ and a family of scaling functions $\Phi_{1,\omega}$ for $\omega > -\frac{1}{2}$ such that

$$c_j(t) \sim \varsigma(t)^{-a}\Phi(j\varsigma(t)^{-b})$$

holds with $a = \frac{1-\omega}{2+\omega}$ and $b = 1$.

Analogous results also hold for $\omega = -\frac{1}{2}$.

For $\omega < -\frac{1}{2}$ some partial results, and numerical evidence, suggests that this behaviour does not hold.

Introdução

Neste trabalho são analisados alguns aspectos do comportamento asimptótico dos sistemas de um número infinito de equações diferenciais ordinárias que modelam a cinética de partículas de coagulação.

Com início no estudo pioneiro de Smoluchowski [66, 67] sobre coagulação de partículas coloidais animadas de movimento Browniano, na segunda década do Séc. XX, surgiram uma variedade de equações cinéticas, genericamente designadas por equações de coagulação ou de coagulação-fragmentação, com bons resultados na modelização de um vasto domínio de fenómenos naturais a várias escalas, que vão desde o fitoplancton ao fumo, colóides, aerosóis, planetoides e transições de fase em ligas metálicas (veja-se, e.g., [33, 61, 64, 65]).

Um testemunho claro da importância dos modelos de coagulação consiste nas experiências realizadas em órbita a bordo da Estação Espacial Internacional. Estas experiências foram realizadas no âmbito do programa de investigação *Coarsening in solid liquid mixtures-1* e *-2* [22, 23] destinado a analisar o problema do crescimento das fases descrito pela teoria de Lifshitz e Slyozov [51] e Wagner [68] (LSW) sobre a maturação de Ostwald. De referir a importância deste modelo para o conhecimento das propriedades de evolução do crescimento dos grãos das estruturas granulares em ligas metálicas binárias. Recentemente verificou-se que a teoria LSW da maturação de Ostwald está intimamente relacionada [14, 42, 60, 63] com o modelo de crescimento de agregados de Becker-Döring [8, 61], o qual é descrito por um sistema de equações de tipo coagulação-fragmentação.

As múltiplas aplicações destas equações, a par da natureza das próprias equações, reflectiram-se nas últimas décadas num crescente interesse na investigação dos problemas matemáticos (Análise, Probabilidades, Análise Numérica, Física Matemática) por elas colocados. Devido ao âmbito muito específico deste trabalho, não se revela oportuno rever aqui todas as diversas direcções de investigação e a vasta literatura resultante. Sobre estas, contudo, não podemos deixar de fazer uma referência aos recentes trabalhos de revisão [3, 9, 44, 70], com uma excelente bibliografia sobre o tipo de sistemas em análise, ou mesmo os trabalhos de revisão menos actuais como [28, 29], mesmo assim, ainda de interesse.

A par da descrição dos sistemas de equações diferenciais em análise neste trabalho, ao longo deste capítulo introdutório, e sempre que oportuno, faremos referência à literatura imprescindível para o desenvolvimento e compreensão do trabalho apresentado nos capítulos seguintes.

As equações de coagulação-fragmentação

Neste trabalho consideramos sistemas com um grande número de partículas, sendo cada partícula constituída por um determinado número de uma unidade fundamental indivisível designada por monómero. Assume-se que cada monómero tem massa igual a 1, de tal modo que cada agregado formado por j monómeros tem massa igual a j. Tais partículas serão designadas por j-agregados. Assume-se que o sistema é bem misturado, pelo que as concentrações de interesse são espacialmente homogéneas, e podem ser descritas por um vector (c_j) de concentrações, sem dependência espacial, em que o índice j assume todos os possíveis valores de massa dos agregados.

Adicionalmente, assume-se que os agregados podem ligar-se de modo a formar agregados maiores. Sob a hipótese de *fragmentação*, os agregados podem ainda fragmentar-se em agregados menores. Sem menosprezar os modelos que envolvem reacções mais complexas, neste trabalho consideramos apenas as reacções *binárias*. Já no âmbito da interacção entre agregados, considera-se que é verificada a lei de acção de massas da cinética química. Deste modo temos associadas duas taxas (coeficientes ou núcleos), designadamente, de coagulação e de fragmentação, e que devem sintetizar toda a informação relevante sobre as reacções cinéticas em modelação. Tradicionalmente, estes coeficientes são apenas uma função das *massas* de cada um dos dois agregados envolvidos na reacção, mas refira-se que situações mais complexas podem ainda surgir [70, Section 4.3].

Com estes pressupostos, considere-se uma sucessão $(c_j(t))_{j=1}^{\infty}$ cujos elementos $c_j(t)$ são a concentração de agregados de massa j no instante t. Pretende-se analisar a evolução de $c_j(t)$ quando $t \to +\infty$. Por razões que serão mencionadas mais à frente, fisicamente impõe-se estudar esta convergência para a topologia forte fixada num espaço de Banach adequado, a convergência pontual em j (i.e., para cada j fixo), ou ainda analisar a convergência quando j também converge para $+\infty$. Neste último caso, a convergência de j para $+\infty$ relaciona-se com o modo de convergência de t para $+\infty$.

A evolução temporal de $(c_j(t))_{j=1}^{\infty}$ depende do tipo de processo cinético elementar a que cada agregado é submetido. Tipicamente estes processos são os abaixo indicados, onde se assume que $j \geq 2$:

(i) dois agregados mais pequenos coligam-se para formar um j-agregado

$$[(j-k)\text{-agregado}\,] + [k\text{-agregado}\,] \longrightarrow [j\text{-agregado}\,], \quad k = 1, \ldots, j-1$$

(ii) um j-agregado fragmenta-se em dois mais pequenos

$$[j\text{-agregado}\,] \longrightarrow [(j-k)\text{-agregado}\,] + [k\text{-agregado}\,], \quad k = 1, \ldots, j-1$$

(iii) um j-agregado coliga-se com outro agregado

$$[j\text{-agregado}\,] + [k\text{-agregado}\,] \longrightarrow [(j+k)\text{-agregado}\,], \quad k = 1, 2, \ldots$$

(iv) um agregado maior fragmenta-se

$$[(j+k)\text{-agregado}\,] \longrightarrow [j\text{-agregado}\,] + [k\text{-agregado}\,], \quad k = 1, 2, \ldots$$

Assumindo válida a lei da acção de massas da cinética química, denote-se por $a_{j,k}$ a constante cinética para a reacção de coagulação entre um j-agregado e um

k-agregado e por $b_{j,k}$ a taxa de fragmentação de um $(j+k)$-agregado em um j-agregado e um k-agregado. Conclui-se imediatamente que a contribuição de cada um dos processos (**i**)–(**iv**) na evolução temporal de c_j é dada, respectivamente, por

(i) $\dfrac{\mathrm{d}c_j}{\mathrm{d}t} = a_{j-k,k}c_{j-k}c_k\,,$

(ii) $\dfrac{\mathrm{d}c_j}{\mathrm{d}t} = -b_{j-k,k}c_j\,,$

(iii) $\dfrac{\mathrm{d}c_j}{\mathrm{d}t} = -a_{j,k}c_jc_k\,,$

(iv) $\dfrac{\mathrm{d}c_j}{\mathrm{d}t} = b_{j,k}c_{j+k}\,.$

Para $j=1$, obviamente que não se têm as contribuições **(i)** e **(ii)**.

Consequentemente, obtém-se assim o sistema de coagulação-fragmentação

$$\frac{\mathrm{d}}{\mathrm{d}t}c_j = \begin{cases} +\dfrac{1}{2}\displaystyle\sum_{k=1}^{j-1} a_{j-k,k}c_{j-k}c_k & \text{Ganho por coagulação} \\ -\dfrac{1}{2}\displaystyle\sum_{k=1}^{j-1} b_{j-k,k}c_j & \text{Perda por fragmentação} \\ -\displaystyle\sum_{k=1}^{\infty} a_{j,k}c_jc_k & \text{Perda por coagulação} \\ +\displaystyle\sum_{k=1}^{\infty} b_{j,k}c_{j+k} & \text{Ganho por fragmentação.} \end{cases} \tag{1}$$

Simplificando a notação, a equação (1) pode escrever-se como,

$$\dot{c}_j = \frac{1}{2}\sum_{k=1}^{j-1} W_{j-k,k} - \sum_{k=1}^{\infty} W_{j,k}\,, \qquad j \geqslant 1, \tag{2}$$

com $\dot{c}_j = \frac{\mathrm{d}}{\mathrm{d}t}c_j$, e

$$W_{i,j} = W_{i,j}(c) \coloneqq a_{i,j}\,c_i\,c_j - b_{i,j}\,c_{i+j}\,, \quad i,j \geqslant 1.$$

De um ponto de vista físico, é natural supor que os coeficientes $a_{i,j}$ e $b_{i,j}$ são ambos não-negativos e simétricos: $a_{i,j} = a_{j,i} \geqslant 0$, e $b_{i,j} = b_{j,i} \geqslant 0, \forall i, j \geqslant 1$. Contudo, a hipótese fundamental sobre estes coeficientes é que devem conter toda a informação relativa ao mecanismo subjacente ao processo de agregação. Na Tabela 1 [18, Table 1] estão listadas algumas das taxas de coagulação que habitualmente são mais consideradas na literatura científica.

Table 1: Alguns núcleos de coagulação $a_{j,k}$ comuns nas aplicações.

$a_{j,k}$	Comentário
1	Coagulação Browniana aproximada, Polimerização de cadeias lineares
$j+k$	Polimerização de cadeias ramificadas, Caso limite de coagulação gravitacional
jk	Polimerização de cadeias ramificadas
$j^{-2/3}+k^{-2/3}$	Crescimento difusional de grãos cristalinos metálicos
$(j^{1/3}+k^{1/3})(j^{-1/3}+k^{-1/3})$	Coagulação Browniana (regime contínuo)
$(j^{1/3}+k^{1/3})^2(j^{-1}+k^{-1})^{1/2}$	Coagulação Browniana (regime molecular livre)
$(j^{1/3}+k^{1/3})^3$	Coagulação tangencial (perfís de velocidade lineares)
$(j^{1/3}+k^{1/3})^7$	Coagulação tangencial (perfís de velocidade não lineares)
$(j^{1/3}+k^{1/3})^2\lvert j^{1/3}-k^{1/3}\rvert$	Deposição gravitacional

Relativamente aos coeficientes de fragmentação, e do ponto de vista de uma fundamentação física, a informação disponível é bastante menor. Algumas hipóteses têm sido avançadas, mas as mais relevantes têm apontado para uma condição de balanço microscópico (ou detalhado) [5, 13].

A maior parte dos coeficientes tabelados na Tabela 1, assim como outros que surgem na literatura específica sobre aplicações, têm o inconveniente de terem origem em sistemas que, à data presente, são de análise rigorosa difícil. Este facto justifica porque na grande maioria das equações investigadas as taxas consideradas serem de tipo particularmente simples (como as três primeiras descritas na Tabela 1), ou serem majoradas por tais taxas elementares.

Quer por questões matemáticas, quer físicas, de entre as várias possíveis classes de coeficientes evidenciam-se três tipos: fragmentação pura ($a_{i,j} \equiv 0$), coagulação pura ou Smoluchovski ($b_{i,j} \equiv 0$) e o sistema de Becker-Döring ($a_{i,j} = b_{i,j} \equiv 0$ se $i \wedge j \geq N$, para algum $N \geq 1$). De abordagem claramente mais acessível que um sistema de coagulação-fragmentação completo (1), estes casos destacam-se pela importância manifestada em várias aplicações e por presentemente, e apesar da sua aparente simplicidade, ainda encerrarem alguns desafios matemáticos muito apreciáveis.

Uma classe diferente de sistemas, mas também importante, são os obtidos como uma versão *contínua* de (2), considerando que a massa j de um j-agregado pode variar continuamente entre 0 e $+\infty$. Neste caso, a massa será designada por x. Formalmente, a diferença entre os sistemas assim obtidos e os sistemas discretos (2) está na substituição das somas por integrais e, em lugar de um sistema infinito de equações diferenciais ordinárias, tem-se agora uma equação integro-diferencial parcial não-local:

$$\frac{\partial c}{\partial t}(t,x) = \frac{1}{2}\int_0^x W(x-y,y)dy - \int_0^{+\infty} W(x,y)dy. \tag{3}$$

Aqui, $W(x,y) := a(x,y)c(t,x)c(t,y) - b(x,y)c(t,x+y)$, onde as funções a e b são os correspondentes naturais dos coeficientes, respectivamente, de coagulação e de fragmentação do caso discreto.

É razoável questionármo-nos sobre a relação entre os modelos discretos e os contínuos. Este problema é presentemente uma área activa de investigação, [43], mas, não se relacionando directamente com o tema do presente trabalho, não nos iremos debruçar sobre ele.

Neste trabalho analisaremos alguns aspectos do comportamento assimptótico dos sistemas discretos de Smoluchowski com interacções do tipo Becker-Döring, também designado por modelo *aditivo* [10, 34, 38]. Neste contexto, dar-se-à particular ênfase ao comportamento de auto-semelhança das soluções. Sobre este comportamento específico alguma literatura tem sido publicada para as versões discreta e contínua das equações de Smoluchowski. Uma breve revisão dos resultados conhecidos para ambas as versões e das contribuições deste trabalho para melhorar o conhecimento sobre o caso discreto, serão o assunto das secções seguintes deste capítulo introdutório.

Preliminares matemáticos

Enquadramento

O tratamento matemático de um sistema de dimensão infinita (2) naturalmente requer algum cuidado com a escolha dos espaços para formulação do problema e com a própria definição do que se entende por solução de um tal sistema.

O espaço de Banach básico utilizado na análise de (2) é o espaço das *sucessões com massa finita*, ou seja, o espaço linear $X := \left\{(c_j) \in \mathbb{R}^{\mathbb{N}} : (jc_j) \in \ell^1\right\}$ para a norma $\|(c_j)\| := \|(jc_j)\|_{\ell^1}$. Recordando que cada c_j é a concentração de j-agregados, resulta assim que jc_j é proporcional à massa dos j-agregados. Deste modo, $\|(c_j)\| = \sum_{j=1}^{\infty} j|c_j|$ é proporcional à massa total do sistema. Isto justifica a naturalidade da escolha do espaço X para o estudo da equação (2), assim como a sua designação.

Nalgumas aplicações específicas ganham relevo os espaços X^α, $\alpha \geq 0$, definidos por $X^\alpha := \left\{(c_j) \in \mathbb{R}^{\mathbb{N}} : \|(c_j)\|_\alpha := \|(j^\alpha c_j)\|_{\ell^1} < +\infty\right\}$. Tem-se então definida uma escala compacta, normal e regular [15] de espaços de Banach $\{X^\alpha, \|\cdot\|_\alpha\}_{\alpha\geq 0}$. Destas propriedades refira-se a compacidade das imersões com particular relevo técnico nalgumas aplicações. É claro que para $\alpha = 1$ tem-se $X^1 = X$.

As definições de solução mais comuns na literatura são essencialmente de dois tipos: *solução suave*, ou seja, uma solução contínua da versão integral de (2) [4, 5], ou, menos frequentemente, *solução fraca*, entendida no sentido usual da teoria das equações diferenciais parciais [43]. Resultados de regularidade sobre soluções suaves, ou fracas, permitem concluir sobre a existência de regularidade suficiente para dar sentido às derivadas que surgem no lado esquerdo da equação (2), obtendo-se, deste modo, uma *solução forte*. Este procedimento decorre de modo semelhante ao que habitualmente é efectuado na teoria das equações diferenciais ordinárias em dimensão finita.

De modo genérico, os resultados de existência e unicidade de solução para as equações (2) ou (3), ou sobre a regularidade das soluções, dependem das propriedades assumidas sobre os coeficientes cinéticos a e b, nomeadamente, condições de *crescimento* ou hipóteses da *estrutura* destes coeficientes. Actualmente há um vasto trabalho matemático realizado sobre estes problemas, que vai desde os trabalhos pioneiros de Melzak [56] e McLeod [53, 54, 55], passando pelos trabalhos de Aizenman e Bak [2], e White [71], até aos trabalhos seminais de Ball, Carr e Penrose [5], e de Ball e Carr [4], que despoletaram muito do actual interesse matemático por esta área de investigação. De facto, para o propósito do presente trabalho, os resultados provados em [5] sobre existência, unicidade e regularidade têm aplicação e utilidade directas.

Comportamento assimptótico e auto-semelhança

O estudo do comportamento assimptótico das soluções das equações (2) ou (3) é actualmente assunto de grande interesse e de forte actividade de investigação.

De um ponto de vista físico, é de esperar que quando $t \to +\infty$ as soluções globais de (2) ou (3) convirjam para uma solução independente do tempo. Mais, havendo conservação de massa em cada processo cinético elementar descrito na página 4, é razoável prever que qualquer solução do sistema conserve a massa. Por outras palavras, a massa total (norma) de uma solução $(c_j(t))$ deve permanecer invariante, i.e., $\|(c_j(t))\| = \rho, \forall t$, para alguma constante ρ (a massa inicial).

De facto, ao longo de várias décadas numerosos estudos baseados em cálculos e argumentos meramente formais indicavam que esta expectativa era demasiado simplista. Esta ideia era reforçada pelo conhecimento de soluções para as equações de Smoluchowski com coeficiente de coagulação $a_{j,k} = jk$ e para condições iniciais muito particulares (monodispersas, e.g., $c_j(0) = \delta_{j,1}$). Nestas soluções verifica-se conservação de massa até um certo instante T_g, registando-se depois desse instante um decaimento regular da massa para zero. Este comportamento é conhecido pelo *fenómeno de gelificação*, e, fisicamente, deve-se a uma transição de fase que ocorre no sistema originando a formação de um agregado "infinito" (o *gel*) que não é passível de ser modelado por um sistema de coagulação-fragmentação do tipo considerado acima e é o responsável pela perda de massa do sistema de agregados. Matematicamente, a caracterização deste fenómeno foi um problema que se manteve em aberto durante longo tempo e, apesar de alguns progressos [4, 11, 12, 16, 47, 53, 54], só muito recentemente é que foi satisfatoriamente solucionado [30, 31, 36].

No âmbito do comportamento assimptótico das soluções, a convergência para estados de equilíbrio e a conservação (ou não) de massa estão longe de serem as únicas grandes questões de interesse. Outras duas, ambas com manifesta importância física, relacionam-se com a elucidação dos detalhes da convergência para o estado limite, nomeadamente, o problema da ocorrência de comportamento meta-estável nas equações Becker-Döring [35, 61, 62], e a questão do comportamento auto-semelhante em sistemas de Smoluchowski e relacionados.

Quando se estuda a evolução da distribuição de agregados com o tempo (na ausência de fragmentação), somos confrontados com o problema de esclarecer qual o comportamento das soluções para grandes tempos e tamanhos de agregados. Pretendemos esclarecer se, para uma determinada gama de condições iniciais, as soluções convergirão, eventualmente, para uma solução exibindo invariância de escala. Consequentemente, um problema central no estudo do comportamento assimptótico de soluções de equações de coagulação é a sua convergência para um perfil auto-semelhante ψ, quando j e $t \to +\infty$

$$c_j(t) \sim \varsigma(t)^{-a}\Phi(j\varsigma(t)^{-b}),$$

onde a e b são expoentes positivos apropriados, $\varsigma(t)$ é um rescalamento do tempo (com $\varsigma(t) \to +\infty$ quando $t \to +\infty$) e a variável de escalamento $j/\varsigma(t)^b$ é mantida fixa.

Conjectura-se que este comportamento ocorra para uma grande classe de condições iniciais quando os coeficientes de coagulação (simétricos) são homogéneos, ou seja, $a_{k,j} = a_{j,k} = K(j,k)$, com K uma função homogénea, para algum coeficiente de homogeneidade λ, isto é, $K(ux, uy) = u^\lambda K(x, y)$.

Este tipo de problemas foi inicialmente introduzido e tratado por Friedlander e colaboradores [33] nos anos sessenta e setenta. Foram intensamente estudados, de um ponto de vista de modelação matemática formal, por van Dongen e Ernst no anos oitenta, [26, 27].

Em [27], van Dongen e Ernst estudaram núcleos homogéneos com a condição adicional de crescimento $K(i, j) \sim i^{\mu} j^{\nu}$, $(j \gg i, \lambda = \mu + \nu)$. Estes estudos distinguiram várias classes de coeficientes, dependendo do sinal do μ. Além disto, por considerações de carácter físico, imposeram $\nu \leqslant 1$ e $\lambda \leqslant 2$. Argumentando que a velocidade de reacção j^{ν} de um j-agregado com j grande $(j \gg i)$ não deve aumentar mais rapidamente que o seu volume, concluiram que $\nu \leqslant 1$. Analogamente, como para agregados de tamanho semelhante entre si o núcleo $K(j, j) = j^{\lambda} K(1, 1)$ não deverá crescer mais rapidamente que j^2, deverá ter-se $\lambda \leqslant 2$. Um aspecto importante é a introdução da distinção entre núcleos *não-gelificantes* e *gelificantes*, relacionados com o problema da não conservação de massa que mencionámos anteriormente, e para os quais o tamanho médio dos agregados $\varsigma(t)$ permanece finito para todos os tempos $t < +\infty$ no primeiro caso, ao passo que no segundo caso, quando a gelificação ocorre, tem-se divergência de $\varsigma(t)$ num determinado instante finito de tempo T_c, o *ponto de gelificação*. Deste modo, o comportamento de auto-semelhança quando $\varsigma(t) \to +\infty$ corresponde a $t \to T_c$ no caso de gelificação e a $t \to +\infty$ no caso de conservação da massa.

Leyvraz [49, 50] estudou o problema do escalamento próximo de, mas antes do, ponto de gelificação, onde o tamanho típico dos agregados diverge para infinito, originando o aparecimento do agregado *gigante*. As dificuldades desta análise motivaram uma atenção redobrada em relação a soluções exactas e a métodos numéricos.

Para soluções que conservam a massa, um dos primeiros resultados demonstrados de modo matematicamente rigoroso foi obtido por Kreer e Penrose [37], baseando-se numa ideia de Lushnikov [52], para o caso da equação de Smoluchowski discreta (1) com núcleo de coagulação constante, e para a sua versão contínua (3).

Para o caso discreto, assumindo que as condições iniciais satisfazem $c_1(0) > 0$ e são de decrescimento exponencial, mais precisamente, $0 \leqslant c_j(0) \leqslant A(1 + \Delta)^{-j}$, para A e Δ constantes positivas independentes de j, eles usaram a função geradora $\Phi(z,t) \coloneqq \sum_{j=1}^{\infty} c_j(t) z^j$ a fim de obterem uma expressão integral explícita para $c_j(t)$. Estudando a função geradora $\Phi(z,t)$, Kreer e Penrose concluiram que

$$\lim_{\substack{j,t\to+\infty \\ \xi=j/t \text{ fixo}}} t^2 c_j(t) = \frac{1}{\rho} e^{-\xi/\rho},$$

uniformemente para ξ em subconjuntos compactos de $\mathbb{R}^+$, onde $\rho > 0$ é a massa da condição inicial (e, como as soluções conservam a massa, é também a densidade da solução em todos os restantes instantes).

Subsequentemente, da Costa [17] melhorou a restrição sobre a condição inicial relaxando a imposição de positividade de $c_1(0)$, obtendo o resultado:

$$\text{se } \; 0 \leqslant c_j(0) \leqslant A(1+\Delta)^{-j} \text{ então } \lim_{\substack{j,t\to+\infty \\ \xi=j/t \text{ fixo} \\ j\in\mathcal{J}}} t^2 c_j(t) = \frac{q}{\rho} e^{-\xi/\rho},$$

onde $\mathcal{J}$ é o conjunto dos índices j para os quais $c_j(t) > 0$, e $q = \text{mdc}\{j \colon c_j(0) > 0\}$.

Diversas demonstrações foram posteriormente obtidas por Aldous [3], usando um "argumento probabilístico simples", e Deaconu e Tanré [25] usando também uma abordagem probabilista (ambos para convergência L^1-fraca). Deaconu e Tanré [25, Teoremas 3.9 e 3.11] obtiveram também um resultado importante: uma transformação que relaciona as soluções do caso aditivo com as do caso multiplicativo.

Recentemente, a investigação sobre este tópico, diversamente designado por *teoria do escalamento*, *escalamento dinâmico* ou *auto-semelhança*, tem sido alvo de uma crescente atenção, tendo surgido um apreciável número de contribuições importantes na área da modelação matemática, tais como as de Brilliantov e Krapivsky [10], Wattis e colaboradores [24, 69], e, naturalmente, o detalhado e completo artigo de revisão de Leyvraz [48] sobre a teoria do escalamento e modelos exactos. Do ponto de vista da análise matemática, artigos recentes importantes são os de Menon e Pego [57, 58, 59], da Costa et al.[21], e Laurençot e colaboradores [6, 39, 41].

Estes estudos estão ainda longe de fornecerem uma compreensão completa do fenómeno do escalamento. Em particular, usam coeficientes de coagulação bastante restritivos, a saber, constantes, aditivo e multiplicativo, e a análise baseia-se em métodos que parecem ser verdadeiramente dependentes da estrutura dos coeficientes, tais como funções geradoras, transformadas de Laplace e mudanças de variáveis. Apesar destas desvantagens, a análise dos sistemas com estes coeficientes restritivos está muito longe de ser trivial e é rica em pontos de grande delicadeza técnica e em ligações insuspeitadas com outras áreas da matemática, como os magníficos artigos de Menon e Pego claramente mostram.

Uma contribuição de relevo para a nossa compreensão do que acontece com modelos com coeficientes de coagulação mais realistas é o recente artigo [32], por Fournier e Laurençot, no qual eles provam a existência de soluções auto-semelhantes rapidamente decrescentes no infinito, para uma vasta classe de núcleos de coagulação homogéneos. Apesar de não estabelecerem resultados sobre a estabilidade destas soluções auto-semelhantes, ou seja, sobre a possibilidade de uma qualquer solução genérica convergir para uma auto-semelhante e, se tal se verificar, para qual (o tipo de questões tratado em [37, 17, 69] e, mais notavelmente, em [57, 58, 59]), este artigo foi uma contribuição muito significativa no sentido de melhorar a nossa compreensão acerca do comportamento das soluções do sistema de Smoluchowski.

O problema da estabilidade dos perfís auto-semelhantes é, de facto, um que permanece largamente em aberto, atendendo a que os resultados existentes e acima assinalados são válidos para núcleos particularmente simples e usam técnicas, como as funções geradoras ou as transformadas de Laplace, que têm poucas hipóteses de serem aplicáveis a situções mais gerais. Uma abordagem alternativa, por Laurençot e Mischler [45], prova a convergência para soluções auto-semelhantes da equação de Smoluchowski com coeficientes constantes, através da escrita da equação em variáveis auto-semelhantes seguida da identificação de vários funcionais de Liapunov. O mesmo tipo de abordagem tem sido tentado para núcleos mais gerais, até agora sem sucesso [40].

O interesse no comportamento assimptótico das soluções de sistemas de coagulação alargou-se para áreas inesperadas, tais como a teoria da probabilidade, motivada pelo influente artigo de revisão de Aldous [3]. Escrito há dez anos, centrase numa interpretação probabilística do modelo de campo médio fornecido pela equação de Smoluchowski, usando a dualidade com processos de ramificação, Aldous concentrou-se nos três núcleos mais estudados (constante, aditivo e multiplicativo) e nas suas relações com ideias clássicas em probabilidade (teoremas limite, leis estáveis), e enunciou dez problemas em aberto, muitos deles relativos à unicidade e à dependência em relação a condições iniciais para soluções de escalamento. O volume de investigação motivado por este artigo e pelos seus problemas em aberto é verdadeiramente notável (para uma monografia sobre desenvolvimentos recentes vêja-se o livro de Bertoin [9]).

Na restante parte desta introdução focaremos a nossa atenção no sistema que será o tema do resto deste trabalho: o modelo aditivo com uma fonte de monómeros.

Auto-semelhança no modelo aditivo com introdução de monómeros

Neste trabalho serão consideradas as equações (2) para o caso particular de coagulação pura e em que as únicas interacções relevantes são aquelas entre agregados e monómeros (e *não* do tipo agregado-agregado). Como referimos anteriormente, a evolução do sistema pode ser descrita pelo esquema cinético:

$$(j\text{-agregado}) + (1\text{-agregado}) \xrightarrow{a_{j,1}} ((j+1)\text{-agregado}), \quad j \in \mathbb{N}^+.$$

Definindo os coeficientes $a_1 := \frac{1}{2}a_{1,1}$ e $a_j := a_{j,1}$ para $j \geqslant 2$, este esquema cinético corresponde ao sistema de equações de coagulação

$$\begin{cases} \dot{c}_1 = -a_1 c_1^2 - c_1 \displaystyle\sum_{j=1}^{\infty} a_j c_j \\ \dot{c}_j = a_{j-1} c_1 c_{j-1} - a_j c_1 c_j, \quad j \geqslant 2. \end{cases} \tag{4}$$

Neste contexto, e de modo pouco rigoroso, é de esperar que, e devido ao papel desempenhado pelos monómeros, uma vez esgotados estes, todo o processo pare. Obviamente, o estado do sistema neste ponto de paragem dependerá da distribuição inicial de agregados no sistema.

O problema apenas se torna interessante nos casos em que um qualquer mecanismo impede que, em tempo finito, não haja esgotamento de monómeros no meio. Uma situação em que tal ocorre é quando há a possibilidade dos agregados se fragmentarem (i.e., a dinâmica do sistema torna-se reversível), tornando-se na dinâmica clássica do sistema de Becker-Döring [5]. Outra possibilidade é a inclusão de uma fonte externa de monómeros por adição de um termo $J_1(t)$ ao lado direito da primeira equação em (4).

Nesta última situação, o tipo de função a ser introduzida naturalmente dependerá do tipo de sistema que se pretende modelar, e diversos contextos têm sido considerados na literatura. Entre eles, refira-se o caso de $J_1(t)$ ser identicamente igual aos termos que surgem no membro direito da primeira equação em (4), reduzindo então a equação em c_1 em (4) a $\dot{c}_1 = 0$. Fisicamente este caso corresponde ao acoplamento do sistema de agregados com um reservatório infinito de monómeros, o que conduz a que a sua concentração seja constante ao longo do tempo. Isto corresponde, de facto, à teoria original de Becker-Döring sobre meta-estabilidade [62, 63].

Alternativamente, pode introduzir-se uma fonte de monómeros $J_1(t)$ que seja independente do estado do sistema. Esta hipótese é razoável num grande número de aplicações, incluindo os modelos de polimerização e de crescimento "epitaxial" [7]. O caso mais simples é quando $J_1(t)$ é uma constante independente do tempo e é um caso que se revela extremamente útil nas aplicações. Como mais à frente se verá, este caso foi recentemente tratado em [21].

Outro caso que se pode considerar é uma fonte do tipo lei de potência

$$J_1(t) = \alpha t^{\omega}, \qquad \text{com} \quad \alpha > 0 \text{ e } \omega \in \mathbb{R}.$$

Para as aplicações, este caso possivelmente não é tão interessante como o anterior (reobtido para $\omega = 0$) mas, mesmo assim, revela-se um caso bastante interessante para análise, já que o método de compactificação e as técnicas de redução à variedade central utilizadas em [21] manifestamente não se aplicam ao caso de uma lei de potências. Uma análise preliminar deste caso foi recentemente realizada em

[69]. O *ansatz* aí apresentado será utilizado neste trabalho para estudar rigorosamente o modelo com coeficientes de coagulação constantes, $a_j \equiv 1$, e com uma fonte de monómeros do tipo lei de potências, $J_1(t) = \alpha t^\omega$, i.e.,

$$\begin{cases} \dot{c}_1 = \alpha t^\omega - c_1^2 - c_1 \sum_{j=1}^{\infty} c_j \\ \dot{c}_j = c_1 c_{j-1} - c_1 c_j, \quad j \geqslant 2, \end{cases} \tag{5}$$

para $\alpha > 0$ e ω constantes.

Sobre o sistema de equações (5) serão analisados dois aspectos relacionados com a dinâmica do sistema. Primeiro, o comportamento pontual das soluções quando $t \to +\infty$ e o comportamento da quantidade total de agregados definido por $\sum_{j=1}^{\infty} c_j$. O segundo aspecto prende-se com a ocorrência de comportamentos auto-semelhantes.

Para este efeito, e numa primeira fase, o sistema de dimensão infinita (5) é reduzido a um novo problema quase explicitamente resolúvel. Isto significa o seguinte. Considerando o número total de agregados $c_0(t)$ como uma nova variável macroscópica,

$$c_0(t) = \sum_{j=1}^{\infty} c_j(t),$$

por derivação formal conclui-se que esta nova variável é solução da equação de evolução

$$\dot{c}_0 = \alpha t^\omega - c_0 c_1.$$

Isto permite reescrever o sistema (5) na forma fechada

$$\begin{cases} \dot{c}_0 = \alpha t^\omega - c_0 c_1, \\ \dot{c}_1 = \alpha t^\omega - c_0 c_1 - c_1^2, \\ \dot{c}_j = c_1 c_{j-1} - c_1 c_j, \quad j \geqslant 2. \end{cases} \tag{6}$$

Esta nova formulação do problema tem uma clara vantagem técnica: isola a descrição completa da dinâmica dos monómeros e do número total de agregados (duas primeiras equações) dos restantes componentes do sistema (última equação).

Esta separação permite, em particular, realizar o estudo do primeiro aspecto sobre a dinâmica acima referido, independentemente do comportamento dos restantes elementos constituintes do sistema.

Obtida uma solução do sistema

$$\begin{cases} \dot{c}_0 = \alpha t^\omega - c_0 c_1 \\ \dot{c}_1 = \alpha t^\omega - c_0 c_1 - c_1^2, \end{cases} \tag{7}$$

numa segunda fase rescala-se o tempo

$$\varsigma(t) := \varsigma_0 + \int_{t_0}^{t} c_1(s)\, ds, \tag{8}$$

para ς_0 uma constante positiva, e então consideram-se as novas variáveis resultantes

$$\widetilde{c}_j(\varsigma) := c_j(t(\varsigma)), \tag{9}$$

onde $t(\varsigma)$ é a função inversa de $\varsigma(t)$. De notar que, no caso $c_1(t) > 0$, todo este procedimento está bem definido e ς é uma função crescente em t.

Para estas novas variáveis, as equações em c_j $(j \geq 2)$ descritas em (6) reescrevem-se agora na forma

$$\widetilde{c}_j{}' = \widetilde{c}_{j-1} - \widetilde{c}_j, \qquad j \geqslant 2,$$

em que a linha que surge no lado esquerdo designa a derivada $\frac{\mathrm{d}}{\mathrm{d}\varsigma}$. O sistema de equações diferenciais assim obtido tem a vantagem de ser de tipo linear triangular inferior e, consequentemente, pode ser explicitamente resolvido em $\widetilde{c}_1(\varsigma)$ por substituição directa começando pela equação para $j = 2$ e aplicando recursivamente a fórmula da variação das constantes

$$\widetilde{c}_j(\varsigma) = e^{-\varsigma} \sum_{k=2}^{j} \frac{\varsigma^{j-k}}{(j-k)!} c_k(0) + \frac{1}{(j-2)!} \int_0^{\varsigma} \widetilde{c}_1(\varsigma - s) s^{j-2} e^{-s}\, ds. \tag{10}$$

Relativamente a (10), e na eventualidade de um completo conhecimento sobre a função $\widetilde{c}_1$, esta representação para a componente j da solução em termos de $\widetilde{c}_1(\varsigma)$ permite obter informação detalhada sobre o comportamento assimptótico das soluções do sistema inicial (5).

O estudo sobre as funções $\widetilde{c}_j(\varsigma)$ será essencial para prosseguir de encontro a um dos principais objectivos deste trabalho. Como anteriormente foi referido, o segundo aspecto a ser analisado sobre a dinâmica do sistema prende-se com a ocorrência de comportamentos auto-semelhantes. Como já mencionámos (cf. pág. 9), isto significa que todas as soluções aproximam-se de uma única solução. Por outras palavras, analisar-se-à a existência de duas funções, Φ e ς, e de duas constantes positivas, a e b, tais que, para j e t suficientemente grandes, se obtenha

$$c_j(t) \sim \varsigma(t)^{-a}\Phi(j\varsigma(t)^{-b}). \tag{11}$$

Para esta análise torna-se importante conhecer o comportamento de $\widetilde{c}_1(\varsigma)$, não apenas quando $\varsigma \to +\infty$, mas também a taxa de convergência efectiva de modo a obter-se uma relação do tipo (11). Para o caso autónomo $\omega = 0$ estudado em [21], esta informação é obtida pelo estudo de (7) utilizando regiões invariantes, uma mudança de variáveis de fase e da variável independente sugeridas pelo método de compactificação de Poincaré, e pelo método das variedades centrais.

Esta abordagem não é válida para ω não nulo genérico. Para estes casos, teremos que utilizar um caminho "menos directo" que começa por utilizar um *ansatz* para uma mudança de variável conveniente, sugerida por [69, Table 2] e obtida através de uma análise assimptótica. De sublinhar ainda, que a mudança de variável considerada está relacionada com o termo dominante da análise assimptótica de [69], e depende explicitamente das regiões consideradas para ω.

Estudaremos os casos $\omega > -1/2$ (Capítulo 2), $\omega = -1/2$ (Capítulo 4) e $\omega = -1$ (Capítulo 5) usando o *ansatz* fornecido pela mudança de variáveis. Os casos $-1 < \omega < -1/2$ e $\omega < -1$ serão tratados no Capítulo 5 usando uma abordagem diferente, baseada no estudo das propriedades de monotonia das soluções.

Começamos por analisar o caso $\omega > -1/2$ no Capítulo 2, o qual inclui o caso particular $\omega = 0$, já discutido em [21], e que agora pode ser encarado num contexto mais amplo. Os resultados relevantes retirados do nosso conhecimento sobre o comportamento de $\widetilde{c}_j$ permitem-nos concluir a existência da função

$$\varsigma(t) \sim t^{\frac{\omega+2}{3}},$$

e de uma familia de funções de escalamento $\Phi_{1,\omega}$, tais que (11) se verifica com $a = \frac{1-\omega}{2+\omega}$ e $b = 1$.

Os gráficos de $\eta \mapsto \Phi_{1,\omega}(\eta)$ são apresentados na Figura 2.7, página 68.

Da observação que $\Phi_{1,\omega}$ fica descontínua em $\eta = 1$ para $\omega \leqslant 1$, e ilimitada em qualquer conjuntou $(1-\varepsilon, 1)$ quando $\omega < 1$, é razoável inferir que o escalamento considerado poderá não ser apropriado em torno desse ponto. Com uma diferente variável auto-semelhante conseguimos provar a existência de uma função de escalamento diferente $\Phi_{2,\omega}$, mais adequada ao estudo do que se passa em torno do ponto singular de $\Phi_{1,\omega}$.

Essencialmente, temos, para j e t suficientemente grandes,

$$c_j(t) \sim \varsigma(t)^{-a/2}\Phi_{2,\omega}\left(\frac{j-\varsigma(t)}{\sqrt{\varsigma(t)}}\right), \tag{12}$$

com a e $\varsigma(t)$ definidas como anteriormente.

Os gráficos das novas funções de escalamento $\Phi_{2,\omega}$ são apresentados na Figura 2.8, na página 74.

No Capítulo 3 consideramos uma extensão dos resultados do Capítulo 2, para fontes de monómeros do tipo

$$J_1(t) = \alpha t^{\omega}(1+\varepsilon(t)),$$

onde $\varepsilon(\cdot)$ é uma função contínua satisfazendo $\varepsilon(t) \to 0$ quando $t \to +\infty$. A maioria, mas não todos, dos resultados do Capítulo anterior pode ser generalizada assumindo apenas esta hipótese geral sobre ε. Isto é verdade, em particular, no que diz respeito à existência de perfís auto-semelhantes. Os resultados relativos à taxa de convergência da quantidade $c_1 \sum_{j=1}^{\infty} c_j$ necessitam de uma hipótese adicional acerca da taxa de decaimento da perturbação $\varepsilon(t)$.

No Capítulo 4 analizamos o caso $\omega = -\frac{1}{2}$, para o qual, de acordo com [69], esperamos que as soluções (c_0, c_1) de (7) se comportem como

$$\begin{cases} c_0(t) \sim \left(3\alpha^2\right)^{\frac{1}{3}} (\log t)^{\frac{1}{3}} \\ c_1(t) \sim \left(\dfrac{\alpha}{3}\right)^{\frac{1}{3}} t^{-\frac{1}{2}} (\log t)^{-\frac{1}{3}}, \text{ quando } t \to +\infty. \end{cases}$$

A presença dos termos logarítmicos resulta num certo número de complicações técnicas, mas o método que foi utilizado para o caso $\omega > -\frac{1}{2}$ é ainda aplicável e os resultados que obtemos são semelhantes, exceptuando o facto de que mostramos a *não* existência de um análogo de Φ_2. Mais precisamente, mostramos que quando $j, \varsigma \to +\infty$ e quando $\xi = \frac{j-\varsigma}{\sqrt{\varsigma}}$, o limite de

$$\varsigma^{\frac{1}{2}} (\log \varsigma)^{\frac{2}{3}} \tilde{c}_j(\varsigma)$$

é infinito.

No último Capítulo consideramos o caso $\omega < -\frac{1}{2}$ e estudamos separadamente os casos $\omega = -1$, $\omega > -1$ e por fim $\omega < -1$. Para $\omega = -1$ usamos um *ansatz* semelhante ao utilizado no caso $\omega \geqslant -\frac{1}{2}$. Acima e abaixo de $\omega = -1$ o *ansatz* resulta em dificuldades que não se conseguiram ultrapassar, devidas a um delicado balanço entre contribuições com sinais diferentes, pelo que se desenvolveu uma abordagem alternativa, baseada numa abordagem directa de (7), e consistindo num cuidado estudo de propriedades de monotonia das soluções em certas regiões de $\mathbb{R}^2$ utilizando curvas auxiliares apropriadas. Não nos foi possível provar se as soluções se comportam, ou não, de um modo auto-semelhante, mas baseado em evidência numérica, a existência de auto-semelhança nestes casos parece estar excluída.

Chapter 1

Introduction

In this work we study some aspects of the long time behaviour of a system of an infinite number of ordinary differential equations modelling the kinetics of particle coagulation.

Since the pioneering work of Smoluchowski [66, 67], who studied the coagulation of colloid particles under the assumption of Brownian motion, in the second decade of the twentieth century, a variety of kinetic equations, under the common designation of coagulation, or coagulation-fragmentation, equations have been successfully used to model a wide range of phenomena over a huge range of scales, from phytoplankton to smoke, clouds, colloids, aerosols, planetesimals, and phase transitions in alloys (see, e.g., [33, 61, 64, 65]).

A clear measure of the importance of coagulation models can be seen by the experiments carried in orbit aboard the International Space Station (ISS) under the *Coarsening in solid liquid mixtures-1 and -2* programs [22, 23] to investigate the long-time coarsening in the Lifshitz and Slyozov [51] and Wagner [68] theory of Ostwald ripening. This model is of a vital importance when trying to understand the properties of coarsening of the granular structure in binary alloys over an extended period of time. It has been shown recently that the LSW theory of Ostwald ripening is closely related with the Becker-Döring [8, 61]model of cluster growth [14, 42, 60, 63], which is a kinetic system of the coagulation-fragmentation type.

The importance of these equations for the applied sciences has been reflected, in the last few decades, in an upsurge of interest in its mathematical theory among the mathematical (analysts, probabilists, and numerical analysts), as well as in the scientific modelling communities. It is clearly out of place to review, here, all the aspects of the, at present, already huge mathematical literature on this type of systems. A number of recent mathematical reviews are in existence and provide an excellent guide to the literature for someone now entering the field [3, 9, 44, 46, 70]. Even not so recent mathematical reviews such as [28, 29] are still of interest, and can profitably be read today.

In the remaining part of this *Introduction* we start by describing the differential equations system and by reviewing the part of the literature which is relevant to the work to be presented in the remaining chapters, which will be summarized in the last section of the current chapter.

1.1 The coagulation-fragmentation equations

We consider a system composed of a large number of particles, each particle consisting of an integer number of a fundamental unit called monomer; we assume this indivisible unit to have mass 1, so that a j-cluster, i.e, a particle formed by j-monomers, will have mass j. We assume the system to be, somehow, well mixed, so that it can be described by a spatially independent vector of concentrations (c_j), where the subscript j runs through all the possible cluster masses.

Furthermore we assume these clusters can bind together to form larger clusters, and, if we assume *fragmentation*, they can also break-up giving rise to smaller clusters. We assume that we only have *binary* reactions, although models with more complicated reaction schemes have been considered. The cluster interaction is assumed to follow the mass action law of chemical kinetics, giving rise to a coagulation and a fragmentation *rate coefficients* (or kernels), that must contain all the information about the reaction kinetics we are modeling. Typically these coefficient are only a function of the *masses* of the two clusters involved in the reaction, but more complicated situations also arises [70, Section 4.3].

We introduce $(c_j(t))_{j=1}^{\infty}$, the sequence whose elements are the concentration of clusters of mass j at some time t, and we want to study the evolution of $c_j(t)$ as $t \to +\infty$, either in the strong topology of an appropriate Banach space, or pointwise in j (i.e., for each fixed j), or when j also converges to $+\infty$ in some way related to the convergence of t. The physical relevance of these distinct approaches will be pointed out below.

The time evolution of $(c_j(t))_{j=1}^{\infty}$ is dependent of the type of elementary kinetic processes that each cluster is supposed to undergo. Typically, these processes are the following (we assume next that $j \geq 2$):

(i) two smaller cluster can coalesce to form a j-cluster

$$[(j-k)\text{-cluster}\,] \;+\; [k\text{-cluster}\,] \longrightarrow [j\text{-cluster}\,], \qquad k = 1, \dots, j-1$$

(ii) a j-cluster can fragment into two smaller one

$$[j\text{-cluster}\,] \longrightarrow [(j-k)\text{-cluster}\,] \;+\; [k\text{-cluster}\,], \qquad k = 1, \dots, j-1$$

(iii) a j-cluster can merge with another cluster

$$[j\text{-cluster}\,] \;+\; [k\text{-cluster}\,] \longrightarrow [(j+k)\text{-cluster}\,], \qquad k = 1, 2, \dots$$

(iv) a larger cluster can break-up

$$[(j+k)\text{-cluster}\,] \longrightarrow [j\text{-cluster}\,] \;+\; [k\text{-cluster}\,], \qquad k = 1, 2, \dots$$

Assuming the mass action law of chemical kinetics, and letting the rate coefficient for the coagulation reaction between a j-cluster and a k-cluster be denoted by $a_{j,k}$, and that for the break up of a $(j+k)$-cluster into a j-cluster and a k-cluster be $b_{j,k}$, we immediately conclude that the contributions of the processes **(i)**-(**iv**) to the time evolution of c_j are given by, respectively,

(i) $\dfrac{\mathrm{d}c_j}{\mathrm{d}t} = a_{j-k,k}c_{j-k}c_k\,,$

(ii) $\dfrac{\mathrm{d}c_j}{\mathrm{d}t} = -b_{j-k,k}c_j\,,$

(iii) $\dfrac{\mathrm{d}c_j}{\mathrm{d}t} = -a_{j,k}c_jc_k\,,$

(iv) $\dfrac{\mathrm{d}c_j}{\mathrm{d}t} = b_{j,k}c_{j+k}\,.$

Noticing that for $j = 1$ we don't have contributions **(i)** and **(ii)**, the discrete coagulation-fragmentation equations can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}c_j = \begin{cases} +\dfrac{1}{2}\displaystyle\sum_{k=1}^{j-1} a_{j-k,k}c_{j-k}c_k & \text{Gain by coagulation} \\ -\dfrac{1}{2}\displaystyle\sum_{k=1}^{j-1} b_{j-k,k}c_j & \text{Loss by fragmentation} \\ -\displaystyle\sum_{k=1}^{\infty} a_{j,k}c_jc_k & \text{Loss by coagulation} \\ +\displaystyle\sum_{k=1}^{\infty} b_{j,k}c_{j+k} & \text{Gain by fragmentation} \end{cases} \tag{1.1}$$

Or, using a more practical notation,

$$\dot{c}_j = \frac{\mathrm{d}}{\mathrm{d}t}c_j = \frac{1}{2}\sum_{k=1}^{j-1} W_{j-k,k} - \sum_{k=1}^{\infty} W_{j,k}\,, \quad j \geqslant 1, \tag{1.2}$$

where

$$W_{i,j} = W_{i,j}(c) \coloneqq a_{i,j}\,c_i\,c_j - b_{i,j}\,c_{i+j}\,, \quad i,j \geqslant 1.$$

On physical grounds it is reasonable to impose that the coefficients $a_{i,j}$ and $b_{i,j}$ always satisfy a *symmetry* and *positivity* condition: $a_{i,j} = a_{j,i} \geqslant 0$ and $b_{i,j} = b_{j,i} \geqslant 0$, $\forall i, j \geqslant 1$. The fundamental assumption is that all our knowledge of the underlying mechanisms of the clustering process is contained in the rate coefficients. In Table 1.1 [18, Table 1] we list a few of the most commonly used coagulation kernels in the applied literature. For the fragmentation kernels a lot less information seems to be available on physical grounds. A number of assump-

Table 1.1: Some commonly used coagulation kernels $a_{j,k}$.

$a_{j,k}$	Comment
1	Approximate Brownian coagulation, Linear-chain polymerization
$j+k$	Branched-chain polymerization, Limit case of gravitation coagulation
jk	Branched-chain polymerization
$j^{-2/3}+k^{-2/3}$	Diffusion-controlled growth of supported metal crystallites
$(j^{1/3}+k^{1/3})(j^{-1/3}+k^{-1/3})$	Brownian coagulation (continuum regime)
$(j^{1/3}+k^{1/3})^2(j^{-1}+k^{-1})^{1/2}$	Brownian coagulation (free molecular regime)
$(j^{1/3}+k^{1/3})^3$	Shear (linear velocity profile)
$(j^{1/3}+k^{1/3})^7$	Shear (nonlinear velocity profile)
$(j^{1/3}+k^{1/3})^2\lvert j^{1/3}-k^{1/3}\rvert$	Gravitational settling

tions have been used, the most important of which seems to be the fulfillment of a detailed balance condition [5, 13].

Most of the rate coefficients presented in Table 1.1, and others like them that show up in the applied literature, have the unfortunate feature of resulting in systems that, at present, seem too hard to analyse rigorously. As a result, most of the mathematical studies of these equations assume kernels of a particularly simple kind (such as the first three in Table 1.1), or kernels bounded by such simple kernels.

Both from a mathematical and a physical point of view, three other special classes of coefficients have earned a considerable deal of effort: the pure fragmentation ($a_{i,j} \equiv 0$), the pure coagulation, or Smoluchowski, ($b_{i,j} \equiv 0$), and the Becker-Döring system ($a_{i,j} = b_{i,j} \equiv 0$ if $i \wedge j \geq N,$ for some $N \geq 1$). These classes of systems are not only easier to analyse than the full coagulation-fragmentation system (1.1), but some of the questions are still formidably challenging, and are interesting in themselves as valid models in several scientific applications.

A different class of system is the *continuous* version of (1.2), obtained by considering that the mass j of a j-cluster can be allowed to vary continuously between

0 and $+\infty$ (and, hence, it is denoted by x). Formally, the difference to the system (1.2) is that now one has integrals instead of sums and the infinite system of ordinary differential equations is now a non-local integro-partial-differential equation,

$$\frac{\partial c}{\partial t}(t,x) = \frac{1}{2}\int_0^x W(x-y,y)dy - \int_0^{+\infty} W(x,y)dy, \tag{1.3}$$

where $W(x,y) \coloneqq a(x,y)c(t,x)c(t,y) - b(x,y)c(t,x+y)$, and the functions a and b have natural correspondences with the rate coefficients of the discrete case (1.2).

Although it is reasonable to expect a natural relation between the discrete and the continuous models, the mathematically rigorous relation between them is still a matter of current research [43] that shall not concern us here.

In this work we study some aspects of the long time behaviour of a discrete Smoluchowski system, with Becker-Döring type interactions, the so called *addition* model [10, 34, 38]. In particular we are going to focus on the so called *self-similar* behaviour of solutions. Studies of this type of behaviour have been published both for the discrete and for the continuous version of the Smoluchowski equations, and below we briefly outline the current state of the mathematical knowledge in both cases. Otherwise, we will exclusively focus on the discrete system for the remaining of this chapter.

1.2 Mathematical preliminaries

1.2.1 Basic Setting

The mathematical analysis of the infinite dimensional system (1.2) requires that some care must be taken with the choice of the spaces and in the proper definition of the solution.

The basic space used in studies of (1.2) is the space of sequences with *finite mass* $X \coloneqq \{(c_j) \in \mathbb{R}^{\mathbb{N}} : (jc_j) \in \ell^1\}$, with the norm $\|(c_j)\| \coloneqq \|(jc_j)\|_{\ell^1}$. Endowed with this norm the space X is a Banach space. The fact that this is a natural space in which to study (1.2), and the reason for the name of its elements, is clear if we remember that c_j is the concentration of j-clusters , and so jc_j is proportional to the mass of the j-clusters . Hence $\|(c_j)\| = \sum_{j=1}^{\infty} j|c_j|$ is (proportional to) the total mass of the system.

Some spaces analogous to X are also important in certain cases. These are the spaces defined by $X^\alpha := \left\{(c_j) \in \mathbb{R}^{\mathbb{N}} : \|(c_j)\|_\alpha := \|(j^\alpha c_j)\|_{\ell^1} < +\infty\right\}$. Note that $X = X^1$. The family $\{X_\alpha, \|\cdot\|_\alpha\}_{\alpha\geq 0}$ is a regular, normal, and compact scale of Banach spaces [15, Theorem 1.2.1]. In particular the compacity of the embedding plays a crucial role in certain studies.

The definitions of solution that are most common in the literature are either what is sometimes called a *mild solution,* i.e., essentially a continuous solution of the integrated version of (1.2) [4, 5], or, more rarely, a *weak solution,* in the sense usually given to this term in the theory of partial differential equations [43]. Regularity results allow for the upgrading of a mild, or a weak, solution to a function with sufficient differentiability for the derivatives in the left-hand side of (1.2) to make sense, and thus to get a solution in a stronger sense, in similar fashion to what is normally done in the theory of (finite dimensional) ordinary differential equations [4, 5].

Results about existence, uniqueness, and regularity of solutions to either (1.2) or (1.3) depend on the assumptions about the *structure* and the *growth* behaviour of the kinetic coefficients a and b. Currently, there exists a large body of work about these problems, that started with the pioneering works of Melzak [56] and McLeod [53, 54, 55], continued with those of Aizenman and Bak [2], and White [71], and attained its maturity with the seminal works of Ball, Carr and Penrose [5], and of Ball and Carr [4], which formed a basis to much of the current interest and developments in the field. In fact, for the purposes of the current work, the results proved in [5] about existence, uniqueness, and regularity are immediately applicable, either directly or after some trivial modification.

1.2.2 The long-time behaviour of solutions and self-similarity

The study of the long-time behaviour of solutions to (1.2) or (1.3) is an extremely rich and active area of current research.

From a physical point of view, it is expected that global solutions should converge to a time independent solution as $t \to +\infty$. Furthermore, since mass is conserved in each of the elementary processes in page 23, it is reasonable to expect that the same should hold for the solutions of the system, i.e., the total mass (norm) of the solution should remain constant under time evolution: $\|(c_j(t))\| = \rho_0, \forall t,$ for some constant ρ_0 (the initial mass).

Actually, for many decades a number of formal arguments existed in the literature that indicated this expectation to be rather naive, and pointed to the fact that it should not hold true in general. This was reinforced by the knowledge of solutions of the Smoluchowski equations with coagulation kernel $a_{j,k} = jk$ and particularly chosen initial data (monodispersed, e.g., $c_j(0) = \delta_{j,1}$) that behaved in a different way, conserving mass up to a given time T_g and showing a regular mass decay to zero after T_g. This behaviour took the name of *gelation phenomenon* and was physically interpreted as due to the occurrence of a phase transition in the system: the formation of an “infinite” cluster not modeled by the coagulation-fragmentation system (the *gel*) that is responsible for the removal of the mass of the cluster system. Mathematically, the general characterization of this phenomenon was, during many decades, an important open problem, and, although some progress have slowly been made [4, 11, 12, 16, 47, 53, 54], only very recently was the problem successfully tackled [30, 31, 36].

The convergence to equilibria and the conservation of mass (or lack thereof) are hardly the only interesting issues about the long-time behaviour of solutions. Two other issues, both of considerable physical significance, are related with the details of the approach to the final state, namely, the problem of occurrence of metastable behaviour in the Becker-Döring equations [35, 61, 62], and the question of self-similar behaviour in Smoluchowski's and related systems. To this last problem we will now turn our attention.

When studying the cluster size distribution as a function of time (in the absence of fragmentation), we are confronted with the behaviour of solutions for large times and large monomer sizes. We want to know whether solutions within a certain range of initial conditions will eventually approach a *scale invariant* solution. Thus, a central problem in the study of the long-time behaviour of solutions to the coagulation equations is their convergence to some similarity profile Φ, when j and $t \to +\infty$

$$c_j(t) \sim \varsigma(t)^{-a}\Phi(j\varsigma(t)^{-b})$$

where a and b are appropriate positive exponents, $\varsigma(t)$ is a scaling function (with $\varsigma(t) \to +\infty$ as $t \to +\infty$) and the scaling argument $j/\varsigma(t)^b$ is kept fixed.

This is conjectured to occur, for a large class of initial conditions, in the case of (symmetric) homogeneous kinetic coefficients, i.e., those satisfying $a_{k,j} = a_{j,k} = K(j,k)$, with the function K being homogeneous with some homogeneity exponent λ, i.e., $K(ux, uy) = u^{\lambda}K(x, y)$.

These type of questions were first presented and addressed by Friedlander and co-workers [33] in the 1960s and 1970s. It was intensely studied, from a mathematical modelling point of view, by van Dongen and Ernst in the 1980s, [26, 27].

In [27], van Dongen and Ernst studied homogeneous kernels with the additional growth condition $K(i, j) \sim i^{\mu}j^{\nu}$, $(j \gg i, \lambda = \mu + \nu)$. They distinguished between several classes based on the sign of μ. Furthermore, on physical considerations they imposed $\nu \leqslant 1$ and $\lambda \leqslant 2$. It was argued that, since the reaction rate j^{ν} of a large j-monomer (i.e., $j \gg i$) should not increase faster than its volume, we must have $\nu \leqslant 1$, and also that since for clusters of equal size, $K(j, j) = j^{\lambda}K(1, 1)$ should not increase faster than j^2 we must have $\lambda \leqslant 2$. An important aspect is the introduction of *nongelling* and *gelling* kernels, related with the above mentioned problem of mass conservation, where in the former the mean cluster size $\varsigma(t)$ remains finite for all times $t < \infty$ and for the latter, where a gelation transition occurs, characterized by the divergence of $\varsigma(t)$ occurs at a finite time T_c, the *gelpoint*. Accordingly, the scaling behaviour as $\varsigma(t) \to +\infty$ is meant as $t \to T_c$ in the gelling case, and as $t \to +\infty$ in the nongelling case.

Leyvraz [49, 50] studied the scaling theory near the gelpoint, just before the transition, where the typical cluster size diverges, originating the appearance of a *giant* cluster. The difficulties in this case have lead to a closer look at exact solutions and numerical methods in order to get a better understanding.

For density conserving solutions, one of the first results to be proved rigorously (from a mathematical standpoint), was obtained by Kreer and Penrose [37], based on an idea of Lushnikov [52], for the discrete Smoluchowski equation (1.2) with constant coagulation kernel $a_{j,k} = 2$ and for its continuum analogue (1.3). Assuming that the initial data satisfy $c_1(0) > 0$ and are exponentially decreasing, more precisely $0 \leqslant c_j(0) \leqslant A(1+\Delta)^{-j}$ for some positive constants A and Δ, independent of j, they used the generating function $\Psi(z,t) \coloneqq \sum_{j=1}^{\infty} c_j(t) z^j$ to obtain an explicit integral expression for $c_j(t)$. By studying the generating function $\Psi(z,t)$, Kreer and Penrose conclude that

$$\lim_{\substack{j,t\to+\infty \\ \xi=j/t \text{ fixed}}} t^2 c_j(t) = \frac{1}{\rho}\, e^{-\xi/\rho},$$

uniformly for ξ in compact subsets of $\mathbb{R}^+$, where $\rho > 0$ is the mass of the initial condition (and, since solutions are mass conserving, of the solution for all times).

Subsequently, da Costa [17] improved upon the condition on the initial data by relaxing the positivity requirement of $c_1(0)$, obtaining the following result:

$$\text{if } \ 0 \leqslant c_j(0) \leqslant A(1+\Delta)^{-j} \text{ then } \quad \lim_{\substack{j,t\to+\infty \\ \xi=j/t \text{ fixed} \\ j\in\mathcal{J}}} t^2 c_j(t) = \frac{q}{\rho}\, e^{-\xi/\rho},$$

where $\mathcal{J}$ is the set of subscripts j for which $c_j(t) > 0$, and $q = \gcd\{j : c_j(0) > 0\}$.

Different proofs have since been given by Aldous [3], using an "easy probability argument", and Deaconu and Tanré [25] using also a probabilistic approach (both for L^1 weak convergence). Deaconu and Tanré [25, Thms.3.9 and 3.11] obtain also an important result: a transformation which connects the solution of the additive case with the one of the multiplicative case.

In recent years research on this topic, under the various names *scaling theory*, *dynamical scaling* or *self-similarity*, has seen an increasing growth trend, with a number of important contributions on the mathematical modelling approach, such as those by Brilliantov and Krapivsky [10], Wattis and co-workers [24, 69], and, of course, the very thorough review article of Leyvraz [48] on scaling theory and exactly solved models. From the mathematical analysis point of view, important recent articles are those by Menon and Pego [57, 58, 59], da Costa et al.[21], and Laurençot and co-workers [6, 39, 41].

These studies still fall short of providing a complete understanding of the scaling phenomenon, in particular, they use rather restrictive types of coagulation kernels, namely, constant, additive, and multiplicative kernels, and their analysis are mainly based in methods that seem to be truly dependent of the structure of the kernel, such as the use of generating functions, Laplace transforms, or change of variables. In spite of these drawbacks, the analysis of these restrictive kernels are far from trivial and are not short of technically very delicate points and unsuspected connections to other areas of mathematics, as the wonderful papers of Menon and Pego clearly show.

A major recent contribution to our knowledge of what happens with more realistic types of coagulation kernels is provided by the article [32], by Fournier and Laurençot, in which they proved the existence of self-similar solutions decaying rapidly at infinity for a wide class of homogeneous coagulation kernels. Although they could not provide information about the stability of these self-similar solutions, that is, knowledge about the possibility of a generic solution to eventually converge to a self-similar one, and, if such is the case, to which one (the type of questions rigorously addressed in [37, 17, 69] and, most notably, in [57, 58, 59]), this article is indeed a remarkable step towards a better understanding of the behaviour of solutions to the Smoluchowski system.

The problem of stability of the similarity profiles is, indeed, a largely open one, the known results, pointed out above, being for particularly simple kernels and using devices unlikely to be effective in more general situations, such as generating functions and Laplace transforms. In an alternative approach, by Laurençot and Mischler [45], give a proof of the convergence to self-similar profiles for solutions to the Smoluchowski coagulation equation with constant coagulation kernel by writing Smoluchowski's coagulation equation in self-similar variables and identifying several Liapunov functionals. The same type of approach has being tried for more general kernels, so far without success [40].

The interest in the long-time behaviour of solutions to the coagulation systems has spread to unexpected areas, like theoretical and applied probability, sparked by the influential review article by Aldous [3]. Written ten years ago, it focused on a "probabilistic" interpretation of the mean-field model of Smoluchowski coagulation equation, using duality with branching-type processes. Aldous focused on the three most studied kernels (constant, additive and multiplicative), and on its relations with classical ideas in probability (stable laws, limit theorems), he put forward ten open problems, most of them focusing on uniqueness and dependence on initial conditions for scaling solutions, and the quantity of research motivated by this article has been truly amazing (for a monograph on recent developments see the book by Bertoin [9]).

In the remaining of this introduction we will focus on the system which is the subject of this thesis: the addition model with input of monomers.

1.2.3 Self-similarity in the addition model with input of monomers

Throughout this work we will consider a particular case of (1.2) without fragmentation, i.e., $b_{i,j} \equiv 0, \forall i, j$, and such that the only relevant interactions are those between clusters and monomers (and *no* interactions of cluster-cluster type occur); as mentioned already, the evolution of the cluster population can be described by the following kinetic scheme

$$(j\text{-cluster}) + (1\text{-cluster}) \xrightarrow{a_{j,1}} ((j+1)\text{-cluster}), \quad j \in \mathbb{N}^+.$$

Defining the coefficients $a_1 := \frac{1}{2}a_{1,1}$ and $a_j := a_{j,1}$ if $j \geqslant 2$, this scheme corresponds to the following coagulation kinetic equations

$$\begin{cases} \dot{c}_1 = -a_1c_1^2 - c_1\displaystyle\sum_{j=1}^{\infty} a_jc_j \\ \dot{c}_j = a_{j-1}c_1c_{j-1} - a_jc_1c_j, \quad j \geqslant 2. \end{cases} \tag{1.4}$$

Due to the special role played by the monomers in this kinetic scheme, we expect the dynamics of the differential equation to freeze when the system runs out of monomers, and the final state will clearly depend on the initial distribution of clusters sizes.

The problem only becomes interesting if some mechanism is introduced that avoids the complete depletion of monomers in finite time. One such case, widely considered in the literature, is the possibility of clusters to fragment (i.e., the kinetic scheme to be reversible) which gives rise to the classical dynamic Becker-Döring system [5]; another possibility is to externally provide a source of monomers by adding a source term $J_1(t)$ to the right hand side of the c_1-equation in (1.4).

The particular type of input function to be considered clearly depends on the particular situation to be modelled. Several possibilities have been considered in the literature. Among them, we can point the case where $J_1(t)$ exactly balances the remaining terms in the right-hand side of the c_1-equation, in which case that differential equation reduces to $\dot{c}_1 = 0$; this case corresponds, physically, to the coupling of the cluster system with an infinite particle bath of monomers that allows their concentration to be constant in time, and, in fact, constitutes the original Becker-Döring theory of metastability [62, 63].

Another way to externally supply monomers is to define the input term $J_1(t)$ independently of the state of the system. This is a quite reasonable assumption in a number of applications, including in simple models of polymerization and of epitaxial growth [7]. The easiest hypothesis about $J_1(t)$, which turns out to be very useful in applications, is to make it a time independent constant. This case was considered recently in [21] (see below).

Another model case for the input function is to consider it a power law

$$J_1(t) = \alpha t^{\omega}, \qquad \text{with} \quad \alpha > 0 \text{ and } \omega \in \mathbb{R}.$$

This case is possibly not as interesting for the applications as the time independent case ($\omega = 0$), but it is still a case deserving consideration, and constitutes an interesting mathematical challenge, since the approach used in [21] is based on a compactification method and centre manifold reduction techniques, that clearly are not available for the power law case.

A formal analysis of this case was recently performed in [69], and we will use the ansatz provided by that study to rigorously analyse the addition model with constant rate coefficients $a_j \equiv 1$ and a power law input of monomers $J_1(t) = \alpha t^{\omega}$, namely

$$\begin{cases} \dot{c}_1 = \alpha t^{\omega} - c_1^2 - c_1 \displaystyle\sum_{j=1}^{\infty} c_j \\ \dot{c}_j = c_1 c_{j-1} - c_1 c_j, \quad j \geqslant 2, \end{cases} \tag{1.5}$$

where $\alpha > 0$ and ω are constant parameters.

We will study basically two aspects of the dynamical behaviour of solutions to (1.5).

First, we want to establish the componentwise behaviour of the solution as $t \to +\infty$, as well as the behaviour of the total amount of clusters present (the *bulk*), defined by $\sum_{j=1}^{\infty} c_j$.

The second aspect of the dynamics we are interested in is the occurrence of similarity behaviour.

Our first step consists in transforming the infinite dimensional system (1.5) into an almost exactly solvable problem.

Introducing the total number of clusters as a new macroscopic variable $c_0(t)$ defined by

$$c_0(t) = \sum_{j=1}^{\infty} c_j(t),$$

and formally differentiating termwise, we conclude that c_0 satisfies the evolution equation

$$\dot{c}_0 = \alpha t^{\omega} - c_0 c_1.$$

The system (1.5) can then be written, in closed form, as

$$\begin{cases} \dot{c}_0 = \alpha t^{\omega} - c_0 c_1, \\ \dot{c}_1 = \alpha t^{\omega} - c_0 c_1 - c_1^2, \\ \dot{c}_j = c_1 c_{j-1} - c_1 c_j, \quad j \geqslant 2, \end{cases} \tag{1.6}$$

and from the reduced system we observe that the equations governing both the monomer dynamics and the total number of clusters are actually a nonautonomous bidimensional (c_0, c_1)-system

$$\begin{cases} \dot{c}_0 = \alpha t^{\omega} - c_0 c_1 \\ \dot{c}_1 = \alpha t^{\omega} - c_0 c_1 - c_1^2, \end{cases} \tag{1.7}$$

the dynamics of which can be studied in a way totally independent of the remaining components of the infinite dimensional system.

Given a solution of (1.7), we introduce a new time scale

$$\varsigma(t) \coloneqq \varsigma_0 + \int_{t_0}^{t} c_1(s)\, ds, \tag{1.8}$$

where ς_0 is a positive constant, and then we consider the new phase variables

$$\widetilde{c}_j(\varsigma) \coloneqq c_j(t(\varsigma)), \tag{1.9}$$

where $t(\varsigma)$ is the inverse function of $\varsigma(t)$.

When $c_1(t) > 0$, these are well defined and ς is an increasing function of t.

In the new variables, the c_j–equations in (1.6) now become

$$\widetilde{c}_j{}' = \widetilde{c}_{j-1} - \widetilde{c}_j, \qquad j \geqslant 2,$$

where $(\cdot)' = \frac{\mathrm{d}}{\mathrm{d}\varsigma}$.

This system of differential equations turns out to be a lower triangular linear system and thus can be explicitly solved in terms of the function $\widetilde{c}_1(\varsigma)$ starting from the equation for $j = 2$ and applying the variation of constants formula recursively:

$$\widetilde{c}_j(\varsigma) = e^{-\varsigma} \sum_{k=2}^{j} \frac{\varsigma^{j-k}}{(j-k)!} c_k(0) + \frac{1}{(j-2)!} \int_0^{\varsigma} \widetilde{c}_1(\varsigma - s) s^{j-2} e^{-s}\, ds. \tag{1.10}$$

Once a precise knowledge of the behaviour of $\widetilde{c}_1$ is available, this representation formula for the $j-$component of the solution in terms of $\widetilde{c}_1(\varsigma)$ allows us to prove the rather detailed information about the long time evolution of solutions to (1.5).

The study of the behaviour of $\widetilde{c}_j(\varsigma)$ will be essential in the pursuit of our main interest. As we mentioned in page 33 the other aspect we are interested in concerns the existence of self-similar solutions. As we pointed out (c.f. page 28) this means that the cluster size distribution approaches a common profile at large times, more exactly, there exist functions Φ and ς, and positive constants a and b, such that, for sufficiently large j and t,

$$c_j(t) \sim \varsigma(t)^{-a} \Phi(j\varsigma(t)^{-b}). \tag{1.11}$$

What we need to know about the behaviour of $\widetilde{c}_1(\varsigma)$ is not only its limit as $\varsigma \to +\infty$ but the actual rate of convergence in order to obtain a relation like the one in (1.11). For the autonomous $\omega = 0$ case studied in [21] this information was obtained from the study of (1.7) using invariant regions, a change of phase and independent variables suggested by Poincaré's compactification, and a centre manifold analysis.

This approach is not available for general nonzero ω case, and so, here, we use an indirect path, which starts by making an *ansatz* for a convenient change of variables, suggested by [69, Table 2] and obtained via formal asymptotics.

The change of variables we consider is linked to the asymptotic leading order analysis performed in [69], and depends explicitly on the range of ω we consider.

We will study the cases $\omega > -1/2$ (Chapter 2), $\omega = -1/2$ (Chapter 4) and $\omega = -1$ (Chapter 5) using the ansatz provided by an appropriate change of variables. The cases $-1 < \omega < -1/2$ and $\omega < -1$ will be treated in Chapter 5 using a different approach, based on the analysis of the monotonicity behaviour of solutions.

We start by analysing the case $\omega > -1/2$ in Chapter 2, which includes the particular case $\omega = 0$ already discussed in [21], and can now be seen as part of a more general picture.

The relevant results obtained from our knowledge of the behaviour of $\widetilde{c}_j$ allow us to establish the existence of a function

$$\varsigma(t) \sim t^{\frac{\omega+2}{3}},$$

and a family of scaling functions $\Phi_{1,\omega}$, such that (1.11) holds with $a = \frac{1-\omega}{2+\omega}$ and $b = 1$. The graphs of $\eta \mapsto \Phi_{1,\omega}(\eta)$ are presented in Figure 2.7, page 68. From the observation that $\Phi_{1,\omega}$ becomes discontinuous at $\eta = 1$ for $\omega \leqslant 1$, and unbounded in any set $(1-\varepsilon, 1)$ when $\omega < 1$, it is reasonable to infer that the scaling under consideration may not be the appropriate one around that point, and so we use a different scaling function, obtaining $\Phi_{2,\omega}$.

We essentially have, for sufficiently large j and t,

$$c_j(t) \sim \varsigma(t)^{-a/2}\Phi_{2,\omega}\left(\frac{j-\varsigma(t)}{\sqrt{\varsigma(t)}}\right), \tag{1.12}$$

with a and $\varsigma(t)$ as before. The graphs of the new scaling functions $\Phi_{2,\omega}$ are presented in Figure 2.8, on page 74.

In Chapter 3 we consider an extension of the results obtained in Chapter 2, to monomer input

$$J_1(t) = \alpha t^{\omega}(1 + \varepsilon(t)),$$

where $\varepsilon(\cdot)$ is a continuous function satisfying $\varepsilon(t) \to 0$ as $t \to +\infty$. Most, but not all, of the results of the previous Chapter follow with just this general assumption on ε. This is true, in particular, of those concerning convergence to a self-similar profile. Those about the rate of convergence of the bulk quantity $\sum_{j=1}^{\infty} c_j$ will need an extra assumption on the decay rate of the perturbation $\varepsilon(t)$.

In Chapter 4 we analyse the case $\omega = -\frac{1}{2}$, for which according to [69], we expect solutions (c_0, c_1) to (1.7) to behave like

$$\begin{cases} c_0(t) \sim \left(3\alpha^2\right)^{\frac{1}{3}} (\log t)^{\frac{1}{3}} \\ c_1(t) \sim \left(\dfrac{\alpha}{3}\right)^{\frac{1}{3}} t^{-\frac{1}{2}} (\log t)^{-\frac{1}{3}}, \text{ as } t \to +\infty. \end{cases}$$

The presence of log terms gives rise to a number of technical complications, but the method we used previously for $\omega > -\frac{1}{2}$ still applies, and the results obtained are also similar, but in this case we prove that an analog to Φ_2 does *not* exist. More precisely, we show that the limit of

$$\varsigma^{\frac{1}{2}} (\log \varsigma)^{\frac{2}{3}} \tilde{c}_j(\varsigma),$$

when $j, \varsigma \to +\infty$ with $\xi = \frac{j-\varsigma}{\sqrt{\varsigma}}$, does not exist.

In the last Chapter we consider the case $\omega < -\frac{1}{2}$, and we study separately the cases $\omega = -1$, $\omega > -1$ and then the case $\omega < -1$. For $\omega = -1$ we use a similar ansatz to the one we used for $\omega \geqslant -\frac{1}{2}$. Above and below $\omega = -1$, the ansatz used previously leads to some unforeseen difficulties in the delicate balance between positive and negative terms, and so we develop a different approach, based on a direct analysis of (1.7), by carefully studying the monotonicity behaviour of solutions in specific regions of $\mathbb{R}^2$ using some auxiliary curves. We have been unable to show whether solutions exhibit self-similarity or not, but based on numerical evidence the existence of self-similar behaviour seems to be excluded.

Chapter 2

Dynamics for the case $\omega > -\frac{1}{2}$

In this chapter we present the results obtained in collaboration with F. P. da Costa, to appear in 2007, in the *Journal of Dynamics and Differential Equations*, under the title *Dynamics of a non-autonomous ODE system occurring in coagulation theory*, published on-line in 2007 [20].

2.1 Introduction

In this chapter we study the addition model with constant rate coefficients $a_j \equiv 1$ and a power law input of monomers $J_1(t) = \alpha t^{\omega}$ defined by (1.5). The case $\omega \leqslant -\frac{1}{2}$, which based on [69] is expected to give a different result, will be studied in the next chapters.

There are basically two aspects of the dynamical behaviour of solutions to (1.5) that concern us.

First, we want to establish the componentwise behaviour of the solution as $t \to +\infty$, as well as the behaviour of the total amount of clusters present (the *bulk*), defined by $\sum_{j=1}^{\infty} c_j$. This will be stated in Theorem 2.12, Section 2.3, which roughly says that, as $t \to +\infty$,

$$c_j(t) \sim t^{\frac{\omega-1}{3}}, \quad \sum_{j=1}^{\infty} c_j(t) \sim t^{\frac{2\omega+1}{3}} \quad \text{and} \quad \alpha t^{\omega} - c_1(t) \sum_{j=1}^{\infty} c_j(t) \sim t^{2\frac{\omega-1}{3}}.$$

The second aspect of the dynamics we are interested in is the occurrence of similarity behaviour. The relevant results are stated and proved in Theorem 2.13, Section 2.4, and basically establish the existence of a function $\varsigma(t) \sim t^{\frac{\omega+2}{3}}$ and a family of scaling functions $\Phi_{1,\omega}$. Considering that all our $\Phi_{1,\omega}$ have a singular point at precisely $\eta = 1$, independently of ω (although the type of singularity does depend on ω), whereas the singular point of the corresponding functions in [69] vary with ω, we think out result presents a simpler description of the similarity behaviour.

Noticing that $\Phi_{1,\omega}$ becomes discontinuous at $\eta = 1$ for $\omega \leqslant 1$, and unbounded in any set $(1-\varepsilon, 1)$ when $\omega < 1$, it is reasonable to suppose that the scaling under consideration may not be the more appropriate one around that point. A different scaling variable to deal with this case was successfully developed in [21] for the case $\omega = 0$. It was also applied in [69] to the present case where a different scaling function was deduced. The remainder of Section 2.4, in particular Theorem 2.14, states and rigorously proves this different scaling behaviour, and in the process greatly simplifies the original writing of the scaling function in [69, Eq. (3.29)], which resorted to Kummer's hypergeometric functions, by showing it to be the rather pleasent looking function $\Phi_{2,\omega}$ in (2.41) (that reduces to the one obtained in [21] when $\omega = 0$). Note that they provide a kind of inner expansion of the singularity of the scaling functions $\Phi_{1,\omega}$ around $\eta = 1$.

The proofs of these results in Sections 2.3 and 2.4 are achieved due to essentially two arguments that help transforming the infinite dimensional system (1.5) into an almost exactly solvable problem.

The first of them is the fact that by introducing the total number of clusters as a new macroscopic variable $c_0(t)$ defined by $c_0(t) = \sum_{j=1}^{\infty} c_j(t)$, and formally differentiating termwise, we conclude that c_0 satisfies the evolution equation $\dot{c}_0 = \alpha t^{\omega} - c_0 c_1$. Thus, system (1.5) can, at least formally, be written, in closed form, as (1.6).

In fact, the equivalence between the dynamics of (1.5) and that of (1.6) with initial data in the hyperplane $\left\{(c_0, c_1, \ldots) \in \mathbb{R}^{\mathbb{N}} : c_0 - \sum_{j=1}^{\infty} c_j = 0\right\}$ can be proved as in Section 2 of [21].

From the reduced system (1.6) we observe that the equations governing both the monomer dynamics and the total number of clusters are actually the nonautonomous bidimensional (c_0, c_1)-system (1.7). Its dynamics can be studied in a way totally independent of the remaining components of the infinite dimensional system.

The second argument we mentioned above has to do with a change of time scale.

Given a solution to (1.7), we introduce a new time scale (1.8), where ς_0 is a positive constant, and we consider the new phase variables (1.9), where $t(\varsigma)$ is the inverse function of $\varsigma(t)$. When $c_1(t) > 0$, these are well defined and ς is an increasing function of t. In the new variables, the c_j-equations in (1.6) become

$$\widetilde{c}_j\,' = \widetilde{c}_{j-1} - \widetilde{c}_j, \qquad j \geqslant 2,$$

where $(\cdot)' = \frac{\mathrm{d}}{\mathrm{d}\varsigma}$.

This system of differential equations is a lower triangular linear system and thus can be explicitly solved in terms of the function $\widetilde{c}_1(\varsigma)$ starting from the equation for $j = 2$ and applying the variation of constants formula recursively, we obtain (1.10)

$$\widetilde{c}_j(\varsigma) = e^{-\varsigma} \sum_{k=2}^{j} \frac{\varsigma^{j-k}}{(j-k)!} c_k(0) + \frac{1}{(j-2)!} \int_0^{\varsigma} \widetilde{c}_1(\varsigma - s) s^{j-2} e^{-s} \, \mathrm{d}s.$$

Once a precise knowledge of the behaviour of $\widetilde{c}_1$ is available, this representation formula for the $j-$component of the solution in terms of $\widetilde{c}_1(\varsigma)$ allows us to prove the rather detailed information about the long time evolution of solutions to (1.5) that is presented in Sections 2.3 and 2.4.

We start by making an *ansatz* for a convenient change of variables, suggested by

[69, Table 2] and obtained via formal asymptotics.

According to [69] we expect solutions (c_0, c_1) of system (1.7) with $\alpha > 0$ and $\omega > -\frac{1}{2}$ to behave like

$$\begin{cases} c_0(t) \sim \left(\dfrac{3\alpha^2}{1+2\omega}\right)^{\frac{1}{3}} t^{\frac{1+2\omega}{3}} \\ c_1(t) \sim \left(\dfrac{\alpha(1+2\omega)}{3}\right)^{\frac{1}{3}} t^{\frac{\omega-1}{3}}, \end{cases}$$

as $t \to +\infty$.

This suggests the functions

$$\begin{cases} C_0(t) \coloneqq \left(\dfrac{1+2\omega}{3\alpha^2}\right)^{\frac{1}{3}} t^{-\frac{1+2\omega}{3}} c_0(t) \\ C_1(t) \coloneqq \left(\dfrac{3}{\alpha(1+2\omega)}\right)^{\frac{1}{3}} t^{\frac{1-\omega}{3}} c_1(t), \end{cases} \tag{2.1}$$

might be expected to converge to 1 as $t \to +\infty$, and reciprocally, if this happens then c_0 and c_1 behave as stated. To prove this convergence behaviour of (C_0, C_1) we need an equation for its evolution which is readily obtained as follows: differentiating (2.1), substituting into (1.7), changing the time scale $t \mapsto \tau$ so that $\frac{\mathrm{d}\tau}{\mathrm{d}t} = \left(\frac{3\alpha^2}{1+2\omega}\right)^{\frac{1}{3}} t^{\frac{1+2\omega}{3}}$, defining

$$\begin{cases} x(\tau) \coloneqq C_1(t(\tau)) \\ y(\tau) \coloneqq C_0(t(\tau)), \end{cases} \tag{2.2}$$

and denoting $\frac{\mathrm{d}}{\mathrm{d}\tau}(\cdot)$ by $(\cdot)'$, we obtain

$$\begin{cases} x' = 1 - xy - A\tau^{-\frac{1}{2}}x^2 + B\tau^{-1}x \\ y' = (1-xy)A\tau^{-\frac{1}{2}} - A^2\tau^{-1}y\,, \end{cases} \tag{2.3}$$

where

$$A \coloneqq \left(\frac{1+2\omega}{4+2\omega}\right)^{\frac{1}{2}}, \quad \text{and} \quad B \coloneqq \frac{1-\omega}{4+2\omega}. \tag{2.4}$$

To prove that non-negative solutions to (2.3) converge to $(1, 1)$ as $\tau \to +\infty$ is the purpose of Section 2.2. In order to prove convergence we proceed as follows: first we prove, in Lemma 2.2, that $\mathbb{R}^+ \times \mathbb{R}^+$ is a positively invariant set for (2.3); then a relative boundedness result is obtained: we establish in Lemma 2.3 that, for non-negative solutions (x, y), boundedness of x [or y] is equivalent to boundedness away from zero of y [or x]; and then, in Lemma 2.4, we conclude that indeed both x and y are bounded (and bounded away from zero) functions of τ. The proofs of all these three lemmas are achieved by the use of appropriate differential inequalities.

After these boundedness results we use the evolution of two auxiliary functions, $h(\tau) \coloneqq x(\tau)y(\tau)$ and $b(\tau) = y(\tau) - A\tau^{-\frac{1}{2}}x(\tau)$, to locate, for every orbit, its $\omega-$limit set as a (bounded) arc of the hyperbola $xy = 1$ containing the point $(1, 1)$ (Lemmas 2.7 and 2.8) and, using suitable differential inequalities akin to those employed in the boundedness proof (Lemma 2.4), we finally establish that the arc actually degenerates into the single point $(1, 1)$ (Lemma 2.9).

Using this convergence it is possible to refine the estimates for h presented in Lemma 2.7 and actually conclude that, as $\tau \to +\infty$, $h(\tau) \sim 1 - A\tau^{-\frac{1}{2}}$. This estimate, obtained in Lemma 2.10, concludes the information required for the proofs of the results on the dynamics of (1.5) in Sections 2.3 and 2.4 we described above.

The approach just outlined, when compared with the more natural one used in the autonomous $\omega = 0$ case, has the great disadvantage of, besides being lengthier, requiring an *a priori* knowledge about the behaviour to be observed in order to be able to define the appropriate change of variables. This intrinsic drawback is in this case mitigated by our knowledge of the stated behaviour from the formally computed asymptotic results presented in [69, Table 2].

2.2 The bidimensional non-autonomous ODE governing monomer and bulk dynamics

We are interested in nonnegative solutions to (1.7) and so, from hereon, solutions will actually mean nonnegative solutions. The main result of this section is the following

Theorem 2.1 *Let $\alpha > 0$ and $\omega > -\frac{1}{2}$ be constants, and let (c_0, c_1) be any solution of* (1.7)*. Then, as $t \to +\infty$, we have*

(i) $$\left(\frac{1+2\omega}{3\alpha^2}\right)^{\frac{1}{3}} t^{-\frac{1+2\omega}{3}} c_0(t) \longrightarrow 1,$$

(ii) $$\left(\frac{3}{\alpha(1+2\omega)}\right)^{\frac{1}{3}} t^{\frac{1-\omega}{3}} c_1(t) \longrightarrow 1,$$

(iii) $$\left(\frac{3}{\alpha(1+2\omega)}\right)^{\frac{2}{3}} t^{2\frac{1-\omega}{3}} (\alpha t^{\omega} - c_0(t)c_1(t)) \longrightarrow 1.$$

From the discussion in the Introduction we prove this result by working with (2.3) and proving that its solutions converge to $(1, 1)$ as $\tau \to +\infty$.

We start by the obvious observation that (2.3) is asymptotically autonomous. Although in general this would be a potentially fruitful property, the fact that the set of equilibria of the asymptotic limit system

$$\begin{cases} x' = 1 - xy \\ y' = 0, \end{cases}$$

form a continuum, and thus the equilibria are non hyperbolic, does not seem to provide a handy way of using the limit system to get information about the long time behaviour of solutions to the original non-autonomous system (2.3).

As a consequence, we shall *not* use the asymptotic limit of (2.3).Instead, as described in the Introduction, we study (2.3) using the information on the boundedness of solutions provided by certain differential inequalities derived from (2.3) in some regions of the phase plane (x, y), together with information provided by the evolution of two auxiliary functions, to locate the possible ω−limit set of the solutions, and then to prove this set must consist of the single point $(1, 1)$. The somewhat lengthy analysis will be decoupled into a series of lemmas in a way that will highlight the main steps of the argument.

Our first goal is to establish that all (positive) solutions to (2.3) have both components bounded and bounded away from zero. To this end, we start by proving solutions cannot cease being positive.

Lemma 2.2 *The first quadrant* $\{x \geqslant 0, y \geqslant 0\}$ *is positively invariant for* (2.3).

Proof. This is obvious since in $\{(0, y) : y \geqslant 0\}$ we have, from (2.3), $x' = 1 > 0$ for all τ, and so solutions cannot escape through the y−axis. Similarly, in $\{(x, 0) : x \geqslant 0\}$ it holds $y' = A\tau^{-\frac{1}{2}} > 0$, which imply solutions cannot escape through the x−axis either. ∎

Next we prove a relative boundedness behaviour of the x and y components of the solution that will be useful in the sequel.

Lemma 2.3 *Let* (x, y) *be any solution to* (2.3) *with positive initial data. Then, the following equivalences hold:*

- **(i)** y *is bounded* $\Longleftrightarrow$ x *is bounded away from zero.*
- **(ii)** y *is bounded away from zero* $\Longleftrightarrow$ x *is bounded.*

Proof. We start by proving that

$$y \text{ is bounded} \Longrightarrow x \text{ is bounded away from zero.}$$

When $\tau \in (\tau_0, 1]$ (if this set is non empty) the solution $x(\tau)$ is necessarily bounded away from zero since otherwise there would exist at least a point $\tilde{\tau} \in (\tau_0, 1]$ for which $x(\tilde{\tau}) = 0$, and thus $x'(\tilde{\tau}) = 1 > 0$, which is impossible because then $x(\tau) < 0$, for $\tau < \tilde{\tau}$ sufficiently close to $\tilde{\tau}$, contradicting Lemma 2.2. When

$\tau > 1$, let $U_y > 0$ be an upper bound for y. For the case $\omega \geqslant 1$, for which $B \leqslant 0$, the first equation in (2.3) gives

$$x' \geqslant 1 - xU_y - A\tau^{-\frac{1}{2}}x^2 - |B|\tau^{-1}x > 1 - \left(U_y + |B|\right)x - Ax^2.$$

In the case $\omega \in \left(-\frac{1}{2}, 1\right)$, we have $B > 0$, and hence (2.3) results in $x' \geqslant 1 - xU_y - A\tau^{-\frac{1}{2}}x^2 + B\tau^{-1}x > 1 - xU_y - Ax^2$. In either case, standard results on differential inequalities imply $x(\tau)$ is bounded away from zero.

To prove the reverse result,

$$x \text{ is bounded away from zero} \Longrightarrow y \text{ is bounded},$$

we notice that the equation for y in (2.3) is linear in y. In fact it can be written as

$$y' = -\left(x + A\tau^{-\frac{1}{2}}\right)A\tau^{-\frac{1}{2}}y + A\tau^{-\frac{1}{2}}.$$

Changing to a new time scale $\tau \mapsto \theta$ with $\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = A\tau^{-\frac{1}{2}}$, the equation for y becomes

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = -\left(x + \frac{2A^2}{\theta}\right)y + 1.$$

Since this a linear equation with differentiable coefficients, it is obvious the solution is bounded for θ in every bounded set, so the problem reduces to the study of $y(\theta)$ when $\theta \to +\infty$. By the variation of constants formula we get

$$y(\theta) = y(\theta_0)e^{-\int_{\theta_0}^{\theta}\left(x(\sigma)+\frac{2A^2}{\sigma}\right)\mathrm{d}\sigma} + \int_{\theta_0}^{\theta} e^{-\int_{s}^{\theta}\left(x(\sigma)+\frac{2A^2}{\sigma}\right)\mathrm{d}\sigma}\,\mathrm{d}s. \tag{2.5}$$

Now let $L_x > 0$ be a lower bound for x. Then, we have

$$e^{-\int_{\theta_0}^{\theta}\left(x(\sigma)+\frac{2A^2}{\sigma}\right)\mathrm{d}\sigma} \leqslant e^{-L_x(\theta-\theta_0)}e^{2A^2\log(\theta_0/\theta)} = \left(\frac{\theta_0}{\theta}\right)^{2A^2} e^{-L_x(\theta-\theta_0)} \longrightarrow 0,$$

as $\theta \to +\infty$. Also,

$$\int_{\theta_0}^{\theta} e^{-\int_s^\theta \left(x(\sigma)+\frac{2A^2}{\sigma}\right)\mathrm{d}\sigma} \mathrm{d}s \leqslant \int_{\theta_0}^{\theta} e^{-L_x(\theta-s)}\,\mathrm{d}s = \frac{1-e^{-L_x(\theta-\theta_0)}}{L_x} < \frac{1}{L_x},$$

and plugging these into (2.5) we conclude the boundedness of y, concluding the proof of part (**i**).

Let us now consider the case

$$y \text{ is bounded away from zero} \Longrightarrow x \text{ is bounded.}$$

Let $L_y > 0$ be a positive lower bound for y. Then,

$$x' \leqslant 1 - xL_y - A\tau^{-\frac{1}{2}}x^2 + B\tau^{-1}x < 1 - \left(L_y - B\tau^{-1}\right)x.$$

When $\omega \geqslant 1$ we have $B \leqslant 0$ and thus $x' < 1 - L_y x$ and by standard differential inequalities results we conclude the boundedness of x:

$$x(\tau) \leqslant x(\tau_0)e^{-L_y(\tau-\tau_0)} + \int_{\tau_0}^{\tau} e^{-L_y(\tau-s)}\,\mathrm{d}s = o(1) + \frac{1-e^{-L_y(\tau-\tau_0)}}{L_y} < o(1) + \frac{1}{L_y}$$

as $\tau \to +\infty$.

When $\omega \in \left(-\frac{1}{2}, 1\right)$ we have $B > 0$. For this case, let $\tilde{\tau} = 2B/L_y$. For $\tau > \tilde{\tau}$ it follows that $L_y - B\tau^{-1} > L_y - B\tilde{\tau}^{-1} = L_y/2$, and thus $x' < 1 - \frac{L_y}{2}x$. Now, for $\tau \in (\tau_0, \tilde{\tau})$, we have $x(\tau) \leqslant u(\tau)$ where u is the solution of $u' = 1 - \left(L_y - B\tau^{-1}\right)u$ with $u(\tau_0) = x(\tau_0)$, which is obviously a bounded function in the compact interval $[\tau_0, \tilde{\tau}]$. For $\tau \geqslant \tilde{\tau}$ we just repeat the argument employed in the case $\omega \geqslant 1$ with L_y changed to $L_y/2$.

Finally, it remains to be proved that

$$x \text{ is bounded} \Longrightarrow y \text{ is bounded away from zero.}$$

Let U_x be an upper bound for x. By (2.5), the variation of constants formula for y, we can write

$$\begin{aligned} y(\theta) &\geqslant \int_{\theta_0}^{\theta} e^{-U_x(\theta-s)} e^{-\int_s^\theta \frac{2A^2}{\sigma}\,\mathrm{d}\sigma}\,\mathrm{d}s \\ &\geqslant \int_{\theta_0}^{\theta} e^{-(U_x+\frac{2A^2}{\theta_0})(\theta-s)}\,\mathrm{d}s \\ &= \frac{1}{U_x+\frac{2A^2}{\theta_0}}\left(1-e^{-(U_x+\frac{2A^2}{\theta_0})(\theta-\theta_0)}\right) \xrightarrow[\theta\to+\infty]{} \frac{1}{U_x+\frac{2A^2}{\theta_0}} > 0. \end{aligned}$$

Hence, for all θ sufficiently large, we certainly have y bounded away from zero, and for θ in compact intervals y is clearly bounded away from zero being the sum of two differentiable positive functions.

This concludes the proof of Lemma 2.3. ∎

The equivalences of Lemma 2.3 roughly imply that the only possibilities left for solutions to escape to infinity are along the coordinate axis. In the next lemma we prove this does not occur, and solutions are, in fact, bounded.

Lemma 2.4 *Every solution* (x, y) *to* (2.3) *with positive initial data is bounded.*

Proof. We separate the proof into two steps, studying first the boundedness of y, and then that of x.

Step 1: the boundedness of $y(\tau)$.

It is easy to conclude that the only possibility for the orbit to have an unbounded y component is that it leaves the region $\Omega_- \coloneqq \{(x, y) : xy \geqslant 1\}$. In fact, for $xy \geqslant 1$ we have, from the second equation in (2.3),

$$y' = \left((1-xy) - A\tau^{-\frac{1}{2}}y\right)A\tau^{-\frac{1}{2}} \leqslant -A^2\tau^{-1}y < 0,$$

and so $y(\tau)$ cannot diverge to $+\infty$ while staying in Ω_-. We now prove that it cannot diverge to $+\infty$ when the orbit is in $\Omega_+ \coloneqq \{(x, y) : 0 < xy < 1\}$ either, and so the orbit must have a bounded y component. Let $L > 1$ be fixed arbitrarily and define $\tau_L := 2^{-2}\left(L^2 - L^{-1}\right)^{-2}$. If $y(\tau) < L$ for all sufficiently large τ, then y is bounded and there is nothing to prove. Suppose there exists a $\tau_0 > \max\{1, \tau_L\}$ such that $y(\tau_0) = L$ and $P_{\tau_0} = (x(\tau_0), y(\tau_0)) \in \Omega_+$. Thus we have $y(\tau_0) > L^2 x(\tau_0)$ (see Figure 2.1).

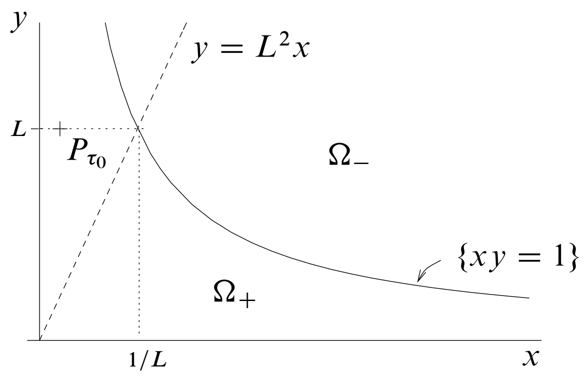

Figure 2.1: Situation described in the text for a point of an orbit having $y(\tau) \to +\infty$ as $\tau \to +\infty$.

Since (2.4) imply that $-BA^{-1} < 1/2$, it follows that, for all $\tau \geqslant \tau_0$ for which the solution satisfies $x < L^{-1}$ and $y > L^2 x$, which clearly covers the region of interest, the following inequality holds true

$$\begin{aligned} y' &= \left(1 - xy - A\tau^{-\frac{1}{2}} y\right) A\tau^{-\frac{1}{2}} \\ &= \left(x' + A\tau^{-\frac{1}{2}} x^2 - B\tau^{-1} x - A\tau^{-\frac{1}{2}} y\right) A\tau^{-\frac{1}{2}} \\ &< \left(x' + (x - L^2 - BA^{-1}\tau^{-\frac{1}{2}}) x A\tau^{-\frac{1}{2}}\right) A\tau^{-\frac{1}{2}} \\ &< \left(x' + (L^{-1} - L^2 + \tfrac{1}{2}\tau^{-\frac{1}{2}}) x A\tau^{-\frac{1}{2}}\right) A\tau^{-\frac{1}{2}} \\ &< x' A\tau^{-\frac{1}{2}}, \end{aligned} \tag{2.6}$$

where the last inequality arises from our lower bound on τ_0.

Now, either $x' \leqslant 0$, in which case $y' < 0$ and y cannot increase, and thus it is bounded above, or $x' > 0$, in which case (2.6) and the inequalities $\tau \geqslant \tau_0 > \max\{1, \tau_L\}$ imply that $y' < x'$ and thus $\frac{\mathrm{d}y}{\mathrm{d}x} < 1$. Observe that this holds independently of L and of the point P_{τ_0}.

This inequality shows that, the orbit's slope being uniformly bounded above by a positive constant, the orbit itself cannot rise too high in Ω_+: the y component of the solution is always bounded above by $S_L := \frac{L+\sqrt{L^2+4}}{2}$ (see Figure 2.2).

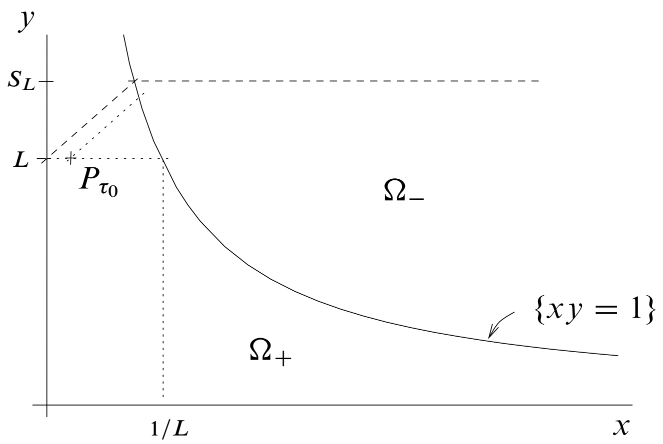

Figure 2.2: Bounding region for an orbit satisfying $y(\tau_0) = L > 1$ in some point $P_{\tau_0} \in \Omega_+$. Note that $y' < 0$ in the closure of Ω_-. S_L is the ordinate of the intersection point of the straight line $y = x + L$ with the hyperbola $xy = 1$.

Step 2: boundedness of $x(\tau)$.

Suppose there exists an orbit such that $x(\tau) \to +\infty$ as $\tau \to +\infty$. First observe that, by Lemma 2.3 (ii), we must have $y(\tau)$ not bounded away from zero for all sufficiently large τ. In fact, the orbit must eventually enter Ω_+: if the orbit stays in Ω_- then $x(\tau)y(\tau) > 1$ and thus, for all sufficiently large times,

$$\begin{aligned} x' &= 1 - xy - A\tau^{-\frac{1}{2}}x^2 + B\tau^{-1}x \\ &= 1 - xy + \left(-x + BA^{-1}\tau^{-\frac{1}{2}}\right) A\tau^{-\frac{1}{2}}x \\ &< 0, \end{aligned}$$

where the term in parentheses is negative since, by assumption, $x(\tau) \to +\infty$ when $\tau \to +\infty$. The same argument shows the orbit cannot leave Ω_+ infinitely often if $x(\tau)$ is diverging to $+\infty$ because, if so, we could take $\tilde{\tau}$ and $x(\tilde{\tau})$ as large as wanted, for which the orbit was entering Ω_- coming from Ω_+. At that point, we would have, from the argument above, $x'(\tilde{\tau}) < 0$ and so, in order for the orbit to enter Ω_- we must have $y'(\tilde{\tau}) > 0$ (see Figure 2.3).

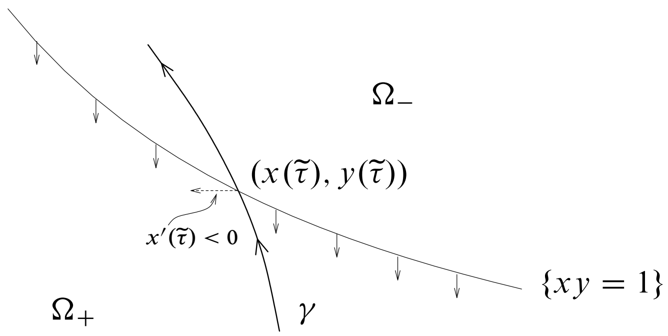

Figure 2.3: The impossibility of an orbit γ to travel from Ω_+ to Ω_- for all large enough $\tilde{\tau}$ and $x(\tilde{\tau})$.

But this is impossible since equation (2.3) implies $y'(\tilde{\tau}) = -y(\tilde{\tau})A^2\tilde{\tau}^{-1} < 0$ for all points $(x(\tilde{\tau}), y(\tilde{\tau}))$ in the hyperbola $\{xy = 1\}$ and $\tilde{\tau} > 0$. Consequently, the orbit for which $x(\tau)$ diverges to $+\infty$ must remain in Ω_+ after some sufficiently large time and thus $y(\tau) \to 0$ as $\tau \to +\infty$. Now, for $\tau > 0$, define $g_\tau(x, y) := x + A\tau^{-\frac{1}{2}} - \frac{1}{y}$, and consider the curves in Ω_+ defined by the level sets $\{g_\tau = 0\}$. These curves form a sequence of hyperbolas monotonically converging to the hyperbola $\{xy = 1\}$ as $\tau \to +\infty$, uniformly in compact subsets of $\mathbb{R}^{2+}$. From (2.3) we deduce that, for all sufficiently large τ,

$$A\tau^{-\frac{1}{2}}x' - y' = -\left(x - \frac{y}{x} - \frac{B}{A}\tau^{-\frac{1}{2}}\right) A^2\tau^{-1}x < 0. \tag{2.7}$$

This obviously entails that, for all τ large enough,

$$y' > x'A\tau^{-\frac{1}{2}}. \tag{2.8}$$

Hence, for all sufficiently large τ, if $x' > 0$ then also $y' > 0$, and so it is clear from the geometry of the curves $g_\tau = 0$ that the orbit under consideration must cross any one of these curves from below, i.e., from $\{g_\tau(x, y) < 0\}$ into $\{g_\tau(x, y) > 0\}$, or, equivalently, g_τ must increase along the orbit. Now, from (2.3) and the definition of g_τ, we deduce

$$g_\tau = 0 \Rightarrow A\tau^{-\frac{1}{2}} y g_\tau(x, y) = 0 \Leftrightarrow y' = 0 \Rightarrow$$
$$x' = A\tau^{-\frac{1}{2}}\Big(y - x^2 + \frac{B}{A}\tau^{-\frac{1}{2}}x\Big), \quad (2.9)$$

furthermore, since by (2.7) the last equation entails $x' < 0$ if $g_\tau = 0$, we also have $\frac{\mathrm{d}g_\tau}{\mathrm{d}\tau} = x' - \frac{1}{2}A\tau^{-\frac{3}{2}} + \frac{y'}{y^2}$, and from these expressions we can conclude that, if $g_\tau = 0$ then $\frac{\mathrm{d}g_\tau}{\mathrm{d}\tau} = x' - \frac{1}{2}A\tau^{-\frac{3}{2}} < 0$ (since by (2.7) the equation (2.9) entails $x' < 0$ if $g_\tau = 0$), which obviously contradicts the result about the crossing of $g_\tau = 0$ from below that was obtained before. Thus we must necessarily have $y' < 0$, and so, by (2.8), $x' < 0$ and hence the boundedness above of $x(\tau)$. This concludes the proof of Lemma 2.4. ∎

An immediate consequence of Lemmas 2.3 and 2.4 is the following

Corollary 2.5 *Every solution to* (2.3) *with positive initial data is bounded and bounded away from zero.*

Remark 2.6 *The conclusions of Lemmas 2.3 and 2.4 (and, hence, also of Corollary 2.5) still hold true if the initial condition is non-negative, provided we consider, for the boundedness away from zero, times τ in $[\tau_0^*, \infty)$ for every $\tau_0^* > \tau_0$, where τ_0^* is the new initial time. This follows easily from the proof of Lemma 2.3 by taking $\big(\tau_0^*, x(\tau_0^*; \tau_0, x_0, y_0), y(\tau_0^*; \tau_0, x_0, y_0)\big)$ the new initial data, where now, from the proof of Lemma 2.2, both $x(\tau_0^*)$ and $y(\tau_0^*)$ are positive.*

Having succeeded in proving every orbit of (2.3) is bounded and bounded away from zero, we can now start to identify its $\omega-$limit set. The first step towards this goal is the following lemma, which guarantees the $\omega-$limit set of every orbit is contained in the hyperbola $\{xy = 1\}$. This will be achieved by considering the function

$$h(\tau) := x(\tau)y(\tau), \quad (2.10)$$

which is a very natural quantity to study if we keep in mind the role played by the hyperbola $\{xy = 1\}$ in the proof of Lemma 2.4, as well as the form of the equations (2.3).

Let (x, y) be a solution of (2.3), then it is easy to conclude that $h(\tau)$ solves the equation

$$h' = \left(y + A\tau^{-\frac{1}{2}}x\right) - \left(y + 2A\tau^{-\frac{1}{2}}x + (A^2 - B)\tau^{-1}\right)h. \tag{2.11}$$

Observe that, rather surprisingly, this equation is linear in h. This allows the use of the variation of constants formula in order to gain precise knowledge of the behaviour of h. In fact, in the next lemma we not only prove that $h(\tau)$ converges to 1 as $\tau \to +\infty$, but that $h(\tau) \to 1^-$, i.e., for all sufficiently large τ the orbit remains in Ω_+. Actually, our proof establishes a bit more: it gives the rate of convergence of $h(\tau)$ to 1. Although this level of detail is not necessary at this point, we shall need it later on, in Lemma 2.10, in order to prove part `(iii)` of Theorem 2.1.

Lemma 2.7 *Let* (x, y) *be any solution to* (2.3). *Let* h *be any solution to* (2.11). *Then* $h(\tau) \to 1^-$ *as* $\tau \to +\infty$.

Proof. To simplify notation we introduce $\alpha(\tau) \coloneqq y(\tau) + 2A\tau^{-\frac{1}{2}}x(\tau) + (A^2 - B)\tau^{-1}$. The variation of constants formula applied to (2.11) results in

$$h(\tau) = h(\tau_0)e^{-\int_{\tau_0}^{\tau}\alpha(s)\,\mathrm{d}s} + \int_{\tau_0}^{\tau}\left(y(s) + As^{-\frac{1}{2}}x(s)\right)e^{-\int_s^{\tau}\alpha(\theta)\,\mathrm{d}\theta}\mathrm{d}s. \tag{2.12}$$

We can write the second term in the right-hand side of this expression as follows

$$\begin{aligned}
&\int_{\tau_0}^{\tau}\left(y(s) + As^{-\frac{1}{2}}x(s)\right)e^{-\int_s^{\tau}\alpha(\theta)\,\mathrm{d}\theta}\mathrm{d}s = \\
&= \; e^{-\int_{\tau_0}^{\tau}\alpha(s)\,\mathrm{d}s}\int_{\tau_0}^{\tau}\frac{\mathrm{d}}{\mathrm{d}s}\left(e^{\int_{\tau_0}^{s}\alpha(\theta)\,\mathrm{d}\theta}\right)\mathrm{d}s - \int_{\tau_0}^{\tau}\left(\frac{Ax(s)}{s^{1/2}} + \frac{A^2 - B}{s}\right)e^{-\int_s^{\tau}\alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s \\
&= \; 1 - e^{-\int_{\tau_0}^{\tau}\alpha(\theta)\,\mathrm{d}\theta} - \int_{\tau_0}^{\tau}\left(\frac{Ax(s)}{s^{1/2}} + \frac{A^2 - B}{s}\right)e^{-\int_s^{\tau}\alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s
\end{aligned}$$

and use this to rewrite (2.12) in the form

$$\frac{1-h(\tau)}{A\tau^{-\frac{1}{2}}} = \frac{(1-h(\tau_0))}{A\tau^{-\frac{1}{2}}} e^{-\int_{\tau_0}^{\tau}\alpha(s)\,\mathrm{d}s} + \frac{\tau^{\frac{1}{2}}}{A}\int_{\tau_0}^{\tau}\left(\frac{Ax(s)}{s^{1/2}} + \frac{A^2-B}{s}\right) e^{-\int_s^{\tau}\alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s. \tag{2.13}$$

It is important to observe that the first term in the right-hand side of this expression is negative if the initial data for (2.3) is in Ω_-. Also, by (2.4), we have $\mathrm{sgn}(A^2 - B) = \mathrm{sgn}(\omega)$ and so the second term in the integral can be a negative function. We shall prove that both these possibly negative contributions converge to zero as $\tau \to +\infty$, while the remaining one, which is always positive, is bounded and bounded away from zero.

By the boundedness away from zero of y we can write $y(\tau) \geqslant L_y > 0$ for some constant L_y, and get

$$\begin{aligned} e^{-\int_{\tau_0}^{\tau}\alpha(s)\,\mathrm{d}s} &= \left(\frac{\tau_0}{\tau}\right)^{A^2-B} e^{-\int_{\tau_0}^{\tau}\left(y(s)+2As^{-1/2}x(s)\right)\mathrm{d}s} \leqslant \left(\frac{\tau_0}{\tau}\right)^{A^2-B} e^{-\int_{\tau_0}^{\tau}y(s)\,\mathrm{d}s} \\ &\leqslant \left(\frac{\tau_0}{\tau}\right)^{A^2-B} e^{-L_y(\tau-\tau_0)}. \end{aligned} \tag{2.14}$$

Thus, as $\tau \to +\infty$, the first term in the right-hand side of (2.13) converges to zero.

We next establish that

$$\tau^{\frac{1}{2}}\int_{\tau_0}^{\tau} s^{-1}e^{-\int_s^{\tau}\alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s \to 0 \quad \text{as} \quad \tau \to +\infty. \tag{2.15}$$

Fix $\beta \in (0,1)$ and write the integral in (2.15) as $\int_{\tau_0}^{\tau-\tau^{\beta}} + \int_{\tau-\tau^{\beta}}^{\tau}$. The first inequality in (2.14) implies that the first of these integrals can be estimated as

$$\tau^{\frac{1}{2}}\int_{\tau_0}^{\tau-\tau^{\beta}} s^{-1}e^{-\int_s^{\tau}\alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s <$$

$$
\begin{aligned}
&< \tau^{-\frac{1}{2}} \int_{\tau_0}^{\tau-\tau^{\beta}} \left(\frac{s}{\tau}\right)^{A^2-B-1} e^{-\int_s^\tau y(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s \\
&< \tau^{-\frac{1}{2}} \max\left\{1, \left(\frac{\tau_0}{\tau}\right)^{A^2-B-1}\right\} \int_{\tau_0}^{\tau-\tau^{\beta}} e^{-L_y(\tau-s)}\,\mathrm{d}s \\
&= \tau^{-\frac{1}{2}} \max\left\{1, \left(\frac{\tau_0}{\tau}\right)^{A^2-B-1}\right\} \frac{1}{L_y}\left(e^{-L_y\tau^{\beta}} - e^{-L_y(\tau-\tau_0)}\right). \qquad (2.16)
\end{aligned}
$$

For the second of the integrals above we have, again by (2.14),

$$
\begin{aligned}
&\tau^{\frac{1}{2}} \int_{\tau-\tau^{\beta}}^{\tau} s^{-1} e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s < \\
&\quad < \tau^{-\frac{1}{2}} \int_{\tau-\tau^{\beta}}^{\tau} \left(\frac{s}{\tau}\right)^{A^2-B-1} e^{-\int_s^\tau y(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s \\
&\quad < \tau^{-\frac{1}{2}} \max\left\{1, \left(1-\tau^{\beta-1}\right)^{A^2-B-1}\right\} \int_{\tau-\tau^{\beta}}^{\tau} e^{-L_y(\tau-s)}\,\mathrm{d}s \\
&\quad = \tau^{-\frac{1}{2}} \max\left\{1, \left(1-\tau^{\beta-1}\right)^{A^2-B-1}\right\} \frac{1}{L_y}\left(1 - e^{-L_y\tau^{\beta}}\right). \qquad (2.17)
\end{aligned}
$$

Clearly, both (2.16) and (2.17) converge to zero as $\tau \to +\infty$, thus proving (2.15). To complete the proof we just need to conclude that there exists constants L and U such that, for all τ sufficiently large,

$$
0 < L \leqslant \tau^{\frac{1}{2}} \int_{\tau_0}^{\tau} \frac{x(s)}{s^{\frac{1}{2}}} e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s \leqslant U < +\infty. \qquad (2.18)
$$

Using the definition of $\alpha(\cdot)$ and the equation for x in (2.3) we have

$$
\alpha(\theta) = \frac{1-x'}{x} + \frac{Ax}{\theta^{\frac{1}{2}}} + \frac{A^2}{\theta}
$$

$$= -\frac{d}{d\theta}\log x(\theta) + \frac{1}{x} + \frac{Ax}{\theta^{\frac{1}{2}}} + \frac{A^2}{\theta},$$

whence

$$e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta} = \frac{x(\tau)}{x(s)}\left(\frac{s}{\tau}\right)^{A^2} \exp\left[-\int\limits_s^\tau \left(\frac{1}{x(\theta)} + \frac{Ax(\theta)}{\theta^{1/2}}\right)\mathrm{d}\theta\right].$$

Plugging this into the integral in (2.18), and taking into account the fact that, by Corollary 2.5, there exists constants $0 < L_x \leqslant U_x$ such that $L_x \leqslant x(\tau) \leqslant U_x$, we conclude that

$$L_x J(\tau) \leqslant \tau^{\frac{1}{2}} \int\limits_{\tau_0}^{\tau} \frac{x(s)}{s^{\frac{1}{2}}} e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s \leqslant U_x J(\tau),$$

where

$$J(\tau) := \int\limits_{\tau_0}^{\tau} \left(\frac{s}{\tau}\right)^{A^2-\frac{1}{2}} e^{-\int_s^\tau \left(\frac{1}{x(\theta)} + \frac{Ax(\theta)}{\theta^{1/2}}\right)\mathrm{d}\theta}\,\mathrm{d}s,$$

and so it is sufficient to establish the bound (2.18) for $J(\tau)$.

As in the proof of (2.15) we start by fixing $\beta \in (0, 1)$ and writing the integral as a sum of a “small” and a “large” τ contribution, with $\tau - \tau^\beta$ as the threshold size. For the “small” size contribution we have, analogously to (2.16),

$$\begin{aligned}
&\int\limits_{\tau_0}^{\tau-\tau^\beta} \left(\frac{s}{\tau}\right)^{A^2-\frac{1}{2}} e^{-\int_s^\tau \left(\frac{1}{x(\theta)} + \frac{Ax(\theta)}{\theta^{1/2}}\right)\mathrm{d}\theta}\,\mathrm{d}s < \\
&< \int\limits_{\tau_0}^{\tau-\tau^\beta} \left(\frac{s}{\tau}\right)^{A^2-\frac{1}{2}} e^{-\ell_x(\tau-s)}\,\mathrm{d}s \\
&< \max\left\{1, \left(\frac{\tau_0}{\tau}\right)^{A^2-\frac{1}{2}}\right\} \frac{1}{\ell_x}\left(e^{-\ell_x\tau^\beta} - e^{-\ell_x(\tau-\tau_0)}\right)
\end{aligned} \tag{2.19}$$

where $\ell_x \coloneqq \inf_{\theta \in [\tau_0, \infty)} \left(\frac{1}{x(\theta)} + \frac{Ax(\theta)}{\theta^{1/2}} \right)$ is a positive constant, by Corollary 2.5 (and Remark 2.6). Clearly (2.19) converges to zero as $\tau \to +\infty$. For the "large" τ contribution, observe that, for $s \in [\tau - \tau^\beta, \tau]$,

$$\min \left\{ 1, \left(1 - \tau^{\beta - 1} \right)^{A^2 - \frac{1}{2}} \right\} \leqslant \left(\frac{s}{\tau} \right)^{A^2 - \frac{1}{2}} \leqslant \max \left\{ 1, \left(1 - \tau^{\beta - 1} \right)^{A^2 - \frac{1}{2}} \right\},$$

and, since both the lower and the upper bounds converge to 1 as $\tau \to +\infty$, to complete the proof it is sufficient to conclude that (2.18) is valid for the integral

$$\int_{\tau - \tau^\beta}^{\tau} e^{-\int_s^\tau \left(\frac{1}{x(\theta)} + \frac{Ax(\theta)}{\theta^{1/2}} \right) \mathrm{d}\theta} \, \mathrm{d}s. \tag{2.20}$$

With $\ell_x > 0$ defined above, and $u_x \coloneqq \sup_{\theta \in [\tau_0, \infty)} \left(\frac{1}{x(\theta)} + \frac{Ax(\theta)}{\theta^{1/2}} \right) < +\infty$, we immediately conclude that

$$\frac{1 - e^{-u_x \tau_0^\beta}}{u_x} < \frac{1 - e^{-u_x \tau^\beta}}{u_x} \leqslant \int_{\tau - \tau^\beta}^{\tau} e^{-\int_s^\tau \left(\frac{1}{x(\theta)} + \frac{Ax(\theta)}{\theta^{1/2}} \right) \mathrm{d}\theta} \, \mathrm{d}s \leqslant \frac{1 - e^{-\ell_x \tau^\beta}}{\ell_x} < \frac{1}{\ell_x},$$

which concludes the proof. ∎

Having identified the $\omega-$limit set of every orbit as a subset of the hyperbola $\{xy = 1\}$ and knowing, by Corollary 2.5, that this set must be a bounded one, the next result provides a more precise knowledge of its location by showing it must be an (eventually degenerate) arc of the hyperbola containing the point $(1, 1)$. This is achieved by the study of the dynamics of another auxiliary function, namely

$$b(\tau) \coloneqq y(\tau) - A\tau^{-\frac{1}{2}} x(\tau). \tag{2.21}$$

This function is a somewhat less natural choice than $h(\tau)$ used previously. The rationale for its definition is, nevertheless, not difficult to understand: the sets $\{h = \text{const}\}$ provide a foliation of $(\mathbb{R}^+)^2$ and, since Lemma 2.7 shows the time behaviour of h to be rather tame, it is reasonable to look for another variable, call it

$b = b(\tau)$, so that every point (x, y) is uniquely described by the new coordinates (h, b). One possible definition, among others, is to have b as the ordinate of the straight line $y = x + b$. However, due to the form of (2.3), the equation for this b becomes rather unyielding. A better choice, from the point of view of handling the dynamic equation for the new variable, is the slight modification of this idea provided by (2.21).

It is easy to check that if (x, y) is a solution of (2.3) then $b(\tau)$ defined above solves the equation

$$b' = (-b + x^2)A^2\tau^{-1}. \tag{2.22}$$

Changing the time scale $\tau \mapsto \zeta$ with $\frac{d\zeta}{d\tau} = A^2\tau^{-1}$, and still denoting by b and x those functions in the new independent variable, the equation for b becomes

$$\frac{db}{d\zeta} = -b + x^2. \tag{2.23}$$

Again, the fact that this evolution equation is linear, and furthermore that it does not depend on y, allow us to use it to gain the sought for information about the location of the $\omega-$limit set.

Lemma 2.8 *Let* (x, y) *be any solution to* (2.3)*. Let* $x_* := \underline{\lim}_{\tau\to+\infty} x(\tau)$*,* $x^* := \overline{\lim}_{\tau\to+\infty} x(\tau)$*, and similarly for* y_* *and* y^**. Then, the following inequalities are satisfied*

$$x_* \leqslant 1 \leqslant x^* \quad \text{and} \quad y_* \leqslant 1 \leqslant y^*.$$

Proof. Let b be the function defined in (2.21). Fix any $\beta \in (0, 1)$ and integrate (2.23) in $[\zeta - \zeta^\beta, \zeta]$ to obtain

$$b(\zeta) = b(\zeta - \zeta^\beta)e^{-\zeta^\beta} + \int_{\zeta-\zeta^\beta}^{\zeta} x^2(s)e^{-(\zeta-s)}\,\mathrm{d}s. \tag{2.24}$$

The first term in the right-hand side of (2.24) converges to zero as $\zeta \to +\infty$ because the exponential converges to zero, and (2.21) together with the boundedness of x and y entail the boundedness of $b(\zeta - \zeta^\beta)$. The integral in (2.24) can be bounded above using the supremum of x^2 in the region of integration to obtain, as $\zeta \to +\infty$,

$$\begin{aligned} b(\zeta) &\leqslant o(1) + \left(\sup_{s \in [\zeta - \zeta^\beta, \zeta]} x^2(s) \right) \int_{\zeta - \zeta^\beta}^{\zeta} e^{-(\zeta - s)} \, \mathrm{d}s \\ &\leqslant o(1) + \left(1 - e^{-\zeta^\beta}\right) \sup_{s \in [\zeta - \zeta^\beta, \infty)} x^2(s), \end{aligned}$$

and taking $\overline{\lim}_{\zeta \to +\infty}$ we obtain

$$\overline{\lim_{\zeta \to +\infty}} b(\zeta) \leqslant \left(x^*\right)^2 . \tag{2.25}$$

Now, taking again (2.24), using this time the bound with the infimum of x^2 in the region of integration and taking $\underline{\lim}_{\zeta \to +\infty}$ we get the corresponding inequality

$$\underline{\lim}_{\zeta \to +\infty} b(\zeta) \geqslant (x_*)^2 . \tag{2.26}$$

From the definition of b and the boundedness of x we also know that

$$\overline{\lim_{\zeta \to +\infty}} b(\zeta) = y^*, \qquad \underline{\lim}_{\zeta \to +\infty} b(\zeta) = y_*. \tag{2.27}$$

Furthermore, since by Lemma 2.7 we know that $h(\zeta) \to 1$ as $\zeta \to +\infty$, we also have

$$y^* = \frac{1}{x_*} \quad \text{and} \quad y_* = \frac{1}{x^*}. \tag{2.28}$$

Now, using (2.25), (2.27), (2.28), and $x_* \leqslant x^*$, we conclude that

$$\frac{1}{x^*} \leqslant \frac{1}{x_*} = y^* = \overline{\lim_{\zeta \to +\infty}} b(\zeta) \leqslant \left(x^*\right)^2 ,$$

and so $x^* \geqslant 1$. Similarly, we have the corresponding result using (2.26), namely $x_* \leqslant 1$. From these inequalities and (2.28) we conclude the proof. ∎

We are now left with the need to prove the equalities in the conclusion of Lemma 2.8 do, in fact, hold. This will be done next

Lemma 2.9 *Let* (x, y) *be any solution to* (2.3). *Then, with the notation of Lemma 2.8,*

$$x_* = 1 = x^* \quad \text{and} \quad y_* = 1 = y^*.$$

Proof. We shall assume $x_* < 1 < x^*$ and draw a contradiction. First notice that in this situation every orbit γ will eventually behave like the one depicted in Figure 4. Pick any $(\tilde{x}, \tilde{y})$ in the hyperbola $\{xy = 1\}$ located strictly between

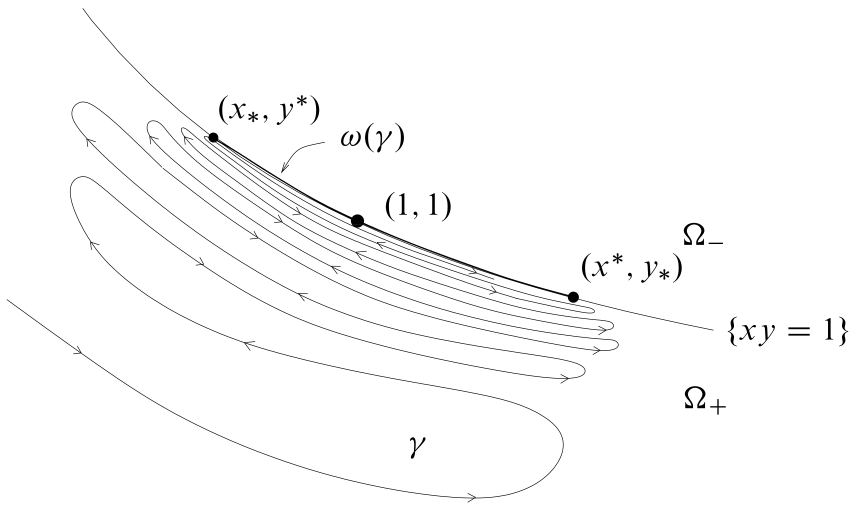

Figure 2.4: Long time behaviour of the orbits if $x_* < 1 < x^*$.

$(1, 1)$ and (x_*, y^*). Since $(\tilde{x}, \tilde{y}) \in \omega(\gamma)$, and from the geometry of the situation, we can choose a sequence $\tau_n \uparrow +\infty$ and a corresponding sequence of points $P_n = (x(\tau_n), y(\tau_n)) \in \gamma$ satisfying $P_n \xrightarrow[n\to+\infty]{} (\tilde{x}, \tilde{y})$, and $x'(\tau_n) < 0$ and $y'(\tau_n) > 0$ for all n. Furthermore, since $x(\tau_n) \to \tilde{x} < 1$ and $y(\tau_n) \to \tilde{y} > 1$, we can, without loss of generality, consider a sequence satisfying $\frac{y(\tau_n)}{x(\tau_n)} > L^2 > 1$ for all n, and for some constant L independent of n (see Figure 2.4). We easily conclude that

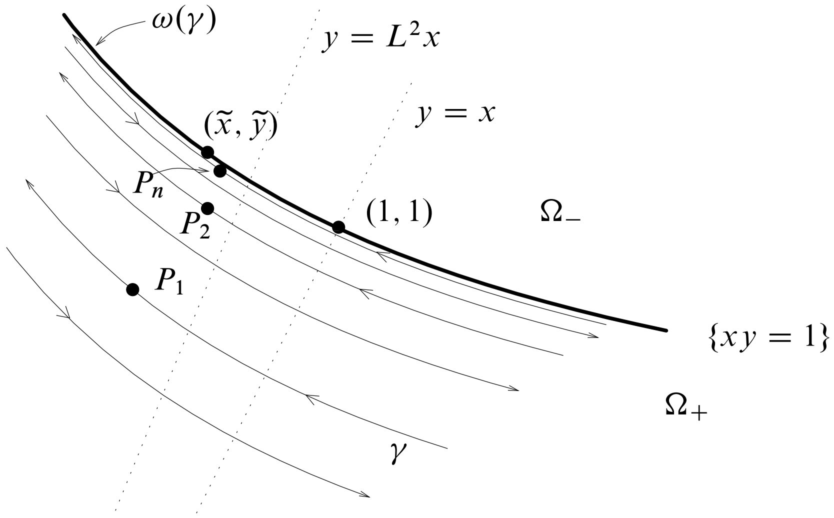

Figure 2.5: Sequence of points (P_n) on γ converging to $(\widetilde{x}, \widetilde{y}) \in \omega(\gamma)$.

inequality (2.6) holds true for the coordinates of the points P_n, namely

$$y'(\tau_n) < x'(\tau_n) A \tau_n^{-\frac{1}{2}}.$$

But then $x'(\tau_n) < 0 \Rightarrow y'(\tau_n) < 0$ contradicting our assumption about the P_n. This contradiction shows there is no sequence satisfying those conditions, which can only be so if we cannot choose $(\widetilde{x}, \widetilde{y})$ as stated, i.e., if $(x_*, y^*) = (1, 1)$.

Now choose any point $(\hat{x}, \hat{y})$ in $\{xy = 1\}$ strictly between $(1, 1)$ and (x^*, y_*). Again we can choose a sequence of times $\nu_n \uparrow +\infty$ and a corresponding sequence

$$Q_n = (x(\nu_n), y(\nu_n)) \in \gamma \quad \text{satisfying} \quad Q_n \xrightarrow[n \to +\infty]{} (\hat{x}, \hat{y}),$$

but this time such that $x'(\nu_n) > 0$ and $y'(\nu_n) < 0$ for all n. Without loss of generality we consider $x(\nu_n) > 1$ for all n (see Figure 2.5). To proceed we need to obtain an inequality relating $x'(\nu_n)$ and $y'(\nu_n)$ from which to obtain a contradiction. From (2.3) and $x(\nu_n) > 1 \Rightarrow x(\nu_n)^2 > x(\nu_n)$ we conclude that, for all n sufficiently large

$$y' = \left(1 - xy - A\nu_n^{-\frac{1}{2}} y\right) A\nu_n^{-\frac{1}{2}}$$

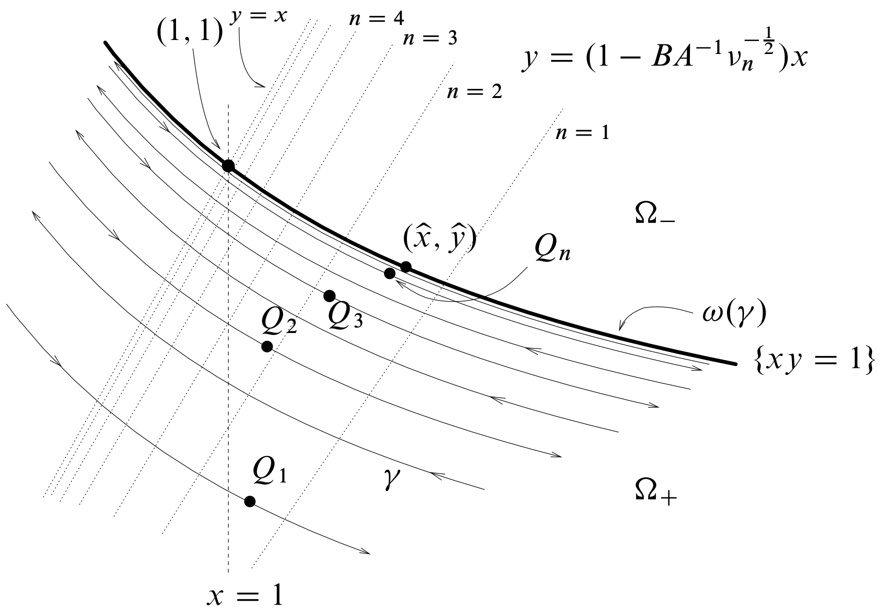

Figure 2.6: Sequence of points (Q_n) on γ converging to $(\hat{x}, \hat{y}) \in \omega(\gamma)$.

$$
\begin{aligned}
&= \left(x' + A\nu_n^{-\frac{1}{2}}x^2 - B\nu_n^{-1}x - A\nu_n^{-\frac{1}{2}}y\right) A\nu_n^{-\frac{1}{2}} \\
&> \left(x' + (A\nu_n^{-\frac{1}{2}} - B\nu_n^{-1})x - A\nu_n^{-\frac{1}{2}}y\right) A\nu_n^{-\frac{1}{2}} \\
&= \left(x' + [(1 - BA^{-1}\nu_n^{-\frac{1}{2}})x - y]A\nu_n^{-\frac{1}{2}}\right) A\nu_n^{-\frac{1}{2}} \\
&> x' A\nu_n^{-\frac{1}{2}},
\end{aligned}
\tag{2.29}
$$

where we wrote x for $x(\nu_n)$ and y for $y(\nu_n)$ in order not to overload the notation. For the last inequality two observations are in order: if $\omega \geqslant 1$ then $B \leqslant 0$ and thus, for all n, the straight lines $y = \left(1 - BA^{-1}\nu_n^{-\frac{1}{2}}\right)x$ are above $y = x$, and so $y(\nu_n) < \left(1 - BA^{-1}\nu_n^{-\frac{1}{2}}\right)x(\nu_n)$, which imply the last inequality in (2.29); if $\omega \in \left(-\frac{1}{2}, 1\right)$ then $B > 0$, in which case the straight lines $y = \left(1 - BA^{-1}\nu_n^{-\frac{1}{2}}\right)x$, although below $y = x$, converge to this line uniformly in compact sets, and thus, for all sufficiently large values of n, we again have the inequality (2.29). Whence, we conclude from (2.29) and the assumption $x'(\nu_n) > 0$ that also $y'(\nu_n) > 0$, a contradiction that shows there is no sequence satisfying the conditions, which in turn imply no such $(\hat{x}, \hat{y})$ exists, i.e., that $(x^*, y_*) = (1, 1)$. This concludes the proof. ∎

At this point we have all the ingredients to actually complete the proof of parts (**i**) and (**ii**) of Theorem 2.1. Part (**iii**) requires a more precise knowledge of how $h(\tau)$ converges to 1 as $\tau \to +\infty$, and this will be the object of the next and final lemma of this section.

Lemma 2.10 *Let* (x, y) *be any solution to* (2.3) *and let* $h = xy$. *Then we have*

$$\lim_{\tau \to +\infty} \frac{1 - h(\tau)}{A\tau^{-\frac{1}{2}}} = 1.$$

Proof. The proof is the same as in Lemma 2.7 down to the estimate (2.18), which now must be changed to

$$\tau^{\frac{1}{2}} \int_{\tau_0}^{\tau} \frac{x(s)}{s^{\frac{1}{2}}} e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s \longrightarrow 1 \quad \text{as} \quad \tau \to +\infty, \tag{2.30}$$

the analysis of which also follows the same lines as in the proof of Lemma 2.7, but now we make use of $x(\tau) = 1 + o(1)$ to relate the integral in (2.30) with $J(\tau)$. In the end, corresponding to the estimate of (2.20), we now need to prove that

$$\int_{\tau-\tau^\beta}^{\tau} e^{-\int_s^\tau \left(\frac{1}{x(\theta)} + \frac{Ax(\theta)}{\theta^{1/2}}\right)\mathrm{d}\theta}\,\mathrm{d}s \longrightarrow 1 \quad \text{as} \quad \tau \to +\infty, \tag{2.31}$$

and for this we shall use the convergence of $x(\tau)$ to 1 provided by Lemma 2.9. From this convergence we can write that

$$\forall_{\varepsilon>0}, \exists_{T^*} : \forall_\theta, \theta > T^* \Rightarrow 1 - \frac{\varepsilon}{2} < \frac{1}{x(\theta)} < 1 + \frac{\varepsilon}{2},$$

and also that

$$\forall_{\varepsilon>0}, \exists_{T^{**}} : \forall_\theta, \theta > T^{**} \Rightarrow -\frac{\varepsilon}{2} < \frac{Ax(\theta)}{\theta^{\frac{1}{2}}} < \frac{\varepsilon}{2}.$$

Consequently, for all $\theta > T := \max\{T^*, T^{**}\}$, we have

$$1 - \varepsilon < \frac{1}{x(\theta)} + \frac{Ax(\theta)}{\theta^{\frac{1}{2}}} < 1 + \varepsilon,$$

which implies that, by considering τ so large that $\tau - \tau^{\beta} > T$, we have $\theta \geqslant s \geqslant \tau - \tau^{\beta} > T$ and so the integral in (2.31) can be estimated as

$$\frac{1}{1+\varepsilon}\left(1 - e^{-(1+\varepsilon)\tau^{\beta}}\right) \leqslant \int_{\tau-\tau^{\beta}}^{\tau} e^{-\int_s^{\tau}\left(\frac{1}{x(\theta)} + \frac{Ax(\theta)}{\theta^{1/2}}\right)\mathrm{d}\theta}\,\mathrm{d}s \leqslant \frac{1}{1-\varepsilon}\left(1 - e^{-(1-\varepsilon)\tau^{\beta}}\right).$$

Passing to the limit $\tau \to +\infty$, and by the arbitrariness of $\varepsilon > 0$, we immediately conclude (2.31). ∎

We are now in position to prove Theorem 2.1 :

Proof of Theorem 2.1: Lemma 2.9 means that $(x(\tau), y(\tau)) \to (1,1)$ as $\tau \to +\infty$ and thus, by (2.1)-(2.2), we conclude statements (**i**) and (**ii**) of the theorem hold true. To prove (**iii**), first remember the relation between the two time scales: $\frac{\mathrm{d}\tau}{\mathrm{d}t} = \left(\frac{3\alpha^2}{1+2\omega}\right)^{\frac{1}{3}} t^{\frac{1+2\omega}{3}}$. After integration this obviously entails, as $\tau \to +\infty$ (or as $t \to +\infty$),

$$\tau^{-\frac{1}{2}} t^{\frac{2+\omega}{3}} \left(\frac{3\alpha^2}{1+2\omega}\right)^{\frac{1}{6}} \left(\frac{3}{4+2\omega}\right)^{\frac{1}{2}} = 1 + o(1). \tag{2.32}$$

Now, using (2.1)-(2.2), (2.4), and (2.32), we get, after a few algebraic manipulations,

$$\frac{1-h(\tau)}{A\tau^{-\frac{1}{2}}} = \left(\frac{3}{\alpha(1+2\omega)}\right)^{\frac{2}{3}} t^{2\frac{1-\omega}{3}} \big(\alpha t^{\omega} - c_0(t)c_1(t)\big)(1 + o(1)) \text{ as } \tau, t \to +\infty.$$

By Lemma 2.10 the left-hand side of this expression is $1 + o(1)$ and this completes the proof. ∎

2.3 Long time behaviour

From now on we consider the time scale introduced in (1.8). The first result we need is the following auxiliary statement concerning the relation between the original $t-$scale and the new $\varsigma-$scale defined by (1.8).

In order to simplify the notation, we first define the following time independent scaling factor

$$Q(\omega) := \left(\frac{3}{\alpha(1+2\omega)}\right)^{\frac{1}{2+\omega}} \left(\frac{2+\omega}{3}\right)^{r}, \qquad \text{where } r := \frac{1-\omega}{2+\omega}. \tag{2.33}$$

Proposition 2.11 *With* (c_j), ς, *and* $(\widetilde{c}_j)$ *given in* (1.8) *and* (1.9), *and* $Q(\omega)$ *given by* (2.33), *the following holds true:*

(i) $\displaystyle\lim_{t\to+\infty} \frac{2+\omega}{3}\left(\frac{3}{\alpha(1+2\omega)}\right)^{1/3} t^{-\frac{2+\omega}{3}}\varsigma(t) = 1,$

(ii) $\displaystyle\lim_{\varsigma\to+\infty} Q(\omega)\varsigma^{r}\,\widetilde{c}_1(\varsigma) = 1.$

Proof. From Theorem 2.1 (**ii**) we have the following bounds

$$\forall\varepsilon>0, \exists T = T(\varepsilon) : \forall t > T(\varepsilon), 1-\varepsilon < \left(\frac{3}{\alpha(1+2\omega)}\right)^{\frac{1}{3}} t^{\frac{1-\omega}{3}} c_1(t) < 1+\varepsilon\,.$$

First consider the upper bound: we have

$$c_1(t) < (1+\varepsilon)\left(\frac{3}{\alpha(1+2\omega)}\right)^{-\frac{1}{3}} t^{-\frac{1-\omega}{3}},$$

and hence integrating between t and t_0 and substituting into (1.8) we obtain

$$\begin{aligned}\varsigma(t)-\varsigma_0 = \int_{t_0}^{t} c_1(s)\,\mathrm{d}s &< (1+\varepsilon)\left(\frac{3}{\alpha(1+2\omega)}\right)^{-\frac{1}{3}} \int_{t_0}^{t} s^{-\frac{1-\omega}{3}}\,\mathrm{d}s \\ &= (1+\varepsilon)\left(\frac{3}{\alpha(1+2\omega)}\right)^{-\frac{1}{3}} \frac{3}{2+\omega}\left(t^{\frac{2+\omega}{3}} - t_0^{\frac{2+\omega}{3}}\right).\end{aligned}$$

Multiplying both sides of this inequality by $\left(\frac{3}{\alpha(1+2\omega)}\right)^{\frac{1}{3}}\frac{2+\omega}{3}t^{-\frac{2+\omega}{3}}$ and passing to the limit as $t\to+\infty$ results in

$$\overline{\lim_{t\to+\infty}}\left(\frac{3}{\alpha(1+2\omega)}\right)^{\frac{1}{3}}\frac{2+\omega}{3}t^{-\frac{2+\omega}{3}}\varsigma(t)\leqslant(1+\varepsilon).$$

Now considering the lower bound, and the same argument we obtain the reverse bound:

$$\underline{\lim_{t\to+\infty}}\left(\frac{3}{\alpha(1+2\omega)}\right)^{\frac{1}{3}}\frac{2+\omega}{3}t^{-\frac{2+\omega}{3}}\varsigma(t)\geqslant(1-\varepsilon).$$

The two results thus obtained, together with the arbitrariness of ε, prove part (**i**). To get part (**ii**), again start from Theorem 2.1 (**ii**), and use part (**i**) above to write, as $t,\varsigma\to+\infty$,

$$\begin{aligned}
1+o(1) &= \left(\frac{3}{\alpha(1+2\omega)}\right)^{\frac{1}{3}}t^{\frac{1-\omega}{3}}c_1(t)\\
&= \left(\frac{3}{\alpha(1+2\omega)}\right)^{\frac{1}{3}}\left[\left(\frac{2+\omega}{3}\left(\frac{3}{\alpha(1+2\omega)}\right)^{\frac{1}{3}}\varsigma\right)^{\frac{3}{2+\omega}}\right]^{\frac{1-\omega}{3}}\widetilde{c}_1(\varsigma)\\
&= Q(\omega)\varsigma^r\widetilde{c}_1(\varsigma),
\end{aligned}$$

which concludes the proof. ∎

We can now use the expression (1.10) for the $\widetilde{c}_j(\varsigma)$ component of the solution, and the information provided by Proposition 2.11 to prove the following result about the asymptotic behaviour of solutions to (1.5):

Theorem 2.12 *Let* (c_j) *be any non-negative solution of* (1.5) *with initial data satisfying* $c_0(0)=\sum_{j=1}^{\infty}c_j(0)<\infty$. *Then, as* $t\to+\infty$, *we have*

(**i**) $\left(\frac{3}{\alpha(1+2\omega)}\right)^{\frac{1}{3}}t^{\frac{1-\omega}{3}}c_j(t)\longrightarrow 1\qquad$ for all $j\geqslant 1$,

(ii) $\left(\dfrac{1+2\omega}{3\alpha^2}\right)^{\frac{1}{3}} t^{-\frac{1+2\omega}{3}} \sum_{j=1}^{\infty} c_j(t) \longrightarrow 1,$

(iii) $\left(\dfrac{3}{\alpha(1+2\omega)}\right)^{\frac{2}{3}} t^{2\frac{1-\omega}{3}} \left(\alpha t^{\omega} - c_1(t)\sum_{j=1}^{\infty} c_j(t)\right) \longrightarrow 1.$

Proof. For $j = 1$, **(i)** is just **(ii)** from Theorem 2.1.

For $j \geqslant 2$ we can use the expression (1.10) for the $\tilde{c}_j$:

$$\tilde{c}_j(\varsigma) = e^{-\varsigma} \sum_{k=2}^{j} \frac{\varsigma^{j-k}}{(j-k)!} c_k(0) + \frac{1}{(j-2)!} \int_0^{\varsigma} \tilde{c}_1(\varsigma - s) s^{j-2} e^{-s}\, \mathrm{d}s.$$

Multiplying both sides of (1.10) by $Q(\omega)\varsigma^r$ we have to estimate the two terms on the right hand side.

Since j is fixed, the first one can be written, as $\varsigma \to +\infty$

$$Q(\omega)\varsigma^r e^{-\varsigma} \sum_{k=2}^{j} \frac{\varsigma^{j-k}}{(j-k)!} c_k(0) = \mathcal{O}\left(\varsigma^{j-1} e^{-\varsigma}\right) = o(e^{-\lambda\varsigma}),$$

for every $\lambda < 1$.

To study the second term, we start by changing variables in the integral in order to integrate over the fixed interval $[0,1]$, $s \mapsto y = s/\varsigma$, and we define the function $\psi_\omega(\cdot) := Q(\omega)\,(\cdot)^r\, \tilde{c}_1(\cdot)$, which is a continuous function, and is $1 + o(1)$ at infinity by **(ii)** in Proposition 2.11, and hence there is a constant M_{ψ_ω} such that $0 \leqslant \psi_\omega(s) \leqslant M_{\psi_\omega}$. Let $0 < \varepsilon < 1$ be fixed, and write the integral as $\int_0^{1-\varepsilon} + \int_{1-\varepsilon}^1$. The integral over the second interval can be estimated as

$$\varsigma^{j-1} \int_{1-\varepsilon}^{1} \frac{\psi_\omega(\varsigma(1-y))y^{j-2}}{(1-y)^r} e^{-\varsigma y}\, \mathrm{d}y \leqslant$$

$$\leqslant M_{\psi_\omega} \varsigma^{j-1} e^{-(1-\varepsilon)\varsigma} \int_{1-\varepsilon}^{1} \frac{y^{j-2}}{(1-y)^r}\, \mathrm{d}y < \frac{1}{1-r} \varepsilon^{1-r} M_{\psi_\omega} \varsigma^{j-1} e^{-(1-\varepsilon)\varsigma},$$

and so it is also exponentially small as $\varsigma \to +\infty$.
To understand the behaviour of the integral over $[0, 1-\varepsilon]$ we make use of Watson's Lemma [1, pp.427–428].

Since
$$\frac{y^{j-2}}{(1-y)^r} = y^{j-2}\left(1 + \sum_{j=1}^{\infty} \frac{y^j}{j!} \prod_{k=0}^{j-1}(r+k)\right),$$
which is convergent for $|y| < 1$, by Watson's Lemma we obtain

$$\int_0^{1-\varepsilon} \frac{y^{j-2}}{(1-y)^r} e^{-\varsigma y}\, \mathrm{d}y = \frac{\Gamma(j-1)}{\varsigma^{j-1}} + \mathcal{O}\left(\frac{1}{\varsigma^j}\right), \text{ as } \varsigma \to +\infty.$$

These three exponentially small terms allow us to conclude that

$$\lim_{\varsigma \to +\infty} Q(\omega)\varsigma^r\ \widetilde{c}_j(\varsigma) = 1, \text{for all}\, j \geqslant 2,$$

which is equivalent to (**i**) as is clear from the proof of Proposition 2.11(**i**).

Cases (**ii**) and (**iii**) are those already proved in Theorem 2.1 (**i**) and (**iii**) respectively. ∎

2.4 Convergence to self-similarity

We can now turn to the results concerning convergence of solutions to self-similar profiles. Let $\Phi_{1,\omega} : \mathbb{R}^+ \setminus \{1\} \to \mathbb{R}$ be defined by

$$\Phi_{1,\omega}(\eta) \coloneqq \begin{cases} (1-\eta)^{\frac{\omega-1}{\omega+2}} & \text{if } \eta < 1, \\ 0 & \text{if } \eta > 1. \end{cases} \tag{2.34}$$

In Figure 2.7 we present the graphs of functions $\Phi_{1,\omega}$ for various values of ω. Note these functions can be continuously extended to $\eta = 1$ if and only if $\omega > 1$.

Our first result states that the functions $\Phi_{1,\omega}$ are the similarity profiles of the solutions to (1.5) along non-characteristic directions.

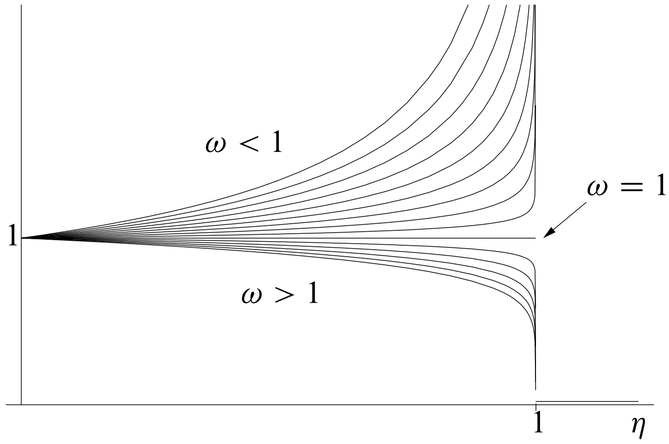

Figure 2.7: Graphs of $\Phi_{1,\omega}$ for $\omega = 1$ and for values of ω below and above 1 in steps of 0.1

Theorem 2.13 *Let* (c_j) *be any non-negative solution of* (1.5) *with initial data satisfying* $\exists\rho > 0, \mu > r : \forall j, c_j(0) \leqslant \rho/j^\mu$, *where again* $r := \frac{1-\omega}{2+\omega}$. *Let* $\varsigma(t)$ *and* $\widetilde{c}_j(\varsigma)$ *be as in* (1.8) *and* (1.9), *respectively, and let* $Q(\omega)$ *be given by (2.33). Then,*

$$\lim_{\substack{j,\varsigma\to+\infty \\ \eta=j/\varsigma \text{ fixed} \\ \eta\neq 1}} Q(\omega)\varsigma^r\,\widetilde{c}_j(\varsigma) = \Phi_{1,\omega}(\eta).$$

Proof. The proof of Theorem 2.13 follows exactly the same steps as the corresponding result about Φ_1 in the $\omega = 0$ case [21], and we will just show the most important steps of the proof, pointing out the differences.

We consider first the case of monomeric initial conditions $c_j(0) = c_1(0)\delta_{1,j}$. Starting from the representation expression for $\widetilde{c}_j$ given by (1.10) and multiplying by $g(\varsigma) := Q(\omega)\varsigma^r$, we can define φ_1 on $[2,\infty)\times[0,\infty)$ by

$$\varphi_1(x,\varsigma) := \frac{g(\varsigma)}{\Gamma(x-1)}\int_0^\varsigma \widetilde{c}_1(\varsigma - s)s^{x-2}e^{-s}\,\mathrm{d}s. \tag{2.35}$$

When $x \geqslant 2$ is an integer, the function φ_1 clearly satisfies $\varphi_1(x,\varsigma) = g(\varsigma)\widetilde{c}_x$, and we shall use φ_1 instead of the definition of $\widetilde{c}_j$. Using Stirling's asymptotic formula $\Gamma(x) = e^{-x}x^{x-\frac{1}{2}}\sqrt{2\pi}\left(1 + \mathcal{O}(x^{-1})\right)$ as $x \to \infty$, the recursive relation

$\Gamma(x-1) = \Gamma(x)/(x-1)$, letting $\eta \coloneqq x/\varsigma$, and changing variable $s \mapsto y = s/\varsigma$, so that the integration region becomes $(0,1)$, and finally using Proposition 2.11 (**ii**), we obtain as $\varsigma \to +\infty$,

$$\varphi_{1,\omega}(\eta\varsigma,\varsigma) = \frac{1}{\sqrt{2\pi}}\eta^{\frac{3}{2}-\eta\varsigma}\varsigma^{\frac{1}{2}}\left(1+\mathcal{O}\left(\varsigma^{-1}\right)\right)\int_0^1 \psi_\omega(\varsigma(1-y))\frac{e^{\varsigma(\eta\log y - y + \eta)}}{y^2(1-y)^r}\,\mathrm{d}y, \tag{2.36}$$

where $\psi_\omega(\cdot) \coloneqq Q(\omega)(\cdot)^r\widetilde{c}_1(\cdot)$.

To prove that $\varphi_{1,\omega}$ converges to $\Phi_{1,\omega}$ when $\varsigma \to +\infty$ (for fixed η) we proceed as in [21, Section 5.1], and we next give the main idea of the proof.

The case $\eta > 1$ is easy, since the integral in (2.36) can be estimated using the bound $y^{-2}e^{\varsigma(\eta\log y - y+\eta)} < e^{-\varsigma}e^{\varsigma\eta}$. This follows from the fact that for $y \in (0,1]$, $\eta > 1$ and $\varsigma > 2/(\eta-1)$ it holds that $y^{-2}e^{\varsigma(\eta\log y - y)} = e^{(\varsigma\eta-2)\log y - \varsigma y} < e^{-\varsigma}$. Using the $L^1(0,1)$–integrability of $\theta \mapsto \theta^{-r}$ arising from the fact that $r < 1$ for $\omega > -\frac{1}{2}$, we obtain

$$\begin{aligned}\int_0^1 \psi_\omega(\varsigma(1-y))\frac{y^{-2}e^{\varsigma(\eta\log y - y+\eta)}}{(1-y)^r}\,\mathrm{d}y &\leqslant M_{\psi_\omega}e^{-\varsigma+\eta\varsigma}\int_0^1 (1-y)^{-r}\,\mathrm{d}y \\ &= M_{\psi_\omega}e^{-\varsigma+\eta\varsigma}C_r.\end{aligned}$$

Combining this exponential with the one in (2.36), $\eta^{-\eta\varsigma}$, as $\varsigma \to +\infty$ we obtain the desired convergence to zero.

When $\eta \in (0,1)$ we decompose the integral in (2.36) as $\int_0^\varepsilon + \int_\varepsilon^{1-\varepsilon} + \int_{1-\varepsilon}^1$ with $\varepsilon < \min\left\{\eta e^{-1}, 1-\eta\right\}$.

The contributions of the first and last integrals can be estimated as in the case $\eta > 1$.

Next, we have to prove that, as $\varsigma \to +\infty$, the integral on $(\varepsilon, 1-\varepsilon)$ converges to $\Phi_{1,\omega}(\eta)$ (where $\Phi_{1,\omega}(\eta) = (1-\eta)^{-r}$ if $\eta < 1$). This follows from Proposition 2.11 (**ii**), which implies that $\psi_\omega(\theta) = 1 + o(1)$ as $\theta \to +\infty$, when ω is fixed. Using the same kind of argument as before we are left to estimate

$$\eta^{\frac{3}{2}-\eta\varsigma}\varsigma^{r} J(\eta,\varsigma) \quad \text{where} \quad J(\eta,\varsigma) \coloneqq \int_{\varepsilon}^{1-\varepsilon} \frac{1}{y^2(1-y)^r} e^{-\varsigma\phi(y)}\, \mathrm{d}y,$$

and $\phi : (0,1) \to \mathbb{R}$ is defined by $\phi(y) \coloneqq y - \eta\log y - \eta$.

Since this function is smooth and has a unique minimum, attained at $y = \eta \in (\varepsilon, 1-\varepsilon)$ with value $\phi(\eta) = -\eta\log\eta$ and $\phi''(\eta) = \eta^{-1}$, we can apply Laplace's method for the asymptotic evaluation of integrals [1, pg. 431] to $J(\eta,\varsigma)$.

We obtain, as $\varsigma \to +\infty$,

$$J(\eta,\varsigma) = \int_{\varepsilon}^{1-\varepsilon} \frac{e^{-\varsigma(y-\eta\log y-\eta)}}{(1-y)^r y^2}\, \mathrm{d}y = e^{\varsigma\eta\log\eta} \frac{1}{\eta^2(1-\eta)^r}\sqrt{\frac{2\pi}{\varsigma/\eta}} + \mathcal{O}\left(\frac{e^{\varsigma\eta\log\eta}}{\varsigma^{3/2}}\right). \tag{2.37}$$

Now from (2.36) and (2.37), we obtain for $\eta < 1$,

$$\begin{aligned}\varphi_{1,\omega}(\eta\varsigma,\varsigma) &= \frac{1}{\sqrt{2\pi}}\eta^{\frac{3}{2}-\eta\varsigma}\varsigma^{\frac{1}{2}} e^{\varsigma\eta\log\eta}\frac{1}{\eta^2(1-\eta)^r}\sqrt{\frac{2\pi}{\varsigma/\eta}} + \mathcal{O}\left(\varsigma^{-1}\right)\\ &= \frac{1}{(1-\eta)^r}\left(1 + o(1)\right), \text{ as } \varsigma \to +\infty,\end{aligned}$$

thus proving convergence to $\Phi_{1,\omega}$ for monomeric initial data.

If the initial condition is not monomeric we have the contribution arising from the sum term in the right hand side of (1.10).

Multiplying it by $Q(\omega)\varsigma^r$ we now have to prove that

$$\lim_{\substack{j,\varsigma\to+\infty\\ \eta=j/\varsigma \text{ fixed}\\ \eta\neq 1}} Q(\omega)\varsigma^r\, e^{-\varsigma}\sum_{k=2}^{j}\frac{\varsigma^{j-k}}{(j-k)!}c_k(0) = 0. \tag{2.38}$$

Changing the summation variable $k \mapsto \ell \coloneqq j-k$, using the bound on the initial data, and using the similarity variable to write $\varsigma = j\nu$, with $\nu \coloneqq \eta^{-1}$, the limit

(2.38) can be estimated by the limit when $j \to \infty$ of the function

$$\varphi_{2,\omega}(\nu, j) \coloneqq (j\nu)^r e^{-j\nu} \sum_{\ell=0}^{j-2} \frac{(j\nu)^\ell}{\ell!(j-\ell)^\mu}.$$

Our goal is to prove that $\varphi_2(\nu, j) \to 0$ as $j \to \infty$, for all positive $\nu \neq 1$.

We can adapt the results in the study of φ_2 done in [21, Section 5.2], noticing that we only need to multiply all the estimates in [21, Section 5.2] by $(j\nu)^{r-1/2}$. The estimates show that now in order for φ_2 to converge to zero we need to consider initial data satisfying $c_j(0) \leqslant \rho/j^\mu$, but in this case with $\mu > r$.

We shall study the cases $\nu > 1$ and $\nu < 1$ separately.

Consider first the case $\nu > 1$.

Change the summation variable $k \mapsto \ell \coloneqq j - k$. It is sufficient, for this range of ν, to bound φ_2 as follows

$$\begin{aligned} \varphi_2(\nu, j) &= (j\nu)^r\ e^{-j\nu} \sum_{\ell=0}^{j-2} \frac{(j\nu)^\ell}{\ell!(j-\ell)^\mu} \\ &\leqslant \frac{1}{2^\mu}(j\nu)^r\ e^{-j\nu} \sum_{\ell=0}^{j-2} \frac{(j\nu)^\ell}{\ell!}. \end{aligned} \tag{2.39}$$

Considering the sequence $u_\ell \coloneqq \frac{(j\nu)^\ell}{\ell!}$, and studying the sign of

$$u_{\ell+1} - u_\ell = \frac{(j\nu)^\ell}{\ell!}\left(\frac{j\nu}{\ell+1} - 1\right),$$

we conclude the maximum of u_ℓ is attained at $\ell = \lfloor j\nu \rfloor > j\nu - 1 > j - 1 > j - 2$. As $j \to \infty$, we can thus estimate the right-hand side of (2.39):

$$\begin{aligned} &\frac{1}{2^\mu}(j\nu)^r\ e^{-j\nu} \sum_{\ell=0}^{j-2} \frac{(j\nu)^\ell}{\ell!} \\ &\quad < \frac{1}{2^\mu}(j\nu)^r\ e^{-j\nu}(j-1)\frac{(j\nu)^{j-2}}{(j-2)!} \end{aligned}$$

$$= \frac{\nu^{r-2}}{2^{\mu}\sqrt{2\pi}} \left(\frac{j-1}{j}\right)^2 e^{-j(\nu-1-\log\nu)+(r+1/2)\log j}(1+o(1)) \tag{2.40}$$

where we use Stirling's approximation in the second equality. Since $\nu > 1$, we have $\nu - 1 - \log\nu > 0$, and so $-j(\nu - 1 - \log\nu) + (r + 1/2)\log j \to -\infty$ as $j \to \infty$ from which we conclude (2.40) goes to zero as $j \to \infty$, and we obtain the result we seek in the case $\nu > 1$.

For the second part of the proof, a finer approach is required, which consists in estimating the small and the large ℓ contributions to φ_2 separately.

Let $\beta \in \left(\nu e^{1-\nu}, \min\{\nu e, 1\}\right)$ be fixed, and write

$$\begin{aligned}\varphi_2(\nu, j) &= (j\nu)^r\, e^{-j\nu} \\ &\times \left(\sum_{0\le\ell\le\beta j} \frac{(j\nu)^\ell}{\ell!(j-\ell)^\mu} + \sum_{\beta j<\ell\le j-2} \frac{(j\nu)^\ell}{\ell!(j-\ell)^\mu}\right) \\ &=: S_1(j) + S_2(j).\end{aligned}$$

For the first sum we have

$$\begin{aligned}S_1(j) &\leqslant (j\nu)^r\, e^{-j\nu} \sum_{0\leqslant\ell\leqslant\beta j} \frac{(j\nu)^\ell}{\ell!(j-\beta j)^\mu} \\ &\leqslant (j\nu)^r\, e^{-j\nu} \frac{1}{j^\mu(1-\beta)^\mu} e^{j\nu} \\ &= \frac{\nu^r}{(1-\beta)^\mu} j^{r-\mu} \longrightarrow 0 \text{ as } j \to \infty, \ \text{ if } \mu > r.\end{aligned}$$

By Stirling's expansion we can write, for all sufficiently large ℓ, $\ell! \geqslant e^{-\ell}\ell^{\ell+\frac{1}{2}}$. Therefore, since in S_2 we have $(j-\ell)^\mu \geqslant 2^\mu \geqslant 1$, we can use these bounds to estimate that term and conclude it converges to 0 as $j \to \infty$. This conludes the proof. ∎

When $\omega \in \left(-\frac{1}{2}, 1\right]$ the similarity profiles $\Phi_{1,\omega}$ have a jump at $\eta = 1$. In these cases it is natural to look for a different similarity variable and scaling that can provide a better description of the behaviour of solutions along the characteristic direction corresponding to the jump position $\eta = 1$. As pointed out in the Introduction, one such similarity variable is $\xi \in \mathbb{R}$ defined by $j = \varsigma + \xi\sqrt{\varsigma}$, with ς given, as in Theorem 2.13, by expression (1.8).

Let $\Phi_{2,\omega} : \mathbb{R} \to \mathbb{R}$ be defined by

$$\Phi_{2,\omega}(\xi) := e^{-\frac{1}{2}\xi^2} \int_0^{+\infty} y^{\frac{3\omega}{\omega+2}} e^{-\xi y^2 - \frac{1}{2}y^4}\,\mathrm{d}y. \tag{2.41}$$

In Figure 2.8 we present the graphs of $\Phi_{2,\omega}$ for several values of ω in $\left(-\frac{1}{2}, 1\right]$.

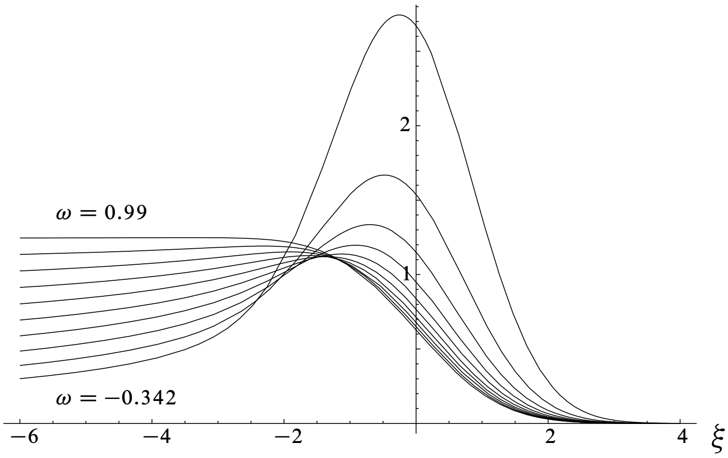

Figure 2.8: Graphs of $\Phi_{2,\omega}$ for values of ω from -0.342 to 0.99 in steps of 0.148

Our final result proves that the functions $\Phi_{2,\omega}$ are also similarity profiles of the solutions to (1.5). Observe that not only the similarity variable has changed, but the time scale is also different: the power of ς is half that of Theorem 2.13.

Theorem 2.14 *Let* (c_j) *be any non-negative solution of* (1.5) *with monomeric initial data. Let* $\varsigma(t)$, $\widetilde{c}_j(\varsigma)$, $Q(\omega)$, *and* r *be as in Theorem 2.13. Then,*

$$\lim_{\substack{j,\varsigma\to+\infty \\ \xi=\frac{j-\varsigma}{\sqrt{\varsigma}} \text{ fixed} \\ \xi\in\mathbb{R}}} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} Q(\omega)\, \varsigma^{\frac{r}{2}}\, \widetilde{c}_j(\varsigma) = \Phi_{2,\omega}(\xi).$$

We shall consider only the main steps of the proof, as the proof of this result is analogous to the one for $\omega = 0$ [21, Section 6].

Proof. Since we now consider only the monomeric case it means we only have the integral term in the representation formula for $\widetilde{c}_j$; multiplying the integral in (1.10) by

$$\left(\frac{\pi}{2}\right)^{\frac{1}{2}} Q(\omega)\, \varsigma^{\frac{r}{2}},$$

and defining the function φ_3 in $[2,\infty) \times [0,\infty)$ by

$$\varphi_3(x,\varsigma) \coloneqq \sqrt{\frac{\pi}{2}}\, \frac{Q(\omega)\, \varsigma^{\frac{r}{2}}}{\Gamma(x-1)} \int_0^{\varsigma} \widetilde{c}_1(\varsigma - s) s^{j-2}\, e^{-s}\, \mathrm{d}s,$$

and using the similarity variable $\xi = \frac{j-\varsigma}{\sqrt{\varsigma}} \left(= \frac{x-\varsigma}{\sqrt{\varsigma}}\right)$ we can rewrite φ_3 as

$$\varphi_3(\varsigma + \xi\sqrt{\varsigma}, \varsigma) = \sqrt{\frac{\pi}{2}}\, \frac{Q(\omega)\, \varsigma^{\frac{r}{2}}}{\Gamma(\varsigma + \xi\sqrt{\varsigma} - 1)} \int_0^{\varsigma} \widetilde{c}_1(\varsigma - s)\, s^{\varsigma + \xi\sqrt{\varsigma} - 2}\, e^{-s}\, \mathrm{d}s.$$

If $2 \leqslant x = j \in \mathbb{N}$, we have $\varphi_3(j,\varsigma) = \sqrt{\frac{\pi}{2}}\, Q(\omega)\, \varsigma^{\frac{r}{2}} \widetilde{c}_j(\varsigma)$, and hence we need to evaluate the limit

$$\lim_{\varsigma \to +\infty} \varphi_3(\varsigma + \xi\sqrt{\varsigma}, \varsigma). \tag{2.42}$$

Changing the integration variable

$$s \mapsto w \coloneqq \sqrt{\frac{\varsigma - s}{\sqrt{\varsigma}}} = \sqrt{\sqrt{\varsigma} - \frac{s}{\sqrt{\varsigma}}},$$

in such a way that $\varsigma - s = \sqrt{\varsigma}\, w^2$ and $\mathrm{d}s = -2\sqrt{\varsigma} w\ \mathrm{d}w$, we obtain

$$\varphi_3(\varsigma + \xi\sqrt{\varsigma}, \varsigma) = \sqrt{\frac{\pi}{2}}\, \frac{Q(\omega)\varsigma^{r/2}}{\Gamma(\varsigma + \xi\sqrt{\varsigma} - 1)} \times \int_0^{\varsigma^{\frac{1}{4}}} \widetilde{c}_1(\sqrt{\varsigma}w^2)(\varsigma - \sqrt{\varsigma}w^2)^{\varsigma + \xi\sqrt{\varsigma} - 2} e^{-\varsigma + \sqrt{\varsigma}w^2} 2\sqrt{\varsigma} w\, \mathrm{d}w. \tag{2.43}$$

We now use our knowledge about the large time behaviour of $\widetilde{c}_1$ to compute the limit of φ_3 as $\varsigma \to +\infty$; using the definition of ψ_ω to write (2.43) as

$$\begin{aligned}
\varphi_3(\varsigma+\xi\sqrt{\varsigma},\varsigma) &= \frac{\sqrt{2\pi}\,\varsigma^{r/2}}{\Gamma(\varsigma+\xi\sqrt{\varsigma}-1)}\int_0^{\varsigma^{\frac14}}(\sqrt{\varsigma}w^2)^{-r} \\
&\quad\times\psi(\sqrt{\varsigma}w^2)(\varsigma-\sqrt{\varsigma}w^2)^{\varsigma+\xi\sqrt{\varsigma}-2}e^{-\varsigma+\sqrt{\varsigma}w^2}\sqrt{\varsigma}w\,\mathrm{d}w \\
&= \frac{\sqrt{2\pi}\,\varsigma^{1/2}}{\Gamma(\varsigma+\xi\sqrt{\varsigma}-1)}\int_0^{\varsigma^{\frac14}}\psi(\sqrt{\varsigma}w^2) \\
&\quad\times(\varsigma-\sqrt{\varsigma}w^2)^{\varsigma+\xi\sqrt{\varsigma}-2}e^{-\varsigma+\sqrt{\varsigma}w^2}w^{1-2r}\,\mathrm{d}w \\
&= \frac{\sqrt{2\pi}\,\varsigma^{\varsigma+\xi\sqrt{\varsigma}-3/2}\,e^{-\varsigma}}{\Gamma(\varsigma+\xi\sqrt{\varsigma}-1)}\int_0^{\varsigma^{\frac14}}\psi(\sqrt{\varsigma}w^2) \\
&\quad\times\left(1-\frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2}e^{\sqrt{\varsigma}w^2}w^{1-2r}\,\mathrm{d}w,
\end{aligned}$$

and using Stirling's asymptotic expansion for the Gamma function we can write

$$1/\,\Gamma(\varsigma+\xi\sqrt{\varsigma}-1) = \frac{1}{\sqrt{2\pi}}\frac{e^{\varsigma+\xi\sqrt{\varsigma}}}{(\varsigma+\xi\sqrt{\varsigma})^{\varsigma+\xi\sqrt{\varsigma}-3/2}}(1+o(1)),\ \text{as}\ \varsigma\to+\infty$$

and hence φ_3 can be written, as $\varsigma \to +\infty$,

$$\begin{aligned}
\varphi_3(\varsigma+\xi\sqrt{\varsigma},\varsigma) &= \frac{\varsigma^{\varsigma+\xi\sqrt{\varsigma}-3/2}\,e^{\xi\sqrt{\varsigma}}}{(\varsigma+\xi\sqrt{\varsigma})^{\varsigma+\xi\sqrt{\varsigma}-3/2}}(1+o(1)) \\
&\quad\times\int_0^{\varsigma^{\frac14}}\psi(\sqrt{\varsigma}w^2)\left(1-\frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2}e^{\sqrt{\varsigma}w^2}w^{1-2r}\,\mathrm{d}w.
\end{aligned}\tag{2.44}$$

To estimate the multiplicative prefactor in (2.44) as $\varsigma \to +\infty$ we can write it as

$$\frac{\varsigma^{\varsigma+\xi\sqrt{\varsigma}-3/2}\,e^{\xi\sqrt{\varsigma}}}{(\varsigma+\xi\sqrt{\varsigma})^{\varsigma+\xi\sqrt{\varsigma}-3/2}} = e^{\xi\sqrt{\varsigma}}\left(\frac{\varsigma}{\varsigma+\xi\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}}\left(1+\frac{\xi}{\sqrt{\varsigma}}\right)^{3/2}$$

$$
\begin{aligned}
&= e^{\xi\sqrt{\varsigma}} \frac{1}{(1+\xi/\sqrt{\varsigma})^{\varsigma}} \frac{1}{\left((1+\xi/\sqrt{\varsigma})^{\sqrt{\varsigma}}\right)^{\xi}} (1+o(1)) \\
&= \left(\frac{e}{(1+\xi/\sqrt{\varsigma})^{\sqrt{\varsigma}/\xi}}\right)^{\xi\sqrt{\varsigma}} e^{-\xi^2}(1+o(1)) \qquad (2.45) \\
&= e^{\xi^2/2} e^{-\xi^2}(1+o(1)) \\
&= e^{-\xi^2/2}(1+o(1)),
\end{aligned}
$$

where in (2.45) to compute the limit as $\varsigma \to +\infty$ we use the change of variables $\varsigma \mapsto x := \frac{\xi}{\sqrt{\varsigma}}$ to obtain $\left(\frac{e}{(1+x)^{1/x}}\right)^{\xi^2/x}$ and then we apply L'Hôpital's rule twice. Using this last expression we can write (2.44) as $\varsigma \to +\infty$ in the following way

$$
\begin{aligned}
\varphi_3(\varsigma + \xi\sqrt{\varsigma}, \varsigma) &= e^{-\xi^2/2}(1+o(1)) \qquad (2.46) \\
&\times \int_0^{\varsigma^{\frac{1}{4}}} \psi(\sqrt{\varsigma} w^2)\left(1 - \frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2} e^{\sqrt{\varsigma}w^2} w^{1-2r}\, \mathrm{d}w.
\end{aligned}
$$

As in the case where $\omega = 0$ we proceed with a study of the integral term in (2.46) by considering w (and hence $\sqrt{\varsigma}w^2$) close to zero, and w away from zero, and showing that the integral, for small values of w, can be made arbitrarily small, while the remaining integral converges as $\varsigma \to +\infty$.

The only difference from the results in [21, Section 6.1] is that now both integrals contain the multiplicative factor y^{1-2r} in the integrand function. Observe that for $\omega > -1/2$, we have $1-2r > -1$. As we are computing the limit as $\varsigma \to +\infty$, and this factor does not depend on ς, the asymptotic estimates in [21] apply verbatim.

Remembering that $r = \dfrac{1-\omega}{\omega+2}$, and hence $1-2r = \dfrac{3\omega}{\omega+2}$, we obtain the stated result about $\Phi_{2,\omega}$. ∎

Chapter 3

The perturbed input case

Part of the results in this Chapter were obtained in collaboration with F. P. da Costa and J. T. Pinto, and published in 2007, in *Applied and Industrial Mathematics in Italy II, Selected Contributions from the 8th SIMAI Conference*, under the title *Convergence to self-similarity in an addition model with power-like time-dependent input of monomers* [19].

As in the previous Chapter we consider the system

$$\begin{cases} \dot{c}_1 = J_1(t) - c_1^2 - c_1 \sum_{j=1}^{\infty} c_j \\ \dot{c}_j = c_1 c_{j-1} - c_1 c_j, \quad j \geq 2. \end{cases} \tag{3.1}$$

We provide an extension of the results obtained in previous Chapter to monomer input of the type $J_1(t) = \alpha t^{\omega}(1 + \varepsilon(t))$ where $\varepsilon(\cdot)$ is a continuous function satisfying $\varepsilon(t) \to 0$ as $t \to +\infty$. Most, but not all, of the results of Chapter 2 follow with just this general assumption on ε. This is true, in particular, of those concerning convergence to a self-similar profile. Those about the rate of convergence of the bulk quantity $\sum_{j=1}^{\infty} c_j$ will need an extra assumption on the decay rate of the perturbation $\varepsilon(t)$.

3.1 General Approach and Statement of Results

We shall consider the following general assumption for the input of monomers $J_1(t)$:

(H1) $J_1(t) = (1+\varepsilon(t))\alpha t^{\omega}$, where $\alpha > 0$, $\omega > -\frac{1}{2}$, and $\varepsilon(t)$ is a continuous function satisfying $\varepsilon(t) \to 0$ as $t \to +\infty$.

The case where $\varepsilon(t) \equiv 0$ was considered in Chapter 2 and the particular case where, additionally to that, we had time independent input $\omega = 0$ was first studied by da Costa et al. [21]. Both these studies contributed to the rigorous justification of formal results by Wattis [69]. The case of polynomial-like monomer input term considered here has never been investigated before, not even at a formal level.

The main idea behind our approach is that, since $\varepsilon(t) \to 0$ as $t \to +\infty$, the presence of this perturbation term should not be felt at large times, and so the results in Chapter 2 should remain valid. In particular, we should expect the following to hold true:

Theorem 3.1 *Assume* ***(H1)*** *holds, and let* (c_0, c_1) *be any solution of*

$$\begin{cases} \dot{c}_0 = J_1(t) - c_0c_1 \\ \dot{c}_1 = J_1(t) - c_0c_1 - c_1^2. \end{cases} \tag{3.2}$$

Then, as $t \to +\infty$, *we have*

(i) $\left(\dfrac{1+2\omega}{3\alpha^2}\right)^{\frac{1}{3}} t^{-\frac{1+2\omega}{3}} c_0(t) \longrightarrow 1,$

(ii) $\left(\dfrac{3}{(1+2\omega)\alpha}\right)^{\frac{1}{3}} t^{\frac{1-\omega}{3}} c_1(t) \longrightarrow 1.$

As mentioned in the Introduction it is convenient to introduce the new time scale defined by (1.8):

$$\varsigma(t) \coloneqq \varsigma_0 + \int_{t_0}^{t} c_1(s)\,\mathrm{d}s,$$

where ς_0 is a positive constant, and to consider the new phase variables defined by (1.9):

$$\widetilde{c}_j(\varsigma) := c_j(t(\varsigma)),$$

where $t(\varsigma)$ is the inverse function of $\varsigma(t)$. Using the result in Theorem 3.1 (**ii**) and translating it to the phase variable $\widetilde{c}_1$ in the time scale ς we can use it in expression (1.10) for $\widetilde{c}_j(\varsigma)$:

$$\widetilde{c}_j(\varsigma) = e^{-\varsigma} \sum_{k=2}^{j} \frac{\varsigma^{j-k}}{(j-k)!} c_k(0) + \frac{1}{(j-2)!} \int_0^{\varsigma} \widetilde{c}_1(\varsigma - s) s^{j-2} e^{-s}\, \mathrm{d}s,$$

to obtain information on the long time behaviour of the $j-$cluster concentration $\widetilde{c}_j(\varsigma)$, in particular concluding that the behaviour in Theorem 3.1 (**ii**) holds true with $c_1(t)$ substituted by $c_j(t)$, for every $j \geq 1$.

Furthermore, and more interesting from a dynamical point of view, the information about $\widetilde{c}_1$ allows us to use (1.10) in order to prove that solutions to (3.1) converge to the similarity profiles in Chapter 2 as $j, t \to +\infty$.

In fact, defining

$$Q(\omega) := \left(\frac{3}{(1+2\omega)\alpha}\right)^{\frac{1}{2+\omega}} \left(\frac{2+\omega}{3}\right)^{r} \qquad \text{where } r := \frac{1-\omega}{2+\omega},$$

the following holds true:

Theorem 3.2 *Assume (**H1**). Let* (c_j) *be any solution of* (3.1) *with initial data* $c_j(0) \in \ell_1$. *Let* $\varsigma(t)$ *and* $\widetilde{c}_j(\varsigma)$ *be as above. Then the statements of Theorems 2.13 and 2.14 still hold.*

The proof of Theorem 3.2 is in no way dependent on the function $\varepsilon(t)$, since it is based on the analysis of (1.10) which does not involve any information concerning the input of monomers other than that carried over by the $\widetilde{c}_1(\varsigma)$. The proofs are similar to the corresponding ones in Chapter 2.

We now concentrate on the proof of Theorem 3.1, and specially in those parts of the proof that differ from the proof of the corresponding result in Chapter 2.

3.2 On the proof of Theorem 3.1

The general plan of the proof of Theorem 3.1 is analogous to that of the corresponding theorem in Chapter 2.

We start by defining a new time scale $t \mapsto \tau$ so that $\frac{\mathrm{d}\tau}{\mathrm{d}t} = \left(\frac{3\alpha^2}{1+2\omega}\right)^{1/3} t^{\frac{1+2\omega}{3}}$. We then let

$$\begin{cases} x(\tau) := \left(\dfrac{3}{(1+2\omega)\alpha}\right)^{\frac{1}{3}} t(\tau)^{\frac{1-\omega}{3}} c_1(t(\tau)) \\ y(\tau) := \left(\dfrac{1+2\omega}{3\alpha^2}\right)^{\frac{1}{3}} t(\tau)^{-\frac{1+2\omega}{3}} c_0(t(\tau)), \end{cases}$$

and use (**H1**), so that system (3.2) becomes

$$\begin{cases} x' = (1+\varepsilon(\tau) - xy) - A\tau^{-\frac{1}{2}}x^2 + B\tau^{-1}x \\ y' = (1+\varepsilon(\tau) - xy)\cdot A\tau^{-\frac{1}{2}} - A^2\tau^{-1}y\,, \end{cases} \tag{3.3}$$

where A and B are defined as in (2.4), and, in order not to overload the notation we keep denoting by $\varepsilon(\cdot)$ the function $\varepsilon(t(\cdot))$. To prove Theorem 3.1 it is sufficient to prove that all non-negative solutions (x, y) to system (3.3) satisfy $(x(\tau), y(\tau)) \to (1, 1)$ as $\tau \to +\infty$. In order to obtain this result we may start by proving the positivity and relative boundedness of the solution vector (meaning that y [resp. x] is bounded iff x [resp. y] is bounded away from zero); these are proved exactly as in Lemmas 2.2 and 2.3.

Next we need to prove the boundedness of the orbit $(x(\tau), y(\tau))$ when $\tau \to +\infty$. This is where things start to look a bit different from what happened in Chapter 2. By (**H1**) we need only to consider times large enough so that $|\varepsilon(\tau)| < \epsilon$ (where ϵ can be chosen arbitrarily small) if $\tau > T_\epsilon$ say. Observe that from the $y-$equation in system (3.3) we have $y' < 0$ if $(x, y) \in \Omega_\epsilon^- := \{1 + \epsilon - xy < 0\}$ and so $y(\tau)$ can only escape to $+\infty$ if the orbit ultimately remains in $\Omega_\epsilon^+ := \mathbb{R}^{2+} \setminus \Omega_\epsilon^-$. But to show that this cannot occur an argument similar to the one used in step 1 of the proof of Lemma 2.4 is enough, with the natural changes of the geometrical setting provided by Figure 3.1.

Without loss of generality we can assume the initial point P_{τ_0} is in Ω_ϵ^+ with $\tau_0 >$

$\max\{1, \tau_{L,\epsilon}\}$, where $\tau_{L,\epsilon} := \left(2\left(\frac{L^2}{1+\epsilon} - \frac{1+\epsilon}{L}\right)\right)^{-2}$, and $L > \sqrt[3]{1+\epsilon}$ is the ordinate of P_{τ_0}.

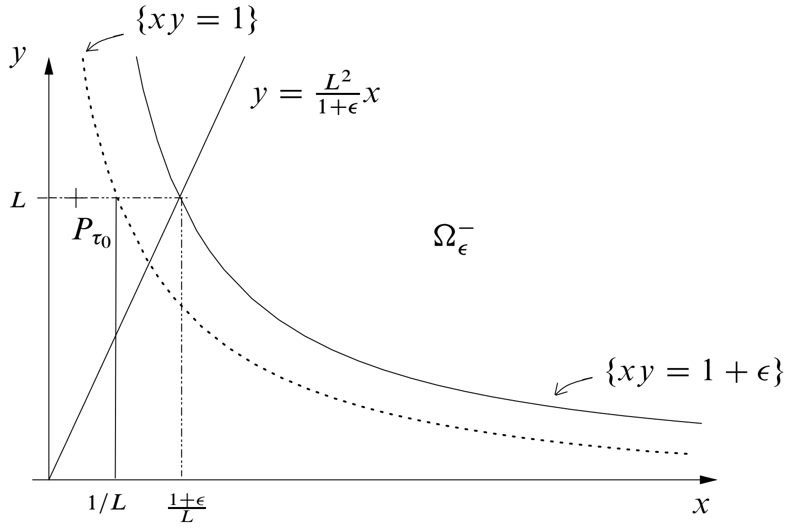

Figure 3.1: Geometric setting for the analysis of the possibility of an orbit starting at P_{τ_0} to escape to infinity due to $y(\tau) \to +\infty$ as $\tau \to +\infty$.

Using the differential equations (3.3) we easily conclude that, in the region of Figure 3.1 where $y > \frac{L^2}{1+\epsilon}x$ and $x < \frac{1+\epsilon}{L}$, the following holds true:

$$y' < \left(x' + \left(\frac{1+\epsilon}{L} - \frac{L^2}{1+\epsilon} - BA^{-1}\tau^{-\frac{1}{2}}\right) xA\tau^{-\frac{1}{2}}\right) A\tau^{-\frac{1}{2}} < x'A\tau^{-\frac{1}{2}} < x',$$

where the last two inequalities hold true for all $\tau > \tau_0 > \max\{1, \tau_{L,\epsilon}\}$. But this is the same inequality as in Chapter 2 and from this uniform bound on the slope $\frac{\mathrm{d}y}{\mathrm{d}x}$ of the orbit, and the behaviour of y' in Ω_ϵ^-, we immediately conclude $y(\tau)$ cannot diverge to $+\infty$ as $\tau \to +\infty$.

The analysis of what happens for large x, and in particular the impossibility for $x(\tau)$ to become unbounded as $\tau \to +\infty$, requires in the present case an altogether different approach to that used in Chapter 2. There we used a method based on the behaviour of the orbit on the level sets of an auxiliary function approaching the (fixed) hyperbola $\{xy = 1\}$. This essentially meant a change to a kind of moving reference system. Now this method is not likely to work in this case without unnatural (and unreasonable) restrictions on the perturbation $\varepsilon(\tau)$. Hence, we now use a novel approach. Let us suppose an orbit of (3.3) satisfies $x(\tau_n) \to +\infty$, for some sequence τ_n such that $\tau_n \to +\infty$. Then this orbit must eventually leave $\Omega^-_{\varepsilon(\tau)} := \{1 + \varepsilon(\tau) < xy\}$ since otherwise we would have

$$x' = 1 + \varepsilon(\tau) - xy - A\tau^{-\frac{1}{2}}x^2 + B\tau^{-1}x < A\tau^{-\frac{1}{2}}x(-x + BA^{-1}\tau^{-\frac{1}{2}}) < 0,$$

for all sufficiently large τ, a contradiction. An easy argument, similar to the one used in Chapter 2 shows that for all large times, the orbit cannot enter $\Omega^-_{\varepsilon(\tau)}$. Furthermore, for all large enough τ it easily follows from (3.3) that

$$A\tau^{-\frac{1}{2}}x' - y' = -\left(x - \frac{y}{x} - \frac{B}{A}\tau^{-\frac{1}{2}}\right)A^2\tau^{-1}x < 0. \tag{3.4}$$

Now suppose $x' > 0$ for all sufficiently large τ. From (3.4) this entails $y' > A\tau^{-\frac{1}{2}}x' > 0$, which leads to a contradiction since the orbit would eventually enter $\Omega^-_{\varepsilon(\tau)}$ for large enough τ. So, in order to have $x(\tau_n) \to +\infty$ as $\tau_n \to +\infty$, there must exist an infinite sequence of time intervals in which $x' < 0$, and in at least some subintervals we must have $y' < 0$, since otherwise the orbit would enter in $\Omega^-_{\varepsilon(\tau)}$ anyway, or it would remain bounded. The differential inequality (3.4) implies that the situation depicted in Figure 3.2 is essentially the only possible one.

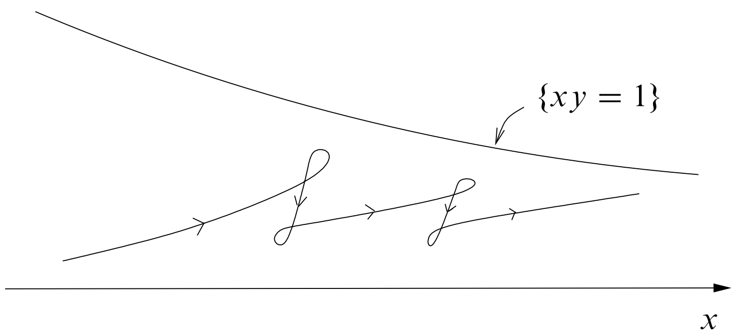

Figure 3.2: The only possibility to have an orbit with $x(\tau) \to +\infty$ as $\tau \to +\infty$. Note that (3.4) implies that $y'(\tau) > 0$ whenever $x'(\tau) > 0$.

Observe that for the orbit to decrease in the y component it must first start decreasing in x, since (3.4) forces $A\tau^{-\frac{1}{2}}x' < y'$. This explains why the orbit should essentially be as shown in Figure 3.2, with that type of self-crossings. But this is impossible. To see why, let us concentrate our attention in Figure 3.3, where a portion of the orbit of Figure 3.2 at the start of one of its $y-$descent sections is enlarged showing a self-crossing P that occurs at two instants, τ_1 and τ_2, with $\tau_1 < \tau_2$.

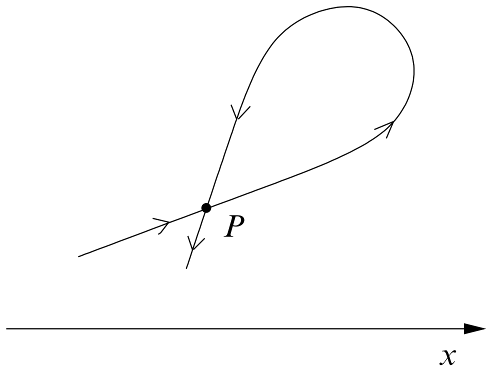

Figure 3.3: Situation described in the text for a self-intersecting point of an orbit having $x(\tau) \to +\infty$ as $\tau \to +\infty$.

At τ_1 we have $x'(\tau_1) > 0$ and (3.4) gives $\frac{\mathrm{d}y}{\mathrm{d}x}(\tau_1) > A\tau_1^{-\frac{1}{2}}$, and at τ_2 we have $x'(\tau_2) < 0$, which implies that (3.4) now means $\frac{\mathrm{d}y}{\mathrm{d}x}(\tau_2) < A\tau_2^{-\frac{1}{2}}$. But since $\tau_2 > \tau_1$, we conclude that $\frac{\mathrm{d}y}{\mathrm{d}x}(\tau_2) < A\tau_2^{-\frac{1}{2}} < A\tau_1^{-\frac{1}{2}} < \frac{\mathrm{d}y}{\mathrm{d}x}(\tau_1)$ in contradiction with what we concluded above should happen at a self-crossing point. This concludes the proof that (x, y) must remain bounded (and bounded away from zero).

With this information we can now start to identify the orbit's $\omega-$limit, by showing that the $\omega-$limit set of any orbit is contained in the hyperbola $\{xy = 1\}$. In order to obtain this result we consider an auxiliary function $h(\tau) \coloneqq x(\tau)y(\tau)$; as was done in Chapter 2, if (x, y) satisfies (3.4) then $h(\tau)$ satisfies the following linear (in h) differential equation:

$$h'(\tau) = (1 + \varepsilon(\tau))\left(y + A\tau^{-\frac{1}{2}}x\right) - \alpha(\tau)h(\tau), \tag{3.5}$$

where

$$\alpha(\tau) \coloneqq y(\tau) + 2A\tau^{-\frac{1}{2}}x(\tau) + (A^2 - B)\tau^{-1}. \tag{3.6}$$

Using the variation of constants formula we can write the solution to (3.5) as

$$h(\tau) = h(\tau_0)e^{-\int_{\tau_0}^{\tau}\alpha(s)\,\mathrm{d}s} + \int\limits_{\tau_0}^{\tau}(1+\varepsilon(s))\left(y(s)+\frac{Ax(s)}{s^{\frac{1}{2}}}\right)e^{-\int_s^{\tau}\alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s. \quad (3.7)$$

We need to show that $h \to 1$. Using Equation (3.7), we show that the first term goes to zero and that the second one goes to 1, as $\tau \to +\infty$. Since y is bounded away from zero, there exists a constant $L_y > 0$ such that $y(\tau) \geqslant L_y$, and so

$$\begin{aligned} e^{-\int_{\tau_0}^{\tau}\alpha(s)\,\mathrm{d}s} &= \left(\frac{\tau_0}{\tau}\right)^{A^2-B} e^{-\int_{\tau_0}^{\tau}\left(y(s)+2As^{-1/2}x(s)\right)\mathrm{d}s} \\ &\leqslant \left(\frac{\tau_0}{\tau}\right)^{A^2-B} e^{-\int_{\tau_0}^{\tau}y(s)\,\mathrm{d}s} \leqslant \left(\frac{\tau_0}{\tau}\right)^{A^2-B} e^{-L_y(\tau-\tau_0)}, \end{aligned}$$

showing that the first term in (3.7) goes to zero (exponentially) as $\tau \to +\infty$.

Next we show that the second term in (3.7) goes to 1 as $\tau \to +\infty$. Using again the fact that $|\varepsilon(\tau)| < \epsilon$ if $\tau > T_\epsilon$, we can estimate the second term in (3.7) by splitting the integral as $\int_{\tau_0}^{\tau} = \int_{\tau_0}^{T_\epsilon} + \int_{T_\epsilon}^{\tau}$. The integral over $[\tau_0, T_\epsilon]$ converge to zero as $\tau \to \infty$, since we are integrating an exponentially decaying function on a fixed compact interval. On $[T_\epsilon, \tau]$, we can write

$$(1-\epsilon)\int\limits_{T_\epsilon}^{\tau} g(s,\tau)\,\mathrm{d}s \leqslant \int\limits_{T_\epsilon}^{\tau}(1+\varepsilon(\tau))g(s,\tau)\,\mathrm{d}s \leqslant (1+\epsilon)\int\limits_{T_\epsilon}^{\tau} g(s,\tau)\,\mathrm{d}s,$$

where $g(s,\tau) \coloneqq \left(y(s) + As^{-\frac{1}{2}}x(s)\right)e^{-\int_s^{\tau}\alpha(\theta)\,\mathrm{d}\theta}$. Using the results from the proof of Lemma 2.7 we have $\int\limits_{T_\epsilon}^{\tau} g(s,\tau)\,\mathrm{d}s \to 1$ as $\tau \to +\infty$ and this completes the proof of the convergence of h.

Inspired by the approach in Chapter 2, the next steps to locate the $\omega-$limit set of the orbits will be, first to prove that, for each orbit, its $\omega-$limit set, (which, by the previous result about h, is an arc of the hyperbola $\{xy = 1\}$) contains the point $(1,1)$, and second to conclude that that arc degenerates into the single point $(1,1)$. Resorting, as in Chapter 2, to the auxiliary function $b(\tau) \coloneqq y(\tau) - A\tau^{-\frac{1}{2}}x(\tau)$ we discover that its dynamics is governed by the differential equation

$$b' = (-b + x^2)A^2\tau^{-1},$$

which is exactly the same as in Chapter 2, i.e., the dynamic behaviour of b is independent of $\varepsilon(\tau)$ and so all the results proved when $\varepsilon(\tau) \equiv 0$ in Lemmas 2.8 and 2.9 remain valid for the present case. Hence the two steps referred to above can be completed and the proof of Theorem 3.1 is achieved.

3.3 Remarks on the Rate of Convergence

A question that naturally comes to mind at this point is to ask at what rate do solutions converge to their final states. Whatever the mathematical sense we may give this question, its elucidation is bound to include the long time behaviour of the quantity $J_1(t) - c_0(t)c_1(t)$ (if we are thinking in terms of solutions to (3.2)) or of $1 + \varepsilon(\tau) - x(\tau)y(\tau)$ (if we are considering solutions to (3.3)).

In the case $\varepsilon(t) \equiv 0$ treated in Chapter 2 the assumptions in Theorem 3.1 are sufficient to prove that

$$\frac{1 - x(\tau)y(\tau)}{A\tau^{-\frac{1}{2}}} \longrightarrow 1 \quad \text{as} \quad \tau \to +\infty, \tag{3.8}$$

and hence to have, as $t \to +\infty$,

$$\left(\frac{3}{(1+2\omega)\alpha}\right)^{\frac{2}{3}} t^{2\frac{1-\omega}{3}}(\alpha t^{\omega} - c_0(t)c_1(t)) \longrightarrow 1. \tag{3.9}$$

In order to obtain similar results when $\varepsilon(t)$ is not identically zero we need further assumptions on this function besides those stated in (**H1**), namely we need to assume something about its decay rate to zero. The following hypothesis is sufficient to ensure (3.9) holds also in this case.

(H2) The function $\varepsilon(t)$ is continuously differentiable and, as $t \to +\infty$, satisfies $\varepsilon(t) = o(t^{-\frac{2+\omega}{3}})$, and $\dot{\varepsilon}(t) = o(t^{-\frac{1-\omega}{3}})$.

Remark 3.3 *Observe that, by the definition of the time scale τ given in page 78, we easily conclude that $\tau^{-\frac{1}{2}} = \mathcal{O}(t^{-\frac{2+\omega}{3}})$, from which it follows that we must have $\varepsilon(\tau) = o(\tau^{-\frac{1}{2}})$ and also $\varepsilon'(\tau) = \dot{\varepsilon}\frac{\mathrm{d}t}{\mathrm{d}\tau} = o(\tau^{-\frac{1}{2}})$.*

The way the proof of (3.8) proceeds is analogous to the proof of Lemmas 2.7 and 2.10: in the present case the expression for $(1 - h(\tau))/\tau^{-1/2}$ has the additional additive contribution coming from the perturbation term $\varepsilon(\tau)$, namely

$$-\tau^{\frac{1}{2}} \int_{\tau_0}^{\tau} \varepsilon(s) \left(y(s) + As^{-\frac{1}{2}} x(s) \right) e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta} \,\mathrm{d}s, \tag{3.10}$$

with $\alpha(\cdot)$ the function defined in (3.6). The analysis of (3.10) proceeds as follows: first write it as

$$-\tau^{\frac{1}{2}}\int_{\tau_0}^{\tau} \varepsilon(s)\alpha(s)e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s + \tau^{\frac{1}{2}}\int_{\tau_0}^{\tau} \varepsilon(s) \left(\frac{Ax(s)}{s^{1/2}} + \frac{A^2 - B}{s} \right) e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s, \tag{3.11}$$

then observe that the first integral in (3.11) can be written as follows

$$\begin{aligned} \int_{\tau_0}^{\tau} \varepsilon(s)\alpha(s)e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s &= e^{-\int_{\tau_0}^\tau \alpha(s)\,\mathrm{d}s} \int_{\tau_0}^{\tau} \varepsilon(s)\frac{\mathrm{d}}{\mathrm{d}s}\left(e^{\int_{\tau_0}^s \alpha(\theta)\,\mathrm{d}\theta} \right) \mathrm{d}s \\ &= \varepsilon(\tau) - \varepsilon(\tau_0)e^{-\int_{\tau_0}^\tau \alpha(s)\,\mathrm{d}s} - \int_{\tau_0}^{\tau} \varepsilon'(s)e^{\int_{\tau_0}^s \alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s, \end{aligned} \tag{3.12}$$

where the last equality was obtained using integration by parts. Now, the estimates developed in the proof of Lemma 2.7 together with the assumption (**H2**), in the version presented in Remark 3.3, allow us to control the second integral in (3.11) and the right-hand side of (3.12), thus completing the proof of (3.8), and thus also the corresponding version of (3.9) that now reads as

$$\left(\frac{3}{(1+2\omega)\alpha} \right)^{\frac{2}{3}} t^{2\frac{1-\omega}{3}} \left((1+\varepsilon(t))\,\alpha t^\omega - c_0(t)c_1(t) \right) \xrightarrow[t\to+\infty]{} 1.$$

The results in this Chapter include the cases where the input rate of monomers can have a rather erratic behaviour (albeit continuous and asymptotically approaching a power αt^{ω}, with $\omega > -\frac{1}{2}$), providing further indication of the robustness of the convergence to self-similar behaviour in this class of systems.

Chapter 4

The transition input: $\omega = -1/2$

4.1 Introduction

In this chapter we consider the case where $\omega = -1/2$, and hence the system can be written as

$$\begin{cases} \dot{c}_1 = \alpha t^{-1/2} - c_1^2 - c_1 \displaystyle\sum_{j=1}^{\infty} c_j \\ \dot{c}_j = c_1 c_{j-1} - c_1 c_j, \quad j \geqslant 2. \end{cases} \tag{4.1}$$

As in the previous chapters the equations governing both the monomer dynamics and the total number of clusters are actually a nonautonomous bidimensional $(c_0, c_1)-$system

$$\begin{cases} \dot{c}_0 = \alpha t^{-1/2} - c_0 c_1 \\ \dot{c}_1 = \alpha t^{-1/2} - c_0 c_1 - c_1^2 \,. \end{cases} \tag{4.2}$$

According to [69] we expect solutions (c_0, c_1) of system (4.2) with $\alpha > 0$ to behave like

$$\begin{cases} c_0(t) \sim \left(3\alpha^2\right)^{\frac{1}{3}} (\log t)^{\frac{1}{3}} \\ c_1(t) \sim \left(\dfrac{\alpha}{3}\right)^{\frac{1}{3}} t^{-\frac{1}{2}} (\log t)^{-\frac{1}{3}}\,, \text{ as } t \to +\infty. \end{cases}$$

This suggests the functions

$$\begin{cases} C_0(t) \coloneqq & \left(3\alpha^2\right)^{-\frac{1}{3}} (\log t)^{-\frac{1}{3}}\, c_0(t) \\ C_1(t) \coloneqq & \left(\dfrac{\alpha}{3}\right)^{-\frac{1}{3}} t^{\frac{1}{2}}\, (\log t)^{\frac{1}{3}}\, c_1(t), \end{cases} \tag{4.3}$$

might both be expected to converge to 1 as $t \to +\infty$, and reciprocally, if this happens then c_0 and c_1 behave as stated.

To prove this convergence behaviour of (C_0, C_1) we need an equation for its evolution. We begin by differentiating (4.3), and then replacing it into system (4.2).

We then change the time scale $t \mapsto \tau$ by letting

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \left(3\alpha^2\right)^{\frac{1}{3}} (\log t)^{\frac{1}{3}}\,. \tag{4.4}$$

In order to have a well defined change of variables we must have $\dfrac{\mathrm{d}\tau}{\mathrm{d}t} > 0$, and hence $t > 1$.

Defining

$$\begin{cases} x(\tau) \coloneqq C_1(t(\tau)) \\ y(\tau) \coloneqq C_0(t(\tau))\,, \end{cases}$$

and denoting $\dfrac{\mathrm{d}}{\mathrm{d}\tau}$ by $'$ we finally obtain an equation for (x, y):

$$\begin{cases} x' = 1 - xy - \widetilde{c}(\tau)x^2 + \widetilde{d}(\tau)x \\ y' = \widetilde{c}(\tau)(1 - xy - \widetilde{c}(\tau)y)\,, \end{cases} \tag{4.5}$$

where

$$\widetilde{c}(\tau) = c\,(t(\tau)) \coloneqq (9\alpha)^{-1/3}\, (t(\tau))^{-1/2}\, (\log t(\tau))^{-2/3}\,, \tag{4.6}$$

and

$$\widetilde{d}(\tau) = d\,(t(\tau)) \coloneqq \widetilde{c}^{\,2}(\tau)\left(\frac{3}{2}\log t(\tau) + 1\right). \tag{4.7}$$

In order to simplify the notation and to make clear what variable we are working with, we define $\tilde{\ell}(\tau) \coloneqq 3/2 \log t(\tau) + 1$, and so (4.7) now reads

$$\tilde{d}(\tau) = \tilde{c}^2 \tilde{\ell}(\tau), \tag{4.8}$$

and hence we can write (4.5) in the simpler form:

$$\begin{cases} x' = 1 - xy - \tilde{c}(\tau)\, x^2 + \tilde{c}^2(\tau)\, \tilde{\ell}(\tau) x \\ y' = \tilde{c}(\tau)(1 - xy - \tilde{c}(\tau) y)\,. \end{cases} \tag{4.9}$$

Due to the fact that the time scale change (4.4) doesn't give us an explicit expression for t as a function of τ, as happened for $\omega > -1/2$, we need some results about the asymptotic relationship between the two time scales.

4.2 Relationship between the two time scales

A few results we will use are the following

(i) Since $t > 1$ implies that $\dfrac{d\tau}{dt} = \left(3\alpha^2\right)^{\frac{1}{3}} (\log t)^{\frac{1}{3}} > 0$, we have that $\tau(t)$ is a strictly increasing function of t. And clearly, also that $t(\tau)$ is a strictly increasing function of τ.

(ii) This allows us to conclude that $\tau \to +\infty$ as $t \to +\infty$, and vice versa $t \to +\infty$ as $\tau \to +\infty$

(iii) To get a better estimate on the asymptotic behaviour of $\tau(t)$ we can integrate (4.4) by parts to obtain
$\tau(t) = (3\alpha^2)^{\frac{1}{3}} \left(t(\log t)^{1/3} - \frac{1}{3} \int_1^t (\log s)^{-2/3}\, ds \right)$.

(iv) The last expression can be integrated by parts again allowing us to write
$\tau(t) = t(3\alpha^2 \log t)^{\frac{1}{3}} \left(1 + \mathcal{O}(\frac{1}{\log t}) \right)$ as $t \to +\infty$.

(v) Since what we need is the behaviour as $\tau \to +\infty$, we write the last expression as $\tau(t) = t(3\alpha^2 \log t)^{\frac{1}{3}} \left(1 + o(1)\right)$ as $t \to +\infty$.

(vi) This allows us to write $t(\tau) = \tau(3\alpha^2 \log \tau)^{-\frac{1}{3}} (1 + o(1))$ as $\tau \to +\infty$.

(vii) And also to get an estimate on the size of $\log(t(\tau)) = \log\left(\tau(3\alpha^2 \log \tau)^{-\frac{1}{3}} (1 + o(1))\right)$, as $\tau \to +\infty$.

(viii) From here we can get an estimate on the size of

$$\begin{aligned} \widetilde{c}(\tau) = c\,(t(\tau)) &= (9\alpha)^{-1/3}\ (t(\tau))^{-1/2}\ (\log t(\tau))^{-2/3} \\ &= (9\alpha)^{-1/3}\Bigg(\left(\tau(3\alpha^2 \log \tau)^{-\frac{1}{3}} (1 + o(1))\right)^{-1/2} \\ &\quad \times \left(\log\left(\tau(3\alpha^2 \log \tau)^{-\frac{1}{3}} (1 + o(1))\right)\right)^{-2/3}\Bigg), \text{ as } \tau \to +\infty. \end{aligned} \tag{4.10}$$

(ix) We have $\widetilde{c}(\tau) > 0$, since $(9\alpha)^{-1/3}\ (t(\tau))^{-1/2}\ (\log t(\tau))^{-2/3} > 0$.

(x) For all our purposes we will use the following relations:

$$\begin{aligned} \tau(t) &\sim t(\log t)^{1/3} && \text{as } \tau \to +\infty,\ t \to +\infty, \\ t(\tau) &\sim \tau(\log \tau)^{-1/3} && \text{as } \tau \to +\infty,\ t \to +\infty, \\ \log(t(\tau)) &\sim \log \tau && \text{as } \tau \to +\infty,\ t \to +\infty, \\ c(t(\tau)) &\sim (\tau \log \tau)^{-1/2} && \text{as } \tau \to +\infty,\ t \to +\infty. \end{aligned}$$

(xi) From the definition of $\widetilde{d}$ (4.8), we have that $\widetilde{d} \sim \tau^{-1}$.

4.3 The bidimensional system

Since we are only interested in non-negative solutions to (4.2), by *solution* we shall mean *non-negative solution*.

This section's main result is

Theorem 4.1 *Let* $\alpha > 0$*, and* (c_0, c_1) *be any solution of* (4.2)*. Then, as* $t \to +\infty$*, we have*

(i) $\left(3\alpha^2\right)^{-\frac{1}{3}} (\log t)^{-\frac{1}{3}}\, c_0(t) \longrightarrow 1$,

(ii) $\left(\frac{\alpha}{3}\right)^{-\frac{1}{3}} t^{\frac{1}{2}} (\log t)^{\frac{1}{3}} c_1(t) \longrightarrow 1$,

(iii) $\left(\frac{3}{\alpha}\log t\right)^{\frac{2}{3}} t(\alpha t^{-\frac{1}{2}} - c_0 c_1) \longrightarrow 1$.

The proof of this theorem is based on a series of Lemmas, in the same spirit as in Chapter 2. The statements of the lemmas used are the same, and the proofs only differ because we have a log term and also because we do not have an explicit expression for $\tau(t)$.

As a result we will focus on the main differences between these proofs and those of Chapter 2.

We start by showing that non-negative solutions to (4.9) remain non-negative as $\tau \to +\infty$.

Lemma 4.2 *The first quadrant* $\{x \geqslant 0, y \geqslant 0\}$ *is positively invariant for* (4.9).

Proof. In $\{(0, y) : y \geqslant 0\}$ we have, from (4.9), $x' = 1 > 0$ for all τ, and so solutions cannot cross the $y-$axis. In $\{(x, 0) : x \geqslant 0\}$ we have $y' = c(t(\tau)) > 0$, and so solutions cannot cross the $x-$axis either. ∎

Our next result shows how the x and y boundedness are related.

Lemma 4.3 *Let* (x, y) *be any solution to* (4.9) *with positive initial data. Then, the following equivalences hold:*

(i) y *is bounded* $\Longleftrightarrow$ x *is bounded away from zero.*

(ii) y *is bounded away from zero* $\Longleftrightarrow$ x *is bounded.*

Proof. We start by proving that

$$y \text{ is bounded } \Longrightarrow x \text{ is bounded away from zero.}$$

Since by hypothesis y is bounded , there exists $U_y > 0$ such that $y \leqslant U_y$. Then from the first equation in (4.9) we derive

$$x' = 1 - xy - \tilde{c}x^2 + \tilde{d}x$$

$$
\begin{aligned}
&\geqslant 1 - xU_y - \widetilde{c}x^2 + \widetilde{d}x \\
&> 1 - xU_y - \widetilde{c}x^2, \ \text{since } \widetilde{d} > 0.
\end{aligned} \tag{4.11}
$$

We study separately the cases $\tau_0 > \widehat{\tau}$ and $\tau_0 < \widehat{\tau}$. When $\tau \in (\tau_0, \widehat{\tau}]$, in case this set is not empty, the solution is necessarily bounded away from zero since otherwise there would exist at least one point $\tau_1 \in (\tau_0, \widehat{\tau}]$ such that $\widetilde{x}(\tau_1) = 0$, and from (4.9) we would have $\widetilde{x}'(\tau_1) = 1 > 0$ and so $\widetilde{x}$ would have to be negative for $\tau < \tau_1$. But according to Lemma 4.2 this cannot happen, and so from this contradiction we conclude that in $(\tau_0, \widehat{\tau}]$, x is bounded away from zero.

When $\tau_0 > \widehat{\tau}$ we can use (4.11) and the fact that since $\widetilde{c}$ is decreasing we have $\widetilde{c}(\tau) \leqslant \widetilde{c}(\tau_0)$ to write $x' > 1 - U_y x - \widetilde{c}(\tau_0)x^2$. By studying the behaviour of the solution u of the differential equation $u' = 1 - U_y u - \widetilde{c}(\tau_0)u^2$ and using standard results for differential inequalities we can conclude that x is bounded away from zero.

In order to prove that

$$
x \text{ is bounded away from zero} \Longrightarrow y \text{ is bounded},
$$

we can use the $y-$equation in (4.9) to write $y' = \widetilde{c}\,(-y(x + \widetilde{c}) + 1)$, which is a linear equation, and hence by changing time scales $\tau \longmapsto \theta$ in such a way that $\dfrac{\mathrm{d}\theta}{\mathrm{d}\tau} = \widetilde{c}(\tau)$ it becomes

$$
\frac{\mathrm{d}y}{\mathrm{d}\theta} = -y\left(x + \widetilde{\widetilde{c}}\right) + 1, \tag{4.12}
$$

where $\widetilde{\widetilde{c}}(\theta) \coloneqq \widetilde{c}(\tau(\theta))$.

This is a linear equation with differentiable coefficients and hence it is bounded for θ in bounded sets, and so to prove y is bounded we only need to check what happens when $\theta \longrightarrow +\infty$. Since we have a linear equation, we can use the variation of constants formula to write the solution of (4.12) as

$$
y(\theta) = y(\theta_0)e^{-\int_{\theta_0}^{\theta}\left(x(\sigma)+\widetilde{\widetilde{c}}(\sigma)\right)\mathrm{d}\sigma} + \int_{\theta_0}^{\theta} e^{-\int_{s}^{\theta}\left(x(\sigma)+\widetilde{\widetilde{c}}(\sigma)\right)\mathrm{d}\sigma}\mathrm{d}s. \tag{4.13}
$$

Letting $L_x > 0$ be a lower bound for x, we have

$$e^{-\int_{\theta_0}^{\theta}\left(x(\sigma)+\widetilde{\widetilde{c}}(\sigma)\right)\mathrm{d}\sigma} \leqslant e^{-\int_{\theta_0}^{\theta} x(\sigma)\,\mathrm{d}\sigma} e^{-\int_{\theta_0}^{\theta} \widetilde{\widetilde{c}}(\sigma)\,\mathrm{d}\sigma} \leqslant e^{-L_x(\theta-\theta_0)} \cdot 1 \xrightarrow[\theta\to+\infty]{} 0,$$

and also

$$\int_{\theta_0}^{\theta} e^{-\int_s^{\theta}\left(x(\sigma)+\widetilde{\widetilde{c}}(\sigma)\right)\mathrm{d}\sigma}\mathrm{d}s \leqslant \int_{\theta_0}^{\theta} e^{-L_x(\theta-s)}\,\mathrm{d}s = \frac{1-e^{-L_x(\theta-\theta_0)}}{L_x} < \frac{1}{L_x},$$

and combining these two results with (4.13) gives the boundedness of y.

This concludes the proof of part (**i**).

In order to prove (**ii**) we start by proving

$$y \text{ is bounded away from zero} \Longrightarrow x \text{ is bounded}.$$

Since y is bounded away from zero there is a constant $L_y > 0$ such that $L_y \leqslant y$ and so from the $x-$equation in (4.9) we obtain:

$$x' = 1 - xy - \widetilde{c}x^2 + \widetilde{d}x \leqslant 1 - x\left(L_y + \widetilde{c}x - \widetilde{d}\right) < 1 - x(L_y - \widetilde{d}) \quad (4.14)$$

since $\widetilde{c} > 0$.

Now since $\widetilde{d}(\tau) \sim \tau^{-1} \xrightarrow[\tau\to+\infty]{} 0$, there is a $\hat{\tau}$ such that for $\tau > \hat{\tau}$ we have $\widetilde{d}(\tau) < L_y/2$.

And so for $\tau > \hat{\tau}$ (4.14) gives $x' < 1 - L_y/2x$ and hence

$$\begin{aligned} x(\tau) &\leqslant x(\tau_0)e^{\frac{-L_y(\tau-\tau_0)}{2}} + \int_{\tau_0}^{\tau} e^{\frac{-L_y(\tau-s)}{2}}\,\mathrm{d}s \\ &= \mathcal{O}(1) + \frac{2\left(1-e^{\frac{-L_y(\tau-\tau_0)}{2}}\right)}{L_y} \\ &< \mathcal{O}(1) + \frac{2}{L_y}, \text{ as } \tau \to +\infty, \end{aligned}$$

by standard results in differential inequalities, from which we conclude the boundedness of x for $\tau > \hat{\tau}$.

For $\tau \in (\tau_0, \hat{\tau})$, we have from (4.14) that $x' < 1 - x(L_y - \widetilde{d})$ and letting u be the solution of $u' = 1 - x(L_y - \widetilde{d})$ satisfying the same initial condition $x(\tau_0) = u(\tau_0)$ we have $x(\tau) \leqslant u(\tau)$ which is a bounded function on the compact set $[\tau_0, \hat{\tau}]$.

In the last part of this Lemma we show that

$$x \text{ is bounded } \Longrightarrow y \text{ is bounded away from zero.}$$

As before, since x is bounded, there is an upper bound $U_x \geqslant x$, and so using (4.13) we have

$$\begin{aligned}
y(\theta) &\geqslant \int_{\theta_0}^{\theta} e^{-U_x(\theta-s)} e^{-\int_s^\theta \widetilde{\widetilde{c}}(\sigma)\,\mathrm{d}\sigma}\,\mathrm{d}s \\
&\geqslant \int_{\theta_0}^{\theta} e^{-\left(U_x+\widetilde{\widetilde{c}}(\theta_0)\right)(\theta-s)}\,\mathrm{d}s \qquad\qquad \text{(since } c \text{ is decreasing)} \\
&= \frac{1}{U_x + \widetilde{\widetilde{c}}(\theta_0)} \left(1 - e^{-\left(U_x+\widetilde{\widetilde{c}}(\theta_0)\right)(\theta-\theta_0)}\right) \xrightarrow[\theta\to+\infty]{} \frac{1}{U_x + \widetilde{\widetilde{c}}(\theta_0)} > 0\,.
\end{aligned}$$

And so remembering (4.13), we conclude that y is bounded away from zero for large enough θ, and in compact sets, y is bounded away from zero since it is the sum of two positive and differentiable functions.
This ends the proof of Lemma 4.3. ∎

Lemma 4.4 *Every solution to* (4.9) *with positive initial data is bounded.*

Proof. As the geometric setting is the same as in Lemma 2.7, we will not repeat Figures 2.1, 2.2 and 2.3 here, but merely refer to the ones in Chapter 2.

We first show that Lemma 4.4 holds for the $y-$component of the solution.

Let $\Omega_- \coloneqq \{(x,y)\colon x\,y \geqslant 1\}$. Then in Ω_- we have:

$$y'(\tau) = \widetilde{c}(\tau)(1 - xy - \widetilde{c}(\tau)y) \leqslant -\widetilde{c}^2(\tau)y(\tau) < 0\,,$$

and so, in Ω_-, y cannot escape to infinity.

Let us show that in $\Omega_+ \coloneqq \{(x,y)\colon 0 < xy < 1\}$, y cannot escape to infinity either, from which we will conclude that the orbit's y component is bounded.

Pick $\mathrm{L} > 1$ arbitrarily, and set

$$\tau_{\mathrm{L}} \coloneqq \inf\{\tau\colon \widetilde{c}(\tau) < 1\}\ ,\ (\text{since } \widetilde{c}(\tau) \longrightarrow 0 \text{ as } \tau \to +\infty\ ).$$

If $y(\tau) < \mathrm{L}$ for all sufficiently large τ, then we conclude that y is bounded. Otherwise, suppose there is a $\tau_0 > \max\{1, \tau_{\mathrm{L}}\}$, such that $y(\tau_0) = \mathrm{L}$ and $\mathcal{P}_{\tau_0} \in \Omega_+ \coloneqq \{(x,y) : 0 < x\,y < 1\}$ with $\mathcal{P}_{\tau_0} = \mathcal{P}(x(\tau_0), y(\tau_0))$. Then

$$x(\tau_0)y(\tau_0) < 1 \Rightarrow x(\tau_0)\,\mathrm{L} < 1 \Rightarrow x(\tau_0)\,\mathrm{L}^2 < \mathrm{L} = y(\tau_0),$$

and as shown in Figure 2.1 we consider the set

$$\{(x,y)\colon \forall\tau \geqslant \tau_0\,,\, x < \mathrm{L}^{-1} \text{ and } y > \mathrm{L}^2 x\},$$

which covers the region we are interested in studying. Then since $x^2 - y < x^2 - \mathrm{L}^2\,x = x(x - \mathrm{L}^2) < x(1/\mathrm{L} - \mathrm{L}^2)$ the following inequalities hold:

$$\begin{aligned}
y' &= \widetilde{c}\,(1 - xy - \widetilde{c}y) \\
&= \widetilde{c}\left(x' + \widetilde{c}x^2 - \widetilde{d}x - \widetilde{c}y\right) \\
&= \widetilde{c}\left(x' + \widetilde{c}\left(x^2 - y\right) - \widetilde{d}x\right) \\
&< \widetilde{c}\left(x' + \widetilde{c}x\left[1/\mathrm{L} - \mathrm{L}^2 - \frac{\widetilde{d}}{\widetilde{c}}\right]\right) \\
&< \widetilde{c}x'\,,
\end{aligned} \tag{4.15}$$

since $\mathrm{L} > 1 \Rightarrow 1/\mathrm{L} - \mathrm{L}^2 < 0$ and $\widetilde{d}/\widetilde{c} > 0$.

Now either $x'(\tau) \leqslant 0$ and then $y'(\tau) < 0$ also and then we cannot have y escaping to infinity, and hence it is bounded above , or $x'(\tau) > 0$ and in this case we can write $\frac{dy}{dx} = \frac{y'}{x'} < \widetilde{c}(\tau) < 1$, since $\tau \geqslant \tau_0 > \tau_L$. This bound does not depend on L or $\mathcal{P}_{\tau_0}$, and hence the fact that the slope of the orbit is uniformly bounded above by some positive constant (1), guarantees that the orbit cannot go too high in Ω_+, indeed , as is clear from Figure 2.2, y is always bounded by $S_L \coloneqq \frac{L+\sqrt{4+L^2}}{2}$.

Having shown that y is bounded, we now show that x is also bounded. If it was not bounded, we could consider an orbit satisfying $x(\tau) \longrightarrow +\infty$ as $\tau \to +\infty$. By part (**ii**) of Lemma 4.3, we would then have that $y(\tau)$ is not bounded away from zero, for sufficiently large τ. This orbit must eventually enter the set Ω_+, for if it remains in Ω_- , then $x(\tau)y(\tau) > 1$ and so

$$\begin{aligned} x' &= 1 - xy - \widetilde{c}x^2 + \widetilde{d}x \\ &= 1 - xy + \widetilde{c}x\left(-x + \widetilde{d}/\widetilde{c}\right) \\ &< 0\,, \end{aligned}$$

since $1 - xy < 0$, $x \to +\infty$ and $\widetilde{d}/\widetilde{c} \to 0$ as $\tau \to +\infty$. Similarly the orbit cannot leave Ω_+ infinitely often if x is going to infinity, because if it happened, we could take $\widetilde{\tau}$ and $x(\widetilde{\tau})$ as large as we wanted, for which the orbit was entering Ω_- coming from Ω_+. Then at that point, from the argument above, $x'(\widetilde{\tau}) < 0$ and thus for the orbit to enter Ω_- we must have $y'(\widetilde{\tau}) > 0$, as shown in Figure 2.3.

Clearly this is impossible since at $\widetilde{\tau} > 0$ we have $y'(\widetilde{\tau}) = -\widetilde{c}^2(\widetilde{\tau})y(\widetilde{\tau}) < 0$, for points on the hyperbola. As a consequence, this orbit (that goes to ∞) must remain in Ω_+ after some sufficiently large enough time, and since it stays in Ω_+ , we must have $y \to 0$ as $\tau \to +\infty$.

We now consider for $\tau > 0$ the functions $g_\tau(x, y) \coloneqq x + \widetilde{c} - 1/y$ and the curves in Ω_+ defined by the level sets $\{g_\tau = 0\}$. On these curves we have that $\frac{1}{y} = x + \widetilde{c}$, and since $\widetilde{c}(\tau) \to 0$ as $\tau \to +\infty$, this sequence of hyperbolas converges monotonically to the hyperbola $\{x\,y = 1\}$, uniformly in compact sets of $\mathbb{R}^{2+}$, as $\tau \to +\infty$.

Since $x \to +\infty$ and $\widetilde{d}/\widetilde{c} \to 0$, as $\tau \to +\infty$, for all sufficiently large τ we have

$$\widetilde{c}(\tau)x'(\tau) - y'(\tau) = -\widetilde{c}^2(\tau)x(\tau)\left(x(\tau) - \frac{y(\tau)}{x(\tau)} - \frac{\widetilde{d}(\tau)}{\widetilde{c}(\tau)}\right) < 0.$$

And so for all sufficiently large τ

$$y'(\tau) > \widetilde{c}(\tau)x'(\tau). \tag{4.16}$$

And hence,

$$y' = 0 \Rightarrow 1 - xy = \widetilde{c}y \Rightarrow x' = \widetilde{c}y - \widetilde{c}x^2 + \widetilde{d}x = \widetilde{c}x(\frac{y}{x} - x + \frac{\widetilde{d}}{\widetilde{c}}) < 0,$$

and also

$$\frac{\mathrm{d}g_\tau}{\mathrm{d}\tau} = x' + \widetilde{c}\,' + \frac{y'}{y^2} = x' - \widetilde{c}^{\,3}\left(\frac{3}{2}\log t + 1\right) < 0, \text{ when } y' = 0.$$

This clearly contradicts our assertion about the *direction* in which the orbit crosses those lines. ∎

As in Chapter 2, an immediate consequence of Lemmas 4.3 and 4.4 is that solutions to (4.9), with positive initial data, are bounded and bounded away from zero; we also have that the conclusions of Lemmas 4.3 and 4.4 still hold if the initial condition is nonnegative.

In the preceding lemmas we have proved that every orbit of (4.9) is bounded and bounded away from zero. We are now ready to study its $\omega-$limit set. We start by showing that the $\omega-$limit set of every orbit is contained in the hyperbola $\{xy = 1\}$.

As in our analysis of the case $\omega > -1/2$ in Chapter 2, it is convenient to use an auxiliary function:

$$h(\tau) := x(\tau)\, y(\tau).$$

Letting (x, y) be a solution of (4.9) and computing $h'(\tau)$ we can conclude that

$h(\tau)$ solves the following differential equation

$$h'(\tau) = (y(\tau) + \widetilde{c}(\tau)x(\tau)) - \Big(y(\tau) + 2\widetilde{c}(\tau)x(\tau) - (\widetilde{d}(\tau) - \widetilde{c}^2(\tau))\Big) h(\tau), \tag{4.17}$$

which is a linear equation in h, and hence we can use the variation of constants representation formula for its solution in order to get a better understanding of h's behaviour.

Lemma 4.5 *Let* (x, y) *be any solution to* (4.9)*, and let* h *be any solution to* (4.17)*. Then* $h(\tau) \to 1^-$ *as* $\tau \to +\infty$.

Proof. To simplify notation we introduce

$$\alpha(\tau) \coloneqq y(\tau) + 2\widetilde{c}(\tau)x(\tau) - (\widetilde{d}(\tau) - \widetilde{c}^2(\tau)).$$

Applying the variation of constants formula to (4.17) we obtain

$$h(\tau) = h(\tau_0)e^{-\int_{\tau_0}^{\tau} \alpha(s)\,\mathrm{d}s} + \int_{\tau_0}^{\tau} (y(s) + \widetilde{c}(s)\,x(s))\,e^{-\int_s^{\tau} \alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s. \tag{4.18}$$

By writing the integral term in the right hand side of (4.18) as

$$\begin{aligned}
&\int_{\tau_0}^{\tau} (y(s) + \widetilde{c}(s)\,x(s))\,e^{-\int_s^{\tau} \alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s = \\
&= e^{-\int_{\tau_0}^{\tau} \alpha(s)\,\mathrm{d}s} \int_{\tau_0}^{\tau} \frac{\mathrm{d}}{\mathrm{d}s}\Big(e^{\int_{\tau_0}^{s} \alpha(\theta)\,\mathrm{d}\theta}\Big)\,\mathrm{d}s \\
&\qquad - \int_{\tau_0}^{\tau} \Big(\widetilde{c}(s)\,x(s) - (\widetilde{d}(\tau) - \widetilde{c}^2(\tau))\Big)\,e^{-\int_s^{\tau} \alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s \\
&= 1 - e^{-\int_{\tau_0}^{\tau} \alpha(\theta)\,\mathrm{d}\theta} - \int_{\tau_0}^{\tau} \Big(\widetilde{c}(s)\,x(s) - (\widetilde{d}(\tau) - \widetilde{c}^2(\tau))\Big)\,e^{-\int_s^{\tau} \alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s\,,
\end{aligned}$$

we can put (4.18) in a way more suitable for what we want to prove:

$$\begin{aligned}\frac{1-h(\tau)}{\widetilde{c}(\tau)} &= \frac{1}{\widetilde{c}(\tau)}\Bigg((1-h(\tau_0))e^{-\int_{\tau_0}^{\tau}\alpha(s)\,\mathrm{d}s}+\\ &\quad+\int_{\tau_0}^{\tau}(\widetilde{c}(s)x(s)-(\widetilde{d}(\tau)-\widetilde{c}^2(\tau)))e^{-\int_s^{\tau}\alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s\Bigg)\\ &= \frac{1-h(\tau_0)}{\widetilde{c}(\tau)}e^{-\int_{\tau_0}^{\tau}\alpha(s)\,\mathrm{d}s}+\\ &\quad+\frac{1}{\widetilde{c}(\tau)}\int_{\tau_0}^{\tau}\Big(\widetilde{c}(s)x(s)-(\widetilde{d}(\tau)-\widetilde{c}^2(\tau))\Big)\,e^{-\int_s^{\tau}\alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s\,. \qquad (4.19)\end{aligned}$$

It is worth noticing that in the right-hand side of the last equation we can have positive and negative contributions to the final result, but we will show that the first and last term converge to zero as $\tau\to+\infty$ and that the second one is always positive, bounded and bounded away from zero.

The behaviour of $e^{-\int_{\tau_0}^{\tau}\alpha(s)\,\mathrm{d}s}$ as $\tau\to+\infty$ clearly depends on the sign of $\alpha(\tau)=y(\tau)+2\widetilde{c}(\tau)x(\tau)-(\widetilde{d}(\tau)-\widetilde{c}^2(\tau))$, and since x , y and $\widetilde{c}$ are positive we must study the sign of $-(\widetilde{d}(\tau)-\widetilde{c}^2(\tau))$. Using the definition of $\widetilde{c}(\tau)$ and of $\widetilde{d}(\tau)$, this term is equal to $-3/2c^2\log t(\varsigma)$, and so it is always negative as $\tau\to+\infty$. We will show that although the final contribution from this term behaves like τ as $\tau\to+\infty$, we still have $e^{-\int_{\tau_0}^{\tau}\alpha(s)\,\mathrm{d}s}\to 0\,,$ as $\tau\to+\infty$.

Indeed, since y is bounded away from zero, there is a constant $L_y>0$ such that $y(\tau)\geqslant L_y$, and hence, since x and $\widetilde{c}$ are positive,

$$\begin{aligned}e^{-\int_{\tau_0}^{\tau}\alpha(s)\,\mathrm{d}s} &= e^{-\int_{\tau_0}^{\tau}\left(y(s)+2\widetilde{c}(s)x(s)-(\widetilde{d}(\tau)-\widetilde{c}^2(\tau))\right)\,\mathrm{d}s}\\ &\leqslant e^{-L_y(\tau-\tau_0)}e^{-\int_{\tau_0}^{\tau}-\left(\widetilde{d}(\tau)-\widetilde{c}^2(\tau)\right)\,\mathrm{d}s}. \qquad (4.20)\end{aligned}$$

Clearly $e^{-L_y(\tau-\tau_0)}=C_{\tau_0}\,e^{-L_y\tau}\longrightarrow 0$ as $\tau\to+\infty$, and so for large values of τ we need to evaluate $\displaystyle\int_{\tau_0}^{\tau}-\Big(\widetilde{d}(\tau)-\widetilde{c}^2(\tau)\Big)\,\mathrm{d}s$.

Using (4.4) to rewrite this integral in the original t variable we have

$$
\begin{aligned}
\int_{\tau_0}^{\tau} \left(\widetilde{c}^{\,2}(s) - \widetilde{\mathrm{d}}(s)\right) \, \mathrm{d}s &= \int_{\tau_0}^{\tau} \left(c^2(t(s)) - d(t(s))\right) \, \mathrm{d}s \\
&= \int_{\tau_0}^{\tau} \left(c^2(t(s)) - c^2(t(s)) \left(\frac{3}{2}\log t(s) + 1\right)\right) \, \mathrm{d}s \\
&= -\frac{3}{2}\int_{\tau_0}^{\tau} c^2(t(s)) \log t(s) \, \mathrm{d}s \\
&= -\frac{3}{2}\int_{\tau_0}^{\tau} (9\alpha)^{-2/3} \, (t(s))^{-1} \, (\log t(s))^{-4/3} \log t(s) \, \mathrm{d}s \\
&= -\frac{1}{2}\int_{\tau(t_0)}^{\tau(t)} (3\alpha^2)^{-1/3} (\log t(s))^{-1/3} \, (t(s))^{-1} \, \mathrm{d}s \\
&= -\frac{1}{2}\int_{t_0}^{t} u^{-1} \, \mathrm{d}u = -\log\sqrt{\frac{t}{t_0}} = \log\sqrt{\frac{t_0}{t}},
\end{aligned}
\tag{4.21}
$$

and hence,

$$
e^{-\int_{\tau_0}^{\tau}\left(\widetilde{c}^{\,2}(s) - \widetilde{\mathrm{d}}(s)\right) \, \mathrm{d}s} = e^{-\log\sqrt{\frac{t_0}{t}}} = \sqrt{\frac{t}{t_0}} = C_{t_0}\sqrt{t}.
$$

Dividing by $\widetilde{c}(\tau) = c\,(t(\tau)) = (9\alpha)^{-1/3} \, (t(\tau))^{-1/2} \, (\log t(\tau))^{-2/3}$, we obtain

$$
\frac{e^{-\int_{\tau_0}^{\tau}\left(\widetilde{c}^{\,2}(s) - \widetilde{d}(s)\right) \, \mathrm{d}s}}{\widetilde{c}(\tau)} = C'_{\tau_0} t (\log t)^{2/3} \,. \tag{4.22}
$$

Since $t(\tau) \sim \tau(\log\tau)^{-1/3}$ and $\log t \sim \log\tau$ as $\tau \to +\infty$ we can conclude from (4.20) that

$$
\frac{e^{-\int_{\tau_0}^{\tau}\alpha(s) \, \mathrm{d}s}}{\widetilde{c}(\tau)} \leqslant C''_{\tau_0}\, e^{-L_y\tau}\, t(\log t)^{2/3} \sim e^{-L_y\tau}\tau(\log\tau)^{1/3} \longrightarrow 0 \text{ as } \tau \to +\infty. \tag{4.23}
$$

Having shown that the first term in (4.19) goes to zero, as $\tau \to +\infty$, we now show how the second term behaves, namely that

$$\frac{1}{\widetilde{c}(\tau)} \int_{\tau_0}^{\tau} \widetilde{c}(s) x(s) e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\ \mathrm{d}s$$

is bounded above and below, and that

$$\frac{1}{\widetilde{c}(\tau)} \int_{\tau_0}^{\tau} \left(\widetilde{d}(s) - \widetilde{c}^{\,2}(s)\right) e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\ \mathrm{d}s$$

goes to zero, as $\tau \to +\infty$. We start by showing that

$$\frac{1}{\widetilde{c}(\tau)} \int_{\tau_0}^{\tau} \left(\widetilde{d}(s) - \widetilde{c}^{\,2}(s)\right) e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\ \mathrm{d}s \longrightarrow 0 \text{ as } \tau \to +\infty. \tag{4.24}$$

Let $\beta \in (0, 1)$ be fixed, and split the integral over (τ_0, τ) as $\int_{\tau_0}^{\tau-\tau^\beta} + \int_{\tau-\tau^\beta}^{\tau}$.

For the first of these integrals we start by showing that $\widetilde{d} - \widetilde{c}^{\,2}$ is a monotone decreasing function, indeed

$$\begin{aligned} \left(\widetilde{d}(\tau) - \widetilde{c}^{\,2}(\tau)\right)' &= \frac{3}{2} \left(\widetilde{c}^{\,2} \log t\right)' \\ &= -C \frac{\mathrm{d}t}{\mathrm{d}\tau} \frac{1}{t(\log t)^{1/3}} \left(1/t + 1/\log t\right) < 0\,, \end{aligned} \tag{4.25}$$

since $t > 1$ and $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^{-1} > 0$ by (4.4), and C is a positive constant depending only on α.

Hence choosing t_0 such that (4.25) holds, we have

$$\frac{1}{\widetilde{c}(\tau)} \int_{\tau_0}^{\tau-\tau^\beta} \left(\widetilde{d}(s) - \widetilde{c}^{\,2}(s)\right) e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\ \mathrm{d}s <$$

$$
\begin{aligned}
&< \frac{1}{\widetilde{c}(\tau)}(\widetilde{d}(\tau_0) - \widetilde{c}^2(\tau_0)) \int_{\tau_0}^{\tau-\tau^\beta} e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\ \mathrm{d}s \quad (\text{since } d - c^2 \text{ is decreasing}) \\
&< \frac{1}{\widetilde{c}(\tau)}(\widetilde{d}(\tau_0) - \widetilde{c}^2(\tau_0)) \int_{\tau_0}^{\tau-\tau^\beta} e^{-\int_s^\tau y(\theta)\,\mathrm{d}\theta}\ \mathrm{d}s \\
&< \frac{1}{\widetilde{c}(\tau)}(\widetilde{d}(\tau_0) - \widetilde{c}^2(\tau_0)) \int_{\tau_0}^{\tau-\tau^\beta} \sqrt{\frac{t}{t_0}} e^{-L_y(\tau-s)}\ \mathrm{d}s \\
&= \frac{1}{\widetilde{c}(\tau)}(\widetilde{d}(\tau_0) - \widetilde{c}^2(\tau_0)) \sqrt{\frac{t}{t_0}} \frac{1}{L_y} \left(e^{-L_y\tau^\beta} - e^{-L_y(\tau-\tau_0)}\right) \\
&< C\frac{\sqrt{t}}{\widetilde{c}(\tau)} \left(e^{-L_y\tau^\beta} - e^{-L_y(\tau-\tau_0)}\right),
\end{aligned}
$$

and since as $\tau \to +\infty$ we have

$$
\frac{\sqrt{t}}{\widetilde{c}(\tau)} \left(e^{-L_y\tau^\beta} - e^{-L_y(\tau-\tau_0)}\right) \sim \tau(\log\tau)^{2/3} \left(e^{-L_y\tau^\beta} - e^{-L_y(\tau-\tau_0)}\right) \longrightarrow 0,
$$

we conclude

$$
\frac{1}{\widetilde{c}(\tau)} \int_{\tau_0}^{\tau-\tau^\beta} \left(\widetilde{d}(s) - \widetilde{c}^2(s)\right) e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\ \mathrm{d}s \longrightarrow 0 \text{ as } \tau \to +\infty. \tag{4.26}
$$

The integral from the large τ contribution is

$$
\begin{aligned}
&\frac{1}{\widetilde{c}(\tau)} \int_{\tau-\tau^\beta}^{\tau} \left(\widetilde{d}(s) - \widetilde{c}^2(s)\right) e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\ \mathrm{d}s < \\
&\quad < \frac{1}{\widetilde{c}(\tau)} \left(\widetilde{d}(\tau - \tau^\beta) - \widetilde{c}^2(\tau - \tau^\beta)\right) \int_{\tau-\tau^\beta}^{\tau} e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\ \mathrm{d}s \\
&\quad < \frac{1}{\widetilde{c}(\tau)} \left(\widetilde{d}(\tau - \tau^\beta) - \widetilde{c}^2(\tau - \tau^\beta)\right) \int_{\tau-\tau^\beta}^{\tau} \sqrt{\frac{t}{t - t^\beta}} e^{-L_y(\tau-s)}\ \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&< \frac{1}{\widetilde{c}(\tau)}\left(\widetilde{d}(\tau-\tau^{\beta})-\widetilde{c}^{2}(\tau-\tau^{\beta})\right)\sqrt{\frac{t}{t-t^{\beta}}}\int\limits_{\tau-\tau^{\beta}}^{\tau} e^{-L_y(\tau-s)}\,\mathrm{d}s\\
&= \frac{1}{\widetilde{c}(\tau)}\left(\widetilde{d}(\tau-\tau^{\beta})-\widetilde{c}^{2}(\tau-\tau^{\beta})\right)\sqrt{\frac{1}{1-t^{\beta-1}}}\frac{1}{L_y}\left(1-e^{-L_y\tau^{\beta}}\right),
\end{aligned}
$$

from this last equality it is clear that for this integral to converge to zero we need to make sure that $\frac{1}{\widetilde{c}(\tau)}(\widetilde{d}(\tau-\tau^{\beta})-\widetilde{c}^{2}(\tau-\tau^{\beta})) \to 0$ as $\tau \to +\infty$, since the remaining terms converge to $1/L_y$ as $\tau \to +\infty$.

Indeed, since $\widetilde{c}(\tau) \sim (\tau\log\tau)^{-1/2}$ and $\log t \sim \log\tau$ as $\tau \to +\infty$, and

$$
\widetilde{d}-\widetilde{c}^{2} = \frac{3}{2}\widetilde{c}^{2}\log t,
$$

we can write

$$
\begin{aligned}
\frac{\widetilde{d}(\tau-\tau^{\beta})-\widetilde{c}^{2}(\tau-\tau^{\beta})}{\widetilde{c}(\tau)} &= \frac{3}{2}\,\frac{\widetilde{c}^{2}(\tau-\tau^{\beta})\log(t-t^{\beta})}{\widetilde{c}(\tau)}\\
&\sim \frac{\left((\tau-\tau^{\beta})\log(\tau-\tau^{\beta})\right)^{-1}\log(\tau-\tau^{\beta})}{(\tau\log\tau)^{-1/2}}\\
&= \frac{(\tau)^{-1/2}(\log\tau)^{1/2}}{1-\tau^{\beta-1}} \to 0 \text{ as } \tau \to +\infty.
\end{aligned}
$$

Finally we show that there are constants L and U such that, for τ large enough, the following inequalities

$$
0 < L \leqslant \frac{1}{\widetilde{c}(\tau)}\int\limits_{\tau_0}^{\tau} \widetilde{c}(s)x(s)e^{-\int_s^{\tau}\alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s \leqslant U < +\infty \tag{4.27}
$$

hold.

Using equation (4.9) for x' we can write

$$
1-x' = xy+\widetilde{c}(\tau)x^{2}-\widetilde{d}(\tau)x \Longrightarrow \frac{1-x'}{x} = y+\widetilde{c}(\tau)x-\widetilde{d}(\tau),
$$

and using the definition of $\alpha(\cdot)$ we have

$$
\begin{aligned}
\alpha(\theta) &= y(\theta) + 2\widetilde{c}(\theta)x(\theta) - (\widetilde{d}(\theta) - \widetilde{c}^2(\theta)) \\
&= y(\theta) + \widetilde{c}(\theta)x(\theta) - \widetilde{d}(\theta) + \widetilde{c}(\theta)x(\theta) + \widetilde{c}^2(\theta) \\
&= \frac{1-x'}{x} + \widetilde{c}(\theta)x(\theta) + \widetilde{c}^2(\theta) \\
&= -\frac{\mathrm{d}}{\mathrm{d}\theta}\log x(\theta) + \frac{1}{x} + \widetilde{c}(\theta)x(\theta) + \widetilde{c}^2(\theta),
\end{aligned}
$$

and so upon integration

$$
e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta} = \frac{x(\tau)}{x(s)} \exp\left[-\int_s^\tau \left(\frac{1}{x(\theta)} + \widetilde{c}(\theta)x(\theta) + \widetilde{c}^2(\theta)\right)\,\mathrm{d}\theta\right]. \tag{4.28}
$$

We start by evaluating the term that doesn't involve x; using the same argument as in (4.21)

$$
\begin{aligned}
\int_s^\tau \widetilde{c}^2(\theta)\,\mathrm{d}\theta &= \int_s^\tau (9\alpha)^{-2/3}\,(t(\theta))^{-1}\,(\log t(\theta))^{-4/3}\,\mathrm{d}\theta \\
&= \frac{1}{3}\int_s^\tau (3\alpha^2)^{-1/3}\,(t(\theta))^{-1}\,(\log t(\theta))^{-1/3}\,(\log t(\theta))^{-1}\,\mathrm{d}\theta \\
&= \frac{1}{3}\int_s^t u^{-1}(\log u)^{-1}\,\mathrm{d}u = \log\left(\frac{\log t}{\log s}\right)^{1/3},
\end{aligned}
$$

and so we can conclude that

$$
\begin{aligned}
e^{-\int_s^\tau \widetilde{c}^2(\theta)\,\mathrm{d}\theta} &= e^{-\log\left(\frac{\log t}{\log s}\right)^{1/3}} = e^{\log\left(\frac{\log s}{\log t}\right)^{1/3}} \\
&= \left(\frac{\log s}{\log t}\right)^{1/3} \sim (\log\tau)^{-1/3} \text{ as } \tau \to +\infty\,.
\end{aligned}
$$

This enables us to write (4.28) as

$$
e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta} = \frac{x(\tau)}{x(s)}\left(\frac{\log s}{\log t}\right)^{1/3} \exp\left[-\int_s^\tau \left(\frac{1}{x(\theta)} + \widetilde{c}(\theta)x(\theta)\right)\,\mathrm{d}\theta\right],
$$

and remembering our initial expression, and that we want to prove (4.27),

$$\frac{1}{\widetilde{c}(\tau)}\int_{\tau_0}^{\tau}\widetilde{c}(s)x(s)e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\ \mathrm{d}s = \int_{\tau_0}^{\tau}\frac{\widetilde{c}(s)}{\widetilde{c}(\tau)}x(s)\frac{x(\tau)}{x(s)}\left(\frac{\log s}{\log t}\right)^{\frac{1}{3}}$$
$$\exp\left[-\int_s^\tau\left(\frac{1}{x(\theta)}+\widetilde{c}(\theta)x(\theta)\right)\ \mathrm{d}\theta\right]\ \mathrm{d}s$$
$$=\int_{\tau_0}^{\tau}x(\tau)\frac{\widetilde{c}(s)}{\widetilde{c}(\tau)}\left(\frac{\log s}{\log t}\right)^{\frac{1}{3}}e^{-\int_s^\tau\left(\frac{1}{x(\theta)}+\widetilde{c}(\theta)x(\theta)\right)\,\mathrm{d}\theta}\ \mathrm{d}s.$$

Using the fact that x is bounded and bounded away from zero, there are constants L_x and U_x satisfying $0 < L_x \leqslant U_x$ and $L_x \leqslant x(\tau) \leqslant U_x$, allowing us to rewrite (4.27) as

$$L_x K(\tau) \leqslant \frac{1}{\widetilde{c}(\tau)}\int_{\tau_0}^{\tau}\widetilde{c}(s)x(s)e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\ \mathrm{d}s \leqslant U_x K(\tau), \tag{4.29}$$

where

$$K(\tau) \coloneqq \int_{\tau_0}^{\tau}\frac{\widetilde{c}(s)}{\widetilde{c}(\tau)}\left(\frac{\log s}{\log t}\right)^{\frac{1}{3}}e^{-\int_s^\tau\left(\frac{1}{x(\theta)}+\widetilde{c}(\theta)x(\theta)\right)\,\mathrm{d}\theta}\ \mathrm{d}s, \tag{4.30}$$

and so instead of proving the bounds in (4.27) it is enough to prove them for $K(\tau)$:

$$L \leqslant \int_{\tau_0}^{\tau}\frac{\widetilde{c}(s)}{\widetilde{c}(\tau)}\left(\frac{\log s}{\log t}\right)^{\frac{1}{3}}e^{-\int_s^\tau\left(\frac{1}{x(\theta)}+\widetilde{c}(\theta)x(\theta)\right)\,\mathrm{d}\theta}\ \mathrm{d}s \leqslant U. \tag{4.31}$$

As we did for the proof of (4.24) we fix $\beta \in (0,1)$, and split the integral over (τ_0, τ) as $\int_{\tau_0}^{\tau-\tau^\beta} + \int_{\tau-\tau^\beta}^{\tau}$.

We first prove that the integral over $(\tau_0, \tau - \tau^\beta)$ goes to zero as $\tau \to +\infty$; we have

$$\int_{\tau_0}^{\tau-\tau^\beta} \frac{\widetilde{c}(s)}{\widetilde{c}(\tau)}\left(\frac{\log s}{\log t}\right)^{\frac{1}{3}} e^{-\int_s^\tau \left(\frac{1}{x(\theta)}+\widetilde{c}(\theta)x(\theta)\right)\,\mathrm{d}\theta}\,\mathrm{d}s <$$

$$< \frac{\widetilde{c}(\tau_0)}{\widetilde{c}(\tau)}\left(\frac{\log(\tau-\tau^\beta)}{\log t}\right)^{\frac{1}{3}} \frac{1}{\ell_x}\left(e^{-\ell_x\tau^\beta}-e^{-\ell_x(\tau-\tau_0)}\right), \tag{4.32}$$

where

$$\ell_x \coloneqq \inf_{\theta\in[\tau_0,+\infty)}\left(\frac{1}{x(\theta)}+x(\theta)\widetilde{c}(\theta)\right)$$

is a positive constant since x is bounded and bounded away from zero, and $\widetilde{c}$ is decreasing to zero as $\tau\to+\infty$. As for the other terms we have that

$$\frac{\log(\tau-\tau^\beta)}{\log t}\sim\frac{\log(\tau-\tau^\beta)}{\log\tau}=\frac{\log\tau(1-\tau^{\beta-1})}{\log\tau}=1+\frac{\log(1-\tau^{\beta-1})}{\log\tau}\longrightarrow 1,$$

as $\tau\to+\infty$, and $\dfrac{\widetilde{c}(\tau_0)}{\widetilde{c}(\tau)}\sim(\tau\log\tau)^{\frac{1}{2}}$ as $\tau\to+\infty$.

Hence the right hand side of 4.32 satisfies

$$\frac{\widetilde{c}(\tau_0)}{\widetilde{c}(\tau)}\left(\frac{\log(\tau-\tau^\beta)}{\log t}\right)^{\frac{1}{3}} \frac{1}{\ell_x}\left(e^{-\ell_x\tau^\beta}-e^{-\ell_x(\tau-\tau_0)}\right)\sim$$

$$\sim(\tau\log\tau)^{\frac{1}{2}}\left(e^{-\ell_x\tau^\beta}-e^{-\ell_x(\tau-\tau_0)}\right)\longrightarrow 0 \text{ as } \tau\to+\infty.$$

For the integral over $(\tau-\tau^\beta,\tau)$, we have $\forall s\in(\tau-\tau^\beta,\tau)$

$$\frac{\widetilde{c}(\tau)}{\widetilde{c}(\tau)}\left(\frac{\log(\tau-\tau^\beta)}{\log t}\right)^{\frac{1}{3}}\leqslant\frac{\widetilde{c}(s)}{\widetilde{c}(\tau)}\left(\frac{\log s}{\log t}\right)^{\frac{1}{3}}\leqslant\frac{\widetilde{c}(\tau-\tau^\beta)}{\widetilde{c}(\tau)}\left(\frac{\log\tau}{\log t}\right)^{\frac{1}{3}}. \tag{4.33}$$

From what we have seen above the left hand side of (4.33) goes to 1 as $\tau\to+\infty$, and for the right hand side of (4.33) we have that

$$\frac{\widetilde{c}(\tau-\tau^\beta)}{\widetilde{c}(\tau)}\sim\left(\frac{\tau\log\tau}{(\tau-\tau^\beta)\log(\tau-\tau^\beta)}\right)^{\frac{1}{2}}=\left(\frac{1}{1-\tau^{\beta-1}}\right)^{\frac{1}{2}}\left(1+\frac{\log(1-\tau^{\beta-1})}{\log\tau}\right)^{-\frac{1}{2}}$$

and this term also goes to 1 as $\tau \to +\infty$. Hence we can conclude that in order to prove the bounds in (4.31) it is enough to show they hold for the integral

$$\int_{\tau-\tau^\beta}^{\tau} e^{-\int_s^\tau \left(\frac{1}{x(\theta)}+\widetilde{c}(\theta)x(\theta)\right)\,\mathrm{d}\theta}\,\mathrm{d}s.$$

Since $\widetilde{c}(\tau) \longrightarrow 0$ as $\tau \to +\infty$ and x is bounded above and bounded away from zero, we can define

$$u_x \coloneqq \sup_{\theta\in[\tau_0,\infty)} \left(\frac{1}{x(\theta)}+\widetilde{c}(\theta)x(\theta)\right) < +\infty,$$

and so with ℓ_x defined as above, a simple computation shows that we have

$$\frac{1-e^{-u_x\tau_0^\beta}}{u_x} < \frac{1-e^{-u_x\tau^\beta}}{u_x} \leqslant \int_{\tau-\tau^\beta}^{\tau} e^{-\int_s^\tau \left(\frac{1}{x(\theta)}+\widetilde{c}(\theta)x(\theta)\right)\mathrm{d}\theta}\,\mathrm{d}s \leqslant \frac{1-e^{-\ell_x\tau^\beta}}{\ell_x} < \frac{1}{\ell_x},$$

which are exactly the bounds we wanted, thus concluding the proof of Lemma 4.5. ∎

In this last Lemma we showed that xy converges to 1, and thus that the $\omega-$limit set of all orbits is a bounded part of the hyperbola $\{y = 1/x\}$. In order to fully identify it, we will use the same argument as in Chapter 2, by introducing another auxiliary function:

$$b(\tau) \coloneqq y(\tau) - \widetilde{c}(\tau)x(\tau). \tag{4.34}$$

In order to study the dynamics of b, we first need a few results about the derivative of $\widetilde{c}(\tau)$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{c}(\tau) = \frac{\mathrm{d}c}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = -c^3\left(\frac{3}{2}\log t+1\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^{-1}\frac{\mathrm{d}t}{\mathrm{d}\tau} = -c^3\left(\frac{3}{2}\log t+1\right),$$

and of b,

$$b'(\tau) = y'(\tau) - \widetilde{c}\,'(\tau)x(\tau) - \widetilde{c}(\tau)x'(\tau) = c^2(\tau)(-b(\tau)+x^2(\tau)). \tag{4.35}$$

By changing variables $\tau \longrightarrow \theta$ so that $\mathrm{d}\theta = c^2\,\mathrm{d}\tau$, (4.35) now reads as a linear equation for b:

$$\frac{\mathrm{d}b}{\mathrm{d}\theta} = -b + x^2. \tag{4.36}$$

we will use this auxiliary function b in the proof of our next lemma

Lemma 4.6 *Let* (x, y) *be any solution to* (4.9)*. Letting*

$$x_* := \varliminf_{\tau\to+\infty} x(\tau),\ x^* := \varlimsup_{\tau\to+\infty} x(\tau)$$

and similarly for y_* *and* y^**, the following inequalities are satisfied*

$$x_* \leqslant 1 \leqslant x^* \quad \textit{and} \quad y_* \leqslant 1 \leqslant y^*. \tag{4.37}$$

Proof. For a fixed $\beta \in (0,1)$, integrating (4.36) in $[\theta - \theta^\beta, \theta]$ gives

$$b(\theta) = b(\theta - \theta^\beta)e^{-\theta^\beta} + \int_{\theta-\theta^\beta}^{\theta} x^2(s)e^{-(\theta-s)}\,\mathrm{d}s. \tag{4.38}$$

The boundedness of x and y, and the fact that $\widetilde{c}(s) \longrightarrow$ as $s \to +\infty$, shows that $b(\theta - \theta^\beta) = y(\theta - \theta^\beta) - c(\theta - \theta^\beta)x(\theta - \theta^\beta)$ is bounded and so the first term in (4.38) converges to zero. The integral term in (4.38) can be bounded above by using the supremum of x^2

$$b(\theta) \leqslant o(1) + \left(\sup_{s\in[\theta-\theta^\beta,\theta]} x^2(s)\right)\int_{\theta-\theta^\beta}^{\theta} e^{-(\theta-s)}\,\mathrm{d}s \tag{4.39}$$

$$\leqslant o(1) + \left(\sup_{s\in[\theta-\theta^\beta,+\infty)} x^2(s)\right)\left(1 - e^{-\theta^\beta}\right), \tag{4.40}$$

and taking the limsup as $\theta \to +\infty$ gives

$$\varlimsup_{\theta\to+\infty} b(\theta) \leqslant \left(x^*\right)^2. \tag{4.41}$$

Similarly the integral term in (4.38) can be bounded below by using the infimum of x^2

$$b(\theta) \geqslant \mathcal{O}(1) + \left(\inf_{s\in[\theta-\theta^\beta,\theta]} x^2(s) \right) \int_{\theta-\theta^\beta}^{\theta} e^{-(\theta-s)} \, \mathrm{d}s \tag{4.42}$$

$$\geqslant \mathcal{O}(1) + \left(\inf_{s\in[\theta-\theta^\beta,+\infty)} x^2(s) \right) \left(1 - e^{-\theta^\beta}\right), \tag{4.43}$$

and taking the liminf as $\theta \to +\infty$ gives

$$\varliminf_{\theta\to+\infty} b(\theta) \geqslant (x_*)^2 \, . \tag{4.44}$$

Since $b(\tau) \coloneqq y(\tau) - \widetilde{c}(\tau)x(\tau)$, and using the fact that x is bounded and c converges to zero, we also have

$$\varlimsup_{\theta\to+\infty} b(\theta) = y^*, \quad \text{and} \quad \varliminf_{\theta\to+\infty} b(\theta) = y_*. \tag{4.45}$$

From Lemma 4.5, we already know that $h(\theta) = x(\theta)y(\theta) \to 1$ as $\theta \to +\infty$ and hence

$$y^* = \frac{1}{x_*} \quad \text{and} \quad y_* = \frac{1}{x^*}. \tag{4.46}$$

Using the fact that $x_* \leqslant x^*$, (4.46), (4.45) and (4.41) we can conclude

$$\left(\frac{1}{x^*} \leqslant \frac{1}{x_*} = y^* = \varlimsup_{\theta\to+\infty} b(\theta) \leqslant \left(x^*\right)^2 \right) \Longrightarrow \left(1 \leqslant \left(x^*\right)^3\right) \Longrightarrow 1 \leqslant x^*.$$

Similarly, this time using (4.46),(4.45) and (4.44) we can conclude

$$\left(\frac{1}{x_*} = y^* \geqslant y_* = \varliminf_{\theta\to+\infty} b(\theta) \geqslant (x_*)^2 \right) \Longrightarrow \left(1 \geqslant (x_*)^3\right) \Longrightarrow 1 \geqslant x_*.$$

This concludes the proof of the lemma for x, and using (4.46) for y. ∎

In the next Lemma we show that the inequalities (4.37) of Lemma 4.6 are indeed equalities, and the proofs are exactly as in Chapter 2, and we just point out the steps where they are different.

Lemma 4.7 *Let* (x, y) *be any solution to* (4.9)*. Then, with the notation of Lemma 4.6,*

$$x_* = 1 = x^* \quad \text{and} \quad y_* = 1 = y^*. \tag{4.47}$$

Proof. As in the proof of Lemma 2.9, we have the situation described in Figure 2.4, we start by supposing $x_* < 1 < x^*$, and then trying to get a contradiction, and choosing a point $(\tilde{x}, \tilde{y})$ strictly between $(1, 1)$ and (x_*, y^*) belonging to the hyperbola $\{xy = 1\}$. Since $(\tilde{x}, \tilde{y})$ belongs to $\omega(\gamma)$, the $\omega-$limit set of the orbit γ, we can choose an increasing sequence of instants τ_n, $\lim_{n\to+\infty} \tau_n = +\infty$, and corresponding to these τ_n a sequence of points $P_n = (x(\tau_n), y(\tau_n))$ belonging to the orbit γ and satisfying $P_n \xrightarrow[n\to+\infty]{} (\tilde{x}, \tilde{y})$, such that $x'(\tau_n) < 0$ and $y'(\tau_n) > 0$ for all n.

But since $x(\tau_n) \to \tilde{x} < 1$ and $y(\tau_n) \to \tilde{y} > 1$, we can, without loss of generality, choose a sequence that also satisfies $\frac{y(\tau_n)}{x(\tau_n)} > L^2 > 1$ for all n, and for some constant L independent of n as in Figure 2.5.

We now show that in these conditions the points P_n satisfy an inequality similar to (4.15):

$$\begin{aligned} y'(\tau_n) &= \tilde{c}(\tau_n)\Big(x'(\tau_n) + \tilde{c}(\tau_n)x^2(\tau_n) - \tilde{d}(\tau_n)x(\tau_n) - \tilde{c}(\tau_n)y(\tau_n)\Big) \\ &= \tilde{c}(\tau_n)\Big(x'(\tau_n) + \tilde{c}(\tau_n)\big(x^2(\tau_n) - y(\tau_n)\big) - \tilde{d}(\tau_n)x(\tau_n)\Big) \\ &< \tilde{c}(\tau_n)x'(\tau_n)\,, \end{aligned} \tag{4.48}$$

since $x(\tau_n) < y(\tau_n)$ and $x(\tau_n) < 1 \Rightarrow x^2(\tau_n) < x(\tau_n)$. Hence the coordinates of the points P_n satisfy $y'(\tau_n) < \tilde{c}(\tau_n)x'(\tau_n)$, from which we conclude that, since we assumed $x'(\tau_n) < 0$, then we would also have $y'(\tau_n) < 0$, against out hypothesis $y'(\tau_n) > 0$. This contradiction stems from the assumption of the existence of a sequence satisfying those conditions, and hence we must necessarily have $(x_*, y^*) = (1, 1)$.

We now show that we can't have a point on the other side of $(1, 1)$ either. Let us suppose similarly that there exists $(\hat{x}, \hat{y})$ in $\{xy = 1\}$ strictly between $(1, 1)$ and

(x^*, y_*). Again we can choose a sequence of times $\nu_n \uparrow +\infty$ and a corresponding sequence $Q_n = (x(\nu_n), y(\nu_n)) \in \gamma$ satisfying $Q_n \xrightarrow[n\to+\infty]{} (\hat{x}, \hat{y})$, but this time such that $x'(\nu_n) > 0$ and $y'(\nu_n) < 0$ for all n. Without loss of generality we consider $x(\nu_n) > 1$ for all n (see Figure 4.1, although with slightly different notation the geometry is the same as in Figure 2.6).

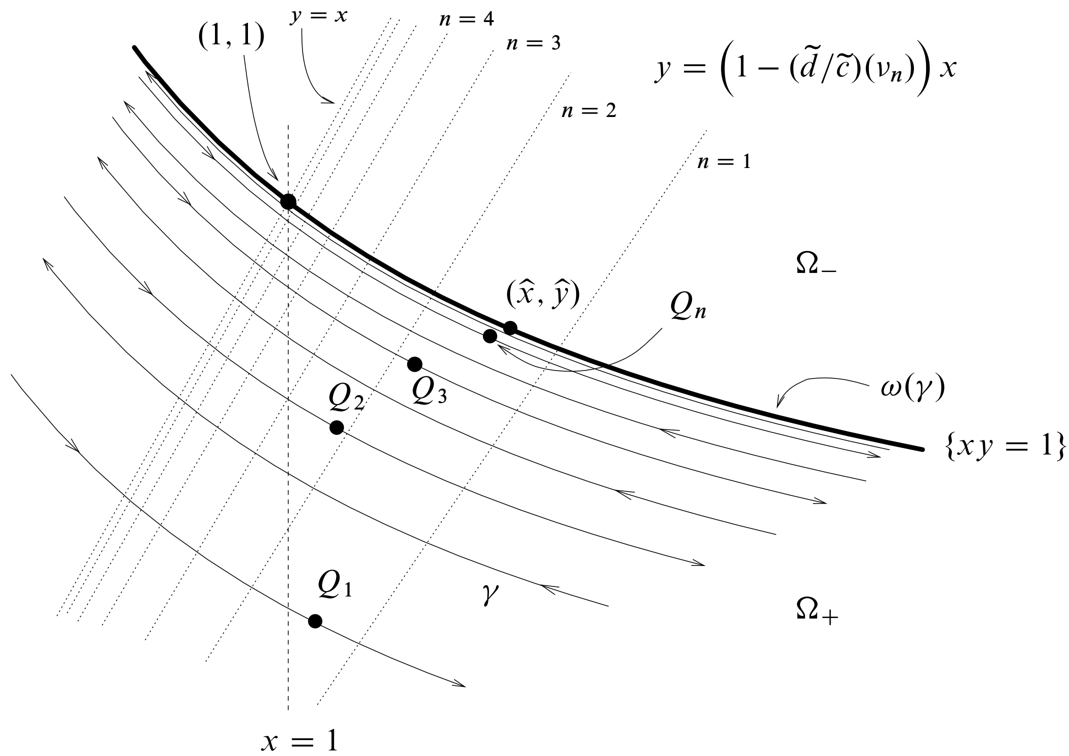

Figure 4.1: Sequence of points (Q_n) on γ converging to $(\hat{x}, \hat{y}) \in \omega(\gamma)$.

In order to obtain a contradiction we need to establish a relation between $x'(\nu_n)$ and $y'(\nu_n)$. As in (4.48) we have

$$\begin{aligned} y'(\nu_n) &= \tilde{c}(\nu_n)\left(x'(\nu_n) + \tilde{c}(\nu_n)x^2(\nu_n) - \tilde{d}(\nu_n)x(\nu_n) - \tilde{c}(\nu_n)y(\nu_n)\right) \\ &= \tilde{c}(\nu_n)\left(x'(\nu_n) + \tilde{c}(\nu_n)\left(x^2(\nu_n) - y(\nu_n)\right) - \tilde{d}(\nu_n)x(\nu_n)\right) \\ &> \tilde{c}(\nu_n)\left(x'(\nu_n) + \tilde{c}(\nu_n)\left(x(\nu_n) - y(\nu_n)\right) - \tilde{d}(\nu_n)x(\nu_n)\right) \\ &= \tilde{c}(\nu_n)\left(x'(\nu_n) + \tilde{c}(\nu_n)\left(x(\nu_n)\left(1 - \frac{\tilde{d}(\nu_n)}{\tilde{c}(\nu_n)}\right) - y(\nu_n)\right)\right) \end{aligned}$$

$$> \widetilde{c}(\nu_n)x'(\nu_n)\,. \tag{4.49}$$

The last inequality is obtained noticing first that

$$\frac{\widetilde{d}}{\widetilde{c}} = \widetilde{c}\left(\frac{3}{2}\log t + 1\right) \sim \left(\frac{\log \tau}{\tau}\right)^{\frac{1}{2}} \to 0 \text{ as } \tau \to +\infty\,,$$

and secondly that the straight lines defined by $y = x\left(1 - \frac{\widetilde{d}(\nu_n)}{\widetilde{c}(\nu_n)}\right)$ are below $y = x$, but converge to it uniformly on compact sets, and hence, for all n sufficiently large, the inequality (4.49) holds. Our hypothesis $x'(\nu_n) > 0$ now implies $y'(\nu_n) > 0$ using (4.49), which contradicts $y'(\nu_n) < 0$. Once again this contradiction means there is no sequence satisfying the conditions imposed, and hence also that no such $(\widehat{x}, \widehat{y})$ in $\{xy = 1\}$ exists, the conclusion being that $(x^*, y_*) = (1, 1)$. These two results conclude the proof. ∎

In Lemma 4.5 we showed that h converges to 1^-, and our next result specifies its the convergence rate.

Lemma 4.8 *Let* (x, y) *be any solution to* (4.9) *and let* $h = xy$. *Then we have*

$$\lim_{\tau\to+\infty} \frac{1 - h(\tau)}{\widetilde{c}(\tau)} = 1.$$

Proof. A closer look at the proof of Lemma 4.5 reveals that instead of the inequalities (4.27) we now need to prove something stronger, namely that

$$\frac{1}{\widetilde{c}(\tau)} \int_{\tau_0}^{\tau} x(s)\widetilde{c}(s)e^{-\int_s^\tau \alpha(\theta)\,\mathrm{d}\theta}\,\mathrm{d}s \longrightarrow 1 \;\text{ as }\; \tau \to +\infty. \tag{4.50}$$

Since we already proved that all terms but one converge to zero, what remains to be proved is that

$$\int_{\tau-\tau^\beta}^{\tau} e^{-\int_s^\tau \left(\frac{1}{x(\theta)} + \widetilde{c}(\theta)x(\theta)\right)\mathrm{d}\theta}\,\mathrm{d}s \longrightarrow 1 \;\text{ as }\; \tau \to +\infty. \tag{4.51}$$

Since $x(\tau) \longrightarrow 1$ as $\tau \to +\infty$,

$$\forall \varepsilon > 0\,,\, \exists T^* : \forall \theta,\, \theta > T^* \Longrightarrow 1 - \frac{\varepsilon}{2} < \frac{1}{x(\theta)} < 1 + \frac{\varepsilon}{2}$$

and

$$\forall \varepsilon > 0\,,\, \exists T^{**} : \forall \theta,\, \theta > T^{**} \Longrightarrow -\frac{\varepsilon}{2} < \widetilde{c}(\theta) x(\theta) < \frac{\varepsilon}{2},$$

since x is bounded and $\widetilde{c}(s) \longrightarrow 0$ as $s \longrightarrow +\infty$. Hence, by choosing $T = \max(T^*, T^{**})$ we have $1 - \varepsilon < \frac{1}{x(\theta)} + \widetilde{c}(\theta)x(\theta) < 1 + \varepsilon$, $\forall_{\theta > T}$ and for τ so large that $\tau - \tau^\beta > T$, we have that for $\theta \geqslant s \geqslant \tau - \tau^\beta > T$, that the integral in (4.51) can be estimated as:

$$\int_{\tau-\tau^\beta}^{\tau} e^{-(1+\varepsilon)(\tau-s)}\,\mathrm{d}s \leqslant \int_{\tau-\tau^\beta}^{\tau} e^{-\int_s^\tau \left(\frac{1}{x(\theta)} + \widetilde{c}(\theta)x(\theta)\right)\mathrm{d}\theta}\,\mathrm{d}s \leqslant \int_{\tau-\tau^\beta}^{\tau} e^{-(1-\varepsilon)(\tau-s)}\,\mathrm{d}s,$$

and

$$\frac{\left(1 - e^{-(1+\varepsilon)\tau^\beta}\right)}{1+\varepsilon} \leqslant \int_{\tau-\tau^\beta}^{\tau} e^{-\int_s^\tau \left(\frac{1}{x(\theta)} + \widetilde{c}(\theta)x(\theta)\right)\mathrm{d}\theta}\,\mathrm{d}s \leqslant \frac{\left(1 - e^{-(1-\varepsilon)\tau^\beta}\right)}{1-\varepsilon}. \quad (4.52)$$

By the arbitrariness of ε we can conclude that the left and the right hand side of (4.52) converge to 1.

This concludes the proof. ∎

4.4 Long time behaviour

Given a solution of (4.2), introduce a new time scale

$$\varsigma(t) \coloneqq \varsigma_0 + \int_{t_0}^{t} c_1(s)\,\mathrm{d}s, \quad (4.53)$$

where ς_0 is a positive constant, and consider the new phase variables

$$\widetilde{c}_j(\varsigma) := c_j(t(\varsigma)), \tag{4.54}$$

where $t(\varsigma)$ is the inverse function of $\varsigma(t)$. When $c_1(t) > 0$, these are well defined and ς is an increasing function of t. In the new variables, the c_j−equations become

$$\widetilde{c}_j\,' = \widetilde{c}_{j-1} - \widetilde{c}_j, \qquad j \geqslant 2,$$

where $' = \frac{\mathrm{d}}{\mathrm{d}\varsigma}$. This system of differential equations is a lower triangular linear system and thus can be explicitly solved in terms of the function $\widetilde{c}_1(\varsigma)$ starting from the equation for $j = 2$ and applying the variation of constants formula recursively:

$$\widetilde{c}_j(\varsigma) = e^{-\varsigma} \sum_{k=2}^{j} \frac{\varsigma^{j-k}}{(j-k)!} c_k(0) + \frac{1}{(j-2)!} \int_0^{\varsigma} \widetilde{c}_1(\varsigma - s) s^{j-2} e^{-s}\, \mathrm{d}s. \tag{4.55}$$

From now on we consider only this new time scale.

We now establish the convergence results of Theorem 4.1 in this new time scale.

By definition we have $\frac{\mathrm{d}\varsigma}{\mathrm{d}t} = c_1(t)$ and we already know from Theorem 4.1 the asymptotic behaviour of c_1, hence we have the following estimates

$$\begin{aligned}
&\forall \varepsilon > 0,\ \exists T = T_\varepsilon \colon \forall t > T_\varepsilon, \quad 1 - \varepsilon < \left(\frac{\alpha}{3}\right)^{-\frac{1}{3}} t^{\frac{1}{2}} (\log t)^{\frac{1}{3}} c_1(t) < 1 + \varepsilon \\
&\Longrightarrow \left(\frac{\alpha}{3}\right)^{\frac{1}{3}} t^{-\frac{1}{2}} (\log t)^{-\frac{1}{3}} (1 - \varepsilon) < c_1(t) < \left(\frac{\alpha}{3}\right)^{\frac{1}{3}} t^{-\frac{1}{2}} (\log t)^{-\frac{1}{3}} (1 + \varepsilon).
\end{aligned}$$

We are thus naturally led to estimate the integral $\int_{t_0}^{t} s^{-1/2} (\log s)^{-1/3}\ \mathrm{d}s$, as $t \to +\infty$. After a few computations we obtain

$$\int_{t_0}^{t} s^{-1/2} (\log s)^{-1/3}\ \mathrm{d}s = 2\, s^{1/2} (\log t)^{-1/3} (1 + o(1)) \text{ as } t \to +\infty. \tag{4.56}$$

We can now use this last expression to obtain the following relation between the 2 time scales:

$$\begin{aligned} \varsigma(t) &= 2\left(\frac{\alpha}{3}\right)^{\frac{1}{3}} t^{\frac{1}{2}} (\log t)^{-\frac{1}{3}} (1 + o(1)) \\ &= \left(\frac{8}{3}\alpha\right)^{\frac{1}{3}} t^{\frac{1}{2}} (\log t)^{-\frac{1}{3}} (1 + o(1)) \quad \text{as } t \to +\infty, \end{aligned} \tag{4.57}$$

or,

$$\lim_{t\to+\infty} \frac{1}{2}\left(\frac{3}{\alpha}\right)^{1/3} t^{-1/2}(\log t)^{1/3}\varsigma(t) = 1. \tag{4.58}$$

And so for $t(\varsigma)$,

$$t(\varsigma) = 2^{2/3}\left(\frac{8}{3}\alpha\right)^{-2/3} \varsigma^2 (\log\varsigma)^{2/3} (1 + o(1)) \tag{4.59}$$

$$= \left(\frac{4}{3}\alpha\right)^{-2/3} \varsigma^2 (\log\varsigma)^{2/3} (1 + o(1)) \quad \text{as } \varsigma \to +\infty, \tag{4.60}$$

or,

$$\lim_{\varsigma\to+\infty} \left(\frac{4}{3}\alpha\right)^{2/3} \varsigma^{-2}(\log\varsigma)^{-2/3}t(\varsigma) = 1. \tag{4.61}$$

We can use (4.60) to obtain the asymptotic behaviour of $\widetilde{c}_1(\varsigma)$ as follows:

$$\begin{aligned} \widetilde{c}_1(\varsigma) &= c_1(t(\varsigma)) \\ &= \left(\frac{\alpha}{3}\right)^{1/3} (t(\varsigma))^{-1/2} (\log t(\varsigma))^{-1/3} (1 + o(1)) \\ &= \left(\frac{\alpha}{3}\right)^{1/3} \left(\left(\frac{4}{3}\alpha\right)^{-2/3} \varsigma^2 (\log\varsigma)^{2/3}\right)^{-1/2} \\ &\qquad \times \left(\log\left(\left(\frac{4}{3}\alpha\right)^{-2/3} \varsigma^2 (\log\varsigma)^{2/3}\right)\right)^{-1/3} (1 + o(1)) \\ &= \left(\frac{2}{9}\alpha^2\right)^{1/3} \varsigma^{-1} (\log\varsigma)^{-2/3} (1 + o(1)) \;\; \text{as } \varsigma \to +\infty, \end{aligned} \tag{4.62}$$

or equivalently

$$\lim_{\varsigma\to+\infty}\left(\frac{2}{9}\alpha^2\right)^{-1/3}\varsigma\;(\log\varsigma)^{2/3}\;\widetilde{c}_1(\varsigma)=1. \tag{4.63}$$

Using the representation of the $\widetilde{c}_j$ given by (4.55) and this last equation we can establish the behaviour of c_j in terms of the original t variable. To this end, letting

$$g(\varsigma):=(1/2)^{1/3}(3/\alpha)^{2/3}\varsigma\;(\log\varsigma)^{2/3}, \tag{4.64}$$

we can write $g(\varsigma)\widetilde{c}_1(\varsigma)=1+o(1)$, as $\varsigma\to+\infty$; we start by multiplying (4.55) by $g(\varsigma)$.

We can write the first term in the right hand side, corresponding to the non-monomeric initial data contribution as

$$\varsigma\;(\log\varsigma)^{2/3}e^{-\varsigma}\sum_{k=2}^{j}\frac{\varsigma^{j-k}}{(j-k)!}c_k(0)=\mathcal{O}\left(\varsigma^{j-1}e^{-\varsigma}\right)=o\left(e^{-\lambda\varsigma}\right)\text{ as }\varsigma\to+\infty,$$

for every $\lambda<1$ and fixed j.

For the second term in the right hand side we start by changing integration variables $s\mapsto y=s/\varsigma$, which allows us to write the integral term as an integral over the fixed interval $[0,1]$, and defining the function

$$\psi(\cdot):=g(\cdot)\,\widetilde{c}_1(\cdot)\,, \tag{4.65}$$

we obtain

$$\begin{aligned}
&\frac{g(\varsigma)}{(j-2)!}\int_0^{\varsigma}\widetilde{c}_1(\varsigma-s)s^{j-2}e^{-s}\,\mathrm{d}s=\\
&\qquad=\left(\frac{9}{2\alpha^2}\right)^{1/3}\frac{\varsigma(\log\varsigma)^{2/3}}{(j-2)!}\int_0^1\widetilde{c}_1(\varsigma(1-y))(y\varsigma)^{j-2}e^{-\varsigma y}\varsigma\,\mathrm{d}y\\
&\qquad=\left(\frac{9}{2\alpha^2}\right)^{1/3}\frac{\varsigma^j(\log\varsigma)^{2/3}}{(j-2)!}\int_0^1\widetilde{c}_1(\varsigma(1-y))\,y^{j-2}e^{-\varsigma y}\,\mathrm{d}y
\end{aligned}$$

$$= \frac{\varsigma^{j-1}(\log\varsigma)^{2/3}}{(j-2)!}\int_0^1 \frac{\psi(\varsigma(1-y))\, y^{j-2}}{(1-y)(\log\varsigma(1-y))^{2/3}}\, e^{-\varsigma y}\, \mathrm{d}y\,.$$

In order to evaluate this integral we split the integral around the singularity at $y = 1$, by writing it as $\int_0^{1-\varepsilon} + \int_{1-\varepsilon}^1$, and then for the second of these integrals since the argument of $\widetilde{c}_1$ is not necessarily large, we know that since $\widetilde{c}_1$ is continuous and goes to zero at infinity, by (4.63), there exists a positive constant M satisfying $0 \leqslant \widetilde{c}_1(x) \leqslant M$ for $x \in [0, +\infty[$ and hence

$$\begin{aligned}\varsigma^j(\log\varsigma)^{2/3}\int_{1-\varepsilon}^1 \widetilde{c}_1(\varsigma(1-y))\, y^{j-2}e^{-\varsigma y}\,\mathrm{d}y &\leqslant \varsigma^j(\log\varsigma)^{2/3}M\cdot 1\int_{1-\varepsilon}^1 e^{-\varsigma y}\,\mathrm{d}y\\ &< M\varsigma^j(\log\varsigma)^{2/3}\frac{1}{\varsigma}\left(e^{-\varsigma(1-\varepsilon)} - e^{-\varsigma}\right)\\ &= M\varsigma^{j-1}(\log\varsigma)^{2/3}e^{-\varsigma}\left(e^{1-\varepsilon}-1\right),\end{aligned}$$

and this term is exponentially small when $\varsigma \to +\infty$.

For the first integral, we now have that $y \in (0, 1-\varepsilon) \Rightarrow \varsigma(1-y) \in (\varsigma\varepsilon, \varsigma)$, and hence for ς sufficiently large, we can use (4.63) and (4.65) to conclude that $\psi = 1 + o(1)$ in the interval we are considering, and thus $\forall\delta_1 > 0\,, \exists\, T(\delta_1)\colon \forall\,\varsigma > T(\delta_1)\,,\ \psi(\varsigma(1-y)) \in [1-\delta_1, 1+\delta_1]$, and hence as $\varsigma \to +\infty$ we have

$$(1-\delta_1)I_j(\varsigma) \leqslant \int_0^{1-\varepsilon} \frac{(\log\varsigma)^{2/3}\psi(\varsigma(1-y))y^{j-2}}{(1-y)(\log\varsigma(1-y))^{2/3}}\, e^{-\varsigma y}\,\mathrm{d}y \leqslant (1+\delta_1)I_j(\varsigma), \tag{4.66}$$

where

$$\begin{aligned}I_j(\varsigma) &\coloneqq \int_0^{1-\varepsilon} \frac{(\log\varsigma)^{2/3}y^{j-2}}{(1-y)(\log\varsigma(1-y))^{2/3}}\, e^{-\varsigma y}\,\mathrm{d}y\\ &= \int_0^{1-\varepsilon}\left(\frac{\log\varsigma}{\log\varsigma(1-y)}\right)^{2/3}\frac{y^{j-2}}{1-y}\, e^{-\varsigma y}\,\mathrm{d}y\end{aligned}$$

$$= \int_0^{1-\varepsilon} \left(\frac{1}{1 + \frac{\log(1-y)}{\log \varsigma}} \right)^{2/3} \frac{y^{j-2}}{1-y} e^{-\varsigma y} \, \mathrm{d}y. \tag{4.67}$$

Since $y \neq 1$, we now have that

$$\log \varsigma \to +\infty \Rightarrow \frac{\log(1-y)}{\log \varsigma} \to 0 \Rightarrow \left(\frac{1}{1 + \frac{\log(1-y)}{\log \varsigma}} \right)^{2/3} \to 1 \text{ as } \varsigma \to +\infty \, ,$$

and so

$$\forall \delta_2 > 0 \, , \exists \, T(\delta_2) \colon \forall \, \varsigma > T(\delta_2) \, , \left(\frac{1}{1 + \frac{\log(1-y)}{\log \varsigma}} \right)^{2/3} \in [1-\delta_2, 1+\delta_2] \, , \forall \, y \neq 1.$$

Hence for ς sufficiently large, it is enough to estimate the integral

$$\int_0^{1-\varepsilon} \frac{y^{j-2}}{1-y} e^{-\varsigma y} \, \mathrm{d}y,$$

which can be computed using Watson's lemma. We start by noting that $\frac{y^{j-2}}{1-y} = y^{j-2} \sum_{j=0}^{\infty} y^j$, which is a convergent series, since $y \in (0, 1-\varepsilon)$, and hence we obtain

$$\int_0^{1-\varepsilon} \frac{y^{j-2}}{1-y} e^{-\varsigma y} \, \mathrm{d}y = \frac{\Gamma(j-1)}{\varsigma^{j-1}} + \mathcal{O}\left(\frac{1}{\varsigma^j} \right), \text{ as } \varsigma \to +\infty \, .$$

Putting this last expression in (4.66) results in

$$\frac{\varsigma^{j-1}}{(j-2)!} \int_0^{1-\varepsilon} \frac{(\log \varsigma)^{2/3} \psi(\varsigma(1-y)) \, y^{j-2}}{(1-y)(\log \varsigma(1-y))^{2/3}} e^{-\varsigma y} \, \mathrm{d}y = 1 + \mathcal{O}(\varsigma^{-1}) \text{ as } \varsigma \to +\infty \, .$$

Gathering these exponentially small contributions and this last result, we have the following generalization of (4.63) : $g(\varsigma)\widetilde{c}_j(\varsigma) = 1 + o(1) \, , \forall \, j \geqslant 1 \, ,$ as $\varsigma \to$

$+\infty$, or in the original t variable, using (4.57)

$$\left(\frac{\alpha}{3}\right)^{-\frac{1}{3}} t^{\frac{1}{2}} (\log t)^{\frac{1}{3}} c_j(t) \longrightarrow 1\,, \forall\, j \geqslant 1\,,\ \text{as}\ t \to +\infty\,.$$

4.5 Self-Similar behaviour

We can now turn to the results concerning convergence of solutions to self-similar profiles. Let $\Phi_{1,-1/2} : \mathbb{R}^+ \setminus \{1\} \to \mathbb{R}$ be defined by (2.34) (page 67). The following result states that the function $\Phi_{1,-1/2}$ is the similarity profile of the solutions to (4.1) along non-characteristic directions.

Theorem 4.9 *Let (c_j) be any non-negative solution of (4.1) with initial data satisfying $\exists\, \rho > 0,\ \mu > 1 :\ \forall\ j,\ c_j(0) \leqslant \rho/j^{\mu}$. Let $\varsigma(t)$ and $\widetilde{c}_j(\varsigma)$ be as in* (4.53) *and* (4.54)*, respectively. Then,*

$$\lim_{\substack{j,\varsigma\to+\infty\\ \eta=j/\varsigma\ \text{fixed}\\ \eta\neq 1}} (1/2)^{1/3}(3/\alpha)^{2/3}\varsigma\,(\log\varsigma)^{2/3}\widetilde{c}_j(\varsigma) = \Phi_{1,-1/2}(\eta).$$

As we did for the $\omega > -1/2$ case, we now study the convergence to similarity profiles. Starting with the monomeric initial data case: in this case we only have the integral term, and we start by multiplying the representation formula for $\widetilde{c}_j$ (given by (4.55)) by $g(\varsigma)$, obtaining

$$g(\varsigma)\widetilde{c}_j(\varsigma) = \frac{g(\varsigma)}{(j-2)!}\int_0^{\varsigma} \widetilde{c}_1(\varsigma - s)s^{j-2}e^{-s}\,\mathrm{d}s.$$

We can define φ_1 on $[2,\infty)\times[0,\infty)$ by

$$\varphi_1(x,\varsigma) := \frac{g(\varsigma)}{\Gamma(x-1)}\int_0^{\varsigma} \widetilde{c}_1(\varsigma - s)s^{x-2}e^{-s}\,\mathrm{d}s. \tag{4.68}$$

When $x \geqslant 2$ is an integer, the function φ_1 clearly satisfies $\varphi_1(x,\varsigma) = g(\varsigma)\widetilde{c}_x$, and we shall use φ_1 instead of the definition of $\widetilde{c}_j$. Using Stirling's asymptotic formula $\Gamma(x) = e^{-x}x^{x-\frac{1}{2}}\sqrt{2\pi}\left(1+\mathcal{O}(x^{-1})\right)$ as $x \to \infty$, the recursive relation $\Gamma(x-1) = \Gamma(x)/(x-1)$, letting $\eta \coloneqq x/\varsigma$, and changing variable $s \mapsto y = s/\varsigma$, we can write, as $\varsigma \to +\infty$,

The proof is the same as in Chapter 2 until we get to

$$\varphi_1(\varsigma\eta,\varsigma) = \frac{1}{\sqrt{2\pi}}\left(\frac{9}{2\alpha^2}\right)^{1/3}\eta^{\frac{3}{2}-\eta\varsigma}\varsigma^{\frac{1}{2}}\varsigma(\log\varsigma)^{2/3} \times\left(1+\mathcal{O}\left(\varsigma^{-1}\right)\right)\int_0^1 \widetilde{c}_1(\varsigma(1-y))\frac{e^{\varsigma(\eta\log y - y+\eta)}}{y^2}\,\mathrm{d}y. \quad (4.69)$$

In order to make clear the asymptotic behaviour of $\widetilde{c}_1(\varsigma)$ we multiply (and divide) inside the previous integral by $g(\varsigma(1-y))$, as defined in (4.64) and (4.65), and we obtain

$$\varphi_1(\varsigma\eta,\varsigma) = \frac{1}{\sqrt{2\pi}}\left(\frac{9}{2\alpha^2}\right)^{1/3}\eta^{\frac{3}{2}-\eta\varsigma}\varsigma^{\frac{1}{2}}\varsigma(\log\varsigma)^{2/3}\left(\frac{9}{2\alpha^2}\right)^{-1/3}\varsigma^{-1} \times\left(1+\mathcal{O}\left(\varsigma^{-1}\right)\right)\int_0^1 \psi(\varsigma(1-y))\frac{e^{\varsigma(\eta\log y-y+\eta)}}{(\log(\varsigma(1-y)))^{2/3}\,(1-y)y^2}\,\mathrm{d}y. \quad (4.70)$$

Simplifying and grouping the log terms we obtain

$$\varphi_1(\varsigma\eta,\varsigma) = \frac{1}{\sqrt{2\pi}}\eta^{\frac{3}{2}-\eta\varsigma}\varsigma^{\frac{1}{2}}\left(1+\mathcal{O}\left(\varsigma^{-1}\right)\right) \times\int_0^1 \psi(\varsigma(1-y))\left(\frac{1}{1+\frac{\log 1-y}{\log\varsigma}}\right)^{2/3}\frac{e^{\varsigma(\eta\log y-y+\eta)}}{(1-y)y^2}\,\mathrm{d}y. \quad (4.71)$$

Rearranging the last expression, the proof reduces to the asymptotic evaluation of the function

$$I(\eta,\varsigma) \coloneqq \varsigma^{\frac{1}{2}}\eta^{-\eta\varsigma}e^{\varsigma\eta}\int_0^1 \psi(\varsigma(1-y))\left(\frac{1}{1+\frac{\log 1-y}{\log\varsigma}}\right)^{2/3}\frac{e^{\varsigma(\eta\log y-y)}}{(1-y)y^2}\,\mathrm{d}y, \quad (4.72)$$

as $\varsigma \to +\infty$.

We start by showing that for $\eta > 1$ we have $I(\eta,\varsigma) \longrightarrow 0$, as $\varsigma \to +\infty$. In order to study the behaviour of φ_1 we start by splitting the interval of integration as $\int_0^1 = \int_0^{1-\varepsilon} + \int_{1-\varepsilon}^1$, since near the point $y = 1$ the argument of $\psi(\varsigma(1-y))$ is no longer large, and so we evaluate the integral over $(0, 1-\varepsilon)$ using (4.71), and over $(1-\varepsilon, 1)$ using $\widetilde{c}_1$ and (4.69).

In $(0, 1-\varepsilon)$ both $\psi(\varsigma(1-y))$ and $\frac{1}{1+\frac{\log 1-y}{\log \varsigma}}$ are $1 + o(1)$ when estimating the integral for large values of ς, and hence to evaluate (4.72) we only have to estimate

$$\begin{aligned}\varsigma^{\frac{1}{2}}\eta^{-\eta\varsigma}e^{\varsigma\eta}\int_0^{1-\varepsilon}\frac{e^{\varsigma(\eta\log y-y)}}{(1-y)y^2}\,\mathrm{d}y &= \varsigma^{\frac{1}{2}}\eta^{-\eta\varsigma}e^{\varsigma\eta}\int_0^{1-\varepsilon}\frac{y^{-2}e^{\varsigma(\eta\log y-y)}}{(1-y)}\,\mathrm{d}y\\ &\leqslant \varsigma^{\frac{1}{2}}\eta^{-\eta\varsigma}e^{\varsigma\eta}\exp\big(\max_{t\in[0,1-\varepsilon]} g_1(t)\big)\int_0^{1-\varepsilon}\frac{1}{1-y}\,\mathrm{d}y\\ &= \varsigma^{\frac{1}{2}}\eta^{-\eta\varsigma}e^{\varsigma\eta}\exp\big(\max_{t\in[0,1-\varepsilon]} g_1(t)\big)\log\varepsilon^{-1}, \quad (4.73)\end{aligned}$$

where $e^{g_1(t)} := t^{-2}e^{\varsigma(\eta\log t-t)} = e^{-2\log t}e^{\varsigma(\eta\log t-t)} = e^{(\varsigma\eta-2)\log t-\varsigma t}$.

For $\varsigma > \frac{2}{\eta-1}$ and $t \leqslant 1$, the function g_1 satisfies

$$g_1'(t) = \frac{\varsigma\eta-2}{t} - \varsigma \geqslant (\varsigma\eta-2)-\varsigma = \varsigma(\eta-1)-2 > 0\,,$$

and hence $g_1(t) \leqslant g_1(1-\varepsilon) = -\varsigma(1-\varepsilon-\eta\log(1-\varepsilon)) - 2\log(1-\varepsilon)$.

Plugging this result back in (4.73) we have

$$\begin{aligned}&\varsigma^{\frac{1}{2}}\eta^{-\eta\varsigma}e^{\varsigma\eta}\exp\big(\max_{t\in[0,1-\varepsilon]} g_1(t)\big)\log\varepsilon^{-1} =\\ &\qquad = \frac{\varsigma^{\frac{1}{2}}\log\varepsilon^{-1}}{(1-\varepsilon)^2}\exp\Big(-\varsigma\big(\eta\log\eta-\eta+(1-\varepsilon)-\eta\log(1-\varepsilon)\big)\Big),\end{aligned}$$

and so it is enough to check that we have $\eta\log\eta-\eta+(1-\varepsilon)-\eta\log(1-\varepsilon) > 0$ for $\varsigma > \frac{2}{\eta-1}$ and $\eta > 1$.

Rewriting this inequality as

$$\eta\log\eta-\eta+(1-\varepsilon)-\eta\log(1-\varepsilon)>0\Leftrightarrow(1-\varepsilon)-\eta>\eta\log\frac{1-\varepsilon}{\eta}$$
$$\Leftrightarrow\frac{1-\varepsilon}{\eta}-1>\log\frac{1-\varepsilon}{\eta},$$

and, letting $z=(1-\varepsilon)/\eta$, this last inequality amounts to $z>\log z+1$ which holds for all $z\neq 1$, and that is the case since $\eta>1\Rightarrow\eta\neq 1-\varepsilon$. This concludes the proof in the interval $(0,1-\varepsilon)$.

In order to show that the integral over $(1-\varepsilon,1)$ also goes to 0 as $\varsigma\to+\infty$ we use a similar argument but this time making use of (4.69). Since $\widetilde{c_1}$ is continuous and goes to 0 as $\varsigma\to+\infty$ it is obviously bounded in $[1-\varepsilon,1]$, and so there is a constant $M>0$ such that $\widetilde{c_1}(\varsigma(1-y))<M$, $\forall y\in[1-\varepsilon,1]$. Hence in this case we have to estimate

$$\begin{aligned}\eta^{-\eta\varsigma}\varsigma^{\frac{3}{2}}(\log\varsigma)^{\frac{2}{3}}\int_{1-\varepsilon}^{1}\frac{e^{\varsigma(\eta\log y-y+\eta)}}{y^2}\,\mathrm{d}y&=\varsigma^{\frac{3}{2}}(\log\varsigma)^{\frac{2}{3}}\int_{1-\varepsilon}^{1}\frac{e^{-\varsigma(\eta\log\eta-\eta\log y+y-\eta)}}{y^2}\,\mathrm{d}y\\&=\varsigma^{\frac{3}{2}}(\log\varsigma)^{\frac{2}{3}}\int_{1-\varepsilon}^{1}\frac{e^{-\varsigma h(y)}}{y^2}\,\mathrm{d}y\\&<\varsigma^{\frac{3}{2}}(\log\varsigma)^{\frac{2}{3}}\exp\big(-\varsigma\min_{t\in[1-\varepsilon,1]}h(t)\big)\int_{1-\varepsilon}^{1}\frac{\mathrm{d}y}{y^2}\\&=\varsigma^{\frac{3}{2}}(\log\varsigma)^{\frac{2}{3}}\frac{\varepsilon}{1-\varepsilon}\exp\big(-\varsigma h(1)\big),\end{aligned}\tag{4.74}$$

where $h(t):=\eta\log\eta-\eta\log t+t-\eta$ has a unique minimum at $t=1$, and since $h(1)=\eta\log\eta+1-\eta$, and $\eta\log\eta+1-\eta>0$ for $\eta\neq 1$ the expression in (4.74) is exponentially small as $\varsigma\to+\infty$. This concludes the proof for $\eta>1$.

For $\eta<1$ we need a similar approach, but the situation being slightly more delicate, since now the (unique) maximum of $\eta\log y-y$ is attained at an interior point $(1>\eta\in(0,1))$, we need to split the integral by writing it as $\int_0^{\varepsilon}+\int_{\varepsilon}^{1-\varepsilon}+\int_{1-\varepsilon}^{1}$.

With g and ψ defined as above, for every $\varepsilon > 0$ we split the integral over $[0,1]$ as $I_1 + I_2 + I_3 \coloneqq \int_0^{\varepsilon} + \int_{\varepsilon}^{1-\varepsilon} + \int_{1-\varepsilon}^{1}$. I_1 , $I_3 \longrightarrow 0$, and the only non zero contribution comes from $\int_{\varepsilon}^{1-\varepsilon}$.

Given $\eta < 1$, we choose $\varepsilon > 0$ in such a way that $\eta \in (\varepsilon, 1-\varepsilon)$, for instance $\varepsilon < \min\{\eta/a, 1-\eta\}$, with $a > 1$.

For the integral over I_1, we now have that both $\psi(\varsigma(1-y))$ and $\frac{1}{1+\frac{\log(1-y)}{\log\varsigma}}$ are $1 + o(1)$ when estimating the integral for large values of ς; and hence to evaluate the integral over I_1 we can use an argument similar to what we used in the $\eta > 1$ case. To evaluate I_1 it is then enough to estimate

$$\varsigma^{\frac{1}{2}}\eta^{-\eta\varsigma}e^{\varsigma\eta}\int_0^{\varepsilon}\frac{e^{\varsigma(\eta\log y - y)}}{(1-y)y^2}\,\mathrm{d}y, \text{ as } \varsigma \to +\infty.$$

Letting $g_1(t) \coloneqq (\varsigma\eta - 2)\log t - \varsigma t$, we have $0 < t < \varepsilon < \eta/a < \eta$ and hence for $\varsigma > \frac{2}{(1-1/a)\eta}$,we can conclude that the function g_1 satisfies

$$g_1'(t) = \frac{\varsigma\eta - 2}{t} - \varsigma = \frac{\varsigma(\eta - t) - 2}{t} > 0\,,\ \text{since } \varsigma > \frac{2}{\eta - \eta/a} > \frac{2}{\eta - t}\,,$$

and hence $g_1(t) \leqslant g_1(\varepsilon) = -\varsigma(\varepsilon - \eta\log\varepsilon) - 2\log\varepsilon$.

We then have the following estimates

$$\begin{aligned}
\varsigma^{\frac{1}{2}}\eta^{-\eta\varsigma}e^{\varsigma\eta}\int_0^{\varepsilon}\frac{e^{\varsigma(\eta\log y - y)}}{(1-y)y^2}\,\mathrm{d}y &= \varsigma^{\frac{1}{2}}\eta^{-\eta\varsigma}e^{\varsigma\eta}\int_0^{\varepsilon}\frac{e^{g_1(y)}}{1-y}\,\mathrm{d}y \\
&\leqslant \varsigma^{\frac{1}{2}}\eta^{-\eta\varsigma}e^{\varsigma\eta}\exp\Big(\max_{t\in[0,\varepsilon]} g_1(t)\Big)\int_0^{\varepsilon}\frac{1}{1-y}\,\mathrm{d}y \\
&= \varsigma^{\frac{1}{2}}\eta^{-\eta\varsigma}e^{\varsigma\eta}e^{g_1(\varepsilon)}\log(1-\varepsilon)^{-1} \\
&= \varsigma^{\frac{1}{2}}\varepsilon^{-2}\log(1-\varepsilon)^{-1}e^{-\varsigma(\eta\log\eta - \eta + \varepsilon - \eta\log\varepsilon)}.
\end{aligned}$$

And so we only need to check that $\eta\log\eta - \eta + \varepsilon - \eta\log\varepsilon > 0$, which is true since this last expression is always positive, expect for $\eta = \varepsilon$ where it is zero, and we chose $\varepsilon < \eta$. Hence we can conclude that $I_1 \to 0$, as $\varsigma \to +\infty$.

For the integral over $[1-\varepsilon, 1]$, we have $0 \leqslant \varsigma(1-y) \leqslant \varsigma\varepsilon$, and hence it is preferable to use equation (4.69), which involves $\widetilde{c}_1$ instead of ψ. Hence for the integral I_3 we have to evaluate

$$\eta^{-\eta\varsigma}\varsigma^{3/2}(\log\varsigma)^{2/3}\int_{1-\varepsilon}^{1}\frac{e^{\varsigma(\eta\log y - y+\eta)}}{y^2}\,\mathrm{d}y\,, \qquad \text{as} \quad \varsigma \to +\infty.$$

This can be done as before, by showing that the function $h(y) := \eta\log\eta - \eta - \eta\log y + y$ is always positive for $y \in [1-\varepsilon, 1]$, remembering that we picked $\varepsilon < 1-\eta$, and hence $y > 1-\varepsilon > \eta$, when evaluating I_3. And so recalling that $h(y) \geqslant 0$, and $h(y) > 0$ for $y \neq \eta$, we can conclude that I_3 is also exponentially small as $\varsigma \to +\infty$.

In order to evaluate the integral I_2, we use again the fact that for $y \in (\varepsilon, 1-\varepsilon)$, we have

$$\log\varsigma \to +\infty \Rightarrow \frac{\log(1-y)}{\log\varsigma} \to 0 \Rightarrow \left(\frac{1}{1+\frac{\log(1-y)}{\log\varsigma}}\right)^{2/3} \to 1 \text{ as } \varsigma \to +\infty\,,$$

and so writing (4.69) as

$$\begin{aligned}
\varphi_1(\varsigma\eta,\varsigma) &= \frac{1}{\sqrt{2\pi}}\left(\frac{9}{2\alpha^2}\right)^{1/3}\eta^{\frac{3}{2}-\eta\varsigma}\varsigma^{\frac{1}{2}\varsigma}(\log\varsigma)^{2/3} \\
&\quad\times\left(1+\mathcal{O}\left(\varsigma^{-1}\right)\right)\int_{\varepsilon}^{1-\varepsilon}\widetilde{c}_1(\varsigma(1-y))\frac{e^{\varsigma(\eta\log y-y+\eta)}}{y^2}\,\mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi}}\eta^{\frac{3}{2}-\eta\varsigma}\varsigma^{\frac{1}{2}\varsigma}(\log\varsigma)^{2/3} \\
&\quad\times\left(1+\mathcal{O}\left(\varsigma^{-1}\right)\right)\int_{\varepsilon}^{1-\varepsilon}\psi(\varsigma(1-y))\frac{e^{\varsigma(\eta\log y-y+\eta)}}{\varsigma(1-y)(\log\varsigma(1-y))^{2/3}y^2}\,\mathrm{d}y \\
&= \frac{1}{\sqrt{2\pi}}\eta^{\frac{3}{2}-\eta\varsigma}\varsigma^{\frac{1}{2}}
\end{aligned}$$

$$\times\left(1+\mathcal{O}\left(\varsigma^{-1}\right)\right)\int_{\varepsilon}^{1-\varepsilon}\psi(\varsigma(1-y))\frac{(\log\varsigma)^{2/3}e^{\varsigma(\eta\log y-y+\eta)}}{(1-y)(\log\varsigma(1-y))^{2/3}y^{2}}\,\mathrm{d}y$$

$$=\frac{1}{\sqrt{2\pi}}\eta^{\frac{3}{2}-\eta\varsigma}\varsigma^{\frac{1}{2}}\left(1+\mathcal{O}\left(\varsigma^{-1}\right)\right)$$

$$\times\int_{\varepsilon}^{1-\varepsilon}\psi(\varsigma(1-y))\left(\frac{1}{1+\frac{\log(1-y)}{\log\varsigma}}\right)^{2/3}\frac{e^{\varsigma(\eta\log y-y+\eta)}}{(1-y)y^{2}}\,\mathrm{d}y.$$

Since in this case $\psi(\varsigma(1-y))=1+o(1)$ and the same holds for $\left(\frac{1}{1+\frac{\log(1-y)}{\log\varsigma}}\right)^{2/3}$, it also holds for their product as $\varsigma\to+\infty$, and we conclude that

$$\forall\delta>0\,,\exists\,T(\delta)\colon\forall\,\varsigma>T(\delta)\,,\;\psi(\varsigma(1-y))\left(\frac{1}{1+\frac{\log(1-y)}{\log\varsigma}}\right)^{2/3}\in[1-\delta,1+\delta].$$

It is then enough to study the integral

$$\eta^{\frac{3}{2}-\eta\varsigma}\varsigma^{\frac{1}{2}}\int_{\varepsilon}^{1-\varepsilon}\frac{e^{\varsigma(\eta\log y-y+\eta)}}{(1-y)y^{2}}\,\mathrm{d}y\qquad\text{as}\qquad\varsigma\to+\infty,$$

since we can write

$$(1-\delta)\eta^{\frac{3}{2}-\eta\varsigma}\varsigma^{\frac{1}{2}}J(\eta,\varsigma)\leqslant I_{2}\leqslant(1+\delta)\eta^{\frac{3}{2}-\eta\varsigma}\varsigma^{\frac{1}{2}}J(\eta,\varsigma)\tag{4.75}$$

where

$$J(\eta,\varsigma):=\int_{\varepsilon}^{1-\varepsilon}\frac{e^{\varsigma(\eta\log y-y+\eta)}}{(1-y)y^{2}}dy,\quad\text{for }\varsigma\text{ sufficiently large.}$$

Applying Laplace's method for the asymptotic evaluation of integrals [1, pg 431] to the integral

$$\int_{\varepsilon}^{1-\varepsilon}\frac{e^{-\varsigma(y-\eta\log y-\eta)}}{(1-y)y^{2}}\,\mathrm{d}y=\int_{\varepsilon}^{1-\varepsilon}\frac{e^{-\varsigma\phi(y)}}{(1-y)y^{2}}\,\mathrm{d}y,\tag{4.76}$$

where $\phi : (0,1) \to \mathbb{R}$ defined by $\phi(y) \coloneqq y - \eta \log y - \eta$, is smooth and has a unique minimum, attained at $y = \eta \in (\varepsilon, 1-\varepsilon)$ with value $\phi(\eta) = -\eta \log \eta$ and $\phi''(\eta) = \eta^{-1}$, we obtain, as $\varsigma \to +\infty$,

$$\int_{\varepsilon}^{1-\varepsilon} \frac{e^{-\varsigma(y-\eta\log y-\eta)}}{(1-y)y^2}\,\mathrm{d}y = e^{\varsigma\eta\log\eta}\frac{1}{\eta^2(1-\eta)}\sqrt{\frac{2\pi}{\varsigma/\eta}} + \mathcal{O}\left(\frac{e^{\varsigma\eta\log\eta}}{\varsigma^{3/2}}\right). \tag{4.77}$$

Now from (4.69) and (4.77), we obtain for $\eta < 1$,

$$\begin{aligned}\varphi_1(\varsigma\eta,\varsigma) &= \frac{1}{\sqrt{2\pi}}\eta^{\frac{3}{2}-\eta\varsigma}\varsigma^{\frac{1}{2}}e^{\varsigma\eta\log\eta}\frac{1}{\eta^2(1-\eta)}\sqrt{\frac{2\pi}{\varsigma/\eta}} + \mathcal{O}\left(\varsigma^{-1}\right) &(4.78)\\ &= \frac{1}{1-\eta}\left(1+o(1)\right), \text{ as } \varsigma \to +\infty. &(4.79)\end{aligned}$$

This concludes the proof in the monomeric case.

If the initial condition is not monomeric we have the contribution arising from the sum term in the right hand side of (4.55).

Multiplying it by $g(\varsigma)$ we now have to prove that

$$\lim_{\substack{j,\varsigma\to+\infty\\ \eta=j/\varsigma \text{ fixed}\\ \eta\neq 1}} g(\varsigma)\, e^{-\varsigma}\sum_{k=2}^{j}\frac{\varsigma^{j-k}}{(j-k)!}c_k(0) = 0. \tag{4.80}$$

Since we want to show the limit is zero, we will drop the constants in the definition of g, and so we only consider $\varsigma(\log\varsigma)^{2/3}$. The proof is based on the same argument used in Chapter 2.

We have

$$\begin{aligned}\varsigma(\log\varsigma)^{2/3}e^{-\varsigma}\sum_{k=2}^{j}\frac{\varsigma^{j-k}}{(j-k)!}c_k(0) &\leqslant (\log j\nu)^{2/3}\rho\, j\nu\, e^{-j\nu}\sum_{k=2}^{j}\frac{(j\nu)^{j-k}}{(j-k)!\,k^{\mu}}\\ &= \rho\varphi_2(\nu, j)\end{aligned}$$

where

$$\varphi_2(j,\nu) := (\log j\nu)^{2/3}\, j\nu\, e^{-j\nu} \sum_{k=2}^{j} \frac{(j\nu)^{j-k}}{(j-k)!\,k^{\mu}}.$$

We can adapt the results in the study of φ_2 done in Chapter 2, where instead of the prefactor $j\nu^r$ in expressions (2.39) and (2.40) we now have to use the term $j\nu(\log j\nu)^{2/3}$. The estimates show that now in order for φ_2 to converge to zero we need to consider initial data satisfying $c_j(0) \leqslant \rho/j^{\mu}$, but in this case with $\mu > 1$. The $\log j$ term growing much slower than $\sqrt{j}$ has no influence on the convergence of φ_2 to zero. This completes the proof of the theorem. ∎

In the case with input αt^{ω} with $\omega > -1/2$, we've seen that for values of $\omega < 1$ the singularity of the self-similar solution $\Phi_{1,\omega}$ can be dealt with by considering a different similarity variable and a different time-scaling , allowing us a sort of inner expansion for the characteristic direction $\eta = 1$, and we obtained a solution $\Phi_{2,\omega}$ satisfying $\widetilde{c}_j(\varsigma) \sim \varsigma^{-r/2}\Phi_{2,\omega}\left(\frac{j-\varsigma}{\sqrt{\varsigma}}\right)$. It is worth noticing that for $\omega > -1/2$ the similarity variable was independent of ω , and the time scaling variable, although $\omega-$dependent was always *half* the one used for $\Phi_{1,-1/2}$.

In the case at hand, we also have a singularity at $\eta = 1$, as we have just seen, and hence it is natural to check if this similarity variable also gives rise to a solution, and if that is the case, one for which $\eta = 1$ is no longer a singularity.

Hence, choosing the similarity variable $\frac{j-\varsigma}{\sqrt{\varsigma}}$ and the scaling

$$\varsigma^{1/2}\left(\log(\varsigma)^{1/2}\right)^{2/3} = (1/2)^{2/3}\varsigma^{1/2}(\log\varsigma)^{2/3}$$

and defining $\Phi_{2,-1/2}$ as in Theorem 2.14, we hope it is equal to the limit

$$\lim_{\substack{j,\varsigma\to+\infty \\ \xi=\frac{j-\varsigma}{\sqrt{\varsigma}}\ \text{fixed} \\ \xi\in\mathbb{R}}} \frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\frac{3}{\alpha}\right)^{\frac{2}{3}} \varsigma^{\frac{1}{2}}\,(\log\varsigma)^{\frac{2}{3}}\,\widetilde{c}_j(\varsigma). \tag{4.81}$$

We now show that this limit does not exist.

Following a strategy similar to the one we used for $\omega > -1/2$, for monomeric initial data we have to estimate

$$\sqrt{\frac{\pi}{2}}\mathcal{C}\,(\log\varsigma)^{2/3}\,\varsigma^{1/2}\widetilde{c}_j(\varsigma) = \sqrt{\frac{\pi}{2}}\frac{\mathcal{C}\,(\log\varsigma)^{2/3}\,\varsigma^{1/2}}{\Gamma(j-1)}\int_0^{\varsigma}\widetilde{c}_1(\varsigma-s)s^{j-2}\,e^{-s}ds,$$

as $\varsigma \to +\infty,\ j \to +\infty,\ \frac{j-\varsigma}{\sqrt{\varsigma}}$ fixed, where $\mathcal{C} \coloneqq \frac{1}{2}\left(\frac{3}{\alpha}\right)^{\frac{2}{3}}$.

As usual we define the function φ_3 in $[2,\infty)\times[0,\infty)$ by

$$\varphi_3(x,\varsigma) \coloneqq \sqrt{\frac{\pi}{2}}\,\frac{\mathcal{C}\,(\log\varsigma)^{2/3}\,\varsigma^{1/2}}{\Gamma(x-1)}\int_0^{\varsigma}\widetilde{c}_1(\varsigma-s)s^{j-2}\,e^{-s}ds,$$

and using the similarity variable $\xi = \frac{j-\varsigma}{\sqrt{\varsigma}}\left(= \frac{x-\varsigma}{\sqrt{\varsigma}}\right)$ we can rewrite φ_3 as

$$\varphi_3(\varsigma+\xi\sqrt{\varsigma},\varsigma) \coloneqq \sqrt{\frac{\pi}{2}}\,\frac{\mathcal{C}\,(\log\varsigma)^{2/3}\,\varsigma^{1/2}}{\Gamma(\varsigma+\xi\sqrt{\varsigma}-1)}\int_0^{\varsigma}\widetilde{c}_1(\varsigma-s)\,s^{\varsigma+\xi\sqrt{\varsigma}-2}\,e^{-s}ds.$$

If $2 \leqslant x = j \in \mathbb{N}$, we have $\varphi_3(j,\varsigma) = \sqrt{\frac{\pi}{2}}\mathcal{C}\,(\log\varsigma)^{2/3}\,\varsigma^{1/2}\widetilde{c}_j(\varsigma)$, and hence we need to evaluate the limit

$$\lim_{\varsigma\to+\infty}\varphi_3(\varsigma+\xi\sqrt{\varsigma},\varsigma). \tag{4.82}$$

Changing variables $s \mapsto w \coloneqq \sqrt{\sqrt{\varsigma}-\frac{s}{\sqrt{\varsigma}}}$, in such a way that $\varsigma - s = \sqrt{\varsigma}\,w^2$ and $\mathrm{d}s = -2\sqrt{\varsigma}w\ \mathrm{d}w$, we obtain

$$\begin{aligned}\varphi_3(\varsigma+\xi\sqrt{\varsigma},\varsigma) &= \sqrt{\frac{\pi}{2}}\,\frac{\mathcal{C}\,(\log\varsigma)^{2/3}\,\varsigma^{1/2}}{\Gamma(\varsigma+\xi\sqrt{\varsigma}-1)}\\ &\quad\times\int_0^{\varsigma^{\frac{1}{4}}}\widetilde{c}_1(\sqrt{\varsigma}w^2)(\varsigma-\sqrt{\varsigma}w^2)^{\varsigma+\xi\sqrt{\varsigma}-2}e^{-\varsigma+\sqrt{\varsigma}w^2}2\sqrt{\varsigma}w\,\mathrm{d}w.\end{aligned} \tag{4.83}$$

Now we would like to use our knowledge about the large time behaviour of $\widetilde{c}_1$ to compute the limit as $\varsigma \to +\infty$; we can use (4.65) to write (4.83) as

$$\begin{aligned}
\varphi_3(\varsigma+\xi\sqrt{\varsigma},\varsigma) &= \frac{\sqrt{2\pi}\,\mathcal{C}\,(\log\varsigma)^{2/3}\,\varsigma^{1/2}}{\Gamma(\varsigma+\xi\sqrt{\varsigma}-1)}\int_0^{\varsigma^{\frac14}}\mathcal{C}^{-1}\big(\log(\sqrt{\varsigma}w^2)\big)^{-2/3}(\sqrt{\varsigma}w^2)^{-1}\\
&\quad\times\psi(\sqrt{\varsigma}w^2)(\varsigma-\sqrt{\varsigma}w^2)^{\varsigma+\xi\sqrt{\varsigma}-2}e^{-\varsigma+\sqrt{\varsigma}w^2}\sqrt{\varsigma}w\,\mathrm{d}w\\
&= \frac{\sqrt{2\pi}\,\varsigma^{1/2}}{\Gamma(\varsigma+\xi\sqrt{\varsigma}-1)}\int_0^{\varsigma^{\frac14}}\left(\frac{1}{1+4\frac{\log w}{\log\varsigma}}\right)^{2/3}\psi(\sqrt{\varsigma}w^2)\\
&\quad\times(\varsigma-\sqrt{\varsigma}w^2)^{\varsigma+\xi\sqrt{\varsigma}-2}e^{-\varsigma+\sqrt{\varsigma}w^2}\frac{1}{w}\,\mathrm{d}w\\
&= \frac{\sqrt{2\pi}\,\varsigma^{\varsigma+\xi\sqrt{\varsigma}-3/2}\,e^{-\varsigma}}{\Gamma(\varsigma+\xi\sqrt{\varsigma}-1)}\int_0^{\varsigma^{\frac14}}\left(\frac{1}{1+4\frac{\log w}{\log\varsigma}}\right)^{2/3}\psi(\sqrt{\varsigma}w^2)\\
&\quad\times\left(1-\frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2}e^{\sqrt{\varsigma}w^2}\frac{1}{w}\,\mathrm{d}w,
\end{aligned}$$

and using Stirling's asymptotic expansion for the Gamma function we can write

$$1/\Gamma(\varsigma+\xi\sqrt{\varsigma}-1) = \frac{1}{\sqrt{2\pi}}\frac{e^{\varsigma+\xi\sqrt{\varsigma}}}{(\varsigma+\xi\sqrt{\varsigma})^{\varsigma+\xi\sqrt{\varsigma}-3/2}}(1+o(1)),\ \text{as}\ \varsigma\to+\infty$$

and hence φ_3 can be written, as $\varsigma\to+\infty$,

$$\begin{aligned}
\varphi_3(\varsigma+\xi\sqrt{\varsigma},\varsigma) &= \frac{\varsigma^{\varsigma+\xi\sqrt{\varsigma}-3/2}\,e^{\xi\sqrt{\varsigma}}}{(\varsigma+\xi\sqrt{\varsigma})^{\varsigma+\xi\sqrt{\varsigma}-3/2}}(1+o(1)) \qquad (4.84)\\
&\quad\times\int_0^{\varsigma^{\frac14}}\left(\frac{1}{1+4\frac{\log w}{\log\varsigma}}\right)^{2/3}\psi(\sqrt{\varsigma}w^2)\left(1-\frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2}e^{\sqrt{\varsigma}w^2}\frac{1}{w}\,\mathrm{d}w.
\end{aligned}$$

To estimate the multiplicative prefactor in (4.84) as $\varsigma\to+\infty$ we can write it as

$$\frac{\varsigma^{\varsigma+\xi\sqrt{\varsigma}-3/2}\,e^{\xi\sqrt{\varsigma}}}{(\varsigma+\xi\sqrt{\varsigma})^{\varsigma+\xi\sqrt{\varsigma}-3/2}} = e^{\xi\sqrt{\varsigma}}\left(\frac{\varsigma}{\varsigma+\xi\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}}\left(1+\frac{\xi}{\sqrt{\varsigma}}\right)^{3/2}$$

$$
\begin{aligned}
&= e^{\xi\sqrt{\varsigma}} \frac{1}{(1+\xi/\sqrt{\varsigma})^{\varsigma}} \frac{1}{\left((1+\xi/\sqrt{\varsigma})^{\sqrt{\varsigma}}\right)^{\xi}} (1+o(1)) \\
&= \left(\frac{e}{(1+\xi/\sqrt{\varsigma})^{\sqrt{\varsigma}/\xi}}\right)^{\xi\sqrt{\varsigma}} e^{-\xi^2}(1+o(1)) \qquad (4.85) \\
&= e^{\xi^2/2} e^{-\xi^2}(1+o(1)) \\
&= e^{-\xi^2/2}(1+o(1)),
\end{aligned}
$$

where in (4.85) to compute the limit as $\varsigma \to +\infty$ we use the change of variables $\varsigma \mapsto x := \frac{\xi}{\sqrt{\varsigma}}$ to obtain $\left(\frac{e}{(1+x)^{1/x}}\right)^{\xi^2/x}$ and then we apply L'Hôpital's rule twice.

Using this last expression we can write (4.84) as $\varsigma \to +\infty$ in the following way

$$
\begin{aligned}
\varphi_3(\varsigma+\xi\sqrt{\varsigma},\varsigma) &= e^{-\xi^2/2}(1+o(1)) \qquad (4.86) \\
&\times \int_0^{\varsigma^{\frac{1}{4}}} \left(\frac{1}{1+4\frac{\log w}{\log\varsigma}}\right)^{2/3} \psi(\sqrt{\varsigma}w^2)\left(1-\frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2} e^{\sqrt{\varsigma}w^2}\frac{1}{w}\,\mathrm{d}w.
\end{aligned}
$$

In the case where $\omega > -1/2$ we proceeded with a study of the integral term in (4.86) by considering w (and hence $\sqrt{\varsigma}w^2$) close to zero, and w away from zero, and showing that the integral, for small values of w, could be made arbitrarily small, while the remaining integral converged as $\varsigma \to +\infty$.

This approach will fail in the case at hand, essentially due to the singularity arising from $1/w$ in the integrand of (4.86).

We now consider $\varepsilon > 0$ arbitrarily small and $1 < T < \varsigma^{1/4}$ and we show that the integral over $[\varepsilon, T]$ can be made arbitrarily large.

We start by splitting the integral over $[0, \varsigma^{1/4}]$ as a sum over 3 intervals

$$
\int_0^{\varsigma^{\frac{1}{4}}} \left(\frac{1}{1+4\frac{\log w}{\log\varsigma}}\right)^{2/3} \psi(\sqrt{\varsigma}w^2)\left(1-\frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2} e^{\sqrt{\varsigma}w^2}\frac{1}{w}\,\mathrm{d}w =
$$

$$\Big(\int_0^{\varepsilon}+\int_{\varepsilon}^{T}+\int_{T}^{\varsigma^{\frac{1}{4}}}\Big)\Big(\frac{1}{1+4\frac{\log w}{\log\varsigma}}\Big)^{2/3}\psi(\sqrt{\varsigma}w^2)\Big(1-\frac{w^2}{\sqrt{\varsigma}}\Big)^{\varsigma+\xi\sqrt{\varsigma}-2}e^{\sqrt{\varsigma}w^2}\frac{1}{w}\,\mathrm{d}w.$$

Both the integrals over $[0,\varepsilon]$ and $[T,\varsigma^{1/4}]$ are non negative and for $w\in[\varepsilon,T]$ we have $\sqrt{\varsigma}w^2\geqslant\sqrt{\varsigma}\varepsilon^2\to+\infty$ as $\varsigma\to+\infty$, and so as we have already seen before $\left(\frac{1}{1+4\frac{\log w}{\log\varsigma}}\right)^{2/3}\psi(\sqrt{\varsigma}w^2)=1+o(1)$ as $\varsigma\to+\infty$.

This means that for $w\in[\varepsilon,T]$ the integral we want to evaluate is asymptotically equal to

$$(1+o(1))\int_{\varepsilon}^{T}\Big(1-\frac{w^2}{\sqrt{\varsigma}}\Big)^{\varsigma+\xi\sqrt{\varsigma}-2}e^{\sqrt{\varsigma}w^2}\frac{1}{w}\,\mathrm{d}w. \tag{4.87}$$

To estimate this last integral we have as $\varsigma\to+\infty$

$$\begin{aligned}\Big(1-\frac{w^2}{\sqrt{\varsigma}}\Big)^{\varsigma+\xi\sqrt{\varsigma}-2}e^{\sqrt{\varsigma}w^2}&=e^{\sqrt{\varsigma}w^2}\Big(1-\frac{w^2}{\sqrt{\varsigma}}\Big)^{-2}\Big(1-\frac{w^2}{\sqrt{\varsigma}}\Big)^{\varsigma}\Big(\Big(1-\frac{w^2}{\sqrt{\varsigma}}\Big)^{\sqrt{\varsigma}}\Big)^{\xi}\\&=\left(\frac{\big(1-w^2/\sqrt{\varsigma}\big)^{\sqrt{\varsigma}}}{e^{-w^2}}\right)^{\sqrt{\varsigma}}e^{-\xi w^2}\big(1+o(1)\big)\\&=e^{-w^4/2}e^{-\xi w^2}(1+o(1)) \qquad (4.88)\\&=e^{-\dfrac{w^4}{2}-\xi w^2}\big(1+o(1)\big), \qquad (4.89)\end{aligned}$$

where (4.88) is obtained by changing variables $\varsigma\mapsto x:=\frac{1}{\sqrt{\varsigma}}$ and applying L'Hôpital's rule.

From (4.89) we conclude that there exists a continuous function $g(w,\varsigma)$ defined for $\varsigma^{1/4}>\varepsilon$ and $w\in[\varepsilon,T]$ and satisfying $1+g(w,\varsigma)\geqslant 0$ and $g(w,\varsigma)\to 0$ as $\varsigma\to+\infty$ for each fixed w, such that

$$\Big(1-\frac{w^2}{\sqrt{\varsigma}}\Big)^{\varsigma+\xi\sqrt{\varsigma}-2}e^{\sqrt{\varsigma}w^2}=e^{-\frac{w^4}{2}-\xi w^2}\big(1+g(w,\varsigma)\big). \tag{4.90}$$

We now estimate the integral

$$\int_{\varepsilon}^{T} e^{-\frac{w^4}{2}-\xi w^2}\big(1+g(w,\varsigma)\big)\frac{1}{w}\,\mathrm{d}w. \tag{4.91}$$

We can use (4.90) to write

$$1+g(w,\varsigma)=\big(1-\frac{w^2}{\sqrt{\varsigma}}\big)^{\varsigma+\xi\sqrt{\varsigma}-2}e^{\sqrt{\varsigma}w^2}e^{\frac{w^4}{2}+\xi w^2}.$$

We can estimate $\big(1-\frac{w^2}{\sqrt{\varsigma}}\big)^{\xi\sqrt{\varsigma}}$ using the series development $\log(1-x)=-x-x^2/2-x^3/3-x^4/4-\cdots$ valid for $x\in[-1,1)$ to write

$$\begin{aligned}
\big(1-\frac{w^2}{\sqrt{\varsigma}}\big)^{\xi\sqrt{\varsigma}} &= \Big(\big(1-\frac{w^2}{\sqrt{\varsigma}}\big)^{\sqrt{\varsigma}}\Big)^{\xi}\\
&= \Big(e^{\sqrt{\varsigma}\log(1-\frac{w^2}{\sqrt{\varsigma}})}\Big)^{\xi}\\
&= \Big(e^{\sqrt{\varsigma}(-\frac{w^2}{\sqrt{\varsigma}}-\frac{1}{2}(\frac{w^2}{\sqrt{\varsigma}})^2-\frac{1}{3}(\frac{w^2}{\sqrt{\varsigma}})^3-\frac{1}{4}(\frac{w^2}{\sqrt{\varsigma}})^4-\cdots)}\Big)^{\xi}\\
&= \Big(e^{-\sqrt{\varsigma}\frac{w^2}{\sqrt{\varsigma}}(1+\frac{1}{2}(\frac{w^2}{\sqrt{\varsigma}})+\frac{1}{3}(\frac{w^2}{\sqrt{\varsigma}})^2+\frac{1}{4}(\frac{w^2}{\sqrt{\varsigma}})^3+\cdots)}\Big)^{\xi}\\
&= \Big(e^{-w^2}e^{-\frac{w^4}{\sqrt{\varsigma}}(\frac{1}{2}+\frac{1}{3}\frac{w^2}{\sqrt{\varsigma}}+\frac{1}{4}(\frac{w^2}{\sqrt{\varsigma}})^2+\cdots)}\Big)^{\xi}\\
&= e^{-w^2\xi}\Big(e^{-\frac{w^4}{\sqrt{\varsigma}}(\frac{1}{2}+\frac{1}{3}\frac{w^2}{\sqrt{\varsigma}}+\frac{1}{4}(\frac{w^2}{\sqrt{\varsigma}})^2+\cdots)}\Big)^{\xi}\\
&> e^{-w^2\xi}e^{-\xi\frac{T^4}{\sqrt{\varsigma}}(\frac{1}{2}+\frac{1}{3}\frac{T^2}{\sqrt{\varsigma}}+\frac{1}{4}(\frac{T^2}{\sqrt{\varsigma}})^2+\cdots)}\geqslant\frac{1}{2}e^{-w^2\xi},
\end{aligned} \tag{4.92}$$

since whether ξ is positive, negative, or zero the argument of the second exponential goes to zero as $\varsigma\to+\infty$.

Using once again the Taylor series for $\log(1-x)$ we can write

$$\begin{aligned}
\Big(e^{w^2}\big(1-\frac{w^2}{\sqrt{\varsigma}}\big)^{\sqrt{\varsigma}}\Big)^{\sqrt{\varsigma}} &= \Big(e^{w^2}e^{-w^2}e^{-\frac{w^4}{\sqrt{\varsigma}}(\frac{1}{2}+\frac{1}{3}\frac{w^2}{\sqrt{\varsigma}}+\frac{1}{4}(\frac{w^2}{\sqrt{\varsigma}})^2+\cdots)}\Big)^{\sqrt{\varsigma}}\\
&= \Big(e^{-\frac{1}{2}\frac{w^4}{\sqrt{\varsigma}}}e^{-\frac{w^4}{\sqrt{\varsigma}}(\frac{1}{3}\frac{w^2}{\sqrt{\varsigma}}+\frac{1}{4}(\frac{w^2}{\sqrt{\varsigma}})^2+\cdots)}\Big)^{\sqrt{\varsigma}}\\
&= e^{-\frac{1}{2}w^4}e^{-\frac{w^6}{\sqrt{\varsigma}}\big(\frac{1}{3}+\frac{1}{4}\frac{w^2}{\sqrt{\varsigma}}+\cdots\big)}
\end{aligned}$$

$$> e^{-\frac{1}{2}w^4} e^{-\frac{T^6}{\sqrt{\varsigma}}\left(\frac{1}{3}+\frac{1}{4}\frac{T^2}{\sqrt{\varsigma}}+\cdots\right)} \geqslant \frac{1}{2}e^{-\frac{1}{2}w^4}, \tag{4.93}$$

since the argument of the second exponential goes to zero as $\varsigma \to +\infty$.

Finally we have

$$\begin{aligned}
1 + g(w,\varsigma) &= \left(1 - \frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2} e^{\sqrt{\varsigma}w^2+\frac{w^4}{2}+\xi w^2} \\
&> \left(1 - \frac{w^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2} e^{\sqrt{\varsigma}w^2} e^{\xi w^2} \quad \left[\text{since } \frac{w^4}{2} > 0\right] \\
&> \left(1 - \frac{T^2}{\sqrt{\varsigma}}\right)^{\varsigma+\xi\sqrt{\varsigma}-2} e^{\sqrt{\varsigma}w^2} e^{\xi w^2} \\
&> \left(\left(1 - \frac{T^2}{\sqrt{\varsigma}}\right)^{\sqrt{\varsigma}}\right)^{\sqrt{\varsigma}+\xi} e^{\sqrt{\varsigma}w^2} e^{\xi w^2} \quad \left[\text{since } \left(1 - \frac{T^2}{\sqrt{\varsigma}}\right)^{-2} > 1\right] \\
&\geqslant \frac{1}{4}e^{-\frac{w^4}{2}},
\end{aligned}$$

and hence the integral in (4.91) can be estimated as

$$\begin{aligned}
\int_{\varepsilon}^{T} e^{-\frac{w^4}{2}-\xi w^2}\big(1 + g(w,\varsigma)\big)\frac{1}{w}\,\mathrm{d}w &\geqslant \frac{1}{4}\int_{\varepsilon}^{T} e^{-w^4-\xi w^2}\frac{1}{w}\,\mathrm{d}w \\
&> L_1 \int_{\varepsilon}^{T} \frac{1}{w}\,\mathrm{d}w,
\end{aligned} \tag{4.94}$$

where $L_1 = L_1(\xi, T) := \frac{1}{4}e^{-T^4-\xi T^2}$.

The integral in (4.94) can be made arbitrarily large, since we can choose $\varepsilon > 0$ suitably small, and hence since the integral in (4.86) is (strictly) larger than the integral in (4.91), this concludes the proof that the limit in (4.82) (and hence in (4.81)) does not exist.

Chapter 5

The slow input cases

This Chapter contains unpublished results obtained in collaboration with F. P. da Costa and J. T. Pinto.

5.1 Introduction

In this chapter we study the case with slow input term $J_1 = \alpha t^{\omega}$ with $\alpha \geqslant 0$, as usual, and $\omega < -1/2$.

$$\begin{cases} x' = \alpha t^{\omega} - xy - x^2 \\ y' = \alpha t^{\omega} - xy. \end{cases} \tag{5.1}$$

5.2 The case $\alpha = 0$

We consider the system

$$\begin{cases} x' = -xy - x^2 \\ y' = -xy. \end{cases} \tag{5.2}$$

Remembering that we are only interested in positive solutions, and that the positive cone of $\mathbb{R}^2$ is invariant under the flow of (5.2), it follows from (5.2) that both x' and y' are negative, and hence both x and y are monotonically decreasing and positive, hence there must exist (x_∞, y_∞) depending on the initial data and non-negative, such that

$$x(t) \longrightarrow x_\infty \quad \text{and} \quad y(t) \longrightarrow y_\infty, \quad \text{as} \quad t \to +\infty.$$

The identification of x_∞ can be easily done, since $x > 0,\ y > 0$ we have from (5.2) that $x' = -xy - x^2 < -x^2$, which upon integration between t_0 and $t > t_0$ gives

$$x(t) < \frac{1}{x(t_0)^{-1} + (t - t_0)} \xrightarrow[t \to +\infty]{} 0,$$

and thus $x_\infty = 0$.

In order to identify y_∞, we start by noticing, from the $y-$equation in (5.2), that y is strictly decreasing. This allows us to change the time scale $t \mapsto y$, and to consider x as a function of y.

From (5.2) we have

$$\frac{\mathrm{d}x}{\mathrm{d}y} = 1 + \frac{x}{y},$$

and defining $u \coloneqq \frac{x}{y}$, we obtain an equation for $u = u(y),\ yu' = 1$ which after integration between y_0 and y gives $u(y) - u(y_0) = \log(y) - \log(y_0)$, which can be written as $x/y - x_0/y_0 = \log(y/y_0)$, or more suitably as

$$\frac{x}{x_0} = \frac{y}{y_0}\left(1 + \frac{y_0}{x_0}\log\frac{y}{y_0}\right), \tag{5.3}$$

where $x_0 = x(y_0)$.

Plotting the graph of $f_\gamma(s) = s(1 + \gamma \log s)$ we conclude from (5.3) and the non–negativity of x and y, and the behaviour of x, that $\dfrac{y(t)}{y_0} \to e^{-x_0/y_0}$ as $t \to +\infty$.

This proves the following result.

Proposition 5.1 *Consider non–negative solutions to* (5.3). *Then the following holds true*

(i) *the function* $\Phi(x, y) \coloneqq \dfrac{x}{y} - \log y$ *is a first integral of* (5.3),

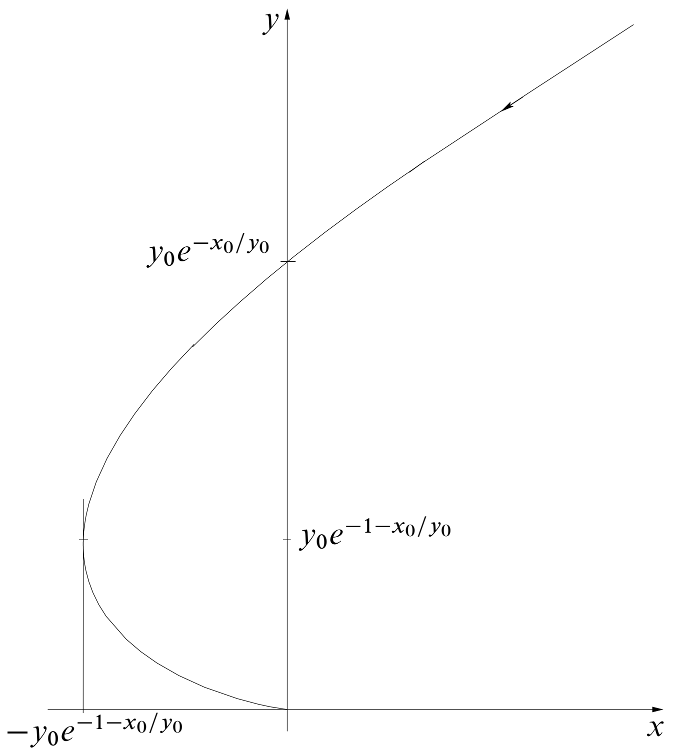

Figure 5.1: Integral curve for the $\alpha = 0$ case.

(ii) *for every initial condition* $(x_0, y_0) \in \mathbb{R}^+ \times \mathbb{R}^+$, *the solution satisfies* $x(t) \to 0,\ y(t) \to y_0 e^{-x_0/y_0}$, *as* $t \to +\infty$.

5.3 The case $\alpha > 0$

This situation is considerably more delicate than the $\alpha = 0$ situation of the last section, since now (5.1) involves balances between positive and negative contributions that are not always easy to handle.

Generally speaking we can cut the positive cone of $\mathbb{R}^2$ into three disjoint (time dependent) sets determined by the sign of the functions in the right-hand side of (5.1).

To fix notation, for $t > 0$, let

$$\begin{aligned}
\Sigma_1(t) &\coloneqq \left\{(x,y) \in \mathbb{R}^{2+} : \alpha t^\omega - xy < 0\right\},\\
\Sigma_2(t) &\coloneqq \left\{(x,y) \in \mathbb{R}^{2+} : \alpha t^\omega - xy - x^2 < 0,\ \alpha t^\omega - xy > 0\right\},\\
\Sigma_3(t) &\coloneqq \left\{(x,y) \in \mathbb{R}^{2+} : \alpha t^\omega - xy - x^2 > 0\right\},
\end{aligned}$$

and let $\Gamma_j(t) \coloneqq \partial\Sigma_j(t) \cap \partial\Sigma_{j+1}(t)$, for $j = 1, 2$.

It follows easily from the definition of the sets Σ_i that a solution (x, y) to (5.1) satisfies $x',\ y' < 0$ in $\Sigma_1(t)$, $x' < 0,\ y' > 0$ in $\Sigma_2(t)$, and $x',\ y' > 0$ in $\Sigma_3(t)$.

We sum up this geometric behaviour in Figure 5.2. It is important to notice that

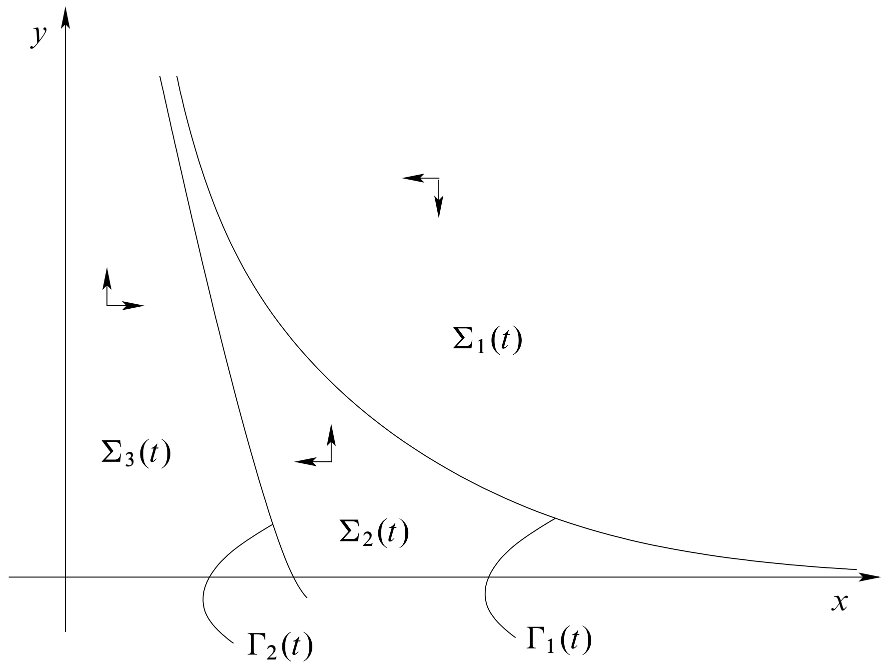

Figure 5.2: Geometry of the sets associated with monotonicity behaviour of solutions to (5.3). The arrows indicate the monotonicity property of the x and y components of the solution vector ($\leftarrow$ means $x(t)$ decreasing, etc.).

the picture in Figure 5.2 is a *dynamic* one, and as $t \to +\infty$, the sets $\Gamma_1(t)$ and $\Gamma_2(t)$ converge to the x and the y axis, and to the y axis, respectively, uniformly in the second case, and uniformly in compact subsets in the first case.

The dynamics of (5.1) is very much dependent on the way the solutions $\varphi(t) =$

$(x(t), y(t))$ visit the sets $\Sigma_j(t)$, and it will be important to know under what circumstances is $\varphi(t)$ able to cross from one $\Sigma_j(t)$ to an adjacent one.

This is fairly easy to answer for $\Sigma_2(t)$ and $\Sigma_3(t)$, and it is done in the next Proposition.

Proposition 5.2 *Let $\varphi(t)$ be a non-negative solution to* (5.1) *with initial condition $\varphi(t_0) \in \Sigma_3(t_0)$.*

Then, there exists a unique $T > t_0$ such that $\varphi(T) \in \Gamma_2(T)$ and $\varphi(t) \in \Sigma_3(t)$ for all $t < T$, and $\varphi(t) \notin \Sigma_3(t)$ for all $t > T$.

Proof. Suppose the solution $\varphi(t)$ remains in $\Sigma_3(t)$ for all $t > T$. This implies that the solution satisfies $x'(t) > 0$ and $y'(t) > 0$ for all $t > t_0$, and thus the solution must remain both in $\Sigma_3(t)$ and in the translation of the positive cone of $\mathbb{R}^2$ with vertex at $\varphi(t_0)$. But this is impossible since $\Gamma_2(t)$ converges uniformly to the $y-$axis and so sweeps the (bounded) part of the cone where $\varphi(t)$ lies in finite time.

Actually it is immediately clear that $\Gamma_2(t)$ passes through $\varphi(t_0)$ at some time $t < (\frac{x_0}{\sqrt{\alpha}})^{2/\omega}$, and so $\varphi(t)$ cannot be in $\Sigma_3(t)$ at this moment in time. Hence there must exist T such that $\varphi(t) \in \Gamma_2(T)$.

At this point $\varphi'(T) = (0, x^2(T))$ and the $y-$component of $\Gamma_2(T)$ is strictly decreasing. So, for a sufficiently close later time $t > T$ the solution $\varphi(t)$ must be in $\Sigma_2(t)$. But now $\varphi(t)$ cannot cross Γ_2 back into $\Sigma_3(t)$: suppose it could cross back and let $t_1 > T$ be the smallest time for which $\varphi(t_1) \in \Gamma_2(t_1)$; then by the definition of Γ_2 we would have $x'(t_1) = 0$ and $y'(t_1) > 0$ while the curve $\Gamma_2(t)$ moves downwards. This means that $\varphi(t)$ cannot enter $\Sigma_3(t)$ for $t > t_1$, a contradiction. This concludes the proof. ∎

Another general feature of the dynamics of (5.1) with $\alpha > 0$ is obtained using the first integral of (5.3) defined in Proposition 5.1.

Proposition 5.3 *Let $\varphi(t)$ be a non-negative solution of* (5.1) *and let*

$$\Phi(x, y) = \frac{x}{y} - \log y, \tag{5.4}$$

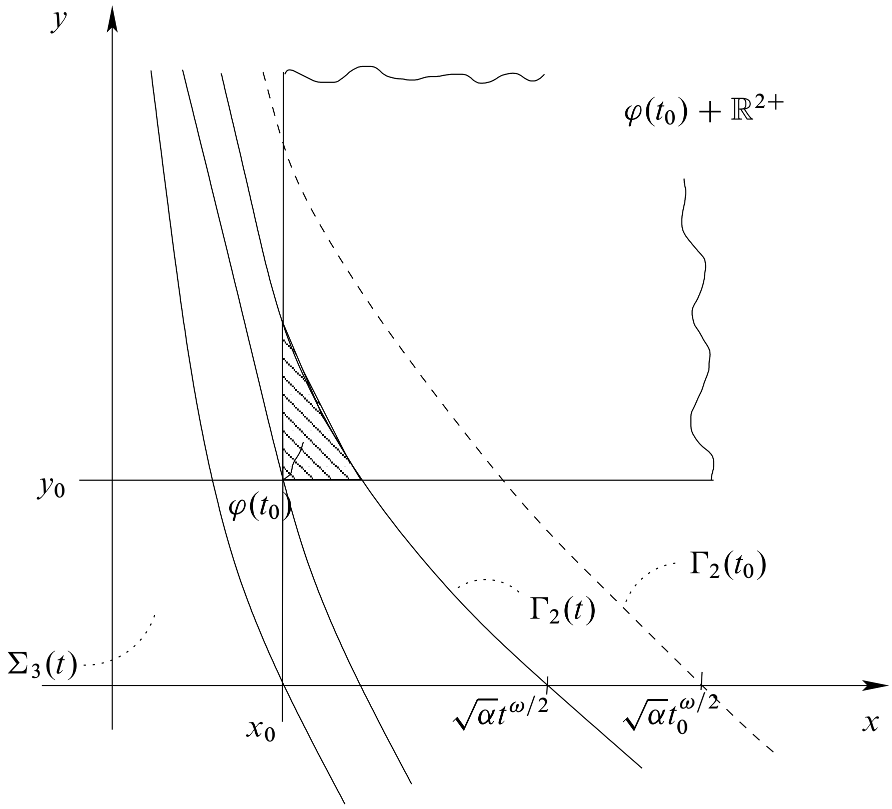

Figure 5.3: The geometry of the problem when $\varphi(t_0) \in \Sigma_3(t_0)$.

then Φ is decreasing along φ, and, as a consequence $\underline{\lim}_{t\to+\infty} y(t) > 0$.

Proof. By computing

$$\begin{aligned}\nabla\Phi(x,y)\cdot\varphi'(t) &= \left(\frac{1}{y}, -\frac{x}{y^2} - \frac{1}{y}\right)\cdot\left(\alpha t^\omega - xy - x^2, \alpha t^\omega - xy\right)\\ &= -\frac{x}{y^2}\,\alpha t^\omega < 0,\end{aligned}$$

we conclude that Φ decreases along solutions. By observing the foliation of $\mathbb{R}^{2+}$ provided by the level sets of Φ, see Figure 5.4, we immediately conclude the lower bound on $y(t)$ holds as stated. ∎

The previous two Propositions sum up all that can be said in general about solutions to (5.1) without further restrictions on the value of ω. In what follows we need to study the cases $\omega \in (-1, -1/2)$, $\omega = -1$, and $\omega < -1$ separately.

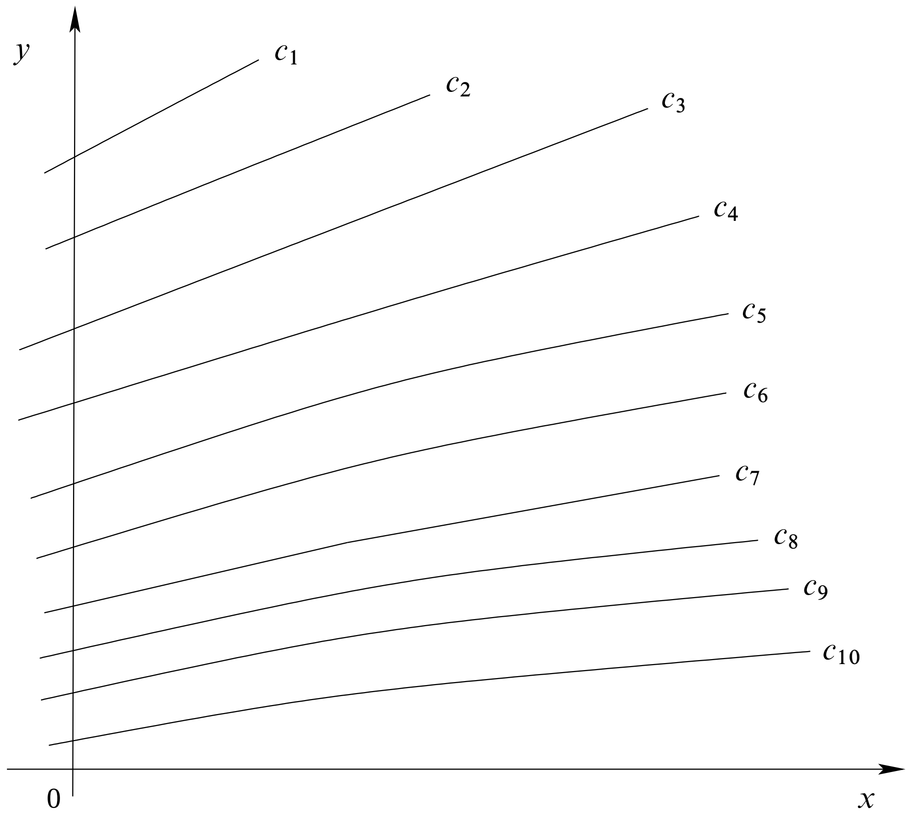

Figure 5.4: Foliation of $\mathbb{R}^{2+}$given by the level sets of Φ, $\Phi(x, y) = c$ (several values of $c = c_j$ with $c_j < c_k$ if $j < k$ are plotted).

The main distinctive feature in these three cases is that solutions $\varphi(t)$ will eventually remain in $\Sigma_2(t)$ in the first case, in $\Sigma_1(t)$ in the third, while the case $\omega = -1$ is a borderline case requiring a different approach, and we treat it last. So the size of ω will determine in which way the orbit will ultimately cross $\Gamma_1(t)$. The last result of this section clarifies how the solutions to (5.1) cross $\Gamma_1(t)$ and will be needed for the remaining sections of this Chapter.

Proposition 5.4 *Let $\varphi(t) = (x(t), y(t))$ be any non-negative solution to* (5.1). *Let $h(t) := x(t)y(t)$.*

Then there exists a unique nodal point

$$N_*(t) = (x_*(t), y_*(t)) := \left(-\frac{\omega}{t}, -\frac{\alpha}{\omega}\, t^{\omega+1}\right) \in \Gamma_1(t)$$

such that $\frac{\mathrm{d}}{\mathrm{d}t}(h(t) - \alpha t^{\omega})$ *is zero at the nodal point, is positive at points of* $\Gamma_1(t)$ *above the nodal point (i.e., with* $x(t) < x_*(t)$ *), and is negative below it.*

Proof. Using the differential equations (5.1) we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}(h(t) - \alpha t^{\omega}) = -(h(t) - \alpha t^{\omega})(2x(t) + y(t)) + \alpha t^{\omega}(-\frac{\omega}{t} - x(t)).$$

At a point on $\Gamma_1(t)$ the first term on the right-hand side of this expression is zero, giving

$$\frac{\mathrm{d}}{\mathrm{d}t}(h(t) - \alpha t^{\omega}) = \alpha t^{\omega}(-\frac{\omega}{t} - x(t)),$$

and this proves the result, since on $\Gamma_1(t)$ we have $y = \alpha t^{\omega}/x$. ∎

Remark 5.5 *We should note that* $\varphi(t)$ *can cross* $\Gamma_1(t)$ *coming from* $\Sigma_2(t)$ *into* $\Sigma_1(t)$ *only above the nodal point* $\varphi_*(t)$*, and, reciprocally, the crossing from* $\Sigma_1(t)$ *into* $\Sigma_2(t)$ *can only occur below* $\varphi_*(t)$*.*

Remark 5.6 *It is important to observe that, being* (5.1) *a non-autonomous system, the situation of an orbit crossing from* $\Sigma_2(t)$ *into* $\Sigma_1(t)$ *does not (necessarily) mean that its* $x-$*component is increasing: in fact, since* $\Gamma_1(t)$ *is actually moving towards the coordinate axes, what the crossing actually means is that the orbit, while also moving towards the* $y-$*axis, is overtaken by the moving* $\Gamma_1(t)$ *. Hence, an orbit* $\varphi(t)$ *that eventually would cross* $\Gamma_1(t)$ *a number of times, as in the scheme in Figure 5.5, does not really correspond to an oscillating solution, but can simply correspond to the situation depicted in Figure 5.6.*

5.3.1 The small input exponent case $\omega \in (-1, -1/2)$

From Proposition 5.2 we already know that every solution to (5.1) must eventually leave $\Sigma_3(t)$ never to return. We now prove that, if $\omega \in (-1, -1/2)$, the same holds true for $\Sigma_1(t)$, and thus solutions must eventually stay in $\Sigma_2(t)$ and converge to a limit point in the $y-$axis with a definite rate. However, this is somewhat less straightforward than the situation of Proposition 5.2.

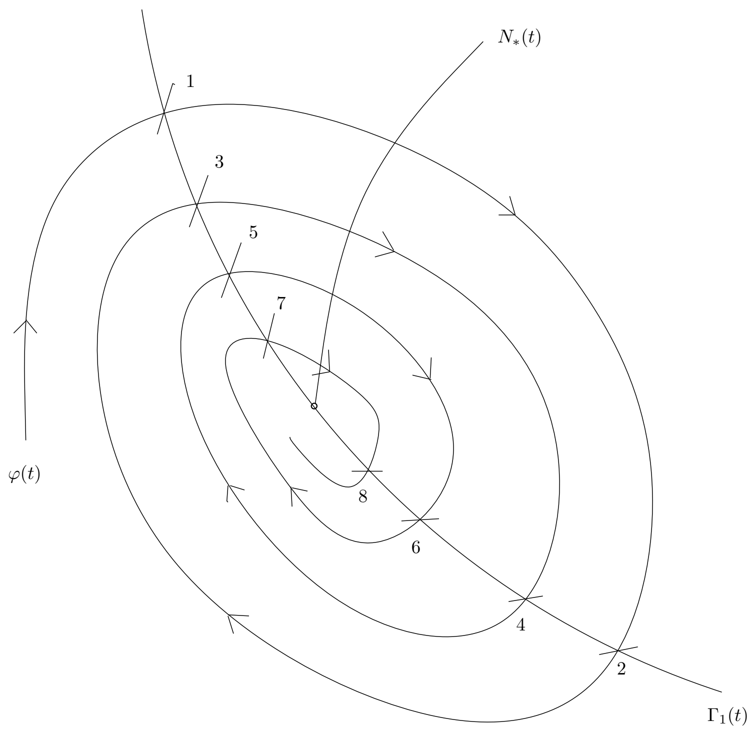

Figure 5.5: A solution repeatedly crossing back and forth between $\Sigma_2(t)$ and $\Sigma_1(t)$, plotted in a coordinate frame centered at the (moving) nodal point $N_*(t)$ and for which $\Gamma_1(t)$ is also fixed.

We start by proving that a solution with initial condition $\varphi(t_0) \in \Sigma_1(t_0)$ cannot remain in $\Sigma_1(t)$ forever.

Proposition 5.7 *Let $\varphi(t)$ be the solution of* (5.1) *with initial conditions $\varphi(t_0) \in \Sigma_1(t_0)$.*

Then, there exists a time $t > t_0$ at which $\varphi(t) \in \Sigma_2(t)$.

Proof. Assume that $\varphi(t)$ remains in $\Sigma_1(t)$ for all $t \geqslant t_0$. Then both $x'(t)$ and $y'(t)$ are negative, and $x'(t) = \alpha t^\omega - x(t)y(t) - x^2(t) = y'(t) - x^2(t) < -x^2(t)$, this implies, as we have already seen, that the following result holds true for all $t > t_0$

$$x(t) < \frac{1}{x(t_0)^{-1} + (t - t_0)} \xrightarrow[t \to +\infty]{} 0. \tag{5.5}$$

Since $y'(t) < 0$ implies that $y(t) < y(t_0)$ for all $t > t_0$, we obviously conclude

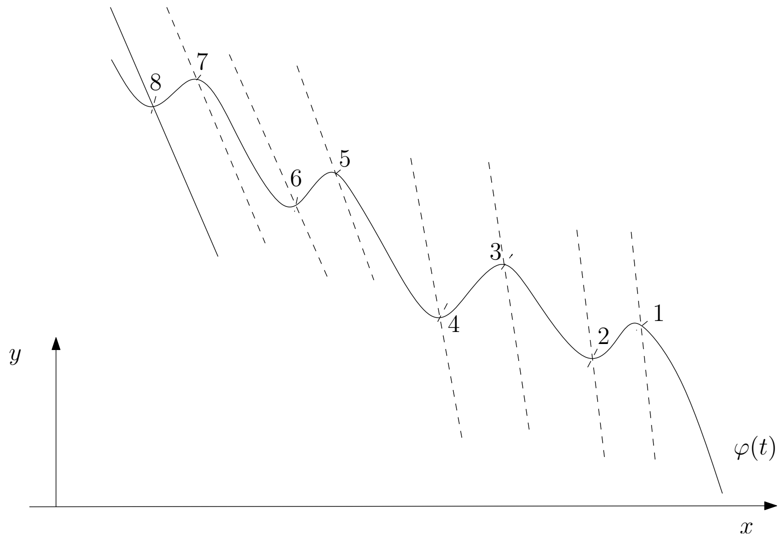

Figure 5.6: Plot of the solution depicted in Figure 5.5 in the fixed cartesian frame. The dashed lines are the lines $\Gamma_1(t)$ at successively higher values of t.

that

$$x(t)y(t) < x(t)y(t_0) < \frac{x(t_0)y(t_0)}{1 + x(t_0)(t - t_0)} < \frac{K}{t},$$

where $K > y(t_0)\max\{1, t_0\, x(t_0)\}$.

But then, if $t > (K/\alpha)^{1/(1+\omega)}$ we have $\alpha t^\omega - x(t)y(t) > \alpha t^\omega - K/t > 0$, which implies that, for those times, $\varphi(t)$ is not in $\Sigma_1(t)$, as assumed. Since by Proposition 5.2 , it can only be in $\Sigma_2(t)$, this proves the result. ∎

Observe that the previous result does not rule out the possibility that the solution $\varphi(t)$ leaves $\Sigma_2(t)$ and enters $\Sigma_1(t)$ at a later time, and proceeds in this way indefinitely.

Note however that, if $\varphi(t)$ does this, it must have the behaviour relative to the nodal point $N_*(t) \in \Gamma_1(t)$ depicted in Figure 5.5, and, as $N_*(t)$ escapes to infinity as $t \to +\infty$, the orbit could not remain bounded in y.

Our next result, Proposition 5.8, excludes this possibility by establishing that $\varphi(t)$ will eventually remain in $\Sigma_2(t)$ for all sufficiently large t.

Still, this does not yet preclude $y(t)$ from being unbounded since in $\Sigma_2(t)$ the derivative $y'(t)$ is positive. The boundedness of y will be established in Proposition 5.9.

Proposition 5.8 *Let $\varphi(t_0) \in \mathbb{R}^{2+}$ and let $\varphi(t)$ be the solution to* (5.1) *with initial condition $\varphi(t_0) \in \Sigma_2(t_0)$.*

Then, there exists $T = T(t_0, \varphi(t_0)) \geqslant t_0$ such that $\varphi(t) \in \Sigma_2(t)$ for all $t > T$.

Proof. Suppose that the orbit does not eventually stay in $\Sigma_2(t)$, then, from Proposition 5.7, we know that it will cross $\Gamma_1(t)$, from $\Sigma_2(t)$ into $\Sigma_1(t)$ in an unbounded sequence of times. But then, by Proposition 5.4, this means that the $y-$component of the solution $\varphi(t)$ must cross the value of the $y-$component of the nodal point $N_*(t) = (x_*(t), y_*(t))$ in a monotonically increasing sequence of times $t_n > t_0$, for which $\varphi(t_n) \in \Sigma_2(t_n)$.

Let r be the smallest integer for which

$$t_r^{1+\omega/2} > -\frac{\omega}{\alpha}\sqrt{-\frac{\alpha\omega}{1+\omega}}.$$

Then, by construction $y_r \coloneqq y(t_r) = y_*(t_r) = -\frac{\alpha}{\omega} t_r^{\omega+1}$.

Since $\Sigma_2(t)$ is an open set there exists $\tau > t_r$ such that $\varphi(t) \in \Sigma_2(t)$ for all $t \in (t_r, \tau)$. We shall prove that for these values of t, we have $y(t) < y_r$. This contradicts the assumption that $\varphi(t)$ does not remain in $\Sigma_2(t)$ for all sufficiently large times, since this inequality implies $y(t) < y_*(t_r)$ and so $\varphi(t)$ cannot cross into $\Sigma_1(t)$, by Proposition 5.4.

Let $t \in (t_r, \tau)$. Then

$$\frac{\alpha t^{\omega/2}}{y_r} - \sqrt{-\frac{\alpha(1+\omega)}{\omega}} < \frac{\alpha t_r^{\omega/2}}{y_r} - \sqrt{-\frac{\alpha(1+\omega)}{\omega}} < 0.$$

Since $\varphi(t) \in \Sigma_2(t)$, we have $y'(t) = \alpha t^\omega - x(t)y(t) > 0$ and so

$$x(t) < \frac{\alpha t^\omega}{y} < \frac{\alpha t^\omega}{y_r} = \frac{\alpha t^{\omega/2}}{y_r} t^{\omega/2} < \sqrt{-\frac{\alpha(1+\omega)}{\omega}}\, t^{\omega/2},$$

which implies $x^2(t) < -\frac{\alpha(1+\omega)}{\omega} t^\omega$. But because the solution $\varphi(t)$ is in $\Sigma_2(t)$ we also have

$$x'(t) = \alpha t^{\omega} - x(t)y(t) - x^2(t) < 0,$$

and hence

$$x(t)y(t) > \alpha t^{\omega} - x^2(t) > \alpha t^{\omega} + \alpha \frac{(1+\omega)}{\omega} t^{\omega},$$

from which it follows that

$$-\alpha \frac{(1+\omega)}{\omega} t^{\omega} - (\alpha t^{\omega} - x(t)y(t)) > 0. \tag{5.6}$$

By the definition of the nodal point in Proposition 5.4 we have $y_*(t) = -\frac{\alpha}{\omega} t^{1+\omega}$, and so

$$\frac{\mathrm{d}y_*}{\mathrm{d}t} = -\alpha \frac{(1+\omega)}{\omega} t^{\omega},$$

which, together with (5.6) and (5.1), implies that

$$\frac{\mathrm{d}y_*}{\mathrm{d}t} - y'(t) > 0. \tag{5.7}$$

Since at t_r we have $y_*(t_r) = y(t_r) = y_r$, then, from (5.7) we have $(y_* - y)' > 0$, and we conclude $y_*(t) > y(t)$. This concludes the proof. ∎

Proposition 5.9 *With the same assumptions of Proposition 5.8, the $y-$component of the solution $\varphi(t)$ is bounded.*

Proof. By Propositions 5.2 and 5.8, we need only to prove that $\varphi(t)$ remains in a bounded subset of $\Sigma_2(t)$. In $\Sigma_2(t)$ the solution satisfies $y'(t) > 0$, and so $x(t) < \frac{\alpha t^{\omega}}{y(t)}$. Since also $x'(t) < 0$, we deduce from (5.1) that

$$y'(t) = x'(t) + x^2(t) < x^2(t) < \alpha^2 \frac{t^{2\omega}}{y^2(t)},$$

from which it follows that

$$y^3(t) < y^3(t_0) - \frac{3\alpha^2}{2\omega + 1} \left(t_0^{2\omega+1} - t^{2\omega+1} \right).$$

Applying $\overline{\lim}_{t\to+\infty}$ to both sides of the last inequality results in

$$\overline{\lim_{t\to+\infty}}\, y^3(t) \leqslant y^3(t_0) - \frac{3\alpha^2}{2\omega+1} t_0^{2\omega+1},$$

from which it follows that y is bounded. This concludes the proof. ∎

The above results allow us now to state and prove the main result of this section.

Proposition 5.10 *Let $\varphi(t) = (x(t), y(t))$ be any non-negative solution to* (5.1) *with initial data $\varphi(t_0) = (t_0, x_0, y_0)$.*

Then, there exists a constant $L = L(t_0, x_0, y_0)$ such that,

$$y(t) \longrightarrow L \quad \textit{and} \quad \alpha^{-1}t^{-\omega}x(t) \longrightarrow L^{-1} \quad \textit{as} \quad t \longrightarrow +\infty.$$

Proof. From Propositions 5.2, 5.8 and 5.9 we can conclude that y is eventually monotonically increasing and bounded above, and so there must exist a constant $L(t_0, x_0, y_0)$ such that $y(t) \to L$ as $t \to +\infty$. By Proposition 5.3 we conclude that $L > 0$.

Actually, as the computation in the proof of Proposition 5.3 shows, since $\Phi(x, y) = \frac{x}{y} - \log y$ is decreasing along solutions, we can conclude that $L > e^{\alpha t_0^{\omega}}$.

Now, for the convergence rate of the $x-$component, we need only to notice that the same Propositions imply that $x'(t) < 0 \Leftrightarrow x^2(t) + y(t)x(t) - \alpha t^{\omega} > 0$ and thus, seeking only positive solutions

$$x(t) > -\frac{1}{2}y(t) + \sqrt{\frac{1}{4}y^2(t) + \alpha t^{\omega}} = \frac{\alpha t^{\omega}}{\frac{1}{2}y(t) + \sqrt{\frac{1}{4}y^2(t) + \alpha t^{\omega}}}.$$

Furthermore, since $y'(t) > 0$, equation (5.1) implies that $x(t) < \frac{\alpha t^{\omega}}{y(t)}$. Hence

$$\frac{2}{y(t) + \sqrt{y^2(t) + 4\alpha t^{\omega}}} < \alpha^{-1}t^{-\omega}x(t) < \frac{1}{y(t)},$$

and applying limits as $t \to +\infty$, remembering that $y(t) \to L$, concludes the proof. ∎

5.3.2 The very small input exponent case $\omega < -1$

In the case $\omega \in (-1, -1/2)$, studied in the previous section, all solutions will eventually end up in $\Sigma_2(t)$. As in that region the solution satisfies $y'(t) > 0$ we need somewhat careful estimates on y in order to conclude its boundedness. On the other hand, the fact that the orbits stay in $\Sigma_2(t)$, and thus between the curves $\Gamma_1(t)$ and $\Gamma_2(t)$, immediately gives us upper and lower bounds on the rate of convergence of $x(t)$ to zero.

For the present case $\omega < -1$, a different situation occurs. We will prove that all solutions will eventually end up in $\Sigma_1(t)$. Since in that region we always have that $y'(t) < 0$, we immediately conclude that y, eventually monotonic decreasing, is bounded, and, from Proposition 5.3, it must converge to a $y_\infty > 0$. But now we only have a natural lower bound on the solution, arising from the boundary $\Gamma_1(t)$ of $\Sigma_1(t)$. This produces a natural lower bound on the rate of convergence of $x(t)$ to zero. An optimal upper bound is a much trickier question we shall discuss later.

We start by a kind of dual to Proposition 5.7 that just states that no solution to (5.1) can stay in $\Sigma_2(t)$ forever.

Proposition 5.11 *Let $\varphi(t)$ be a solution of* (5.1) *with initial datum $\varphi(t_0) \in \Sigma_2(t_0)$. Then, there exists a time $t > t_0$ at which $\varphi(t) \in \Sigma_1(t)$.*

Proof. The proof is quite similar to that of Proposition 5.7; assume $\varphi(t)$ stays in $\Sigma_2(t)$ for all $t \geqslant t_0$.

In that case we would have $y'(t) > 0$ and $x'(t) = \alpha t^\omega - xy - x^2 = y' - x^2 > -x^2$ which implies

$$x(t) > \frac{x(t_0)}{1 + x(t_0)(t - t_0)}.$$

But the positivity of $y'(t)$ implies that $y(t) > y(t_0)$ for all $t > t_0$ and we obviously conclude that

$$x(t)y(t) > x(t)y(t_0) > \frac{x(t_0)y(t_0)}{1 + x(t_0)(t - t_0)},$$

and, since $\omega < -1$, we conclude that, for all t sufficiently large

$$x(t)y(t) - \alpha t^{\omega} > \frac{x(t_0)y(t_0)}{1 + x(t_0)(t - t_0)} - \frac{\alpha}{t^{-\omega}} > 0,$$

which contradicts the assumption that the solution stays in $\Sigma_2(t)$. This concludes the proof. ∎

As was the case for $\omega \in (-1, -1/2)$, under the present conditions we cannot a priori rule out the possibility that the solution visits $\Sigma_2(t)$ repeatedly at an unbounded sequence of later times (t_n). Again, for this to be the case the solution would need to have the behaviour relative to the nodal point $N_*(t) \in \Gamma_1(t)$ as depicted in Figure 5.5.

Our next result prevents this from happening.

Proposition 5.12 *All solutions $\varphi(t)$ to* (5.1) *will stay in $\Sigma_1(t)$ for all $t > t_0$ sufficiently large.*

Proof. Suppose otherwise. Then, there would exist an unbounded sequence of times, (t_n), such that $\varphi(t_n) = (x(t_n), y(t_n)) \in \Sigma_2(t_n)$, $\forall n$.

But then, there would exist another unbounded sequence of times $\tau_n \in (t_n, t_{n+1})$ such that $y(\tau_n) = y_*(\tau_n)$, where $N_*(t) = (x_*(t), y_*(t))$ is the nodal point. Now note that this nodal point $N_*(t)$ is given by $(-\frac{\omega}{t}, -\frac{\alpha}{\omega}t^{\omega+1})$, and since $\omega < -1$, it will converge to $(0, 0)$ as $t \to +\infty$.

But this would mean that the solution $\varphi(t)$ would not be bounded away from zero, which contradicts Proposition 5.3. This concludes the proof. ∎

Proposition 5.13 *Let $\varphi(t) = (x(t), y(t))$ be any non-negative solution of* (5.1) *with initial data $\varphi(t_0) = (t_0, x_0, y_0)$.*

Then, there exists a constant $L = L(t_0, x_0, y_0) > 0$ such that, as $t \to +\infty$

$$y \to L, \quad \varliminf_{t \to +\infty} \alpha^{-1} t^{-\omega} x(t) \geqslant L^{-1}, \text{ and } \varlimsup_{t \to +\infty} \alpha^{-1} t^{-\omega-\varepsilon} x(t) \leqslant L^{-1}, \ \forall \varepsilon > 0.$$

Proof. The convergence of y is trivial. Indeed, by Proposition 5.12 all solutions will eventually stay in $\Sigma_1(t)$. In this region $y'(t) < 0$, and so $y(t)$ is monotonically decreasing (and bounded below by zero), and hence it is convergent.

By Proposition 5.3 the limit must be positive.

The first inequality on the rate of convergence is easily obtained: since $\varphi(t) \in \Sigma_1(t)$, we know that $x(t)y(t) > \alpha t^{\omega}$, and, using the convergence of $y(t)$ as $t \to +\infty$, we obtain the result.

From the fact that $\varphi(t) \in \Sigma_1(t)$ we can only obtain one natural bound, resulting from the bound $\Gamma_1(t)$ provides, and this is what we just used.

To obtain the other inequality we need a more delicate analysis, only part of which was successfully completed thus far.

5.3.3 The case $\omega = -1$

The analysis we performed in the preceding sections fails when $\omega = -1$, and so we try the same approach we used in the $\omega \geqslant -1/2$ cases.

We suppose that $x(t)$ behaves like α/t as $t \to +\infty$, and hence the new variable $X(t) = \alpha^{-1}tx(t)$ behaves like a constant as $t \to +\infty$.

Performing the change of variable $X(t) = \alpha^{-1}tx(t)$, (and $Y(t) = y(t)$), in (5.1) we obtain the system

$$\begin{cases} X' = 1 - XY - \dfrac{\alpha}{t} X \left(X - \dfrac{1}{\alpha} \right) \\ Y' = \dfrac{\alpha}{t} (1 - XY), \end{cases}$$

or, writing everything back in x and y :

$$\begin{cases} x' = 1 - xy - \dfrac{\alpha}{t} x \left(x - \dfrac{1}{\alpha} \right) \\ y' = \dfrac{\alpha}{t} (1 - xy). \end{cases} \tag{5.8}$$

We define the four regions

$$
\begin{aligned}
A &:= \left\{(x, y) \in \mathbb{R}^{2+} \colon \frac{1}{x} > \alpha \text{ and } \frac{1}{x} > y\right\},\\
B &:= \left\{(x, y) \in \mathbb{R}^{2+} \colon \frac{1}{y} < x < \frac{1}{\alpha}\right\},\\
C &:= \left\{(x, y) \in \mathbb{R}^{2+} \colon x > \frac{1}{\alpha} \text{ and } x > \frac{1}{y}\right\},\\
D &:= \left\{(x, y) \in \mathbb{R}^{2+} \colon \frac{1}{\alpha} < x < \frac{1}{y}\right\}.
\end{aligned}
$$

Notice that when $(x(t), y(t)) = (1/\alpha, \alpha)$, which doesn't depend on t, in (5.8) both $x'(t)$ and $y'(t)$ are zero and hence we have an equilibrium point.

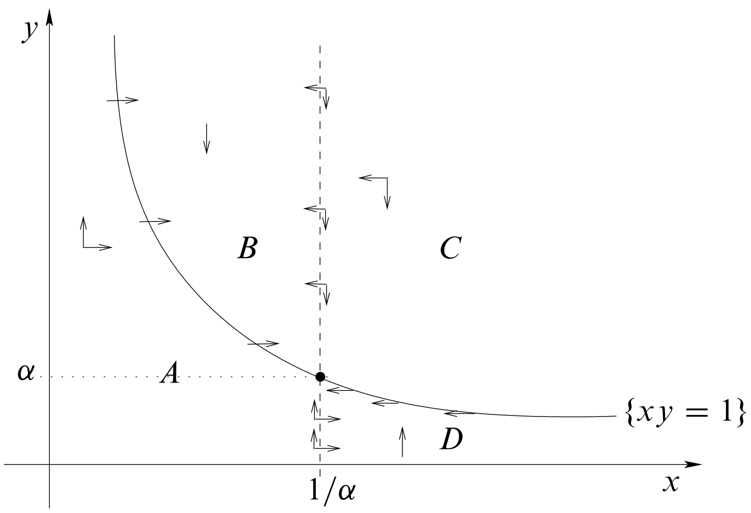

Figure 5.7: Geometry of the sets associated with monotonicity behaviour of solutions to (5.8). The arrows indicate the monotonicity property of the x and y components of the solution vector ($\leftarrow$ means $x(t)$ decreasing, etc.)

Proposition 5.14 *The $y-$component of solutions to* (5.8) *is bounded.*

Proof. In regions B and C in Figure 5.7 we have $y' < 0$, and hence y is bounded. In region D, $y' > 0$ but $y < \alpha$ and hence the only way for the $y-$component of the orbit to escape to $+\infty$, is in region A, since once the orbit leaves A, for B or D, it cannot come back into A again. But in region A, we have $x < 1/\alpha$, and hence x, being increasing, is convergent in A. But by definition, in A we have $y < 1/x$ and hence if x is bounded in A, so is y. This concludes the proof that y is bounded. ∎

The same result also holds for the $x-$component of solutions to (5.8)

Proposition 5.15 *The $x-$component of solutions to* (5.8) *is bounded.*

Proof. In regions A and B we have $x < 1/\alpha$, and so x is bounded in A and B. In region C we have $x' < 0$, and so x is bounded in C, and once the orbit leaves C, for either B or D, it cannot come back in C. Since as mentioned x is bounded in B, the only remaining interesting option is when the orbit leaves C for D. Now in D we have $y' > 0$, with y bounded by the (fixed) hyperbola $\{xy = 1\}$, and hence y converges to some positive limit: $y \to y_\infty$. Considering that in D any horizontal straight line, with $y_\infty > 0$, $\{(x, y_\infty) : 1/\alpha < x < 1/y_\infty\}$, is a bounded set, we conclude that the orbit must have a bounded x-component (since for t in any compact set, the x-component is bounded, and in any set $[\hat{t}, +\infty)$ to the bounded set above, and hence x is also bounded). ∎

Having shown that the solutions are bounded, we now show that they are convergent.

Proposition 5.16 *Solutions to* (5.8) *are convergent.*

Proof. Suppose by contradiction that it does *not* converge. If at some instant the orbit is in region A, then since in A, $x < 1/\alpha$ and x is increasing, the x-component is obviously convergent, to a non-zero limit. Since in A we have $y < 1/x$, and y is increasing it follows that it must converge. Hence if it diverges, then it must necessarily leave region A for region B or D, and if it leaves A it cannot come back.

If the orbit goes into region B, then it cannot escape from region B, since it cannot cross back into A and it cannot go into region C. Although it might fail to converge to the equilibrium point $(1/\alpha, \alpha)$, we know y is decreasing and bounded from below by α in B and hence convergent. Since in B we know that x is bounded above by $1/\alpha$ and below by $1/y$, which converges, it follows that $\frac{\alpha}{t}(x - \frac{1}{\alpha})$ goes to zero as $t \to +\infty$ and hence from the first equation in (5.8) that $1 - xy$ must converge to zero, and hence, y being convergent, that x is convergent.

If the orbit goes into region D we observe that it cannot leave. In region D we have that y is increasing and is bounded by $1/\alpha$, and hence converges; from this we obtain x convergence as in the case of region B.

Finally if we are in region C we have that x is bounded below and x is decreasing and hence converges; in C we know that y decreases and is bounded, and hence converge. If it escapes from C into region B, we already know that it converges there. ∎

5.3.4 Long time behaviour of solutions

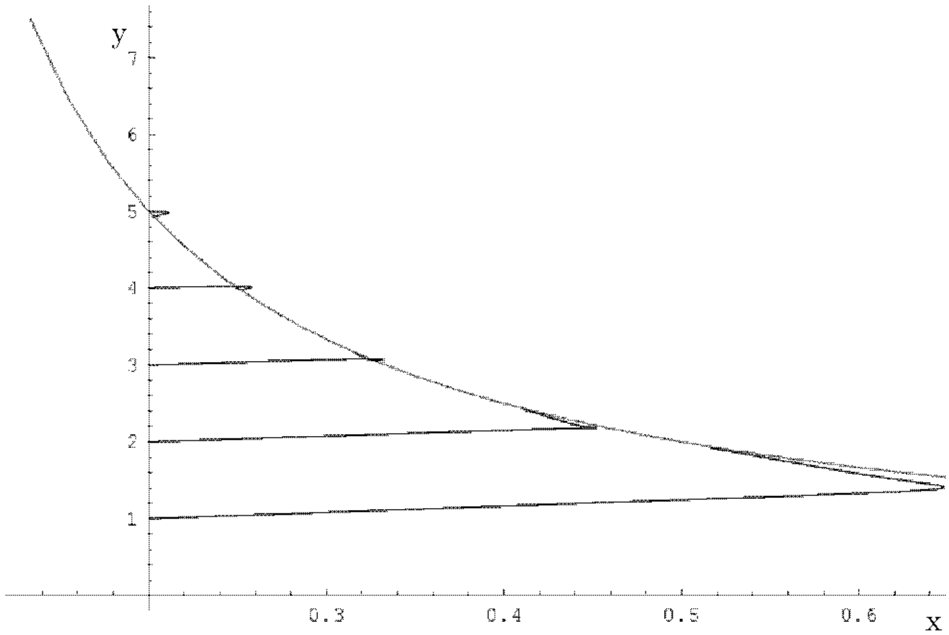

Figure 5.8: Simulation done for $\omega = -0.6$, $\alpha = 1$, $t_0 = 1$, $x_0 = 0.2$, and y_0 varying between 1 and 5 in steps of 1. Also shown is the hyperbola $y = 1/x$.

Having established convergence of solutions of (5.8), we would like to be able to identify their limit. We have already seen in Figure 5.4 the foliation of $\mathbb{R}^{2+}$

provided by the level sets of the function Φ defined by (5.4) that the limit of solutions depends on the initial conditions.

In Figure 5.8 we show a simulation done for $\omega = -0.6$, $\alpha = 1$, $t_0 = 1$, $x_0 = 0.2$, and y_0 varying between 1 and 5 in steps of 1. Also shown is the hyperbola $y = 1/x$.

This, and all the other simulations we ran for $\alpha = 1$ and for different values of ω, point to the *conjecture* that choosing different initial conditions we obtain different limit points; we have not been able (yet) to prove it is the case.

This conjecture also means that there are no self-similar solutions.

Notice that this behaviour markedly constrasts with what we have seen in the previous cases. where we could identify a unique limit point

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