

# Matrix Theory for Independence Algebras

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## Abstract

A universal algebra  $\mathbb{A}$  with underlying set  $A$  is said to be a matroid algebra if  $(A, \langle \cdot \rangle)$ , where  $\langle \cdot \rangle$  denotes the operator *subalgebra generated by*, is a matroid. A matroid algebra is said to be an independence algebra if every mapping  $\alpha : X \rightarrow A$  defined on a minimal generating  $X$  of  $\mathbb{A}$  can be extended to an endomorphism of  $\mathbb{A}$ . These algebras are particularly well-behaved generalizations of vector spaces, and hence they naturally appear in several branches of mathematics, such as model theory, group theory, and semigroup theory.

It is well known that matroid algebras have a well-defined notion of dimension. Let  $\mathbb{A}$  be any independence algebra of finite dimension  $n$ , with at least two elements. Denote by  $\text{End}(\mathbb{A})$  the monoid of endomorphisms of  $\mathbb{A}$ . In the 1970s, Głazek proposed the problem of extending the matrix theory for vector spaces to a class of universal algebras which included independence algebras. In this paper, we answer that problem by developing a theory of matrices for (almost all) finite-dimensional independence algebras.

In the process of solving this, we explain the relation between the classification of independence algebras obtained by Urbanik in the 1960s, and the classification of finite independence algebras up to endomorphism-equivalence obtained by Cameron and Szabó in 2000. (This answers another question by experts on independence algebras.) We also extend the classification of Cameron and Szabó to all independence algebras.

The paper closes with a number of questions for experts on matrix theory, groups, semi-groups, universal algebra, set theory or model theory.

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## 1 Introduction

Let  $A$  be a non-empty set. For any integer  $k \geq 0$ , a  $k$ -ary operation on  $A$  is a function  $f : A^k \rightarrow A$ , where  $A^0 = \{\emptyset\}$  [12, p. 3]. The number  $k$  is called the *arity* of  $f$ . A 0-ary operation on  $A$  is

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called a *nullary* operation. As is customary, we will identify a nullary operation  $f(\emptyset) = c$  with  $c \in A$ . An *operation* on  $A$  is a function that is a  $k$ -ary operation on  $A$  for some  $k \geq 0$ .

A *universal algebra* is a pair  $\mathbb{A} = \langle A; F \rangle$ , where  $A$  is a non-empty set (called the *universe*) and  $F$  is a set of operations on  $A$  (called the *fundamental operations*) [28, p. 8].

A *subalgebra* of a universal algebra  $\mathbb{A} = \langle A; F \rangle$  is any pair  $\langle B; G \rangle$  such that  $B$  is a non-empty subset of  $A$  that is closed under all operations in  $F$ , and  $G$  consists of all operations  $g$  on  $B$  for which there exists  $f \in F$  such that  $g = f \upharpoonright_{B^k}$ , where  $f$  is  $k$ -ary.

An *endomorphism* of  $\mathbb{A}$  is a function  $\alpha : A \rightarrow A$  preserving all fundamental operations of  $\mathbb{A}$ , that is, for all  $k \geq 1$ , if  $f$  is a  $k$ -ary fundamental operation and  $a_1, \dots, a_k \in A$ , then  $\alpha(f(a_1, \dots, a_k)) = f(\alpha(a_1), \dots, \alpha(a_k))$ ; and if  $f(\emptyset) = c$  is a nullary operation in  $F$ , then  $\alpha(c) = c$ . An *automorphism* of  $\mathbb{A}$  is a bijective endomorphism. The set  $\text{End}(\mathbb{A})$  of all endomorphisms of  $\mathbb{A}$  is a monoid under composition of functions. This monoid has the group  $\text{Aut}(\mathbb{A})$  of all automorphisms of  $\mathbb{A}$  as its group of units.

For every  $k \geq 1$  and  $1 \leq i \leq k$ , we will denote by  $p_i^k$  the  $k$ -ary projection on the  $i$ th coordinate, that is,  $p_i^k(x_1, \dots, x_k) = x_i$ . The *clone* of  $\mathbb{A}$  is the smallest set of operations on  $A$  that contains all fundamental operations of  $A$ , all projection operations, and is closed under *generalized composition* of operations [12, p. 79]. We will denote the clone of  $\mathbb{A} = \langle A; F \rangle$  by  $F_{\text{cl}}$ . (We remark that there is an alternative definition which restricts the elements of a clone to non-nullary operations. In that context, our notion of a clone would be called the *extended clone*.)

We say that universal algebras  $\mathbb{A}_1 = \langle A; F_1 \rangle$  and  $\mathbb{A}_2 = \langle A; F_2 \rangle$  are *clone equivalent* if  $(F_1)_{\text{cl}} = (F_2)_{\text{cl}}$  [28, p. 45]. It is straightforward to check that clone equivalent algebras  $\mathbb{A}_1$  and  $\mathbb{A}_2$  have the same endomorphisms. In addition,  $\mathbb{A}_1$  has a subalgebra with universe  $B$  if and only if  $\mathbb{A}_2$  does, and their subalgebras on the same universe are clone-equivalent. Moreover, if  $\mathbb{A}_1$  or  $\mathbb{A}_2$  has a basis, then both algebras have the same bases, and if  $\mathbb{A}_1$  or  $\mathbb{A}_2$  admits the notion of dimension, then both algebras have the same dimension. (The definition of these notions is given below.) In fact, as pointed out in [12, p. 82], in most contexts one can work interchangeably with two algebras that are clone equivalent. Following Marczewski and others [32, 34, 41] who studied these notions in the 1960s, we will consider *clone equivalent algebras to be identical*.

Let  $V$  be a finite-dimensional vector space over a field  $K$ , with  $\dim(V) = n \geq 1$ . It is well known that for each fixed ordered basis  $(e_1, \dots, e_n)$  for  $V$ , there exists an isomorphism between the monoid  $\text{End}(V)$  of endomorphisms (linear operators) of  $V$  and the monoid  $M_n(K)$  of  $n \times n$  matrices over  $K$  with matrix multiplication. In 1979, Glazek introduced a class of universal algebras much larger than the class of vector spaces, and suggested that the theory of matrices for vector spaces be extended to a theory of matrices for that class [25, Problem 4.6].

Here we develop a theory of matrices for the class of finite-dimensional independence algebras. These algebras, which include vector spaces and are included in the class considered in [25], are especially well suited for this type of consideration.

Our theory of matrices is built on a secondary “set of coefficients”, corresponding to the underlying field of a vector space. We make this set we constructed partial binary operations  $+$  and  $\cdot$ , i.e. these operations are not necessarily defined for all pairs of elements. Given an independence algebra and a basis, we identify elements of  $A$  with vectors over the coefficients, and endomorphisms of  $\mathbb{A}$  with matrices over the coefficients. With this identification, the action of endomorphisms and their composition correspond to natural analogues of matrix multiplication.

Let  $\mathbb{A} = \langle A; F \rangle$  be a universal algebra. For a non-empty subset  $X$  of  $A$ , we denote by  $\langle X \rangle$  the smallest (with respect to inclusion) subset of  $A$  such that  $X \subseteq \langle X \rangle$  and  $\langle X \rangle$  is the universe of a subalgebra of  $\mathbb{A}$ . Let  $\text{Con}$  be the set of nullary operations in  $F$ . As in [28, p. 35], we extend the notation  $\langle X \rangle$  to the empty set:  $\langle \emptyset \rangle = \langle \text{Con} \rangle$  if  $\text{Con} \neq \emptyset$ , and  $\langle \emptyset \rangle = \emptyset$  if  $\text{Con} = \emptyset$ . If  $X \subseteq A$ , we call  $\langle X \rangle$  the *closure* of  $X$ . The mapping  $\langle \cdot \rangle : 2^A \rightarrow 2^A; X \mapsto \langle X \rangle$  is called the *closure operator*.

Let  $X$  be a (possibly empty) subset of  $A$ . We say that  $X$  is a *generating set* for  $\mathbb{A}$  if  $\langle X \rangle = A$ ; and that  $X$  is *independent* if for all  $x \in X$ ,  $x \notin \langle X \setminus \{x\} \rangle$ . An independent generating set for  $\mathbb{A}$  is called a *basis* for  $\mathbb{A}$ .

We say that  $\mathbb{A}$  is a *matroid algebra* if it satisfies the *exchange property*: for all  $X \subseteq A$  and  $x, y \in A$ ,

$$x \in \langle X \cup \{y\} \rangle \text{ and } x \notin \langle X \rangle \implies y \in \langle X \cup \{x\} \rangle. \quad (1.1)$$

By standard arguments in matroid theory, we know that every finitely generated matroid algebra  $\mathbb{A}$  has a basis, and all bases for  $\mathbb{A}$  have the same cardinality. The *dimension* of a matroid algebra is the cardinality of one (and hence all) of its bases (for details see [26]).

*Definition 1.1.* A universal algebra  $\mathbb{A}$  is called an *independence algebra* if

1.  $\mathbb{A}$  satisfies the exchange property (1.1), and
2.  $\mathbb{A}$  satisfies the *extension property*, that is, for any basis  $X$  of  $\mathbb{A}$ , if  $\alpha : X \rightarrow A$ , then there is an endomorphism  $\bar{\alpha}$  of  $\mathbb{A}$  such that  $\bar{\alpha}|_X = \alpha$ .

Examples of independence algebras are vector spaces, affine spaces (as defined below), sets, and free  $G$ -sets.

The class of independence algebras was introduced by Gould in 1995 [26]. Her motivation was to understand the properties shared by vector spaces and sets that result in similarities in the structure of their monoids of endomorphisms. As pointed out by Gould, this notion goes back to the 1960s, when the class of  $v^*$ -algebras was introduced by Narkiewicz [34]. (The “ $v$ ” in  $v^*$ -algebras stands for “vector” since  $v^*$ -algebras were primarily seen as generalizations of vector spaces.) In fact,  $v^*$ -algebras can be defined as matroid algebras with the extension property [35], just like independence algebras. Since Gould’s paper [26],  $v^*$ -algebras have been regarded as precisely the same as independence algebras. This is not quite so, as we now explain.

Following Marczewski [31, 32], who introduced the notion of independence in universal algebras, Narkiewicz [34, 35] and Urbanik [39–41], who studied  $v^*$ -algebras, did not consider nullary operations in their definition of a universal algebra. Consequently, they defined the closure operator, which they denoted by  $[\cdot]$ , slightly differently. Let  $\mathbb{A} = \langle A; F \rangle$  be a universal algebra without nullary operations. If  $X \subseteq A$  is not empty, then  $[X]$  is the same as  $\langle X \rangle$ . Let  $\text{Con}^*$  be the set of the images of all unary constant operations contained in the clone  $F_{\text{cl}}$ . Then  $[\emptyset] = \langle \text{Con}^* \rangle$  if  $\text{Con}^* \neq \emptyset$ , and  $[\emptyset] = \emptyset$  if  $\text{Con}^* = \emptyset$ .

The effect of this difference in the definition of a universal algebra is that  $v^*$ -algebras and independence algebras are not *precisely* the same:

1. an independence algebra  $\mathbb{A} = \langle A; F \rangle$  is not a  $v^*$ -algebra if and only if  $F$  contains at least one nullary operation;
2. a  $v^*$ -algebra  $\mathbb{A} = \langle A; F \rangle$  is not an independence algebra if and only if  $|A| \geq 2$  and the clone  $F_{\text{cl}}$  of  $\mathbb{A}$  contains at least one unary constant operation.

However,  $v^*$ -algebras and independence algebras are *essentially* the same, in the following sense. Define a mapping  $\Psi$  from the class of non-trivial ( $|A| \geq 2$ )  $v^*$ -algebras to the class of non-trivial independence algebras by: if  $\mathbb{A} = \langle A; F \rangle$  is any non-trivial  $v^*$ -algebra, then  $\Psi(\mathbb{A}) = \langle A; F' \rangle$ , where  $F'$  is obtained from  $F$  by adding all nullary operations  $g_f(\emptyset) = a$  whenever there exists a constant unary operation  $f(x) = a$  in the clone  $F_{\text{cl}}$ . It is straightforward to check that  $\Psi(\mathbb{A})$  is indeed an independence algebra and that  $\Psi$  is onto (up to clone equivalence).

We now list some basic properties of the operator  $\Psi$ ; a proof of these assertions is in the appendix. Let  $\mathbb{A} = \langle A; F \rangle$ ,  $\mathbb{A}_1$ , and  $\mathbb{A}_2$  be any non-trivial  $v^*$ -algebras, and let  $\Psi(\mathbb{A}) = \langle A; F' \rangle$ .

- (a) except for the nullary operations in  $F'_{\text{cl}}$ , the clones  $F_{\text{cl}}$  and  $F'_{\text{cl}}$  are identical;
- (b) if  $\mathbb{A}_1$  and  $\mathbb{A}_2$  are clone equivalent, then  $\Psi(\mathbb{A}_1)$  and  $\Psi(\mathbb{A}_2)$  are also clone equivalent;
- (c)  $\Psi$  is a one-to-one correspondence (if clone equivalent algebras are regarded as identical);
- (d) for every  $X \subseteq A$ :

- (i)  $[X]$  in  $\mathbb{A}$  is equal to  $\langle X \rangle$  in  $\Psi(\mathbb{A})$ ,
- (ii)  $X$  is independent in  $\mathbb{A}$  if and only if  $X$  is independent in  $\Psi(\mathbb{A})$ ,
- (iii)  $X$  is a basis for  $\mathbb{A}$  if and only if  $X$  is a basis for  $\Psi(\mathbb{A})$ ;
- (e)  $\dim(\mathbb{A}) = \dim(\Psi(\mathbb{A}))$  and  $\text{End}(\mathbb{A}) = \text{End}(\Psi(\mathbb{A}))$ .

Given (a)–(e), we will identify non-trivial  $v^*$ -algebras with non-trivial independence algebras, via the mapping  $\Psi$ . That is, we will call any non-trivial  $v^*$ -algebra  $\mathbb{A}$  an independence algebra, which will mean that we replace  $\mathbb{A}$  with  $\Psi(\mathbb{A})$ . Explicitly, for each constant unary clone function of  $\mathbb{A}$  we add the corresponding nullary function as a fundamental operation.

The mapping  $\Psi$  is not a one-to-one correspondence when applied to the trivial  $v^*$ -algebras. Indeed, we have up to clone equivalence only one such  $v^*$ -algebra:  $\langle \{a\}, \{p_1^1\} \rangle$ . On the other hand, we have two non-equivalent trivial independence algebras:  $\langle \{a\}, \{p_1^1\} \rangle$ , and  $\langle \{a\}, \{g\} \rangle$ , where  $g(\emptyset) = a$ , whose clones differ by the presence of the nullary function  $g$ .

Independence algebras have a structure rich enough to allow classification theorems. A complete classification of independence algebras ( $v^*$ -algebras) was obtained by Urbanik [39–41] in the 1960s. Urbanik described six classes of independence algebras and proved that every independence algebra falls into one of these classes (up to clone equivalence). However, for our matrix theory of independence algebras, we do not need the actual operations in the algebra; the matrix theory depends only on the endomorphism monoid. So a weaker notion of equivalence is appropriate.

*Definition 1.2.* Universal algebras  $\mathbb{A}_1 = \langle A_1, F_1 \rangle$  and  $\mathbb{A}_2 = \langle A_2, F_2 \rangle$  are called *E-equivalent* if there exists a bijection  $\theta : A_1 \rightarrow A_2$  such that the mapping  $\alpha \mapsto \theta \circ \alpha \circ \theta^{-1}$  is an isomorphism from  $\text{End}(\mathbb{A}_1)$  to  $\text{End}(\mathbb{A}_2)$ .

(Here “E” stands for “endomorphism.”) We note that if  $\mathbb{A}_1$  and  $\mathbb{A}_2$  are E-equivalent independence algebras, then  $\theta$  maps the universe of any subalgebra of  $\mathbb{A}_1$  to the universe of some subalgebra of  $\mathbb{A}_2$ , and  $\theta^{-1}$  maps the universe of any subalgebra of  $\mathbb{A}_2$  to the universe of some subalgebra of  $\mathbb{A}_1$ .

Our definition of E-equivalence is not strictly a weakening of clone equivalence, as it also applies to algebras on different universes. We now define a corresponding extension of clone equivalence.

*Definition 1.3.* Let  $\mathbb{A}_1 = \langle A_1, F_1 \rangle$  and  $\mathbb{A}_2 = \langle A_2, F_2 \rangle$  be universal algebras. We say that a bijection  $\tau : A_1 \rightarrow A_2$  is a *clone isomorphism* from  $\mathbb{A}_1$  to  $\mathbb{A}_2$  if  $\{f_\tau : f \in (F_1)_{\text{cl}}\} = (F_2)_{\text{cl}}$ , where for every  $k$ -ary operation  $f \in (F_1)_{\text{cl}}$ ,  $f_\tau$  is a  $k$ -ary operation on  $A_2$  defined by

$$f_\tau(\tau(x_1), \dots, \tau(x_k)) = \tau(f(x_1, \dots, x_k))$$

for all  $x_1, \dots, x_k \in A_1$  (if  $k \geq 1$ ), and  $f_\tau(\emptyset) = \tau(f(\emptyset))$  (if  $k = 0$ ). We say that algebras  $\mathbb{A}_1$  and  $\mathbb{A}_2$  are *clone isomorphic*, written  $\mathbb{A}_1 \cong \mathbb{A}_2$ , if there is a clone isomorphism from  $\mathbb{A}_1$  to  $\mathbb{A}_2$ .

Clearly clone equivalent algebras are clone isomorphic. We remark that the converse, in cases where both concepts are well-defined, is not true. In addition,  $\tau$  is a clone isomorphism if and only if  $f_\tau \in (F_2)_{\text{cl}}, g_{\tau^{-1}} \in (F_1)_{\text{cl}}$  for all  $f \in F_1, g \in F_2$ .

In 2000, Cameron and Szabó [14] described four classes of independence algebras and proved that every *finite* independence algebra is E-equivalent to an independence algebra from one of these classes. We determine the relation between Urbanik’s six classes and Cameron and Szabó’s four classes (Section 2). In particular, we find that every independence algebra from any of Urbanik’s classes is E-equivalent to an algebra from one of Cameron and Szabó’s four classes. Then, for each of the four classes, we will develop a theory of matrices for the finite-dimensional independence algebras from that class.

To summarize, the contributions of this paper are as follows.

1. We clarify the relation between the six classes introduced by Urbanik [39–41], which provide a complete classification of independence algebras, and the four classes introduced by Cameron and Szabó [14], which provide a classification of finite independence algebras up to E-equivalence, thus answering an old question by experts on independence algebras.
2. We extend Cameron and Szabó’s theorem and prove that not just every finite, but an arbitrary independence algebra is E-equivalent to an algebra from one of the four classes defined in [14].
3. For three of the four classes introduced by Cameron and Szabó (and partially for the fourth class), we develop a theory of matrices for the finite-dimensional independence algebras from that class, thus solving a problem proposed by Głazek [25].

As we have already discussed, under the name of  $v^*$ -algebras, independence algebras were introduced and studied by Polish mathematicians in the 1960s. By the end of the 1970s, Głazek wrote a survey paper on these and related algebras, including a bibliography of more than 800 items [25] (see also [9, 10] and the references therein). A little over ten years later, independence algebras naturally appeared in semigroup theory. (For a survey, see [5]; see also [1–3, 8, 14, 16, 17] for some results on independence algebras and semigroups.)

Between the 1960s (when  $v^*$ -algebras were introduced in universal algebra) and the 1990s (when they were rediscovered as independence algebras in semigroup theory), these algebras played an important role in model theory. Givant in the U.S. [18–24] and Palyutin in Russia [36], independently solved an important classification problem in model theory, and their solution involved independence algebras. (For a detailed account of the importance of independence algebras for model theory, see [4] and Point’s enlightening and beautiful AMS Math Review (MR2863435) of this paper.)

By the time Głazek was writing his survey [25], independence algebras appeared again, now in the context of group theory. This was begun by Deza in the 1970s, who defined what he called a *permutation geometry*; this was the analogue, in the meet-semilattice of partial permutations, of a matroid or combinatorial geometry in the lattice of subsets of a set (thinking of a matroid as defined by its lattice of flats). These considerations led to the paper [13], which makes Deza yet another independent founder of independence algebras. Deza called a group that generates a permutation geometry (by taking intersections of elements in the semilattice of partial permutations) a *geometric group*. Although this is not a very good name (there are many reasons why a group is “geometric”), the name has continued to be used, and this led to Maund’s complete determination of finite geometric groups in her thesis in 1987 [33]; this result was used by Cameron and Szabó in their classification.

The paper is organized as follows. In § 2 we clarify the relation between the classifications of Urbanik [39–41] and of Cameron and Szabó [14] (see (a)–(d) after Definition 2.1). As a consequence, we obtain the extension of the latter classification from finite to arbitrary independence algebras (Theorem 2.10). In § 3 we explain what we require of a matrix theory for a given class of independence algebras. In the remaining sections, for each of the four classes of independence algebras, we construct a theory of matrices for that class. We conclude the paper with some open problems.

Throughout this paper, we will assume that every universal algebra considered is not trivial, that is, its universe has at least two elements.

## 2 Classifications

In this section, we compare Urbanik’s classification of all independence algebras [39–41] and Cameron and Szabó’s classification of finite independence algebras up to E-equivalence [14]. We also show, using Urbanik’s classification, that Cameron and Szabó’s classification extends to all independence algebras.

Urbanik [39–41] defined six classes of independence algebras: *group action*, *monoid*, *exceptional*, *quasi-field*, *linear*, and *affine* algebras. He proved that every independence algebra belongs to one of these classes (up to clone equivalence).

Cameron and Szabó [14] defined four classes of independence algebras: *group action*, *sharply 2-transitive group*, *linear*, and *affine* algebras. They proved that every finite independence algebra is E-equivalent to an algebra from one of these classes. They further showed that the subalgebra lattice of any independence algebra, not necessarily finite, is isomorphic to that of an algebra in one of these classes.

To compare these classifications, we introduce the following definition.

**Definition 2.1.** Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be classes of universal algebras. We say that  $\mathcal{C}_1$  is *equal to*  $\mathcal{C}_2$  *up to clone isomorphism* if every algebra in  $\mathcal{C}_i$  is clone isomorphic to some algebra in  $\mathcal{C}_j$ ,  $i, j \in \{1, 2\}$ ,  $i \neq j$ .

We say that  $\mathcal{C}_1$  is *included up to E-equivalence* in  $\mathcal{C}_2$  if every algebra in  $\mathcal{C}_1$  is E-equivalent to some algebra in  $\mathcal{C}_2$ .

We will show that for non-trivial universal algebras:

- (a) up to clone isomorphism, the class of group action algebras of Urbanik is equal to the class of group action algebras of Cameron and Szabó;
- (b) up to E-equivalence, the classes of monoid and exceptional algebras of Urbanik are included in the class of group action algebras;
- (c) the class of sharply 2-transitive group algebras of Cameron and Szabó is included in the class of quasi-field algebras of Urbanik, and the reverse inclusion holds up to E-equivalence.
- (d) the classes of linear and affine algebras of Urbanik are equal (up to clone equivalence), respectively, to the classes of linear and affine algebras of Cameron and Szabó.

### Group action algebras (Urbanik)

Suppose that  $G$  is a group of permutations (acting on the left) of a set  $A$ , and that  $A_0$  is a subset of  $A$  such that: (a) all fixed points of any non-identity  $g \in G$  are in  $A_0$ , and (b) for every  $g \in G$ ,  $g(A_0) \subseteq A_0$ .

Urbanik [41] defined a *group action algebra*, denoted by  $\mathbb{A}^g(A, A_0, G)$ , as a  $v^*$ -algebra with the universe  $A$  and the  $k$ -ary operations  $f(x_1, \dots, x_k) = g(x_j)$  ( $k \geq 1$ ,  $1 \leq j \leq k$ ,  $g \in G$ ), and  $f(x_1, \dots, x_k) = c$  ( $k \geq 1, c \in A_0$ ). It is easy to see that these operations form the entire clone of  $\mathbb{A}(A, A_0, G)$ , and that the clone is generated by the set of unary operations of the forms  $f(x) = g(x)$  and  $f(x) = c$ . Therefore under the correspondence  $\Psi$  described in § 1, and up to clone equivalence, we may replace  $\mathbb{A}(A, A_0, G)$  with  $\langle A; F \rangle$ , where  $F$  consists of the following unary and nullary operations on  $A$ :

$$f_g(x) = g(x) \text{ and } f_c(\emptyset) = c, \quad (2.1)$$

where  $g \in G$ ,  $x \in A$ , and  $c \in A_0$ .

Let  $A_1 = A \setminus A_0$ . Since  $g(A_0) \subseteq A_0$  for every  $g \in G$ , we have  $g(A_0) = A_0$  and  $g(A_1) = A_1$  for every  $g \in G$ . Moreover, for all  $g, h \in G$ , if  $g(x) = h(x)$  for some  $x \in A_1$ , then  $g^{-1}h(x) = x$  and so  $g = h$  (since all fixed points of the non-identity elements of  $G$  are in  $A_0$ ). Therefore  $G$  acts semiregularly on  $A_1$ , that is, it acts faithfully on each orbit. The orbits of the elements of  $A_1$  form a partition of  $A_1$ . Fix a transversal (cross-section)  $X$  of this partition. The next lemma follows immediately from the semiregularity of the action of  $G$  on  $A_1$ .

**Lemma 2.2.** *For each  $z \in A_1$ , there exist a unique  $k_z \in G$  and a unique  $x_z \in X$  such that  $z = k_z(x_z)$ .*

### Group action algebras (Cameron and Szabó)

Let  $G$  be a group,  $C$  a left  $G$ -space (so  $G$  acts on  $C$  by  $(g, c) \mapsto gc$ ), and  $X$  a set such that  $X$ ,  $G$ ,  $C$ , and  $X \times G$  are pairwise disjoint, and  $A = (X \times G) \cup C$  has at least two elements.

Cameron and Szabó [14] defined a *group action algebra*, denoted by  $\mathbb{A}(G, C, X)$ , as a universal algebra  $\langle A; F \rangle$ , where  $F$  consists of the following unary and nullary operations:

$$\lambda_g((x, h)) = (x, gh), \lambda_g(c) = gc, \text{ and } \nu_c(\emptyset) = c, \quad (2.2)$$

where  $g \in G$ ,  $(x, h) \in X \times G$ , and  $c \in C$ .

Both classes consist of independence algebras. We will now prove that these two classes are equal up to clone isomorphism.

**Proposition 2.3.** *Up to clone isomorphism, the class of group action algebras of Urbanik is equal to the class of group action algebras of Cameron and Szabó.*

*Proof.* Let  $\mathbb{A}(A, A_0, G)$  be a group action algebra of Urbanik. As above, we may identify  $\mathbb{A}(A, A_0, G)$  with  $\langle A, F_1 \rangle$ , where  $F_1 = \{f_g : g \in G\} \cup \{f_c : c \in A_0\}$  (see (2.1)). Let  $A_1 = A \setminus A_0$  and as above, fix a transversal  $X$  of the orbits of  $A_1$ . Consider the group action algebra  $\mathbb{A}(G, C, X)$  of Cameron and Szabó, where  $C = A_0$  and  $G$  acts on  $C$  by  $gc = g(c)$ , where  $g \in G$  and  $c \in C$ . Then  $\mathbb{A}(G, C, X) = \langle (X \times G) \cup C; F_2 \rangle$ , where  $F_2 = \{\lambda_g : g \in G\} \cup \{\nu_c : c \in C\}$  (see (2.2)).

We will prove that  $\mathbb{A}(A, A_0, G)$  and  $\mathbb{A}(G, C, X)$  are clone isomorphic. For every  $z \in A_1$ , there are unique  $x_z \in X$  and  $k_z \in G$  such that  $z = k_z(x_z)$  by Lemma 2.2. Define  $\tau : A \rightarrow (X \times G) \cup C$  by:  $\tau(z) = (x_z, k_z)$  if  $z \in A_1$ , and  $\tau(c) = c$  if  $c \in A_0$ . Then  $\tau$  is a bijection since it has inverse  $\tau^{-1}$  defined by:  $\tau^{-1}((x, g)) = g(x)$  if  $(x, g) \in X \times G$ , and  $\tau^{-1}(c) = c$  if  $c \in C$ . Then, for every  $f_c \in F_1$ , where  $c \in A_0$ ,  $(f_c)_\tau(\emptyset) = \tau(f_c(\emptyset)) = \tau(c) = c = \nu_c(\emptyset)$ . Thus  $(f_c)_\tau = \nu_c$  for every  $c \in A_0$ .

Let  $f_g \in F_1$ , where  $g \in G$ . Then, for every  $c \in A_0$ ,  $(f_g)_\tau(\tau(c)) = \tau(f_g(c)) = \tau(g(c)) = g(c) = gc = \lambda_g(c) = \lambda_g(\tau(c))$ . Let  $w \in A \setminus A_0$ . Then  $(f_g)_\tau(\tau(w)) = \tau(f_g(w)) = \tau(g(w)) = (x_z, k_z)$ , where  $z = g(w)$  and  $k_z(x_z) = z$ . On the other hand,  $\lambda_g(\tau(w)) = \lambda_g((x_w, k_w)) = (x_w, gk_w)$ , where  $k_w(x_w) = w$ . Since  $z = g(w)$ , the elements  $z$  and  $w$  lie in the same orbit, and so  $x_z = x_w$ . Thus  $(gk_w)(x_w) = g(k_w(x_w)) = g(w) = z = k_z(x_z)$ , and so  $gk_w = k_z$ , as  $G$  acts semi-regularly. Hence  $(f_g)_\tau(\tau(w)) = \tau(f_g(w)) = (x_z, k_z) = (x_w, gk_w) = \lambda_g(\tau(w))$ . Thus  $(f_g)_\tau = \lambda_g$  for every  $g \in G$ . Therefore,  $\tau$  is a clone isomorphism, and so  $\mathbb{A}(A, A_0, G) \cong \mathbb{A}(G, C, X)$ .

We have proved that every group action algebra of Urbanik is clone isomorphic to some group action algebra of Cameron and Szabó. Conversely, let  $\mathbb{A}(G, C, X)$  be a group action algebra of Cameron and Szabó. In case that  $X = \emptyset$ , we may assume that  $G$  acts faithfully on  $C$  by replacing it with the permutation group it induces on  $C$ . This change results in the same group action algebra and ensures that  $G$  as constructed below will be a group of permutations. Let  $A_0 = C$ ,  $A_1 = X \times G$  and  $A = (X \times G) \cup C$ . View  $G$  as a group of permutations of  $A$ : for  $g \in G$  and  $z \in A$ ,  $g(z) = (x, gh)$  if  $z = (x, h) \in X \times G$ , and  $g(z) = gc$  if  $z = c \in C$ . Consider the group algebra  $\mathbb{A}(A, A_0, G)$  of Urbanik, and note that  $X \times \{1\}$ , where  $1$  is the identity of  $G$ , is a transversal of the partition of  $A_1$  into orbits. Since we can identify  $X$  with  $X \times \{1\}$  in  $\mathbb{A}(G, C, X)$ , it follows from the first part of the proof that  $\mathbb{A}(G, C, X)$  is clone isomorphic to  $\mathbb{A}(A, A_0, G)$ . Hence, every group action algebra of Cameron and Szabó is clone isomorphic to some group action algebra of Urbanik, and so the two classes are equal up to clone isomorphism.  $\square$

The next two classes in Urbanik's classification—the monoid algebras and the exceptional algebra—are included in the class of group action algebras up to E-equivalence.

### Monoid algebras (Urbanik)

Suppose that  $A$  is a monoid such that every non-unit element of  $A$  is a left zero. Urbanik defined a *monoid algebra* as a  $v^*$ -algebra  $\mathbb{A} = \langle A; F \rangle$  such that for every  $f \in F$ ,  $f$  is a  $k$ -ary operation ( $k \geq 1$ ) that satisfies

$$f(a_1 a, \dots, a_k a) = f(a_1, \dots, a_k) a, \quad (2.3)$$

for all  $a, a_1, \dots, a_k \in A$ , and  $F$  contains all unary operations that satisfy (2.3). It is easy to see that the unary operations that satisfy (2.3) are precisely the unary operations  $f$  on  $A$  defined by  $f(x) = bx$ , where  $b$  is an arbitrary fixed element of  $A$ .

**Proposition 2.4.** *Up to E-equivalence, the class of monoid algebras is included in the class of group action algebras.*

*Proof.* Let  $\langle A; F \rangle$  be a monoid algebra. Denote by  $A_0$  the set of left zeros (non-unit elements) of the monoid  $A$ , and by  $G$  the group of units of  $A$ . We can view  $G$  as a group of permutations of  $A$  by putting  $g(a) = ga$  for all  $g \in G$  and  $a \in A$ . Since  $A_0$  is evidently an ideal of  $A$  (that is,  $a \in A, c \in A_0$  imply  $ac, ca \in A_0$ ), it follows that  $g(A_0) \subseteq A_0$  for every  $g \in G$ . Consider the group action algebra  $\mathbb{A}(A, A_0, G)$  (as defined by Urbanik).

We will show that  $\text{End}(\langle A; F \rangle) = \text{End}(\mathbb{A}(A, A_0, G))$ . Suppose  $\alpha \in \text{End}(\langle A; F \rangle)$ . Fix  $b \in A$  and consider the fundamental operation  $f(x) = bx$  in  $\langle A; F \rangle$ . Then since  $\alpha$  preserves  $f$ ,

$$\alpha(bx) = \alpha(f(x)) = f(\alpha(x)) = b\alpha(x), \quad (2.4)$$

for every  $x \in A$ . Let  $f_g$  and  $f_c$ , where  $g \in G$  and  $c \in A_0$ , be fundamental operations in  $\mathbb{A}(A, A_0, G)$  (see (2.1)). Then, by (2.4),  $\alpha(f_g(x)) = \alpha(g(x)) = \alpha(gx) = g\alpha(x) = g(\alpha(x)) = f_g(\alpha(x))$  for every  $x \in A$ . Since  $c$  is a left zero,  $\alpha(c) = \alpha(c1) = c\alpha(1) = c$ . Thus,  $\alpha \in \text{End}(\mathbb{A}(A, A_0, G))$ .

Conversely, suppose  $\alpha \in \text{End}(\mathbb{A}(A, A_0, G))$ . Let  $x \in A$ . Then, for every  $g \in G$ ,  $\alpha(gx) = \alpha(f_g(x)) = f_g(\alpha(x)) = g(\alpha(x)) = g\alpha(x)$ . Further, for every  $c \in A_0$ ,  $\alpha(cx) = \alpha(c) = c\alpha(x)$ . Thus  $\alpha(bx) = b\alpha(x)$  for every  $b \in A$ . Let  $f$  be a  $k$ -ary ( $k \geq 1$ ) fundamental operation in  $\langle A; F \rangle$ . Then, for all  $x_1, \dots, x_k \in A$ ,

$$\begin{aligned} \alpha(f(x_1, \dots, x_k)) &= \alpha(f(x_1 1, \dots, x_k 1)) = \alpha(f(x_1, \dots, x_k) 1) = f(x_1, \dots, x_k) \alpha(1) \\ &= f(x_1 \alpha(1), \dots, x_k \alpha(1)) = f(\alpha(x_1 1), \dots, \alpha(x_k 1)) = f(\alpha(x_1), \dots, \alpha(x_k)). \end{aligned}$$

Thus,  $\alpha \in \text{End}(\langle A; F \rangle)$ . Hence  $\text{End}(\langle A; F \rangle) = \text{End}(\mathbb{A}(A, A_0, G))$ , and so every monoid algebra is E-equivalent to some group action algebra.  $\square$

### Exceptional algebra (Urbanik)

Suppose that  $A$  is a set with four elements. Urbanik defined an *exceptional algebra* as a  $v^*$ -algebra  $\mathbb{A} = \langle A; F \rangle$  with  $F = \{i, q\}$ , where  $i$  is a unary operation on  $A$  and  $q$  is a ternary operation on  $A$  that satisfy the following conditions:

$$i(i(x)) = x, i(x) \neq x, q(x_1, x_2, x_3) = q(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}), q(x, y, i(x)) = y, q(x, y, x) = x, \quad (2.5)$$

for all  $x, y, x_1, x_2, x_3 \in A$  and all permutations  $\sigma$  of  $\{1, 2, 3\}$ . Note that  $i$  is an involution without fixed points, and  $q$  is symmetrical. One can check that  $q$  is uniquely determined by the conditions listed in (2.5), and that, up to isomorphism, there is only one exceptional algebra.

**Proposition 2.5.** *Up to E-equivalence, the exceptional algebra is included in the class of group action algebras.*

*Proof.* Let  $\langle A; F \rangle$  be the exceptional algebra. Then, by the definition of  $i$ ,  $G = \{i, 1\}$  is a group of permutations of  $A$ . Consider the group action algebra  $\mathbb{A}(A, \emptyset, G)$  (as defined by Urbanik). Since  $A_0 = \emptyset$ , the only fundamental operations in  $\mathbb{A}(A, \emptyset, G)$  are  $f_1(x) = x$  and  $f_i(x) = i(x)$ , where  $x \in A$ . We need to show that  $\text{End}(\langle A; F \rangle) = \text{End}(\mathbb{A}(A, \emptyset, G))$ . We clearly have  $\text{End}(\langle A; F \rangle) \subseteq \text{End}(\mathbb{A}(A, \emptyset, G))$  since  $\{f_i\} \subseteq \{i, q\}$  and every function  $\alpha : A \rightarrow A$  preserves  $f_1$ .

For the reverse inclusion let  $\alpha : A \rightarrow A$  preserve  $i$ . It suffices to show that it also preserves  $q$ , that is,  $\alpha(q(x, y, z)) = q(\alpha(x), \alpha(y), \alpha(z))$  for all  $x, y, z \in A$ . This is easy to see if  $x, y, z$  are not pairwise distinct. Otherwise, we may assume that  $z = i(x)$ , obtaining  $\alpha(q(x, y, i(x))) = \alpha(y)$  and  $q(\alpha(x), \alpha(y), \alpha(i(x))) = q(\alpha(x), \alpha(y), i(\alpha(x)))$ . This last expression equals  $\alpha(y)$ , whether or not  $\alpha(y)$  equals one of the other arguments.

Hence  $\text{End}(\langle A; F \rangle) = \text{End}(\mathbb{A}(A, \emptyset, G))$ , and so the independence algebra is E-equivalent to some group action algebra.  $\square$



### Quasi-field algebras (Urbanik)

Let  $A$  be a set on which two binary operations are defined: a multiplication  $(a, b) \mapsto ab$  and a subtraction  $(a, b) \mapsto a - b$ . We say that  $A$  is a *quasi-field* [27] if there is  $0 \in A$  such that  $a0 = 0a = 0$  for every  $a \in A$ ,  $A \setminus \{0\}$  is a group with respect to the multiplication, and for all  $a, b, c \in A$ , the following properties are satisfied:

- (i)  $a - 0 = a$ ,
- (ii)  $a(b - c) = ab - ac$ ,
- (iii)  $a - (a - c) = c$ ,
- (iv)  $a - (b - c) = (a - b) - (a - b)(b - a)^{-1}c$  if  $a \neq b$ .

Suppose that  $A$  is a quasi-field. Urbanik defined a *quasi-field algebra* as a  $v^*$ -algebra  $\mathbb{A} = \langle A; F \rangle$  such that for every  $f \in F$ ,  $f$  is a  $k$ -ary operation with  $k \geq 1$  such that for all  $a, b, a_1, \dots, a_k \in A$ ,

$$f(a - ba_1, \dots, a - ba_k) = a - bf(a_1, \dots, a_k), \quad (2.6)$$

and  $F$  contains all binary operations that satisfy (2.6).

### Sharply 2-transitive group algebras (Cameron and Szabó)

A permutation group  $G$  on a set  $A$  is *sharply 2-transitive* if, for all pairs  $(a_1, a_2)$  and  $(b_1, b_2)$  of  $A \times A$  with  $a_1 \neq a_2$  and  $b_1 \neq b_2$ , there is a *unique* element of  $g \in G$  such that  $g(a_1) = b_1$  and  $g(a_2) = b_2$ .

Let  $G$  be such a permutation group on  $A$ . Suppose  $|A| \geq 3$ . Cameron and Szabó defined a *sharply 2-transitive group algebra* as an algebra  $\mathbb{A} = \langle A; F \rangle$ , where  $F$  is defined as follows. The group  $G$  acts on the set consisting of triples  $(x, y, z)$  of  $A^3$  such that  $x, y, z$  are pairwise distinct. Let  $\{O_i : i \in I\}$  be the set of orbits under this action. Then  $F = \{\mu_i : i \in I\}$ , where, for each  $i \in I$ ,  $\mu_i$  is a binary operation on  $A$  defined as follows. For all  $x, y \in A$ , with  $x \neq y$ ,

$$\begin{aligned} \mu_i(x, x) &= x; \\ \mu_i(x, y) &= \text{the unique } z \in A \text{ such that } (x, y, z) \in O_i. \end{aligned}$$

If  $|A| = 2$ , then  $F$  is defined to be  $\emptyset$ .

Note that in the above definition,  $z$  is unique because by sharp-2-transitivity, only the identity action maps  $(x, y)$  to itself.

Both classes are independence algebras of dimension 2 (see [41, p. 243] and [14, p. 325]).

We will first explain the underlying connection between quasi-fields and the sharply 2-transitive group algebras.

The former were defined by Grätzer as structures with two operations, called subtraction and multiplication, satisfying six axioms; he showed that the linear substitutions  $x \mapsto a - bx$  over a quasi-field form a sharply 2-transitive permutation group, and conversely every sharply 2-transitive permutation group can be represented in this way. It is clear that a 2-sharply transitive group algebra of Cameron and Szabó also determines, and is determined by, the sharply 2-transitive group. So from the point of view of classifying sharply 2-transitive groups the approaches are equivalent, but the independence algebras are not necessarily clone-equivalent.

**Proposition 2.6.** *The class of sharply 2-transitive group algebras is included in the class of quasi-field algebras, up to clone equivalence.*

*Proof.* Let  $G$  be a sharply 2-transitive permutation group on a set  $A$ , and let  $\mathbb{A} = \langle A; F \rangle$  be a sharply 2-transitive group algebra. By [27, Theorem 2], one can define a multiplication and

subtraction on  $A$  such that  $A$  is a quasi-field with these operations, and  $G = \{T_{a,b} : a, b \in A, b \neq 0\}$ , where each  $T_{a,b}$  is a permutation of  $A$  defined by  $T_{a,b}(x) = a - bx$  for every  $x \in A$ .

Consider the quasi-field algebra  $\mathbb{A}_1 = \langle A, F_1 \rangle$ , where  $F_1$  consists of all binary operations that satisfy (2.6). By [41, p. 243],  $F_1 = \{f_w : w \in A\}$ , where  $f_w(x, y) = x - (x - y)w$  for all  $x, y \in A$ . Note that  $f_0(x, y) = x - (x - y)0 = x$  and  $f_1(x, y) = x - (x - y)1 = x - (x - y) = y$  are projections (which are in the clone of  $F_1$ ), so we may assume that  $F_1 = \{f_w : w \in A, w \neq 0, 1\}$ .

We claim that  $F = F_1$ , which implies  $\mathbb{A} = \mathbb{A}_1$ . If  $|A| = 2$ , then  $F = \emptyset = F_1$ . Suppose  $|A| \geq 3$ . Since each orbit  $O_i$  from the definition of a sharply 2-transitive group algebra contains a unique element  $(0, 1, w)$ , where  $w \in A \setminus \{0, 1\}$ , we can index the orbits by the set  $I = A \setminus \{0, 1\}$  in such a way that for every  $w \in I$ ,  $(0, 1, w) \in O_w$ . We now have  $F = \{\mu_w : w \in I\}$  and  $F_1 = \{f_w : w \in I\}$ . Let  $w \in I$ , that is,  $w \in A$  and  $w \neq 0, 1$ . To finish the proof, it suffices to show that  $\mu_w = f_w$ . For every  $x \in A$ ,  $\mu_w(x, x) = x$  and  $f_w(x, x) = x - (x - x)w = x - 0w = x - 0 = x$ . Let  $x, y \in A$  with  $x \neq y$ . Suppose  $\mu_w(x, y) = z$ . Then  $(x, y, z) \in O_w$ , and so  $(T_{a,b}(0), T_{a,b}(1), T_{a,b}(w)) = (x, y, z)$  for some unique  $a, b \in A$  with  $b \neq 0$ . Thus  $a = x$ ,  $a - b = y$ , and  $a - bw = z$ . Hence  $x - b = y$ , which implies  $b = x - y$  [27, (1), p. 29]. Therefore,  $\mu_w(x, y) = z = a - bw = x - (x - y)w = f_w(x, y)$ .  $\square$

The reverse inclusion is true up to E-equivalence.

**Proposition 2.7.** *Up to E-equivalence, the class of quasi-field algebras is included in the class of sharply 2-transitive group algebras.*

*Proof.* Let  $\mathbb{A} = \langle A; F \rangle$  be a quasi-field algebra. By [27, Theorem 1],  $G = \{T_{a,b} : a, b \in A, b \neq 0\}$ , where each  $T_{a,b}$  is a permutation of  $A$  defined by  $T_{a,b}(x) = a - bx$ , for every  $x \in A$ , is a sharply 2-transitive permutation group on  $A$ . Consider the quasi-field algebra  $\mathbb{A}_1 = \langle A; F_1 \rangle$ , where  $F_1$  consists of all binary operations in  $F$ . By the proof of Proposition 2.6,  $\mathbb{A}_1$  is the sharply 2-transitive group algebra defined by  $G$ .

We will show that  $\text{End}(\mathbb{A}) = \text{End}(\mathbb{A}_1)$ . Since  $F_1 \subseteq F$ , we have  $\text{End}(\mathbb{A}) \subseteq \text{End}(\mathbb{A}_1)$ . Now, let  $\alpha \in \text{End}(\mathbb{A}_1)$ . Since in any quasi-field algebra, any two distinct elements form a basis [41, p. 43],  $\{0, 1\}$  is a basis for both  $\mathbb{A}$  and  $\mathbb{A}_1$ . Define  $\beta : \{0, 1\} \rightarrow A$  by  $\beta(0) = \alpha(0)$  and  $\beta(1) = \alpha(1)$ . Then  $\beta$  can be uniquely extended to an endomorphism  $\bar{\beta}$  of  $\mathbb{A}$ . Note that  $\bar{\beta}$  is also an endomorphism of  $\mathbb{A}_1$ . We now have two endomorphisms of  $\mathbb{A}_1$ ,  $\alpha$  and  $\bar{\beta}$ , whose restrictions to the basis  $\{0, 1\}$  for  $\mathbb{A}_1$  are the same. It follows that  $\alpha = \bar{\beta}$ , and so  $\alpha \in \text{End}(\mathbb{A})$ .

Hence  $\text{End}(\mathbb{A}) = \text{End}(\mathbb{A}_1)$ , and so every quasi-field algebra is E-equivalent to some sharply 2-transitive group algebra.  $\square$

### Linear algebras (Urbanik)

Suppose  $A$  is a vector space over a division ring  $K$ , and  $A_0$  is a subspace of  $A$ . Urbanik defined a *linear algebra* as a  $v^*$ -algebra with the universe  $A$  and the operations

$$f(x_1, \dots, x_k) = \sum_{i=1}^k \lambda_i x_i + a, \quad (2.7)$$

where  $k \geq 1$ , each  $\lambda_i \in K$ , and  $a \in A_0$ . It is easy to see that these operations form the clone of the set consisting of the binary operation  $f(x, y) = x + y$ , unary operations  $f_\lambda(x) = \lambda x$ , for each  $\lambda \in K$ , and unary constant operations  $f_a(x) = a$ , for each  $a \in A_0$ .

Applying the correspondence  $\Psi$  described in Section 1 to the linear algebra with  $A$ ,  $A_0$ , and the operations (2.7) we obtain (up to clone equivalence) the algebra  $\langle A; F \rangle$ , where  $F$  consists of the following binary, unary, and nullary operations on  $A$ :

$$f(x, y) = x + y, \quad f_\lambda(x) = \lambda x, \quad \text{and} \quad f_a(\emptyset) = a, \quad (2.8)$$

where  $x, y \in A$ ,  $\lambda \in K$ , and  $a \in A_0$ .

### Linear algebras (Cameron and Szabó)

Let  $V$  be a vector space over a division ring  $K$ , and  $W$  a subspace of  $V$ . Cameron and Szabó defined a *linear algebra*, denoted by  $V[W]$ , as an algebra with the universe  $V$  whose operations are addition (binary), scalar multiplication (one unary operation for each element of  $K$ ), and nullary operations whose values are the elements of  $W$ .

Both classes consist of independence algebras. It is clear by (2.8) that both definitions are identical. Thus we have the following proposition.

**Proposition 2.8.** *The class of linear algebras of Urbanik is equal to the class of linear algebras of Cameron and Szabó, up to clone equivalence.*

### Affine algebras (Urbanik)

Suppose  $A$  is a vector space over a division ring  $K$ , and  $A_0$  is a subspace of  $A$ . Urbanik defined an *affine algebra*  $\mathbb{A}$  as a  $v^*$ -algebra with the universe  $A$  and the operations

$$f(x_1, \dots, x_k) = \sum_{i=1}^k \lambda_i x_i + a, \quad (2.9)$$

where  $k \geq 1$ , each  $\lambda_i \in K$ ,  $\sum_{i=1}^k \lambda_i = 1$ , and  $a \in A_0$ .

Note that all unary functions in the clone of  $\mathbb{A}$  have the form  $f(x) = x + a$ . As we excluded the trivial algebra with  $|A| = 1$ , it follows that  $\Phi(\mathbb{A})$  does not contain any nullary operations.

### Affine algebras (Cameron and Szabó)

Let  $V$  be a vector space over a division ring  $K$ , and  $W$  be a subspace of  $V$ . Cameron and Szabó defined an *affine algebra*, denoted  $\text{Aff}(V)[+W]$ , as an algebra with the universe  $V$  and the following fundamental operations: the unary operations  $\tau_w(x) = x + w$ , for each  $w \in W$ ; the binary operations  $\mu_c(x, y) = x + c(y - x) = (1 - c)x + cy$ , for each  $c \in K \setminus \{0, 1\}$  (defined only when  $|K| \geq 3$ ); and the ternary operation  $\alpha(x, y, z) = x + y + z$  (defined only when  $|K| = 2$ ).

We can extend the definition of the binary operations  $\mu_c(x, y) = (1 - c)x + cy$  by dropping the requirement  $c \neq 0, 1$ . We then have  $\mu_0(x, y) = x$  and  $\mu_1(x, y) = y$ , which are projections, so they do not change the algebra because the projections are in the clone by definition of clone. With this extension, the operations  $\mu_c(x, y) = (1 - c)x + cy$ , where  $c \in K$ , are the same as the operations  $\mu_{c_1, c_2}(x, y) = c_1x_1 + c_2x_2$ , where  $c_1, c_2 \in K$  with  $c_1 + c_2 = 1$ . Moreover, we may add  $\mu_0$  to  $\mu_2$  when  $|K| = 2$  without changing the clone. Therefore,  $\text{Aff}(V)[+W]$  can be defined as the algebra with the universe  $V$  and the fundamental operations

$$\tau_w(x) = x + w, \quad \mu_{c_1, c_2}(x_1, x_2) = c_1x_1 + c_2x_2, \quad \text{and} \quad \alpha(x_1, x_2, x_3) = x_1 + x_2 + x_3, \quad (2.10)$$

where  $w \in W$ ,  $c_1, c_2 \in K$  with  $c_1 + c_2 = 1$ , and  $\alpha$  is defined only when  $|K| = 2$ .

Both classes consist of independence algebras.

**Proposition 2.9.** *The class of affine algebras of Urbanik is equal (up to clone equivalence) to the class of affine algebras of Cameron and Szabó.*

*Proof.* An algebra of each class is defined by a vector space  $A$  over a division ring  $K$ , and a subspace  $A_0$ . Let  $\mathbb{A}_1 = \langle A; F_1 \rangle$  be an affine algebra of Urbanik, where  $F_1$  consists of the operations defined in (2.9), and  $\mathbb{A}_2 = \langle A; F_2 \rangle = \text{Aff}(A)[+A_0]$  be an affine algebra of Cameron and Szabó, where  $F_2$  consists of the operations defined in (2.10). We will prove that these algebras have the same clone.

We have  $(F_2)_{\text{cl}} \subseteq (F_1)_{\text{cl}}$  since  $F_2 \subseteq F_1$ . For  $k \geq 1$  and  $\lambda_1, \dots, \lambda_k \in K$  with  $\lambda_1 + \dots + \lambda_k = 1$ , let  $f_{\lambda_1, \dots, \lambda_k}$  be a  $k$ -ary operation on  $A$  defined by  $f_{\lambda_1, \dots, \lambda_k}(x_1, \dots, x_k) = \sum_{i=1}^k \lambda_i x_i$ . This is an

operation from (2.9), where  $a = 0$ . We will prove, by induction on  $k$ , that each such  $f_{\lambda_1, \dots, \lambda_k}$  belongs to  $(F_2)_{\text{cl}}$ .

Let  $k = 1$ . Then  $\lambda_1 = 1$ , and so  $f_{\lambda_1}(x) = x = \tau_0(x)$  for every  $x \in A$ . Thus  $f_{\lambda_1} = \tau_0 \in F_2 \subseteq (F_2)_{\text{cl}}$ . If  $k = 2$ , then  $f_{\lambda_1, \lambda_2} = \mu_{\lambda_1, \lambda_2} \in F_2 \subseteq (F_2)_{\text{cl}}$ . (See (2.10).)

Let  $k \geq 2$  and suppose that  $f_{c_1, \dots, c_m} \in (F_2)_{\text{cl}}$  for all  $m$ ,  $1 \leq m \leq k$ , and all  $c_1, \dots, c_m \in A$  such that  $c_1 + \dots + c_m = 1$ . Let  $\lambda_1, \dots, \lambda_{k+1} \in A$  with  $\lambda_1 + \dots + \lambda_{k+1} = 1$ . We want to prove that  $f_{\lambda_1, \dots, \lambda_{k+1}} \in (F_2)_{\text{cl}}$ . We consider three cases.

**Case 1.**  $\lambda_{k+1} \neq 1$ .

For each  $i$ ,  $1 \leq i \leq k$ , let  $c_i = (1 - \lambda_{k+1})^{-1} \lambda_i$ . Then

$$c_1 + \dots + c_k = (1 - \lambda_{k+1})^{-1} (\lambda_1 + \dots + \lambda_k) = (1 - \lambda_{k+1})^{-1} (1 - \lambda_{k+1}) = 1.$$

By the inductive hypothesis,  $f_{c_1, \dots, c_k} \in (F_2)_{\text{cl}}$ . Thus the  $(k+1)$ -ary operation  $g$  on  $A$  defined by

$$g(x_1, \dots, x_{k+1}) = f_{c_1, \dots, c_k}(p_1^{k+1}(x_1, \dots, x_{k+1}), \dots, p_k^{k+1}(x_1, \dots, x_{k+1}))$$

is in  $(F_2)_{\text{cl}}$ . We have  $g(x_1, \dots, x_{k+1}) = f_{c_1, \dots, c_k}(x_1, \dots, x_k) = c_1 x_1 + \dots + c_k x_k$ . Further, let  $d_1 = 1 - \lambda_{k+1}$  and  $d_2 = \lambda_{k+1}$ , so  $d_1 + d_2 = 1$ . Then the  $(k+1)$ -ary operation  $s$  on  $A$  defined by

$$s(x_1, \dots, x_{k+1}) = \mu_{d_1, d_2}(g(x_1, \dots, x_{k+1}), p_{k+1}^{k+1}(x_1, \dots, x_{k+1}))$$

is in  $(F_2)_{\text{cl}}$ . We then have

$$\begin{aligned} s(x_1, \dots, x_{k+1}) &= \mu_{d_1, d_2}(c_1 x_1 + \dots + c_k x_k, x_{k+1}) = d_1 c_1 x_1 + \dots + d_1 c_k x_k + d_2 x_{k+1} \\ &= (1 - \lambda_{k+1})(1 - \lambda_{k+1})^{-1} \lambda_1 x_1 + \dots + (1 - \lambda_{k+1})(1 - \lambda_{k+1})^{-1} \lambda_k x_k + \lambda_{k+1} x_{k+1} \\ &= \lambda_1 x_1 + \dots + \lambda_k x_k + \lambda_{k+1} x_{k+1} \\ &= f_{\lambda_1, \dots, \lambda_{k+1}}(x_1, \dots, x_{k+1}), \end{aligned}$$

so  $f_{\lambda_1, \dots, \lambda_{k+1}} \in (F_2)_{\text{cl}}$ .

**Case 2.**  $\lambda_i \neq 1$  for some  $i$ ,  $1 \leq i < k+1$ .

By Case 1, the operation  $f_{\lambda_1, \dots, \lambda_{i-1}, \lambda_{k+1}, \lambda_{i+1}, \dots, \lambda_k, \lambda_i}$  is in  $(F_2)_{\text{cl}}$ . Thus since  $p_j^{k+1}(x_1, \dots, x_{k+1}) = x_j$ , where  $1 \leq j \leq k+1$ , the  $(k+1)$ -ary operation  $h$  on  $A$  defined by

$$h(x_1, \dots, x_{k+1}) = f_{\lambda_1, \dots, \lambda_{i-1}, \lambda_{k+1}, \lambda_{i+1}, \dots, \lambda_k, \lambda_i}(x_1, \dots, x_{i-1}, x_{k+1}, x_{i+1}, \dots, x_k, x_i)$$

is in  $(F_2)_{\text{cl}}$ . It is clear that  $h = f_{\lambda_1, \dots, \lambda_{k+1}}$ , so  $f_{\lambda_1, \dots, \lambda_{k+1}} \in (F_2)_{\text{cl}}$ .

**Case 3.**  $\lambda_1 = \dots = \lambda_{k+1} = 1$ .

We claim that if  $K$  has characteristic 2, then  $f_{1,1,1} \in (F_2)_{\text{cl}}$ . Suppose that  $K$  has characteristic 2. If  $|K| = 2$ , then  $f_{1,1,1} = \alpha \in (F_2)_{\text{cl}}$  (see (2.10)). Suppose  $|K| \geq 3$ . Then there exist  $e \in K$  such that  $e \neq 0, 1$ . Fix such an  $e$  and define  $c = e(1 - e)^{-1}$  and  $d = c^{-1}$  (possible since  $c \neq 0$ ). By Case 1,  $f_{d, 1-d, 0}$  and  $f_{0, e, 1-e}$  are in  $(F_2)_{\text{cl}}$ . Thus the ternary operation  $t$  on  $A$  defined by

$$t(x_1, x_2, x_3) = \mu_{c, 1-c}(f_{d, 1-d, 0}(x_1, x_2, x_3), f_{0, e, 1-e}(x_1, x_2, x_3))$$

is in  $(F_2)_{\text{cl}}$ . We have

$$\begin{aligned} t(x_1, x_2, x_3) &= \mu_{c, 1-c}(dx_1 + (1-d)x_2, ex_2 + (1-e)x_3) \\ &= cdx_1 + c(1-d)x_2 + (1-c)ex_2 + (1-c)(1-e)x_3 \\ &= cc^{-1}x_1 + cx_2 - cc^{-1}x_2 + ex_2 - cex_2 + x_3 - ex_3 - cx_3 + cex_3 \\ &= x_1 + c(1-e)x_2 - x_2 + ex_2 + x_3 - ex_3 - c(1-e)x_3 \\ &= x_1 + 2ex_2 - x_2 + x_3 - 2ex_3 = x_1 + x_2 + x_3, \end{aligned}$$

where the last two equalities are true since  $c(1-e) = e$ ,  $2 = -2 = 0$ , and  $-1 = 1$ . Thus  $t = f_{1,1,1}$ , and so  $f_{1,1,1} \in (F_2)_{\text{cl}}$ . The claim has been proved.

We now have  $f_{-1,1,1} \in (F_2)_{\text{cl}}$  for every  $K$ : by Case 2 if  $-1 \neq 1$ , and by the claim if  $-1 = 1$ . Since  $\lambda_1 = \dots = \lambda_{k+1} = 1$  and  $\lambda_1 + \dots + \lambda_k + \lambda_{k+1} = 1$ , we have  $\lambda_1 + \dots + \lambda_k = 0$ , and so  $\lambda_1 + \dots + \lambda_{k-1} = -1$ . Let  $c_i = -\lambda_i$ , where  $1 \leq i \leq k-1$ . Then  $c_1 + \dots + c_{k-1} = 1$  and  $f_{c_1, \dots, c_{k-1}} \in (F_2)_{\text{cl}}$  by the inductive hypothesis. Thus

$$f_{\lambda_1, \dots, \lambda_{k+1}}(x_1, \dots, x_{k+1}) = f_{-1,1,1}(f_{c_1, \dots, c_{k-1}}(x_1, \dots, x_{k-1}), x_k, x_{k+1}),$$

and so  $f_{\lambda_1, \dots, \lambda_{k+1}} \in (F_2)_{\text{cl}}$ . This concludes the inductive proof.

Let  $f(x_1, \dots, x_k) = \sum_{i=1}^k \lambda_i x_i + a$ , where  $k \geq 1$ ,  $\lambda_1, \dots, \lambda_k \in K$ ,  $\lambda_1 + \dots + \lambda_k = 1$ , and  $a \in A_0$ , be an arbitrary operation in  $F_1$  (see (2.9)). Then  $f(x_1, \dots, x_k) = \tau_a(f_{\lambda_1, \dots, \lambda_k}(x_1, \dots, x_k))$ , and so  $f \in (F_2)_{\text{cl}}$  since  $\tau_a \in F_2$  and  $f_{\lambda_1, \dots, \lambda_k} \in (F_2)_{\text{cl}}$ . We have proved that  $(F_1)_{\text{cl}} \subseteq (F_2)_{\text{cl}}$ .

Thus  $\mathbb{A}_1$  and  $\mathbb{A}_2$  have the same clone. Therefore, the class of affine algebras of Urbanik is equal (up to clone equivalence) to the class of affine algebras of Cameron and Szabó.  $\square$

We have proved statements (a)–(d) after Definition 2.1, which completely clarify the relation between the six classes of independence algebras considered by Urbanik and the four classes of independence algebras considered by Cameron and Szabó. Note that we did not assume that the classes of Cameron and Szabó consist of finite algebras.

Now let  $\mathbb{A}$  be any independence algebra of non-zero dimension. Then  $\mathbb{A}$  belongs to one of the six classes of Urbanik [39–41]. Thus, by (a)–(d),  $\mathbb{A}$  is E-equivalent to an algebra from one of the four classes of Cameron and Szabó. Urbanik does not classify the independence algebras of dimension 0. However, these are clearly E-equivalent to group action algebras with  $A = A_0$ .

Therefore we have an extension of Cameron and Szabó's classification [14, Theorem 1.3] from finite to arbitrary independence algebras.

**Theorem 2.10.** *Any independence algebra  $\mathbb{A} = \langle A; F \rangle$ , with  $|A| \geq 2$ , is E-equivalent to a group action algebra, a 2-transitive group algebra, a linear algebra, or an affine algebra.*

### 3 Matrix systems for independence algebras

Our goal is to define matrices that represent endomorphisms of a given independence algebra. In this section, we will describe what we want to accomplish by introducing the notion of a matrix system for an independence algebra.

Let  $\mathbb{A}$  be a general independence algebra of a given type and let  $n = \dim(\mathbb{A})$ . We fix an ordered basis  $S = (e_1, \dots, e_n)$  of  $\mathbb{A}$ . Our objectives are as follows, where  $n'$  is one of  $n, n+1$ , depending on the type of  $\mathbb{A}$  (we remark that with trivial modifications, it is possible to choose  $n' = n+1$  throughout).

- (A) Select a set  $F$  associated with  $\mathbb{A}$ . Let  $\mathcal{M}_{n' \times n'}(F)$  and  $\mathcal{M}_{n' \times 1}(F)$  be the sets of  $n' \times n'$  and  $n' \times 1$  matrices with entries from  $F$ , respectively. Select a subset  $\mathcal{M}_{n' \times n'}^*(F)$  of  $\mathcal{M}_{n' \times n'}(F)$  and a subset  $\mathcal{M}_{n' \times 1}^*(F)$  of  $\mathcal{M}_{n' \times 1}(F)$ . For example, if  $\mathbb{A} = \mathbb{A}^l = (A, \{0\}, K)$  is a linear independence algebra, then our set  $F$  will be the division ring  $K$ ,  $\mathcal{M}_{n' \times n'}^*(F)$  will be  $\mathcal{M}_{n \times n}(K)$ , and  $\mathcal{M}_{n' \times 1}^*(F)$  will be  $\mathcal{M}_{n \times 1}(K)$ .
- (B) Define an operation  $(M, P) \mapsto M \cdot P$  on  $\mathcal{M}_{n' \times n'}^*(F)$  and a mapping  $(M, v) \mapsto M \cdot v$  from  $\mathcal{M}_{n' \times n'}^*(F) \times \mathcal{M}_{n' \times 1}^*(F) \rightarrow \mathcal{M}_{n' \times 1}^*(F)$  such that for all  $M, P, Q \in \mathcal{M}_{n' \times n'}^*(F)$  and  $v \in \mathcal{M}_{n' \times 1}^*(F)$ ,

$$M \cdot (P \cdot Q) = (M \cdot P) \cdot Q \quad \text{and} \quad M \cdot (N \cdot v) = (M \cdot N) \cdot v.$$

We will refer to both the operation and mapping as *matrix multiplication* and write  $MP$  for  $M \cdot P$  and  $Mv$  for  $M \cdot v$ .

- (C) Define a bijection  $x \mapsto [x]_S$  from  $A$  to  $\mathcal{M}_{n' \times 1}^*(F)$  such that for all  $M \in \mathcal{M}_{n' \times n'}^*(F)$  and  $j \in \{1, \dots, n\}$ ,

$$M[e_j]_S = M_{*j},$$

where  $M_{*j}$  denotes the  $j$ th column of  $M$ . If  $n' = n + 1$ , then all matrices in  $\mathcal{M}_{n' \times n'}^*(F)$  have the same  $n'$ -th column.

For  $x \in A$ , we will call  $[x]_S$  the *coordinate vector* of  $x$  with respect to  $S$ , and write  $[x]$  for  $[x]_S$  if there is no confusion about the basis  $S$ .

- (D) Define a bijection  $\phi \mapsto M_\phi$  from  $\text{End}(\mathbb{A})$  to  $\mathcal{M}_{n' \times n'}^*(F)$  such that for every  $x \in A$ ,

$$[\phi(x)]_S = M_\phi[x]_S.$$

We will say that the matrix  $M_\phi$  *represents*  $\phi$ .

**Definition 3.1.** Let  $\mathbb{A}$  be an independence algebra of dimension  $n$ , and let  $S = (e_1, \dots, e_n)$  be a basis for  $\mathbb{A}$ . A pair  $(F, n')$ , where  $F$  is a set, for which (A)–(D) above are satisfied will be called a *matrix system* for  $\mathbb{A}$  (with respect to the basis  $S$ ).

For example, let  $\mathbb{A}^l = (A, A_0, K)$  be a linear independence algebra with  $A_0 = \{0\}$  and  $K$  a field. We could take  $F = K$  and  $n' = n$ , and define matrix multiplication from (3) as the standard

multiplication of matrices; for every  $x \in A$ , define  $[x]_S = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$ , where  $x = \lambda_1 e_1 + \dots + \lambda_n e_n$ ;

for every  $\phi \in \text{End}(\mathbb{A}^l)$ , define  $M_\phi$  as the  $n \times n$  matrix whose  $j$ th column is  $[\phi(e_j)]_S$ .

By the standard theory of matrices in vector spaces,  $(K, n)$  is a matrix system for  $\mathbb{A}^l$ . Our actual matrix system for linear independence algebras will in general use  $n' = n + 1$ , where the additional coordinate represents the contribution of constants in the case of non-trivial  $A_0$ .

**Lemma 3.2.** *Let  $(F, n')$  be a matrix system for an independence algebra  $\mathbb{A}$ . Let  $\phi \in \text{End}(\mathbb{A})$ . Suppose that  $M \in \mathcal{M}_{n' \times n'}^*(F)$  is such that  $[\phi(x)] = M[x]$  for every  $x \in A$ . Then  $M = M_\phi$ .*

*Proof.* For every  $j \in \{1, \dots, n\}$ ,  $[\phi(e_j)] = M[e_j]$  and  $[\phi(e_j)] = M_\phi[e_j]$ . Thus, by (C),  $M$  and  $M_\phi$  have the same corresponding columns, and so they are equal.  $\square$

**Theorem 3.3.** *Let  $(F, n)$  be a matrix system for an independence algebra  $\mathbb{A}$ . Then:*

- (1) *for all  $\phi, \psi \in \text{End}(\mathbb{A})$ ,  $M_{\phi \circ \psi} = M_\phi M_\psi$ ;*
- (2)  *$\mathcal{M}_{n' \times n'}^*(F)$  with matrix multiplication is a monoid that is isomorphic to  $\text{End}(\mathbb{A})$ .*

*Proof.* Let  $\phi, \psi \in \text{End}(\mathbb{A})$ . For every  $x \in A$ ,

$$[(\phi \circ \psi)(x)] = [\phi(\psi(x))] = M_\phi[\psi(x)] = M_\phi(M_\psi[x]) = (M_\phi M_\psi)[x],$$

where the last equality holds by the associativity condition in (B). Thus  $M_{\phi \circ \psi} = M_\phi M_\psi$  by Lemma 3.2. We have proved (1).

By (B),  $\mathcal{M}_{n' \times n'}^*(F)$  with matrix multiplication is a semigroup. By (D) and (1), the mapping  $\phi \mapsto M_\phi$  from  $\text{End}(\mathbb{A})$  to  $\mathcal{M}_{n' \times n'}^*(F)$  is a semigroup isomorphism. Since  $\text{End}(\mathbb{A})$  is a monoid, (2) follows.  $\square$

In the following sections, we will attempt to matrix systems for each type of independence algebra. We will succeed in three of the four cases, and we will pinpoint what goes wrong in the fourth.

## 4 Matrices for group action algebras

Recall that the three Urbanik classes of group action, monoid and exceptional independence algebras are all E-equivalent to the group action algebras in either definition. In developing our matrix theory, we will use the language of the definition of an Urbanik group action independence algebra, keeping in mind that it is easy to convert all other relevant descriptions.

Thus let  $\mathbb{A}^g = \mathbb{A}(A, A_0, G)$  be a finite-dimensional group action independence algebra and let  $n = \dim(\mathbb{A}^g)$ . Then  $n$  is the number of orbits in  $A_1 = A \setminus A_0$ . Any transversal of the orbits is a basis for  $\mathbb{A}^g$  [41, p. 244]; fix such an ordered basis  $X = (e_1, \dots, e_n)$ . Without loss of generality, let  $G$  be disjoint from  $A$ . We now define a matrix system for  $\mathbb{A}^g$ .

Let  $n' = n + 1$ , and  $0$  be an element not in  $G \cup A_0$ . Set  $G_0 = G \cup A_0 \cup \{0\}$ . Define a multiplication  $\cdot$  and a partial addition  $+$  on  $G_0$  by: for all  $m \in G_0$ ,  $g, h \in G$ , and  $x \in A_0$ ,

$$\begin{aligned} g \cdot h &= hg, \quad m \cdot x = x, \quad m \cdot 0 = 0, \quad x \cdot g = g(x), \quad \text{and } 0 \cdot g = 0, \\ g + 0 &= 0 + g = g, \quad 0 + 0 = 0, \quad \text{and } x + x = x, \end{aligned}$$

where  $hg$  is the product in  $G$  (to motivate the change in the order of the arguments, recall that in standard linear algebra over non-commutative division rings, the matrices have to be taken over the opposite ring; this is a corresponding change to the opposite group). We note that for all  $g, h \in G$  and  $x, y \in A_0$  with  $x \neq y$ ,  $g + h$ ,  $g + x$ ,  $x + g$ ,  $0 + x$ ,  $x + 0$ , and  $x + y$  are undefined.

**Lemma 4.1.** *Let  $1$  be the identity in the group  $G$ . Then  $(G_0, \cdot, 1)$  is a monoid.*

*Proof.* First  $1$  is the identity in  $G_0$  since it is clearly the identity in  $(G, \cdot)$ ,  $1 \cdot 0 = 0 \cdot 1 = 0$ , and for every  $x \in A_0$ ,  $1 \cdot x = x$  and  $x \cdot 1 = 1(x) = x$ . Let  $m, p, q \in G_0$ . We want to prove that  $m \cdot (p \cdot q) = (m \cdot p) \cdot q$ . If  $q = 0$  or  $q = x$  (where  $x \in A_0$ ), then the equality is true since  $0$  and each  $x$  are right zeros.

Suppose that  $q = g \in G$ . If  $p = 0$ , then  $m \cdot (0 \cdot g) = m \cdot 0 = 0$  and  $(m \cdot 0) \cdot g = 0 \cdot g = 0$ . If  $p = x$  (where  $x \in A_0$ ), then  $m \cdot (x \cdot g) = m \cdot g(x) = g(x)$  and  $(m \cdot x) \cdot g = x \cdot g = g(x)$ .

Suppose that  $p = h \in G$ . If  $m = 0$ , then  $0 \cdot (h \cdot g) = 0 \cdot (gh) = 0$  and  $(0 \cdot h) \cdot g = 0 \cdot g = 0$ . If  $m = x$  (where  $x \in A_0$ ), then  $x \cdot (h \cdot g) = x \cdot (gh) = (gh)(x)$  and  $(x \cdot h) \cdot g = h(x) \cdot g = g(h(x)) = (gh)(x)$ .

Finally, suppose that  $m = t \in G$ . Then  $t \cdot (h \cdot g) = t \cdot (gh) = (gh)t$  and  $(t \cdot h) \cdot g = (ht) \cdot g = g(ht) = (gh)t$ . This completes the proof.  $\square$

We set  $F = G_0$ , and  $\mathcal{M}_{n' \times 1}^*(G_0)$  to be the subset of all matrices  $v$  in  $\mathcal{M}_{n' \times 1}(G_0)$  such that  $v$  has exactly one non-zero entry  $l$ , and either

- (i)  $l \in G$  and lies in a row different from  $n'$ , or
- (ii)  $l \in A_0$  and lies in row  $n'$ .

We set  $\mathcal{M}_{n' \times n'}^*(G_0)$  to be the subset of all matrices  $M$  in  $\mathcal{M}_{n' \times n'}(G_0)$  such that the first  $n$  columns of  $M$  are in  $\mathcal{M}_{n' \times 1}^*(G_0)$ , and the last column has entry  $1$  in row  $n'$  and  $0$  elsewhere. We define matrix multiplication as follows: for all  $M = (m_{ij})$ ,  $P = (p_{ij})$  in  $\mathcal{M}_{n' \times n'}^*(G_0)$  and  $v = (v_i)$  in  $\mathcal{M}_{n' \times 1}^*(G_0)$ ,  $M \cdot P = Q = (q_{ij})$  and  $M \cdot v = w = (w_i)$ , where

$$q_{ij} = m_{i1} \cdot p_{1j} + m_{i2} \cdot p_{2j} + \dots + m_{in'} \cdot p_{n'j} \quad \text{and} \quad w_i = m_{i1} \cdot v_1 + m_{i2} \cdot v_2 + \dots + m_{in'} \cdot v_{n'}.$$

Note that we use the standard definition of matrix multiplication applied to the (partial) operations of  $G_0$ .

Let  $i, j \in \{1, \dots, n'\}$ . With notation as in the definition, suppose that the column  $P_{*j}$  has non-zero entry  $y \in A_0$  in row  $n'$ . Then

$$\begin{aligned} q_{ij} &= m_{i1} \cdot p_{1j} + m_{i2} \cdot p_{2j} + \dots + m_{in} \cdot p_{nj} + m_{in'} \cdot p_{n'j} \\ &= m_{i1} \cdot 0 + m_{i2} \cdot 0 + \dots + m_{in} \cdot 0 + m_{in'} \cdot y = m_{i,n'} \cdot y, \end{aligned}$$

which equals  $y$  for  $i = n'$  and 0, otherwise. Suppose that the column  $P_{*j}$  has one entry from  $G$ , say  $p_{kj} = g$ , and all other entries 0. Then

$$q_{ij} = m_{i1} \cdot p_{1j} + m_{i2} \cdot p_{2j} + \cdots + m_{in} \cdot p_{nj} = 0 + \cdots + 0 + m_{ik} \cdot g + 0 + \cdots + 0 = m_{ik} \cdot g.$$

Now if the column  $M_{*k}$  has non-zero entry  $x \in A_0$  in row  $n'$ , then  $q_{n'j} = m_{n'k} \cdot g = x \cdot g = g(x)$ , while  $q_{i,j} = 0$ , for  $i \neq n'$ . Otherwise, the column  $M_{*k}$  has one entry from  $G$ , say  $m_{lk} = h \in G$ , where  $l \neq n'$  and all other entries 0. Then the column  $Q_{*j}$  has one entry from  $G$ , namely  $q_{lj} = m_{lj} \cdot g = h \cdot g = gh$ , and all other entries 0. Finally, it can easily be checked that the column  $Q_{*n'}$  has non-zero entry 1 in row  $n'$ . It follows that matrix multiplication is well-defined on  $\mathcal{M}_{n' \times n'}^*(G_0)$ .

By a similar argument,  $M \cdot v$  is well-defined and lies in  $\mathcal{M}_{n' \times 1}^*(G_0)$ . Since our matrix multiplication is the usual multiplication of matrices, the associativity as stated in (B) can be reduced to the algebraic properties of  $\langle G_0, +, \cdot \rangle$ . However, associativity will follow automatically below once we establish that our the matrix multiplication corresponds to composition (and application) of endomorphisms.

It will be convenient to introduce the following notation. Let  $v \in \mathcal{M}_{n' \times 1}^*(G_0)$ ,  $i \in \{1, \dots, n'\}$  be such that the entry in row  $i$  of  $v$  is equal to  $m \in G \cup A_0$  and all other entries of  $v$  are 0. Then we will write  $v = (m, i)$ . Let  $M \in \mathcal{M}_{n' \times n'}^*(G_0)$  and let  $j \in \{1, \dots, n'\}$ . Let  $i_j \in \{1, \dots, n'\}$  be such that the entry  $(i_j, j)$  of  $M$  is equal to  $m_j \in G \cup A_0$ . Then we will write  $M_{*j} = (m_j, i_j)$ . Finally, we will write  $M = (m_j, i_j)_{1 \leq j \leq n'}$ , or simply  $M = (m_j, i_j)$ , if  $M_{*j} = (m_j, i_j)$ , where  $m_j \in G_0 \setminus \{0\}$ . It is straightforward to check that for all  $M = (m_j, i_j)_{1 \leq j \leq n'}$  and  $P = (p_j, k_j)_{1 \leq j \leq n'}$  in  $\mathcal{M}_{n' \times n'}^*(G_0)$ , and all  $v = (t, k) \in \mathcal{M}_{n' \times 1}^*(G_0)$ ,

$$Mv = (m_j, i_j)(t, k) = (m_k \cdot t, i_k), \quad (4.1)$$

$$MP = (m_j, i_j)(p_j, k_j) = (m_{k_j} \cdot p_j, i_{k_j})_{1 \leq j \leq n'}. \quad (4.2)$$

Let  $x \in A$ . Suppose that  $x \notin A_0$ . Then, by Lemma 2.2, there are unique  $g \in G$  and  $e_i \in S$  such that  $x = g(e_i)$ . We define  $[x]_S$  to be  $(g, i)$ . If  $x \in A_0$ , then we define  $[x]_S$  to be  $(x, n')$ .

**Lemma 4.2.** *The mapping  $x \rightarrow [x]$  is a bijection from  $A$  to  $\mathcal{M}_{n' \times 1}^*(G_0)$ . Moreover, for all  $M \in \mathcal{M}_{n' \times n'}^*(G_0)$  and  $j \in \{1, \dots, n\}$ ,  $M[e_j] = M_{*j}$ .*

*Proof.* Let  $x, y \in A$  with  $[x] = [y]$ . By the definition of  $[x]$ , either  $x, y \in A_0$  or  $x, y \notin A_0$ . Suppose that  $x, y \in A_0$ . Then  $[x] = (x, n')$  and  $[y] = (y, n')$ . Since  $[x] = [y]$ , we have  $x = y$ . Suppose that  $x, y \notin A_0$ . Then  $[x] = (g, i)$  and  $[y] = (h, j)$ , where  $g, h \in G$ ,  $x = g(e_i)$ , and  $y = h(e_j)$ . Since  $[x] = [y]$ , we have  $g = h$  and  $i = j$ , and so  $x = y$ . We have shown that the mapping  $x \rightarrow [x]$  is injective. Let  $v \in \mathcal{M}_{n' \times 1}^*(G_0)$ . If  $v = (x, n')$ , where  $x \in A_0$ , then  $v = [x]$ ; and if  $v = (g, i)$ , where  $g \in G$ , then  $v = [g(e_i)]$ . Hence, the mapping  $x \mapsto [x]$  is a bijection.

Let  $M = (m_k, i_k) \in \mathcal{M}_{n' \times n'}^*(G_0)$  and let  $e_j \in S$ . Then  $e_j = 1(e_j)$ , where 1 is the identity of  $G$ , and so  $[e_j] = (1, j)$ . Then, by (4.1),  $M[e_j] = (m_k, i_k)(1, j) = (m_j \cdot 1, i_j) = (m_j, i_j)$ , and so  $M[e_j]$  is the  $j$ th column of  $M$ .  $\square$

For  $\phi \in \text{End}(\mathbb{A}^g)$ , we define  $M_\phi \in \mathcal{M}_{n' \times n'}(A)$  by  $M_\phi = (m_j, i_j)$ , where  $(m_j, i_j) = [\phi(e_j)]$  for  $j \in \{1, \dots, n\}$ , and by  $M_{*n'} = (1, n')$ .

**Lemma 4.3.** *The mapping  $\phi \mapsto M_\phi$  is a bijection from  $\text{End}(\mathbb{A})$  to  $\mathcal{M}_{n' \times n'}^*(G_0)$ . Moreover, for all  $\phi \in \text{End}(\mathbb{A})$  and  $x \in A$ ,  $[\phi(x)] = M_\phi[x]$ .*

*Proof.* Let  $\phi, \psi \in \text{End}(\mathbb{A}^g)$  with  $M_\phi = M_\psi$ . The for every  $j \in \{1, \dots, n\}$ ,  $[\phi(e_j)] = (M_\phi)_{*j} = (M_\psi)_{*j} = [\psi(e_j)]$ , and so  $\phi(e_j) = \psi(e_j)$  by Lemma 4.2. Hence  $\phi = \psi$  since  $S = \{e_1, \dots, e_n\}$  is a basis for  $\mathbb{A}^g$ . Let  $M = (m_j, i_j) \in \mathcal{M}_{n' \times n'}^*(G_0)$ . Define  $\phi \in \text{End}(\mathbb{A}^g)$  by  $\phi(e_j) = (m_j, i_j)$  for every  $j \in \{1, \dots, n\}$ . Then  $M = M_\phi$  by the definition of  $M_\phi$ . We have proved that the mapping  $\phi \rightarrow M_\phi$  is a bijection.



Let  $\phi \in \text{End}(\mathbb{A}^g)$  and  $x \in A$ . Recall that  $M_\phi = (m_j, i_j)$ , where  $(m_j, i_j) = [\phi(e_j)]$  for every  $j \in \{1, \dots, n\}$ . We want to prove that  $[\phi(x)] = M_\phi[x]$ . Suppose that  $x \in A_0$ . Then there is a constant unary operation  $f$  with image  $x$  in  $\mathbb{A}^g$ . Then  $\phi$  preserves  $f$ , and so for any  $a \in A$ ,  $\phi(x) = \phi(f(a)) = f(\phi(a)) = x$ . Since  $x \in A_0$ , we then have  $[\phi(x)] = [x] = (x, n')$ . Thus, by (4.1),

$$M_\phi[x] = (m_j, i_j)(x, n') = (m_{n'} \cdot x, i_{n'}) = (x, n') = [\phi(x)],$$

where we note that  $i_{n'} = n'$ .

Suppose that  $x \notin A_0$ . Then  $[x] = (g, k)$ , where  $g \in G$  and  $k \in \{1, \dots, n\}$  are unique elements such that  $x = g(e_k)$ . Since the action of  $g$  is a unary operation of  $\mathbb{A}^g$ , we have  $\phi(x) = \phi(g(e_k)) = g(\phi(e_k))$ .

Suppose that  $\phi(e_k) = y \in A_0$ . Then  $g(y)$  is also in  $A_0$  (since  $g(A_0) \subseteq A_0$ ), and so  $[y] = (y, n')$  and  $[\phi(x)] = [g(y)] = (g(y), n')$ . Since  $(m_k, i_k) = (M_\phi)_{*k} = [\phi(e_k)] = [y] = (y, n')$ , we have, by (4.1),

$$M_\phi[x] = (m_j, i_j)(g, k) = (m_k \cdot g, i_k) = (y \cdot g, i_k) = (g(y), i_k) = (g(y), n') = [\phi(x)].$$

Suppose that  $\phi(e_k) = y \notin A_0$ . Then  $(m_k, i_k) = (M_\phi)_{*k} = [\phi(e_k)] = (h, s)$ , where  $h \in G$  and  $s \in \{1, \dots, n\}$  are unique elements such that  $\phi(e_k) = h(e_s)$ . Thus  $\phi(x) = g(\phi(e_k)) = g(h(e_s)) = (gh)(e_s)$ , and so  $[\phi(x)] = (gh, s)$ . By (4.1) again,

$$M_\phi[x] = (m_j, i_j)(g, k) = (m_k \cdot g, i_k) = (h \cdot g, s) = (gh, s) = [\phi(x)].$$

This concludes the proof.  $\square$

We have proved that  $(G_0, n)$ , as defined in this section, is a matrix system for  $\mathbb{A}^g$ .

It is clear the connection of this matrix system and *generalised permutation matrices*.

## 5 Matrices for linear algebras

Let  $\mathbb{A}^l = (A, A_0, K)$  be a linear independence algebra of dimension  $n$  (see (2.7) and (2.8)). We write 0 and  $0_A$  for the zeros of  $K$  and  $A$ , respectively. Any ordered basis  $S = (e_1, \dots, e_n)$  for  $\mathbb{A}^l$  is obtained by choosing a linear basis  $S'$  for the quotient vector space  $A/A_0$ , selecting a representative of each block of  $S'$ , and ordering the selected representatives. We fix such an ordered basis  $S$ . It is easy to see that for every  $x \in A$ , there are unique  $\lambda_1, \dots, \lambda_n \in K$  and  $a \in A_0$  such that

$$x = \lambda_1 e_1 + \dots + \lambda_n e_n + a. \quad (5.1)$$

By (2.7) and (2.8) the monoid  $\text{End}(\mathbb{A}^l)$  consists of the linear transformations  $\varphi$  of the vector space  $A$  such that  $\varphi(a) = a$  for every  $a \in A_0$ .

To define a matrix system for  $\mathbb{A}^l$ , we let  $F = K \cup A_0$ ,  $n' = n + 1$ ,  $\mathcal{M}_{n' \times n'}^*(F)$  to be the set of all  $M \in \mathcal{M}_{n' \times n'}(F)$  such that  $M_{*j} = (\lambda_{1j} \dots \lambda_{nj} a_j)^T$  (if  $1 \leq j \leq n$ ), and  $M_{*n'} = (0 \dots 0 1)^T$ , and  $\mathcal{M}_{n' \times 1}^*(F)$  to be the set of all  $v = (\lambda_1 \dots \lambda_n a) \in \mathcal{M}_{n' \times 1}(F)$ , where  $\lambda_{ij}, \lambda_i \in K$  and  $a_j, a \in A_0$ .

Let  $M, P \in \mathcal{M}_{n' \times n'}(F)$  and  $v \in \mathcal{M}_{n' \times 1}(F)$ . We let the multiplication  $\cdot$  in  $M \cdot P$  and  $M \cdot v$  to be the usual matrix multiplication from linear algebra, where the multiplication  $*$  and addition  $\oplus$  of the entries of the matrices are defined by:  $\lambda_1 * \lambda_2 = \lambda_2 \lambda_1$ ,  $\lambda * a = a * \lambda = \lambda a$ ,  $\lambda_1 \oplus \lambda_2 = \lambda_1 + \lambda_2$ ,  $a_1 \oplus a_2 = a_1 + a_2$ , and  $\lambda \oplus 0_A = 0_A \oplus \lambda = \lambda$ , where  $\lambda_1, \lambda_2, \lambda \in K$ ,  $a_1, a_2, a \in A_0$ . We note that in  $M \cdot P$  and  $M \cdot v$ ,  $a_1 * a_2$ ,  $\lambda \oplus a$ , and  $a \oplus \lambda$  never arise, where  $\lambda \in K$  and  $a_1, a_2, a \in A_0$  with  $a \neq 0_A$ .

For  $x \in A$  and  $\varphi \in \text{End}(\mathbb{A}^l)$ , we let  $[x]_S = (\lambda_1 \dots \lambda_n a)^T$ , where  $\lambda_i$  and  $a$  are as in (5.1), and define  $M_\varphi \in \mathcal{M}_{n' \times n'}^*(F)$  by  $(M_\varphi)_{*j} = [\varphi(e_j)]_S$ , for each  $j \in \{1, \dots, n\}$ , and  $(M_\varphi)_{*n'} = (0 \dots 0 1)^T$ . By routine calculations, we check that for all  $\varphi, \psi \in \text{End}(\mathbb{A}^l)$  and  $x \in A$ ,

$$[\varphi(x)]_S = M_\varphi \cdot [x]_S \quad \text{and} \quad [(\varphi \circ \psi)(x)]_S = (M_\varphi \cdot M_\psi) \cdot [x]_S.$$

It then follows that  $(K \cup A_0, n')$  satisfies (A)–(D), so it is a matrix system for  $\mathbb{A}^l$ .

## 6 Matrices for affine algebras

Let  $\mathbb{A}^a = (A, A_0, K)$  be an affine independence algebra of dimension  $n$  (see (2.9)). (The dimension of the corresponding linear independence algebra  $\mathbb{A}^l = (A, A_0, K)$  is  $n - 1$  [41, page 236].) Any ordered basis  $S = (e_1, \dots, e_n)$  for  $\mathbb{A}^m$  is obtained by choosing an  $n$ -element affine independent set  $S'$  for the quotient space  $A/A_0$ , selecting a representative of each block of  $S'$ , and ordering the selected representatives. We fix such an ordered basis  $S$ . It is easy to see that for every  $x \in A$ , there are unique  $\lambda_1, \dots, \lambda_n \in K$ , with  $\lambda_1 + \dots + \lambda_n = 1$ , and  $a \in A_0$  such that

$$x = \lambda_1 e_1 + \dots + \lambda_n e_n + a. \quad (6.1)$$

By (2.9), the monoid  $\text{End}(\mathbb{A}^a)$  consists of the affine transformations  $\varphi$  of the affine space  $A$  such that  $\varphi(x + a) = \phi(x) + a$  for all  $x \in A$  and  $a \in A_0$ .

To define a matrix system for  $\mathbb{A}^m$ , we let  $F = K \cup A_0$ ,  $n' = n + 1$ ,  $\mathcal{M}_{n' \times n'}^*(F)$  to be the set of all  $M \in \mathcal{M}_{n' \times n'}(F)$  such that  $M_{*j} = (\lambda_{1j} \dots \lambda_{nj} a_j)^T$ , with  $\lambda_{1j} + \dots + \lambda_{nj} = 1$  (if  $1 \leq j \leq n$ ) and  $M_{*n'} = (0 \dots 0 1)^T$ , and  $\mathcal{M}_{n' \times 1}^*(F)$  to be the set of all  $v = (\lambda_1 \dots \lambda_n a) \in \mathcal{M}_{n' \times 1}(F)$ , with  $\lambda_1 + \dots + \lambda_n = 1$ , where  $\lambda_{ij}, \lambda_i \in K$  and  $a_j, a \in A_0$ .

Let  $M, P \in \mathcal{M}_{n' \times n'}(F)$  and  $v \in \mathcal{M}_{n' \times 1}(F)$ . We define the multiplication  $\cdot$  in  $M \cdot P$  and  $M \cdot v$  exactly as in Section 5.

For  $x \in A$  and  $\varphi \in \text{End}(\mathbb{A}^m)$ , we let  $[x]_S = (\lambda_1 \dots \lambda_n a)^T$ , where  $\lambda_i$  and  $a$  are as in (6.1) (so  $\lambda_1 + \dots + \lambda_n = 1$ ), and define  $M_\varphi \in \mathcal{M}_{n' \times n'}^*(F)$  by  $(M_\varphi)_{*j} = [\varphi(e_j)]_S$ , for each  $j \in \{1, \dots, n\}$ , and  $(M_\varphi)_{*n'} = (0 \dots 0 1)^T$ . By routine calculations, exactly the same as in Section 5, we check that for all  $\varphi, \psi \in \text{End}(\mathbb{A}^m)$  and  $x \in A$ ,

$$[\varphi(x)]_S = M_\varphi \cdot [x]_S \quad \text{and} \quad [(\varphi \circ \psi)(x)]_S = (M_\varphi \cdot M_\psi) \cdot [x]_S.$$

It then follows that  $(K \cup A_0, n')$  satisfies (A)–(D), so it is a matrix system for  $\mathbb{A}^m$ .

## 7 Matrices for sharply 2-transitive group algebras

We have failed to develop a matrix theory for independence algebras of this type. However, we can go part of the way; it seemed worth recording the arguments since it throws light on this somewhat unusual case.

In the literature, there are different but closely related generalizations of the equivalence between sharply 2-transitive groups with regular normal subgroups and nearfields. These include Tits' *pseudofields* [38], Wilke's *strong pseudofields* [42], Grätzer's *quasifields* [27] and Karzel's *neardomains* ("Fastbereich") [29, vol. 1, p. 21]. (Note that Grätzer did not name his algebras; the term "quasifield", which does not denote the same class of algebras as the notion of quasifield in projective geometry, is due to Urbanik [41]. Belousov called them *Grätzer algebras* [11].) In all cases, up to left/right conventions for operations and group actions, there is a structure of the form  $(X, \odot, \cdot, 0, 1)$  where  $\odot$  and  $\cdot$  are binary operations and  $0, 1 \in X$ . The element 0 is a right identity element for  $\odot$  and a left zero for  $\cdot$ ,  $(X \setminus \{0\}, \cdot, 1)$  is a group and  $\cdot$  distributes over  $\odot$  on the left. The corresponding sharply 2-transitive group action on  $X$  consists of "affine" mappings of the form  $x \mapsto b \odot (a \cdot x)$ .

Where the structures differ is in their axioms for the operation  $\odot$ . In pseudofields, strong pseudofields and neardomains,  $\odot$  generalizes the addition operation in nearfields. In (Grätzer) quasifields,  $\odot$  generalizes the subtraction operation.

The definitions of near domains and strong pseudofields coincide (up to left/right conventions). Regarding Grätzer's quasifields and (Tits') pseudofields, one can be retrieved from the other using  $x + y := (y - (0 - x))$  (to go from Grätzer to Tits) and  $x - y := ((-y) + x)$  for the converse.

For our present purposes we note the following. "Quasifield independence algebras" are described in terms of Grätzer quasifields, but an examination of the definition shows that what is really required is that the operations in  $F$  commute with the sharply 2-transitive group action.

Thus it does not matter if we follow Urbanik and use Grätzer quasifields or if we use one of the other structures instead. In fact, it seems to be more convenient to use neardomains. As already noted in the introduction to [29], neardomains have an advantage over Tits pseudofields and Grätzer quasifields in that the correspondence between neardomains and sharply 2-transitive groups is one-to-one. (When precisely formulated, the correspondence is actually a categorical equivalence [15].)

A *neardomain*  $(K, +, \cdot, 0, 1)$  is a set  $K$  with two binary operations  $+$ ,  $\cdot$  and elements  $0, 1 \in K$  such that the following properties hold.

1.  $(K, +)$  is a loop with identity element 0 (that is,  $(K, +)$  is a binary structure with identity 0 in which the equations  $a + x = b$  and  $x + a = b$  have unique solutions for all  $(a, b) \in K^2$ );
2. for all  $a, b \in K$ ,  $a + b = 0 \implies b + a = 0$ , that is, in  $(K, +)$  all left-inverses are also right-inverses;
3.  $(K^*, \cdot)$  is a group with identity element 1;
4.  $a(b + c) = ab + ac$  for all  $a, b, c \in K$ ;
5.  $0 \cdot a = 0$  for all  $a \in K$ ;
6. for every  $a, b \in K$ , there exists  $\delta_{a,b} \in K^*$  such that for all  $c \in K$ ,

$$a + (b + c) = (a + b) + \delta_{a,b}c.$$

Here, as usual,  $K^* = K \setminus \{0\}$

A neardomain is a nearfield if and only if  $+$  is associative, that is, precisely when  $\delta_{a,b} = 1$  for all  $a, b$ . For a general neardomain, the additive loop  $(K, +)$  is known variously as a *K-loop* (“K” for “Karzel”) or a Bruck loop [30].

Each  $b \in K^*$  and  $a \in K$  defines an *affine transformation*  $T_{a,b} : K \rightarrow K$  by  $T_{a,b}x = a + bx$ . The composite of two such transformations is another one:

$$(T_{a,b} \circ T_{c,d})x = a + b(c + dx) = a + (bc + bdx) = (a + bc) + \delta_{a,b}bdx = T_{a+bc, \delta_{a,b}bd}x$$

for all  $a, c, x \in K$ ,  $b, d \in K^*$ . Evidently  $T_{0,1}$  is the identity mapping. Further, each  $T_{a,b}$  is a permutation of  $K$  with inverse  $T_{-b^{-1}a, b^{-1}}$  (because the additive loop of a neardomain satisfies the identity  $-x + (x + y) = y$ ). Finally, if  $T_{a,b} = T_{a',b'}$ , then applying both sides to 0 gives  $a = a'$ , while applying both sides to 1 and cancelling  $a$  on the left gives  $b = b'$ .

The *affine group* of  $K$  is  $\text{Aff}(K) = \{T_{a,b} \mid a \in K, b \in K^*\}$ . The above considerations show that this group is isomorphic to  $K \times K^*$  with the following multiplication:

$$(a, b)(c, d) = (a + bc, \delta_{a,b}bd)$$

for all  $a, c \in K$ ,  $b, d \in K^*$ . This type of construction is known as a *quasidirect product*, generalizing the semidirect product of groups [30]. Note that we do not have to check directly the associativity of the multiplication; this follows from the fact that the multiplication represents the composition of affine transformations.

The group  $\text{Aff}(K)$  is sharply 2-transitive, and conversely, every sharply 2-transitive group is isomorphic to the affine group of a (uniquely determined) neardomain. A long standing open problem was whether a proper neardomain exists, that is, a neardomain which is not a nearfield. This is equivalent to the question of whether there exists a sharply 2-transitive group without a regular normal subgroup. This has recently been answered in the affirmative [37]. However, in the finite case, it is elementary to show that every neardomain is a nearfield (that is, every finite sharply 2-transitive group has a regular normal subgroup), and the finite nearfields were classified by Zassenhaus [43].

Returning to independence algebras, we see that we may replace the notion of quasifield independence algebra with that of a *neardomain independence algebra* in which we require that  $f(a + bx) = a + bf(x)$  for all  $f \in F$ ,  $a, x \in K$ ,  $b \in K^*$ .

In the special case that the neardomain is a division ring, we can obtain a matrix system equivalent to that of an affine independence algebra.

Let  $\{e_1, e_2\}$  be a basis (recall that algebras from sharply-2-transitive groups have dimension 2). We set  $n' = 2$ ,  $D = K^{op}$  (the opposite ring of  $K$ ), and identify each element  $a \in A$  uniquely with a column vector  $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in K^2$ , such that

$$a = c_1 e_1 + c_2 e_2 \text{ and } c_1 + c_2 = 1.$$

For each endomorphism on  $A$ , we define a  $(2 \times 2)$ -matrix by Condition (C). With matrix multiplication defined as usual, it is straightforward to check that we indeed obtain a matrix system.

In the general case of a neardomain, it is easy to give a realization that does not relate to specific bases. However, it appears to be necessary to have the associative and both distributive laws in order to obtain a matrix theory that is satisfactory in the sense of adhering to our Conditions (A) to (D).

## 8 Summary

The following table sums up the properties of our matrix theory for the various types of independence algebras.

Independence Algebras	dim	vector size	$(n + 1)$ -st entry	1st - $n$ th entry	conditions
$A^g = \mathbb{A}(A, A_0, G)$ (group action)	$n$	$n + 1$	$A_0 \cup \{0\}$	$G \cup \{0\}$	exactly one non-zero entry
$V[W]$ over $K$ (linear)	$n$	$n + 1$	$W$	$K$	—
$\text{Aff}(V)[+W]$ over $K$ (affine)	$n$	$n + 1$	$W$	$K$	$K$ -entries add to 1
Sharply 2-trans. group alg. $A$ (neardomain is a div. ring)	2	2	n/a	$A$	entries add to 1

## 9 Problems

In this section we present some problems regarding the “linear algebra” of our matrix theory.

*Problem 9.1.* Can a matrix theory for neardomain or nearfield independence algebras be recovered, perhaps by weakening the requirements of the definition?

*Problem 9.2.* Extend the matrix theory developed here to infinite-dimensional independence algebras. Note that the existence of bases for all vector spaces is equivalent to the Axiom of Choice, so some set-theoretic assumption is certainly required.

*Problem 9.3.* Extend to the matrix theory developed here the classic results for the usual matrices (such as the rational or Jordan Canonical Forms, etc.).

*Problem 9.4.* The standard similarity relation on matrices is just one among many possible notions of conjugation for semigroups (see [6, 7]). Let  $\sim$  be a notion of conjugation in semigroups. Describe the  $\sim$ -classes for the new types of matrices introduced here.

*Problem 9.5.* Generalize to the new semigroups of matrices introduced in this paper the results on semigroups of matrices with entries in a field (Green’s relations, automorphisms, congruences, etc.).

## Appendix

In this appendix, we prove several technical properties of the operator  $\Psi$ .

**Lemma 9.6.** *Let  $\mathbb{A} = \langle A; F \rangle$ ,  $\mathbb{A}_1$ , and  $\mathbb{A}_2$  be any non-trivial  $v^*$ -algebras, and let  $\Psi(\mathbb{A}) = \langle A; F' \rangle$ . Then:*

1. *except for the nullary operations in  $F'_{\text{cl}}$ , the clones  $F_{\text{cl}}$  and  $F'_{\text{cl}}$  are identical;*
2. *if  $\mathbb{A}_1$  and  $\mathbb{A}_2$  are clone equivalent, then  $\Psi(\mathbb{A}_1)$  and  $\Psi(\mathbb{A}_2)$  are also clone equivalent;*
3.  *$\Psi$  is a one-to-one correspondence (if clone equivalent algebras are regarded as identical);*
4. *for every  $X \subseteq A$ :*
  - (i)  *$[X]$  in  $\mathbb{A}$  is equal to  $\langle X \rangle$  in  $\Psi(\mathbb{A})$ ,*
  - (ii)  *$X$  is independent in  $\mathbb{A}$  if and only if  $X$  is independent in  $\Psi(\mathbb{A})$ ,*
  - (iii)  *$X$  is a basis for  $\mathbb{A}$  if and only if  $X$  is a basis for  $\Psi(\mathbb{A})$ ;*
5.  *$\dim(\mathbb{A}) = \dim(\Psi(\mathbb{A}))$  and  $\text{End}(\mathbb{A}) = \text{End}(\Psi(\mathbb{A}))$ .*

*Proof.* 1. Clearly  $F_{\text{cl}} \subseteq F'_{\text{cl}}$ , as the  $F \subseteq F'$ . Conversely, let  $h \in F'_{\text{cl}}$  be a non-nullary operation. Then  $h$  can be written as a composition of projections and elements of  $F'$ . If some nullary operation  $g_f$  appears in this composition, then we may replace it with the constant function  $f(x)$  for some argument  $x$  already appearing in the composition ( $x$  is available, as  $h$  is not nullary). It is clear that this changes result in the same function  $h$ . The second composition shows that  $h \in F_{\text{cl}}$ .

2. This follows immediately, because the nullary operations added by the operator  $\Psi$  only depend on the clone of its argument.
3. We construct an inverse operator  $\bar{\Psi}$  as follows: For any non-trivial independence algebra  $\mathbb{A} = \langle A; F \rangle$ , replace every nullary operations  $g \in F$  such that  $g(\emptyset) = a$  with a constant unary operation  $f_g$ , given by  $f_g(x) = a$ . Because  $f_g(x) = \pi_1(g, x)$ , the operations  $f_g$  are in  $F_{\text{cl}}$ . Hence  $\bar{\Psi}(\mathbb{A})$  is obtained from an algebra that is clone-equivalent to  $\mathbb{A}$  by removing all nullary operations, and hence is a  $v^*$ -algebra.

We claim that  $\Psi(\bar{\Psi}(\mathbb{A}))$  is clone equivalent to  $\mathbb{A}$ . Suppose first that  $\mathbb{A}$  has no nullary operations, then  $\bar{\Psi}(\mathbb{A}) = \mathbb{A}$ . As  $\mathbb{A}$  is non-trivial,  $F_{\text{cl}}$  cannot contain any constant nullary clone operations, for otherwise it could not be both a  $v^*$ -algebra and an independence algebra. It follows that  $\Psi(\bar{\Psi}(\mathbb{A})) = \mathbb{A}$ .

Assume otherwise that  $\mathbb{A}$  contains at least one nullary operation  $g_0$ . Clearly,  $\Psi(\bar{\Psi}(\mathbb{A}))$  is obtained from  $\mathbb{A}$  by adding a basic nullary operation  $g_f$  for every constant unary function  $f$  in  $F_{\text{cl}}$  (assuming  $g_f \notin F$  already). However, because  $g_f = f(g_0)$  the  $g_f$  are in  $F_{\text{cl}}$ . Hence  $\Psi(\bar{\Psi}(\mathbb{A}))$  is clone equivalent to  $\mathbb{A}$ .

Conversely, let  $\mathbb{A} = \langle A; F \rangle$  be a non-trivial  $v^*$ -algebra. Then it is easy to see that  $\bar{\Psi}(\Psi(\mathbb{A}))$  is obtained from  $\mathbb{A}$  by adding all constant unary operations from  $F_{\text{cl}}$  as basic operations. It follows that  $\bar{\Psi}(\Psi(\mathbb{A}))$  is clone-equivalent to  $\mathbb{A}$ , as required.

4. Clearly, the last two assertions follow from the first. To show (i), first assume that  $X \neq \emptyset$ . Then  $[X]$  and  $\langle X \rangle$  are defined identical, but with regard to two different algebras. As  $F'$  is obtained from  $F$  by the addition of nullary operation  $f_g$ , we have that  $[X] \subseteq \langle X \rangle$ . Conversely, if  $g_f \in F'$  is a nullary operation, then  $g_f(\emptyset) \in [X]$ , because  $f(x) = g_f(\emptyset)$  for all  $x \in X \neq \emptyset$ . It follows that  $[X] = \langle X \rangle$ .

If  $X = \emptyset$ , then  $[X] = [\text{Con}^*]$ , where  $\text{Con}^*$  is the set of all images of constant unary functions in  $F_{\text{cl}}$ , while  $\langle X \rangle = \langle \text{Con} \rangle$ , where  $\text{Con}$  is the set of images of nullary functions (all of this

holds provided that  $\text{Con}$  and  $\text{Con}^*$  are non-empty). However, by the construction of  $\Psi$ , we have that  $\text{Con}^* = \text{Con}$ , and so  $[X] = \langle X \rangle$  by the case  $X \neq \emptyset$ .

Finally, if  $\text{Con}^* = \emptyset$ , then  $\text{Con} = \emptyset$ , and so  $[X] = \emptyset = \langle X \rangle$ .

5. The first assertion follows directly from 4.(iii).

For the second, it suffices to show that any  $\phi \in \text{End}(\mathbb{A})$  preserves every nullary operation  $g_f \in F'$ . For  $x \in A$  arbitrary, we have

$$\phi(g_f(\emptyset)) = \phi(f(x)) = f(\phi(x)) = g_f(\emptyset),$$

as required. □

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