

MODELLING SILICOSIS: EXISTENCE, UNIQUENESS AND BASIC PROPERTIES OF SOLUTIONS

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ABSTRACT. We present a model for the silicosis disease mechanism following the original proposal by Tran, Jones, and Donaldson (1995) [9], as modified recently by da Costa, Drmota, and Grinfeld (2020) [4]. The model consists in an infinite ordinary differential equation system of coagulation-fragmentation-death type. Results of existence, uniqueness, continuous dependence on the initial data and differentiability of solutions are proved for the initial value problem.

1. INTRODUCTION

Silicosis is an occupational lung disease characterized by inflammation and fibrosis of the lungs caused by the inhalation of crystalline silica dust (also known as quartz or silicon dioxide dust, with chemical composition SiO_2) and can progress to respiratory failure and death [8]. Although it is a preventable disease, it is presently incurable and affects hundreds of thousands of persons, killing an estimated eleven thousand annually (see figures for 2017 in [5]).

In this paper we consider a mathematical model for the process used by the immune system to clear inhaled silica particles from the lungs' alveoli. The model is a slight modification proposed in [4] of one introduced in [9] that, to the best of our knowledge, has not received the attention we think it deserves.

We start by giving a simplified description of the physiological processes, essentially using [9], followed by the presentation of the mathematical model to be studied.

When a silica particle enters the respiratory system down to the pulmonary alveoli (the innermost part of the lungs) its presence can be detected by cells of the immune system called alveoli macrophages, either by random encounters or by chemotaxis generated by the silica particles. Macrophages can then engulf the silica particle in a process known as phagocytosis. Typically, the phagocytosis of organic materials (pathogens, dying or dead cells and cellular debris) result in its destruction by the macrophage with the production of basic molecular fragments

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followed by their expulsion or assimilation [1]. In the case of silica particles, however, there is essentially nothing to be destroyed from a molecular point of view and the role of the macrophages is just to expel their load of silica by travelling up the mucociliary escalator to get eliminated out of the respiratory system [10]. However, due to a mechanism not yet completely understood, silica particles are toxic to the macrophages [6] and lead to their death, which can occur while they are still in the lungs, thus liberating the silica particles back into the respiratory system.

A macrophage can ingest more than one silica particle. The hability of a given macrophage already containing i particles to ingest an additional particle typically decreases with i and in the model in [9] a maximum load capacity of $n_{\max} < \infty$ is assumed *a priori*. Given the toxicity of the silica particles, macrophages with a higher load of silica particles die at an higher rate. Additionally, the capacity of the macrophages to migrate up the mucocilliary escalator is also impaired by an increase in their load of silica particles. It is the balance of these three types of processes that leads to the mathematical model in [9] and that we also consider here with the modifications introduced in [4].

Let us denote by $M_i = M_i(t)$ the concentration of macrophages containing i silica particles (which we will call the i -th cohort) at time t , by $x = x(t)$ the concentration of silica particles, and by r (which, in principle, can be a function of x) the rate of supply of new (empty of silica particles) macrophages. Following [9], we obtain the following equations for the mechanism described above:

$$\begin{aligned} \frac{dM_0}{dt} &= r - k_0 x M_0 - (p_0 + q_0) M_0, \\ \frac{dM_i}{dt} &= k_{i-1} x M_{i-1} - k_i x M_i - (p_i + q_i) M_i, \quad i \geq 1, \end{aligned}$$

where k_i is the rate of phagocytosis of a silica particle by a macrophage already containing i particles, p_i , is the transfer rate of macrophages in the i -th cohort to the mucocilliary escalator, i.e. the rate of their removal from the pulmonary alveoli together with their quartz load, and q_i is the rate of death of the macrophages in the i -th cohort which results in the release of the quartz burden back into the lungs. Unlike in [9] we do not impose an upper limit on the number of particles $n_{\max} < \infty$ a macrophage can contain, the existence of such a load capacity will be a consequence of the assumptions on the rate coefficients k_i and q_i . In [4] the following rate equation for the evolution of the concentration of silica particles in the system under the assumption of an inhalation rate α was postulated, valid under the same assumption about the validity of the mass action law used to obtain the equations for the M_i :

$$\frac{dx}{dt} = \alpha - x \sum_{i=0}^{\infty} k_i M_i + \sum_{i=0}^{\infty} q_i i M_i.$$

A scheme of the processes modelled by the above rate equations is presented in Figure 1.

In [4] the equilibria of this system (with a countable infinite number) of equations was studied. An important modelling goal is, of course, the study of the dynamical stability properties of those equilibria and for this we need to consider solutions which are not equilibria and to study their behaviour as $t \rightarrow +\infty$. Thus,

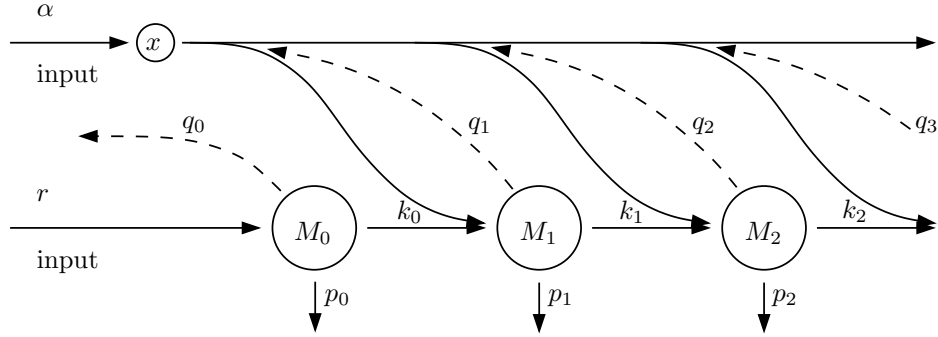


FIGURE 1. Reaction scheme of the mechanism studied in this paper. The input rates of quartz and of macrophages with no quartz particles are α and r , respectively. The concentration of free quartz particles and of macrophages containing j quartz particles are represented by x and M_j , respectively. Macrophages M_j can be destroyed, releasing j quartz particles (dashed lines), or they can be removed by rising in the mucocilliary escalator (vertical downward lines), or they can ingest an additional quartz particle becoming an M_{j+1} macrophage (horizontal rightward arrows).

the existence of solutions of the initial value problem for this system with a countable infinite number of ordinary differential equations is a problem that has to be addressed first. This is the goal of the present paper.

In section 2 we state, in precise mathematical terms, the problem to be studied and the basic approach we shall use, and relate this system to others studied in the literature of cluster growth modelling. In section 3 we consider a finite dimension truncated system that shall be used, in section 4, to prove the existence of solutions to the Cauchy problem, which can be considered the main result of the paper. To prove differentiability of the solutions, in section 6, as well as uniqueness, in section 7, and the semigroup property, in section 8, a number of *a priori* estimates and evolution equations for the solution moments are needed and will be stated and proved in section 5.

2. PRELIMINARIES

The Cauchy problem we will consider in this paper is the following system of a countable number of ordinary differential equations with nonnegative initial conditions

$$\begin{cases} \dot{M}_0 = r - k_0 x M_0 - (p_0 + q_0) M_0, \\ \dot{M}_i = k_{i-1} x M_{i-1} - k_i x M_i - (p_i + q_i) M_i, & i \geq 1, \\ \dot{x} = \alpha - x \sum_{i=0}^{\infty} k_i M_i + \sum_{i=0}^{\infty} i q_i M_i, \\ x(0) = x_0, \quad M_i(0) = M_{0i}, \quad i = 0, 1, 2, \dots \end{cases} \quad (1)$$

This is an infinite dimensional system of ordinary differential equations of the type coagulation-fragmentation-death type and its analysis requires the definition

of an Banach space to work in, as well as an adequate definition of solution. From the biological interpretation of the phase variables x and M_i ($i = 0, 1, \dots$) given in the Introduction we can define the following quantities:

$$\text{Amount of quartz particles: } \mathcal{X} := x + \sum_{i=0}^{\infty} iM_i,$$

$$\text{Amount of macrophages: } \mathcal{M} := \sum_{i=0}^{\infty} M_i,$$

$$\text{Total amount of matter: } \mathcal{U} := \mathcal{X} + \mathcal{M} = x + \sum_{i=0}^{\infty} (i+1)M_i.$$

It is easy to conclude from (1) that, at a formal level,

$$\dot{\mathcal{M}} = \sum_{i=0}^{\infty} \dot{M}_i = \dot{M}_0 + \sum_{i=1}^{\infty} \dot{M}_i = r - \sum_{i=0}^{\infty} (p_i + q_i)M_i,$$

$$\dot{\mathcal{X}} = \dot{x} + \sum_{i=1}^{\infty} i\dot{M}_i = \alpha - \sum_{i=1}^{\infty} ip_iM_i,$$

and

$$\dot{\mathcal{U}} = \dot{\mathcal{X}} + \dot{\mathcal{M}} = (r + \alpha) - \sum_{i=1}^{\infty} ((i+1)p_i + q_i)M_i.$$

Remark 2.1. *These formal computations are justified if, for all $i = 0, 1, 2, \dots$, and all $t \geq 0$, we have, $M_i \geq 0$, $\sum_{i=0}^{\infty} M_i < \infty$, and $\sum_{i=0}^{\infty} (k_i + p_i + q_i)iM_i < \infty$, being these convergences uniform on each interval $[0, T]$.*

Therefore, at least at a formal level, we have the following *a priori* bounds: $\dot{\mathcal{M}} \leq r$, $\dot{\mathcal{X}} \leq \alpha$, and $\dot{\mathcal{U}} \leq r + \alpha$.

These formal computations and a comparison with the case of Becker-Döring equations [2] and with other more general coagulation-fragmentation systems [3] suggest that considering a space of sequences (x, M_0, M_1, \dots) where, for nonnegative sequences, the quantity \mathcal{U} is finite can be expected to be a mathematically reasonable space to work in and, furthermore, will also have biological meaning.

Thus, let us denote an element of $\mathbb{R}^{\mathbb{N}}$ by $y = (y_n) = (x, M_0, M_1, \dots)$. Define

$$\|y\| = |x| + \sum_{i=0}^{\infty} (i+1)|M_i| = |x| + |((i+1)M_i)_{i=0,1,\dots}|_{\ell^1},$$

and $X = \{y = (y_n) : \|y\| < \infty\}$. We say that $y \geq 0$ if and only if $y \in (\mathbb{R}_0^+)^{\mathbb{N}}$ and will denote the nonnegative cone of X by $X_+ := \{y \in X : y \geq 0\}$.

Similarly, comparison with other types of coagulation-fragmentation equations suggests the adoption of a definition of solution that is very close to that used in [2] for the Becker-Döring system. This is not surprising, since an inspection of the model to be considered shows that, as in the Becker-Döring system, there is a phase variable that has a crucially distinct role in the dynamics: in the Becker-Döring system was the concentration of monomers [2], in system (1) it is the concentration of crystalline silica, x . This does not mean that (1) can be automatically transformed into the Becker-Döring system, but it suggests analogies in the definition of solution and in the general approach in the proofs we shall present.

Definition 2.2. Let $0 < T \leq +\infty$. A solution of (1) is a function $y : [0, T) \rightarrow X$ such that:

- (i) $\forall t \in [0, T) \quad y(t) \geq 0$;
- (ii) each $y_n : [0, T) \rightarrow \mathbb{R}$ is continuous and $\sup_{t \in [0, T']} \|y(t)\| < \infty$, for each $T' \in (0, T)$;
- (iii) $\int_0^t x(s) ds < \infty$, $\int_0^t \sum_{i=0}^{\infty} (ip_i + iq_i + k_i) M_i(s) ds < \infty$, for all $t \in [0, T)$;
- (iv) for all $t \in [0, T)$,

$$M_0(t) = M_{00} + rt - \int_0^t [k_0 x(s) M_0(s) - (p_0 + q_0) M_0(s)] ds, \quad (2)$$

$$M_i(t) = M_{0i} + \int_0^t [x(s) k_{i-1} M_{i-1}(s) - x(s) k_i M_i(s) - (p_i + q_i) M_i(s)] ds, \quad i \geq 1, \quad (3)$$

$$x(t) = x_0 + \alpha t + \int_0^t \left[-x(s) \sum_{i=0}^{\infty} k_i M_i(s) + \sum_{i=0}^{\infty} i q_i M_i(s) \right] ds. \quad (4)$$

A standard approach used in many studies of coagulation-fragmentation type equations is to consider a finite n -dimensional truncation of the infinite dimensional system and prove *a priori* bounds for the evolution of quantities associated with the truncated system that are independent of the truncation dimension n . This allows us to take limits as $n \rightarrow \infty$ in the truncated system and, in this way, to prove results for the infinite system. In order to apply the same strategy here, we will present, in the next section, a truncated system and a useful auxiliary time evolution result.

3. THE TRUNCATED SILICOSIS SYSTEM

In (1) consider rate coefficients satisfying $k_i = 0$, for $i \geq n$, and $p_i = q_i = 0$, for $i \geq n + 1$. It is clear from the biological meaning of these constants (and also from the kinetic scheme in Figure 1) that, with these conditions, if we take initial data satisfying $M_i(0) = 0$ if $i \geq n + 1$ then, for these i , $M_i(t)$ will remain zero for all $t > 0$. It is due to this invariance that the above conditions give rise to a finite $(n+2)$ -dimensional approximation of the infinite dimensional system (1). The corresponding Cauchy problem, which we shall use extensively in the next section, can be explicitly written as follows:

$$\begin{cases} \dot{M}_0^n = r - k_0 x^n M_0^n - (p_0 + q_0) M_0^n, \\ \dot{M}_i^n = k_{i-1} x^n M_{i-1}^n - k_i x^n M_i^n - (p_i + q_i) M_i^n, \quad i = 1, \dots, n-1, \\ \dot{M}_n^n = k_{n-1} x^n M_{n-1}^n - (p_n + q_n) M_n^n, \\ \dot{x}^n = \alpha - x^n \sum_{i=0}^{n-1} k_i M_i^n + \sum_{i=0}^n i q_i M_i^n, \\ x^n(0) = x_0, \quad M_i^n(0) = M_{0i}, \quad i = 0, 1, \dots, n. \end{cases} \quad (5)$$

For the truncated versions of \mathcal{M} and \mathcal{X} , \mathcal{M}^n and \mathcal{X}^n say, we have:

$$\begin{aligned}\dot{\mathcal{M}}^n &= \sum_{i=0}^n \dot{M}_i^n = \dot{M}_0^n + \sum_{i=1}^{n-1} \dot{M}_i^n + \dot{M}_n^n = r - \sum_{i=0}^n (p_i + q_i) M_i^n, \\ \dot{\mathcal{X}}^n &= \dot{x}^n + \sum_{i=1}^{n-1} i \dot{M}_i^n + n \dot{M}_n^n = \alpha - \sum_{i=1}^n i p_i M_i^n.\end{aligned}$$

Therefore,

$$\dot{U}^n = r + \alpha - \sum_{i=0}^n (p_i + q_i) M_i^n - \sum_{i=1}^n i p_i M_i^n.$$

For now, let's only assume that the coefficients k_i , p_i and q_i in (5), as well as the constants r and α , are nonnegative.

Since (5) is an initial value problem for a finite dimensional ODE with a smooth vector field, the standard Picard-Lindelöf theorem allows us to conclude that, for each initial condition $y_0^n = (x_0^n, M_0^n, M_1^n, \dots, M_n^n) \in \mathbb{R}^{n+2}$, there is a unique local solution in the classical sense, $y_i^n(\cdot) := (x^n(\cdot), M_0^n(\cdot), M_1^n(\cdot), \dots, M_n^n(\cdot))$, such that $y^n(0) = y_0^n$. Moreover, we have:

Lemma 3.1. *Let $n \in \mathbb{N}_2$. For each $y_0^n \in (\mathbb{R}_0^+)^{n+2}$, the unique local solution of (5) satisfying $y^n(0) = y_0^n$, $y^n(\cdot)$, is extendable to the interval $[0, +\infty)$, and for each $t \geq 0$, $y^n(t) \in (\mathbb{R}_0^+)^{n+2}$. This solution satisfies, for all $t \geq 0$,*

$$\dot{U}^n = r + \alpha - \sum_{i=0}^n (p_i + q_i) M_i^n - \sum_{i=1}^n i p_i M_i^n \leq r + \alpha. \quad (6)$$

Furthermore, assume that $(g_i) \in \mathbb{R}^{n+2}$. Then, for each positive integer $m < n$,

$$\begin{aligned}\sum_{i=m}^n g_i \frac{dM_i^n}{dt} + \sum_{i=m}^n g_i (p_i + q_i) M_i^n &= \\ &= g_m x^n k_{m-1} M_{m-1}^n + \sum_{i=m}^{n-1} (g_{i+1} - g_i) x^n k_i M_i^n. \quad (7)\end{aligned}$$

Proof. Equations (5) have the property that allows us to apply the argument in the proof of Theorem III-4-5 in [7] to prove the nonnegativity property, that is, by adding any $\varepsilon > 0$ to all equations, and being $y^{n,\varepsilon}$ the corresponding solution, for any $t_0 \geq 0$, such that $y^{n,\varepsilon}(t_0) \in (\mathbb{R}_0^+)^{n+2}$ and, for some $j \in \mathbb{N}$, $y_j^{n,\varepsilon}(t_0) = 0$, then, $\dot{y}_j^{n,\varepsilon}(t_0) > 0$.

Since (6) has been proved before, then we can say that y^n is bounded in any bounded subset of its maximal domain of existence. This, together with the smoothness of the vector field in \mathbb{R}^{n+2} , implies that this solution is extendable to $[0, +\infty)$.

Equation (7) is easily obtained by multiplying the equation for each \dot{M}_i^n in (5) by g_i and subsequently adding them together. \square

4. EXISTENCE

We consider the existence theorem for the Cauchy problem (1) for a relevant class of rate coefficients and initial conditions. In the proof we start by obtaining a candidate to a local solution as the limit of a subsequence of solutions of the

truncated systems. This is done by establishing estimates that allow us to apply Ascoli-Arzelà theorem. Then, we pass to the limit in the integral version of the truncated system by using some a priori estimates obtained via Gronwall's inequality. These ideas were originally used in the seminal paper [2] on the Becker-Döring system.

Theorem 4.1. *Let $(g_i)_{i \in \mathbb{N}_0}$ and $(k_i)_{i \in \mathbb{N}_0}$ be nonnegative sequences such that, for some $\delta > 0$, $g_{i+1} - g_i \geq \delta$, $i = 0, 1, 2, \dots$, and furthermore,*

$$(g_{i+1} - g_i)k_i = O(g_i), \quad \text{for } i = 0, 1, 2, \dots$$

Let $(p_i)_{i \in \mathbb{N}_0}$ and $(q_i)_{i \in \mathbb{N}_0}$ be two arbitrary nonnegative sequences.

Then, for $y_0 = (x_0, M_{00}, M_{01}, \dots) \in X_+$, such that, $\sum_{i=0}^{\infty} g_i M_{0i} < \infty$, there exists a solution $y = (x, M_0, M_1, \dots)$ of (1) on $[0, +\infty)$ with $y(0) = y_0$, that satisfies, for any $T \in (0, \infty)$,

$$\sup_{t \in [0, T]} \sum_{i=0}^{\infty} g_i M_i(t) < \infty, \quad \int_0^T \sum_{i=1}^{\infty} g_i (p_i + q_i) M_i(s) ds < \infty. \quad (8)$$

Proof. Let $y^n(0) = (x_0, M_{00}, M_{01}, \dots, M_{0n})$. By Lemma 3.1 there exists a unique solution y^n of (5) defined on $[0, \infty)$, with $y_r^n(t) \geq 0$ for $1 \leq r \leq n+2$, and satisfying $\mathcal{U}^n(t) \leq \mathcal{U}^n(0) + (r + \alpha)t$, for $t \geq 0$.

We shall regard y^n as an element of X_+ , by considering $y_r^n(t) \equiv 0$, if $r > n+2$, or equivalently $M_r^n(t) \equiv 0$ if $r > n$. With this identification we have $\|y^n\| = \mathcal{U}^n$, and therefore, for all $t \geq 0$,

$$\|y^n(t)\| \leq \|y^n(0)\| + (\alpha + r)t \leq \|y_0\| + (\alpha + r)t. \quad (9)$$

Fix $T \in \mathbb{R}^+$ arbitrarily.

Since, for each $i = 0, 1, 2, \dots, n$, we obviously have $(i+1)M_i^n(t) \leq \|y^n(t)\|$, we conclude that, for all $t \in [0, T]$,

$$0 \leq M_i^n(t) \leq \frac{\|y_0\|}{1+i} + \frac{\alpha + r}{1+i}t \leq \frac{\|y_0\|}{1+i} + \frac{\alpha + r}{1+i}T. \quad (10)$$

Now, in order to apply Ascoli-Arzelà theorem, we need estimates on $|\dot{M}_i^n|$, $i = 0, 1, \dots, n$. From (5) we have, for all $t > 0$,

$$\begin{aligned} |\dot{M}_0^n(t)| &\leq r + (k_0 x^n(t) + (p_0 + q_0))M_0^n(t), \\ |\dot{M}_i^n(t)| &\leq k_{i-1}x^n(t)M_{i-1}^n(t) + k_i x^n(t)M_i^n(t) + (p_i + q_i)M_i^n(t), \\ |\dot{M}_n^n(t)| &\leq k_{n-1}x^n(t)M_{n-1}^n(t) + (p_n + q_n)M_n^n(t). \end{aligned} \quad (11)$$

By the estimates (9), (10) and (11) with $t \in [0, T]$, and taking into account that $x^n \leq \|y^n\|$, we conclude

$$\begin{aligned} |\dot{M}_0^n(t)| &\leq r + [k_0(\|y_0\| + (\alpha + r)T) + p_0 + q_0](\|y_0\| + (\alpha + r)T), \\ |\dot{M}_i^n(t)| &\leq \left[\left(\frac{k_{i-1}}{i} + \frac{k_i}{i+1} \right) (\|y_0\| + (\alpha + r)T) + \frac{p_i + q_i}{i+1} \right] \times \\ &\quad \times (\|y_0\| + (\alpha + r)T), \quad \text{for } i = 1, 2, \dots, n. \end{aligned} \quad (12)$$

An important fact to be retained from estimates (12) is that, for each $i = 0, 1, 2, \dots$, $|\dot{M}_i^n|$ is bounded uniformly in n , and therefore the set of functions $M_i^n(\cdot)$, $n =$

$1, 2, \dots$, forms an equibounded and equicontinuous set of functions in $[0, T]$. Therefore, by applying Ascoli-Arzelà theorem, and by a standard diagonalization procedure, we can establish the existence, for each $i = 0, 1, 2, \dots$, of a continuous function $M_i : [0, T] \rightarrow \mathbb{R}$, and a strictly increasing sequence of natural numbers (n_j) , such that,

$$M_i^{n_j} \rightarrow M_i, \text{ uniformly on } [0, T], \text{ as } j \rightarrow \infty.$$

Obviously, $M_i(t) \geq 0$, for all $t \in [0, T]$. On the other hand, since by (6)

$$\sum_{i=0}^N (i+1)M_i(t) = \lim_{j \rightarrow \infty} \sum_{i=0}^N (i+1)M_i^{n_j}(t) \leq \|y_0\| + (\alpha + r)T,$$

for each finite N , we conclude that, for $t \in [0, T]$,

$$\sum_{i=0}^{\infty} (i+1)M_i(t) \leq \|y_0\| + (\alpha + r)T. \quad (13)$$

Considering now the sequence $(x^n(\cdot))$, observe that the equation for x^n in system (5) contains sums which are not clear how to control by estimates similar to the ones above as $n \rightarrow \infty$. Therefore, we adopt the following different approach already used in the proof of Theorem 2.2 in [2]: Since, for $t \in [0, T]$,

$$|x^n(t)| \leq \|y^n(t)\| \leq \|y_0\| + (\alpha + r)T, \quad (14)$$

we can extract a subsequence of (n_j) , which we still designate by (n_j) , such that, for some non-negative $x \in L^\infty(0, T)$,

$$x^{n_j} \xrightarrow{*} x \text{ in } L^\infty(0, T), \quad \text{as } j \rightarrow \infty,$$

which means that

$$\forall \phi \in L^1(0, T), \quad \int_0^T (x^{n_j}(t) - x(t))\phi(t) dt \rightarrow 0, \quad \text{as } j \rightarrow \infty. \quad (15)$$

Now, from (5), we have that, for all $t \geq 0$,

$$M_0^{n_j}(t) = M_{00} + rt - \int_0^t [k_0 x^{n_j}(s) M_0^{n_j}(s) - (p_0 + q_0) M_0^{n_j}(s)] ds. \quad (16)$$

Since, for all $t \in [0, T]$,

$$\int_0^t x^{n_j}(s) M_0^{n_j}(s) ds = \int_0^t x^{n_j}(s) M_0(s) ds + \int_0^t x^{n_j}(s) (M_0^{n_j}(s) - M_0(s)) ds,$$

using the convergence of $M_0^{n_j} \rightarrow M_0$, uniformly on $[0, T]$, and the weak* convergence (15) combined with estimate (14), we can pass to the limit in (16) to obtain, for all $t \in [0, T]$,

$$M_0(t) = M_{00} + rt - \int_0^t [k_0 x(s) M_0(s) - (p_0 + q_0) M_0(s)] ds,$$

thus proving (2) in Definition 2.2. Now, fix $i \in \mathbb{N}$. For j sufficiently large, by (5),

$$\begin{aligned} M_i^{n_j}(t) &= \\ &= M_{0i} + \int_0^t [k_{i-1} x^{n_j}(s) M_{i-1}^{n_j}(s) - k_i x^{n_j}(s) M_i^{n_j}(s) - (p_i + q_i) M_i^{n_j}(s)] ds. \end{aligned} \quad (17)$$

Using the same arguments, now with the uniform convergence $M_i^{n_j} \rightarrow M_i$, on $[0, T]$, $i = 0, 1, 2, \dots$, we can pass to the limit in (17) to obtain

$$M_i(t) = M_{0i} + \int_0^t [k_{i-1}x(s)M_{i-1}(s) - k_i x(s)M_i(s) - (p_i + q_i)M_i(s)] ds,$$

thus proving (3) in Definition 2.2.

In order to prove the other assertions in Definition 2.2 and, in particular, to pass to the limit in the integral version of the last equation of (5) we need some further estimates. Using estimates (13) and (14) in the moments' equation (7) with $m = 1$, and using the hypothesis of the theorem, we obtain¹, after integration, for $t \in [0, T]$,

$$\sum_{i=1}^{\infty} g_i M_i^{n_j}(t) + \int_0^t \sum_{i=1}^{\infty} g_i (p_i + q_i) M_i^{n_j}(s) ds \leq C_1 + C_2 \int_0^t \sum_{i=1}^{\infty} g_i M_i^{n_j}(s) ds, \quad (18)$$

where $C_1 := k_0 g_1 (C_2/C)^2 T + \sum_{i=1}^{\infty} g_i y_{0i}$, $C_2 := \|y_0\| + (\alpha + r)T$, and C is such that $|g_{i+1} - g_i| k_i \leq C g_i$. Then, due to the nonnegativity of (g_i) , (p_i) , (q_i) and $(M_i^{n_j})$, we can apply Gronwall's lemma to the above inequality to deduce that, for all $t \in [0, T]$,

$$\sum_{i=1}^{\infty} g_i M_i^{n_j}(t) + \int_0^t \sum_{i=1}^{\infty} g_i (p_i + q_i) M_i^{n_j}(s) ds \leq C_1 e^{C_2 t}. \quad (19)$$

Now, consider any integer $N \geq 2$. Again, by the nonnegativity referred to above, the bound (19) implies that, for all j ,

$$\sum_{i=1}^N g_i M_i^{n_j}(t) + \int_0^t \sum_{i=1}^N g_i (p_i + q_i) M_i^{n_j}(s) ds \leq C_1 e^{C_2 t},$$

so that, by making $j \rightarrow \infty$, we obtain, due to the uniform convergence $M_i^{n_j} \rightarrow M_i$ in $[0, T]$, for $i = 1, 2, \dots$,

$$\sum_{i=1}^N g_i M_i(t) + \int_0^t \sum_{i=1}^N g_i (p_i + q_i) M_i(s) ds \leq C_1 e^{C_2 t}.$$

Then, taking $N \rightarrow \infty$, we conclude that

$$\sum_{i=1}^{\infty} g_i M_i(t) + \int_0^t \sum_{i=1}^{\infty} g_i (p_i + q_i) M_i(s) ds \leq C_1 e^{C_2 t}, \quad (20)$$

where we have used the monotone convergence theorem for the limit of the integral. This clearly proves assertions (8). Using the assumptions on (g_i) , namely that for all $i = 0, 1, 2, \dots$, we have $\delta k_i \leq (g_{i+1} - g_i) k_i \leq C g_i$, and by (20) we can write

$$\int_0^t \sum_{i=0}^{\infty} k_i M_i(s) ds < \infty, \quad \text{for all } t \in [0, T].$$

Also, from $g_{i+1} - g_i \geq \delta > 0$ we get $g_i \geq g_0 + i\delta$ for $i = 1, 2, \dots$, and hence, by (20),

$$\int_0^t \sum_{i=0}^{\infty} i(p_i + q_i) M_i(s) ds < \infty, \quad \text{for all } t \in [0, T].$$

¹We recall the reader that for the sequences $(M_i^{n_j})$ we are considering in this proof we have $M_i^{n_j} = 0$ if $i > n_j$ and hence all the sums up to ∞ that involve these sequences have, in fact, only a finite number of terms with i up to $i = n_j$.

This proves the second assertion in (iii) in Definition 2.2. The first assertion of (iii) is a direct consequence of (15) and the uniform boundedness of x^{n_j} , $j \in \mathbb{N}$. Since properties (i), (ii) are easily obtained for the coordinates y_i , $i = 2, 3, \dots$, that is, for M_i , $i = 0, 1, 2, \dots$, by the uniform convergence properties, what is still lacking are the corresponding ones for y_1 , that is, for x , and (4) in Definition 2.2.

Our next main goal is to prove that (x_j^n) is a Cauchy sequence in the uniform convergence norm. From the integrated version of (5) and letting $\chi_{i,p}$ be the indicator function of $\{0, 1, \dots, n_p - 1\}$, we can write, for $j > l \in \mathbb{N}$ and $t \in [0, T]$,

$$\begin{aligned} x^{n_j}(t) - x^{n_l}(t) &= - \int_0^t \left(x^{n_j}(s) \sum_{i=0}^{\infty} k_i M_i^{n_j}(s) \chi_{i,j} - x^{n_l}(s) \sum_{i=0}^{\infty} k_i M_i^{n_l}(s) \chi_{i,l} \right) ds \\ &\quad + \int_0^t \left(\sum_{i=0}^{\infty} i q_i M_i^{n_j}(s) - \sum_{i=0}^{\infty} i q_i M_i^{n_l}(s) \right) ds \\ &= I_1 + I_2 + I_3, \end{aligned} \tag{21}$$

where,

$$\begin{aligned} I_1 &:= - \int_0^t (x^{n_j}(s) - x^{n_l}(s)) \sum_{i=0}^{\infty} k_i M_i^{n_j}(s) \chi_{i,j} ds, \\ I_2 &:= - \int_0^t x^{n_l}(s) \sum_{i=0}^{\infty} k_i (M_i^{n_j}(s) - M_i^{n_l}(s)) \chi_{i,j} ds, \\ I_3 &:= \int_0^t \sum_{i=0}^{\infty} i q_i (M_i^{n_j}(s) - M_i^{n_l}(s)) ds. \end{aligned} \tag{22}$$

We will deal first with I_2 and I_3 . Given $N \in \mathbb{N}$, and for $j > l$ sufficiently large, we use the splitting $\sum_{i \geq 0} = \sum_{i=0}^{N-1} + \sum_{i \geq N}$. We start by proving that the parts of I_2 and I_3 that correspond to the sums with $i \geq N$ can be made arbitrarily small by choosing j and l sufficiently large. In order to achieve this we will now deduce a number of adequate estimates.

Using again (7) with $2 \leq m < n_j$ and with $g_i = 1$, for all $i \in \mathbb{N}$, we can write after integration (remembering that $M_i^{n_j} = 0$ for $i > n_j$)

$$\begin{aligned} \sum_{i=m}^{\infty} M_i^{n_j}(t) - \sum_{i=m}^{\infty} M_{0i}^{n_j} &= \\ &= \int_0^t x^{n_j}(s) k_{m-1} M_{m-1}^{n_j}(s) ds - \int_0^t \sum_{i=m}^{\infty} (p_i + q_i) M_i^{n_j}(s) ds. \end{aligned} \tag{23}$$

Next, we pass to the limit $n_j \rightarrow \infty$ in (23). Write the integrand of the first integral in the form

$$x^{n_j}(s) M_{m-1}^{n_j}(s) = x^{n_j}(s) M_{m-1}(s) + x^{n_j}(s) (M_{m-1}^{n_j}(s) - M_{m-1}(s)).$$

By (15),

$$\int_0^t x^{n_j}(s) M_{m-1}(s) ds \rightarrow \int_0^t x(s) M_{m-1}(s) ds, \quad \text{as } j \rightarrow \infty.$$

On the other hand, by estimate (14) and the convergence $M_{m-1}^{n_j} - M_{m-1} \rightarrow 0$, as $n_j \rightarrow \infty$, uniformly on $[0, T]$,

$$\int_0^t x^{n_j}(s) (M_{m-1}^{n_j}(s) - M_{m-1}(s)) ds \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

We can then conclude that

$$\int_0^t x^{n_j}(s) k_{m-1} M_{m-1}^{n_j}(s) ds \rightarrow \int_0^t x(s) k_{m-1} M_{m-1}(s) ds, \quad \text{as } j \rightarrow \infty. \quad (24)$$

For the second integral in the right-hand side of (23) we observe that, for every $\tilde{m} > 0$ and due to (19), the following holds uniformly in n_j and $t \in [0, T]$

$$\begin{aligned} \int_0^t \sum_{i=\tilde{m}}^{\infty} (p_i + q_i) M_i^{n_j}(s) ds &\leq \frac{1}{\tilde{m}} \int_0^t \sum_{i=\tilde{m}}^{\infty} i(p_i + q_i) M_i^{n_j}(s) ds \\ &\leq \frac{1}{\tilde{m}} \int_0^t \sum_{i=1}^{\infty} i(p_i + q_i) M_i^{n_j}(s) ds \\ &\leq \frac{C_1}{\tilde{m}} \int_0^t e^{C_2 s} ds \\ &\leq \frac{C_1}{\tilde{m} C_2} (e^{C_2 T} - 1) \rightarrow 0 \text{ as } \tilde{m} \rightarrow \infty. \end{aligned} \quad (25)$$

For every $\varepsilon > 0$ fix \tilde{m} such that the last right-hand side in (25) is less than $\frac{1}{2}\varepsilon$. By the uniform in t convergence of $M_i^{n_j}$ to M_i we conclude that there exists a $\nu = \nu(\tilde{m})$ such that, for all $t \in [0, T]$ and $n_j > \nu$,

$$\left| \int_0^t \sum_{i=m}^{\tilde{m}-1} (p_i + q_i) M_i^{n_j}(s) ds - \int_0^t \sum_{i=m}^{\tilde{m}-1} (p_i + q_i) M_i(s) ds \right| < \frac{1}{2}\varepsilon. \quad (26)$$

Thus, (25) and (26) imply that for every $\varepsilon > 0$ there exists \tilde{m} and ν such that, for all $t \in [0, T]$ and $n_j > \nu$, we have

$$\left| \int_0^t \sum_{i=m}^{\infty} (p_i + q_i) M_i^{n_j}(s) ds - \int_0^t \sum_{i=m}^{\tilde{m}-1} (p_i + q_i) M_i(s) ds \right| < \varepsilon$$

and, due to the arbitrariness of \tilde{m} , we conclude that, as $n_j \rightarrow \infty$,

$$\int_0^t \sum_{i=m}^{\infty} (p_i + q_i) M_i^{n_j}(s) ds \rightarrow \int_0^t \sum_{i=m}^{\infty} (p_i + q_i) M_i(s) ds. \quad (27)$$

Next we will deal with the convergence of the left-hand side of (23). Again let us consider a fixed $\tilde{m} > m$. Then, for large j so that $n_j > \tilde{m}$, write

$$\sum_{i=m}^{\infty} M_i^{n_j}(t) - \sum_{i=m}^{\infty} M_i(t) = \left(\sum_{i=m}^{\tilde{m}-1} + \sum_{i=\tilde{m}}^{\infty} \right) (M_i^{n_j}(t) - M_i(t)). \quad (28)$$

Estimates (9) and (13) imply that

$$\sum_{i=\tilde{m}}^{\infty} |M_i^{n_j}(t) - M_i(t)| \leq \frac{1}{\tilde{m} + 1} \sum_{i=\tilde{m}}^{\infty} (i+1) (M_i^{n_j}(t) + M_i(t)) \leq \frac{2(\|y_0\| + (\alpha + r)T)}{\tilde{m} + 1},$$

so that, for each $t \in [0, T]$, the series $\sum_{i=\tilde{m}}^{\infty}$ in the right-hand side of (28) converges, uniformly with respect to $j \in \mathbb{N}$, which allows us to conclude that,

$$\lim_{j \rightarrow \infty} \left(\sum_{i=m}^{\tilde{m}-1} + \sum_{i=\tilde{m}}^{\infty} \right) (M_i^{n_j}(t) - M_i(t)) = 0,$$

and so,

$$\lim_{j \rightarrow \infty} \sum_{i=m}^{\infty} M_i^{n_j}(t) = \sum_{i=m}^{\infty} M_i(t). \quad (29)$$

Plugging (24), (27) and (29) into (23) we obtain

$$\sum_{i=m}^{\infty} M_i(t) - \sum_{i=m}^{\infty} M_{0i} = \int_0^t \left(x(s) k_{m-1} M_{m-1}(s) - \sum_{i=m}^{\infty} (p_i + q_i) M_i(s) \right) ds. \quad (30)$$

We shall now use (30) to obtain an estimate that will be crucial to handle the limit of I_2 and I_3 in (22) as $n_j \rightarrow \infty$. For each $m \in \mathbb{N}$ and $t \geq 0$ define

$$f_m(t) := g_m \times \left(\sum_{i=m}^{\infty} M_i(t) - \sum_{i=m}^{\infty} M_{0i} \right), \quad (31)$$

where (g_m) is a positive sequence satisfying the assumptions of the theorem. We can write the product of (30) by g_m in the following form:

$$f_m(t) = g_m \int_0^t \left(k_{m-1} x(s) M_{m-1}(s) - \sum_{i=m}^{\infty} (p_i + q_i) M_i(s) \right) ds. \quad (32)$$

Now, using the assumption $g_{i+1} - g_i \geq \delta > 0$, we have, for $t \in [0, T]$,

$$|f_m(t)| \leq \sum_{i=m}^{\infty} g_i M_i(t) + \sum_{i=m}^{\infty} g_i M_{0i},$$

and, by (20), we conclude that, for each $t \in [0, T]$,

$$|f_m(t)| \leq C, \quad \text{and} \quad \lim_{m \rightarrow \infty} f_m(t) = 0,$$

where $C > 0$ is independent of t and m . Hence, by the bounded convergence theorem, we have

$$\lim_{m \rightarrow \infty} \int_0^T |f_m(t)| dt = 0$$

and so, given $\varepsilon > 0$, there exists $N > 2$ such that

$$\int_0^T |f_N(t)| < \varepsilon \quad \text{and} \quad \sum_{i=N}^{\infty} g_i M_{0i} < \varepsilon.$$

Using this in (32), we obtain, for all $t \in [0, T]$,

$$g_N \int_0^t \int_0^s \left(x(\tau) k_{N-1} M_{N-1}(\tau) - \sum_{i=N}^{\infty} (p_i + q_i) M_i(\tau) \right) d\tau ds < \varepsilon.$$

Now define $f_m^{n_j}(t)$ by (31) using the sequence $(M_i^{n_j})$ instead of (M_i) . Thus, we have the following equality corresponding to (32),

$$f_m^{n_j}(t) = g_m \int_0^t \left(k_{m-1} x^{n_j}(s) M_{m-1}^{n_j}(s) - \sum_{i=m}^{\infty} (p_i + q_i) M_i^{n_j}(s) \right) ds.$$

Using (24) and (27) we conclude that

$$\begin{aligned} & \lim_{n_j \rightarrow \infty} g_m \int_0^t \left(k_{m-1} x^{n_j}(s) M_{m-1}^{n_j}(s) - \sum_{i=m}^{\infty} (p_i + q_i) M_i^{n_j}(s) \right) ds = \\ & = g_m \int_0^t \left(k_{m-1} x(s) M_{m-1}(s) - \sum_{i=m}^{\infty} (p_i + q_i) M_i(s) \right) ds, \end{aligned}$$

and thus $f_m^{n_j}(t) \rightarrow f_m(t)$, pointwise in $[0, T]$. Additionally, from the definition of $f_m^{n_j}(t)$ and (19) we have

$$|f_m^{n_j}(t)| \leq \sum_{i=m}^{\infty} g_i M_i^{n_j}(t) + \sum_{i=m}^{\infty} g_i M_{i0} \leq 2C_1 e^{C_2 T},$$

and consequently, by the dominated convergence theorem,

$$\int_0^t f_m^{n_j}(s) ds \rightarrow \int_0^t f_m(s) ds \quad \text{as } n_j \rightarrow \infty.$$

Hence, for every $\varepsilon > 0$ there exists m_0, j_0 such that, for all $m > m_0$ and $n_j > j_0$ the following hold

$$\begin{aligned} & g_N \int_0^t \int_0^s \left(x^{n_j}(\tau) k_{N-1} M_{N-1}^{n_j}(\tau) - \sum_{i=N}^{n_j} (p_i + q_i) M_i^{n_j}(\tau) \right) d\tau ds = \\ & = \int_0^t f_m^{n_j}(s) ds = \int_0^t f_m(s) ds + \int_0^t (f_m^{n_j}(s) - f_m(s)) ds < 2\varepsilon. \quad (33) \end{aligned}$$

Now, we consider the integrated version of the moments' equation (7) with $m = N$. Taking into consideration the hypothesis of the theorem, namely that $k_i(g_{i+1} - g_i) = O(g_i) \leq K_g g_i$ for some constant K_g independent of i , and recalling (9), we can write, for all $t \in [0, T]$,

$$\begin{aligned} & \int_0^t \left(\sum_{i=N}^{\infty} g_i M_i^{n_j}(s) - \sum_{i=N}^{\infty} g_i M_i^{n_j}(0) + \int_0^s \sum_{i=N}^{\infty} g_i (p_i + q_i) M_i^{n_j}(\tau) d\tau \right) ds = \\ & = \int_0^t \sum_{i=N}^{\infty} g_i M_i^{n_j}(0) ds + g_N \int_0^t \int_0^s k_{N-1} x^{n_j}(\tau) M_{N-1}^{n_j}(\tau) d\tau ds + \\ & \quad + \int_0^t \int_0^s \sum_{i=N}^{\infty} k_i (g_{i+1} - g_i) x^{n_j}(\tau) M_i^{n_j}(\tau) d\tau ds \\ & \leq t \sum_{i=N}^{\infty} g_i M_i^{n_j}(0) + g_N \int_0^t \int_0^s k_{N-1} x^{n_j}(\tau) M_{N-1}^{n_j}(\tau) d\tau ds + \\ & \quad + K_g C_2 \int_0^t \int_0^s \sum_{i=N}^{\infty} g_i x^{n_j}(\tau) M_i^{n_j}(\tau) d\tau ds \end{aligned}$$

where, we recall the reader, $C_2 := \|y_0\| + (\alpha + r)T$. Let

$$z(t) := \int_0^t \left(\sum_{i=N}^{\infty} g_i M_i^{n_j}(s) + \int_0^s \sum_{i=N}^{\infty} g_i (p_i + q_i) M_i^{n_j}(\tau) d\tau \right) ds.$$

The nonnegativity of $M_i^{n_j}$, the inequality just obtained, and (33) imply that, for all $t \in [0, T]$,

$$z(t) \leq t \sum_{i=N}^{\infty} g_i M_i^{n_j}(0) + 2\varepsilon + K_g C_2 \int_0^t z(s) ds.$$

Applying Gronwall's inequality we get, with $K_1 = K_g C_2$ and for all N sufficiently large,

$$\begin{aligned} z(t) &\leq t \sum_{i=N}^{\infty} g_i M_i^{n_j}(0) + 2\varepsilon + K_1 \int_0^t \left(s \sum_{i=N}^{\infty} g_i M_i^{n_j}(0) + 2\varepsilon \right) e^{K_1(t-s)} ds \\ &= K_1^{-1} (e^{K_1 t} - 1) \sum_{i=N}^{\infty} g_i M_i^{n_j}(0) + 2\varepsilon e^{K_1 t} \\ &< \varepsilon(2 + K_1^{-1}) e^{K_1 t}. \end{aligned}$$

Since,

$$\begin{aligned} \int_0^t \int_0^s \sum_{i=N}^{\infty} g_i (p_i + q_i) M_i^{n_j}(\tau) d\tau ds &= \int_0^t (t-s) \sum_{i=N}^{\infty} g_i (p_i + q_i) M_i^{n_j}(s) ds \\ &\geq \frac{t}{2} \int_0^{t/2} \sum_{i=N}^{\infty} g_i (p_i + q_i) M_i^{n_j}(s) ds \end{aligned}$$

we finally conclude that, for sufficiently large fixed N and for all $j \geq j_0$,

$$\int_0^t \sum_{i=N}^{\infty} g_i M_i^{n_j}(s) ds + \frac{t}{2} \int_0^{t/2} \sum_{i=N}^{\infty} g_i (p_i + q_i) M_i^{n_j}(s) ds \leq \varepsilon(2 + K_1^{-1}) e^{K_1 t}, \quad (34)$$

for all $t \in [0, T]$.

We have now enough estimates to deal with the expressions in (22). We consider the splitting $\sum_{i \geq 0} = \sum_{i=0}^{N-1} + \sum_{i \geq N}$. Using the bounds on $M_i^{n_j}$, the uniform convergence $M_i^{n_j} \rightarrow M_i$ on $[0, T]$, for each $i = 0, 1, 2, \dots$, and the weak-star convergence $x^{n_j} \xrightarrow{*} x$, it is easy to check that the $\sum_{i=0}^{N-1}$ parts of I_2, I_3 converge to zero uniformly for $t \in [0, T]$, as $j, l \rightarrow \infty$. On the other hand, by the hypothesis of the theorem we have $k_i = O(g_i)$, and from (34), we conclude that, for $j \geq l \geq j_0$ and all $t \in [0, T]$

$$\begin{aligned} \left| \int_0^t x^{n_l}(s) \sum_{i=N}^{\infty} k_i (M_i^{n_j}(s) - M_i^{n_l}(s)) ds \right| &\leq \\ &\leq 2C_2 \int_0^t \sum_{i=N}^{\infty} k_i (M_i^{n_j}(s) + M_i^{n_l}(s)) ds \leq K_2 \varepsilon, \end{aligned}$$

for some $K_2 > 0$, independent of l, j, t . Furthermore, since $i = O(g_i)$, by (34) we have, for some $K_3 > 0$ independent of l and j ,

$$\left| \int_0^t \sum_{i=0}^{\infty} i q_i (M_i^{n_j}(s) - M_i^{n_l}(s)) ds \right| \leq \int_0^t \sum_{i=0}^{\infty} i q_i (M_i^{n_j}(s) + M_i^{n_l}(s)) ds \leq K_3 \varepsilon,$$

uniformly for $t \in [0, T/2]$. Combining the above statements, we conclude that there is $j_1 \geq j_0$, such that, for $j \geq l \geq j_1$, and all $t \in [0, T/2]$,

$$|I_2(t)| + |I_3(t)| \leq (K_2 + K_3) \varepsilon.$$

But, by the same reasoning, there is $K_4 > 0$ independent of l, j, t such that, for $j \geq l \geq j_1$, and all $t \in [0, T]$,

$$|I_1(t)| = \left| \int_0^t (x^{n_j}(s) - x^{n_l}(s)) \sum_{i=0}^{\infty} k_i M_i^{n_j}(s) \chi_{i,j} ds \right| \leq K_4 \varepsilon \int_0^t |x^{n_j}(s) - x^{n_l}(s)| ds,$$

so, by (21), if $K_5 = K_2 + K_3 + K_4$, for $j \geq l \geq j_1$,

$$|x^{n_j}(t) - x^{n_l}(t)| \leq K_5 \left(\varepsilon + \int_0^t |x^{n_j}(s) - x^{n_l}(s)| ds \right),$$

for all $t \in [0, T/2]$, so that, by Gronwall's inequality, there is $C > 0$, such that

$$\sup_{t \in [0, T/2]} |x^{n_j}(t) - x^{n_l}(t)| \leq C\varepsilon.$$

Due to the arbitrariness of ε we conclude that (x^{n_j}) is a Cauchy sequence in $C([0, T/2])$, and therefore $x^{n_j} \rightarrow x$ uniformly for $t \in [0, T/2]$. This also proves the continuity of x in $[0, T/2]$.

We are now able to pass to the limit in the integrated version of the last equation of (5), that is, in

$$x^{n_j}(t) = x_0 + \alpha t + \int_0^t \left(-x^{n_j}(s) \sum_{i=0}^{\infty} k_i M_i^{n_j}(s) \chi_{i,j} + \sum_{i=0}^{\infty} i q_i M_i^{n_j}(s) \right) ds, \quad (35)$$

where $\chi_{i,j}$ is the indicator function of the set $\{0, 1, \dots, n_j - 1\}$. Splitting again $\sum_{i \geq 0} = \sum_{i=0}^{N-1} + \sum_{i \geq N}$, using (34), and the bound on x^{n_j} , we observe that there is a constant $K_5 > 0$ such that, for $j \geq j_1$,

$$\begin{aligned} \left| \int_0^t \left(-x^{n_j}(s) \sum_{i=N}^{\infty} k_i M_i^{n_j}(s) \chi_{i,j} + \sum_{i=N}^{\infty} i q_i M_i^{n_j}(s) \right) ds \right| &\leq \\ &\leq \int_0^t \left(x^{n_j}(s) \sum_{i=N}^{\infty} k_i M_i^{n_j}(s) \chi_{i,j} + \sum_{i=N}^{\infty} i q_i M_i^{n_j}(s) \right) ds \leq K_5 \varepsilon, \end{aligned} \quad (36)$$

for all $t \in [0, T/2]$. By a procedure already used above, writing in this estimate, $\sum_{i \geq N} = \sum_{i=N}^{l-1} + \sum_{i \geq l}$ and making first $j \rightarrow \infty$ and then $l \rightarrow \infty$, using the uniform convergence of $(M_i^{n_j})$ and (x^{n_j}) , we conclude that

$$\left| \int_0^t \left(-x(s) \sum_{i=N}^{\infty} k_i M_i(s) + \sum_{i=N}^{\infty} i q_i M_i(s) \right) ds \right| \leq K_5 \varepsilon, \quad (37)$$

for all $t \in [0, T/2]$. Thus, from (35) and (36), for $j \geq j_1$,

$$\left| x^{n_j}(t) - x_0 - \alpha t + \int_0^t \left(x^{n_j}(s) \sum_{i=0}^{N-1} k_i M_i^{n_j}(s) - \sum_{i=0}^{N-1} i q_i M_i^{n_j}(s) \right) ds \right| \leq K_5 \varepsilon,$$

for all $t \in [0, T/2]$. Now letting $j \rightarrow \infty$ and using the same uniform convergence property we obtain, from the last inequality,

$$\left| x(t) - x_0 - \alpha t + \int_0^t \left(x(s) \sum_{i=0}^{N-1} k_i M_i(s) - \sum_{i=0}^{N-1} i q_i M_i(s) \right) ds \right| \leq K_5 \varepsilon,$$

and hence, by (37),

$$\left| x(t) - x_0 - \alpha t + \int_0^t \left(x(s) \sum_{i=0}^{\infty} k_i M_i(s) - \sum_{i=0}^{\infty} i q_i M_i(s) \right) ds \right| \leq 2K_5 \varepsilon,$$

for all $t \in [0, T/2]$. Since ε was chosen arbitrarily small we conclude that (4) in Definition 2.2 holds for all $t \in [0, T/2]$.

We have showed that for any given $T > 0$ and for each initial condition $y_0 \in X_+$ there exists a solution y defined in $[0, T/2]$. Since the equations in system (2)–(4) are autonomous this allows us to extend y to $[0, \infty)$. \square

Setting $g_i = i$, for $i = 1, 2, \dots$, in the above theorem we obtain the following result:

Corollary 4.2. *Let $k_i = O(i)$. Then, for each $y_0 = (x_0, M_{01}, M_{02}, \dots) \in X_+$, there exists a solution y of (1) on $[0, +\infty)$ with $y(0) = y_0$.*

5. THE MOMENTS' EQUATION AND SOME A PRIORI RESULTS

The next theorem states an a priori equality which is an integrated version for any solution of (1), of a result stated in Lemma 3.1 for the solutions of the truncated version of the silicosis system.

Theorem 5.1. *Let (g_i) be a real sequence. Let $y := (x, M_0, M_1, \dots)$ be a solution of (1) on some interval $[0, T)$, $0 < T \leq \infty$. Take any pair (t_1, t_2) , such that $0 \leq t_1 < t_2 < T$. Suppose that,*

$$\int_{t_1}^{t_2} \sum_{i=0}^{\infty} |g_{i+1} - g_i| k_i M_i(s) ds < \infty, \quad (38)$$

and, furthermore, one of the following two sets of conditions, (A) or (B), holds:

- (A) $g_i = O(i)$ and $\int_{t_1}^{t_2} \sum_{i=0}^{\infty} g_i (p_i + q_i) M_i(s) ds < \infty$;
 (B) for $p = 1, 2$, $\sum_{i=0}^{\infty} g_i M_i(t_p) < \infty$, and, for sufficient large i , $g_{i+1} \geq g_i \geq 0$.

Then, for $m \geq 1$, the following integrated version of (7) holds for the solution y :

$$\begin{aligned} & \sum_{i=m}^{\infty} g_i M_i(t_2) - \sum_{i=m}^{\infty} g_i M_i(t_1) + \int_{t_1}^{t_2} \sum_{i=m}^{\infty} g_i (p_i + q_i) M_i(s) ds \\ &= \int_{t_1}^{t_2} g_m x(s) k_{m-1} M_{m-1}(s) ds + \int_{t_1}^{t_2} \sum_{i=m}^{\infty} (g_{i+1} - g_i) x(s) k_i M_i(s) ds. \end{aligned} \quad (39)$$

Proof. Take any $n > m \geq 1$. From (3) we easily obtain

$$\begin{aligned} \sum_{i=m}^n g_i M_i(t_2) - \sum_{i=m}^n g_i M_i(t_1) + \int_{t_1}^{t_2} \sum_{i=m}^n g_i (p_i + q_i) M_i(s) ds = \\ = \int_{t_1}^{t_2} g_m x(s) k_{m-1} M_{m-1}(s) ds - \int_{t_1}^{t_2} g_{n+1} x(s) k_n M_n(s) ds \\ + \int_{t_1}^{t_2} \sum_{i=m}^n (g_{i+1} - g_i) x(s) k_i M_i(s) ds. \end{aligned} \quad (40)$$

We now pass to the limit $n \rightarrow \infty$ in both sides. By the boundedness of x and hypothesis (38), we readily conclude that, as $n \rightarrow \infty$,

$$\int_{t_1}^{t_2} \sum_{i=m}^n (g_{i+1} - g_i) x(s) k_i M_i(s) ds \rightarrow \int_{t_1}^{t_2} \sum_{i=m}^{\infty} (g_{i+1} - g_i) x(s) k_i M_i(s) ds. \quad (41)$$

Next we claim that the second integral in the right-hand side of (40) vanishes in this limit: from (ii) and (iii) of Definition 2.2, we readily obtain

$$\int_{t_1}^{t_2} x(s) k_n M_n(s) ds \rightarrow 0, \quad \text{as } n \rightarrow \infty; \quad (42)$$

on the other hand, by setting $g_i = 1$ for all $i = 0, 1, 2, \dots$ in (40), we obtain,

$$\begin{aligned} \sum_{i=m}^n M_i(t_2) - \sum_{i=m}^n M_i(t_1) + \int_{t_1}^{t_2} \sum_{i=m}^n (p_i + q_i) M_i(s) ds = \\ = \int_{t_1}^{t_2} x(s) k_{m-1} M_{m-1}(s) ds - \int_{t_1}^{t_2} x(s) k_n M_n(s) ds, \end{aligned}$$

so that, from (42) and again the properties (ii) and (iii) of Definition 2.2, by letting $n \rightarrow \infty$, we obtain,

$$\begin{aligned} \sum_{i=m}^{\infty} M_i(t_2) - \sum_{i=m}^{\infty} M_i(t_1) + \int_{t_1}^{t_2} \sum_{i=m}^{\infty} (p_i + q_i) M_i(s) ds = \\ = \int_{t_1}^{t_2} x(s) k_{m-1} M_{m-1}(s) ds. \end{aligned} \quad (43)$$

Let us now consider hypothesis (A). This implies that, for $p = 1, 2$,

$$|g_{n+1}| \sum_{i=n+1}^{\infty} M_i(t_p) \leq C(n+1) \sum_{i=n+1}^{\infty} M_i(t_p) \leq C \sum_{i=n+1}^{\infty} i M_i(t_p),$$

for some constant $C > 0$, and thus

$$|g_{n+1}| \sum_{i=n+1}^{\infty} M_i(t_p) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (44)$$

Replacing m by $n+1$ in (43), multiplying both members by $|g_{n+1}|$, letting $n \rightarrow \infty$, and using (iii) of Definition 2.2 together with (44), we conclude that,

$$\int_{t_1}^{t_2} g_{n+1} x(s) k_n M_n(s) ds \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (45)$$

thus proving our claim under the first condition in (A). Using the second condition in (A) together with Definition 2.2, (41), (45), and the bounded convergence theorem, we can pass to the limit $n \rightarrow \infty$ in (40) and prove (39).

Now we prove the claim under the hypothesis (B). This implies that, for some constant $C > 0$,

$$|g_{n+1}| \sum_{i=n+1}^{\infty} M_i(t_p) \leq C \sum_{i=n+1}^{\infty} g_i M_i(t_p),$$

and thus, under these conditions, also (44) and (45) hold true. Therefore, using hypothesis (B), (41), and (45) we conclude that all terms in (40) but the integral in the left-hand side converge as $n \rightarrow \infty$. But then, using the monotone convergence theorem we conclude that in (39) we also have,

$$\int_{t_1}^{t_2} \sum_{i=m}^n g_i(p_i + q_i) M_i(s) ds \rightarrow \int_{t_1}^{t_2} \sum_{i=m}^{\infty} g_i(p_i + q_i) M_i(s) ds,$$

as $n \rightarrow \infty$, thus completing the proof. \square

Given some $y = (x, M_0, M_1, \dots) \in X_+$, recall the definitions of $\mathcal{X}(y)$, $\mathcal{M}(y)$ and $\mathcal{U}(y)$.

Corollary 5.2. *Let $y := (x, M_0, M_1, \dots)$ be a solution of (1) in $[0, T)$, $0 < T \leq \infty$. Then, for all $t \in [0, T)$, the following integrated version of (6) holds for that solution:*

$$\begin{aligned} \mathcal{U}(y(t)) - \mathcal{U}(y(0)) &= \\ &= (r + \alpha)t - \int_0^t \sum_{i=0}^{\infty} (p_i + q_i) M_i(s) ds - \int_0^t \sum_{i=1}^{\infty} i p_i M_i(s) ds. \end{aligned} \quad (46)$$

Also, for every $m \geq 1$,

$$\begin{aligned} \sum_{i=m}^{\infty} M_i(t) - \sum_{i=m}^{\infty} M_i(0) + \int_0^t \sum_{i=m}^{\infty} (p_i + q_i) M_i(s) ds &= \\ &= \int_0^t x(s) k_{m-1} M_{m-1}(s) ds, \end{aligned} \quad (47)$$

and

$$\begin{aligned} \sum_{i=m}^{\infty} i M_i(t) - \sum_{i=m}^{\infty} i M_i(0) + \int_0^t \sum_{i=m}^{\infty} i (p_i + q_i) M_i(s) ds &= \\ &= \int_0^t m x(s) k_{m-1} M_{m-1}(s) ds + \int_0^t \sum_{i=m}^{\infty} x(s) k_i M_i(s) ds. \end{aligned} \quad (48)$$

Proof. Equations (47) and (48) are obtained by setting $g_i = 1$ and $g_i = i$, resp., in the previous theorem. The first choice trivially satisfies all conditions of the theorem. The second, if we recall Definition 2.2, satisfies all statements corresponding to the hypothesis (B). Adding the equation for $x(t)$, (4), to the equation (48) with $m = 1$, we obtain,

$$\mathcal{X}(y(t)) - \mathcal{X}(y(0)) = \alpha t - \int_0^t \sum_{i=1}^{\infty} i p_i M_i(s) ds. \quad (49)$$

On the other hand, adding the equation for $M_0(t)$, (2), to equation (47) with $m = 1$, we obtain,

$$\mathcal{M}(y(t)) - \mathcal{M}(y(0)) = rt - \int_0^t \sum_{i=1}^{\infty} i(p_i + q_i)M_i(s) ds. \quad (50)$$

Finally, adding together (49) and (50) we obtain (46) thus completing the proof. \square

The previous results allow us to directly deduce the following conclusion on the solution regularity:

Corollary 5.3. *Let y be a solution of (1) on an interval $[0, T)$, $0 < T \leq \infty$. Then, $y : [0, T) \rightarrow X$ is continuous and, moreover, the series $\sum_{i=1}^{\infty} iM_i(t)$ is uniformly convergent on compact subintervals of $[0, T)$.*

Proof. By Definition 2.2, x and M_i , $i = 0, 1, 2, \dots$, are continuous real functions on $[0, T)$. For each $n \in \mathbb{N}$, let $\phi_n(t) := x(t) + \sum_{i=0}^n (i+1)M_i(t)$, for all $t \in [0, T)$. Each ϕ_n is a continuous function on $[0, T)$. Take now any $T' \in (0, T)$. By (46) we know that, for all $n \in \mathbb{N}$ and $t \in [0, T']$,

$$\phi_n(t) \leq \mathcal{U}(y(t)) \leq \mathcal{U}(y(0)) + (r + \alpha)T'.$$

But, for each $t \in [0, T)$, the real sequence $(\phi_n(t))$ is increasing, and thus, by Dini's theorem, the series $\sum_{i=0}^{\infty} iM_i(t)$ is uniformly convergent on $[0, T']$ and the second statement of the corollary is proved.

To prove the continuity of $[0, T) \ni t \mapsto y(t) \in X$, we fix any $t_0 \in [0, T)$. Let $T' \in (t_0, T)$ so that the series $\sum_{i=1}^{\infty} iM_i(t)$, and thus also the series $\sum_{i=1}^{\infty} (i+1)M_i(t)$ is uniformly convergent on $[0, T']$. Let $\varepsilon > 0$ be given. Then, there exists $N = N(T') > 1$ such that, $\sup_{t \in [0, T']} \sum_{i=N+1}^{\infty} (i+1)M_i(t) < \varepsilon$. For any $t \in [0, T']$,

$$\begin{aligned} \|y(t) - y(t_0)\| &= |x(t) - x(t_0)| + \left(\sum_{i=0}^N + \sum_{i=N+1}^{\infty} \right) (i+1)|M_i(t) - M_i(t_0)| \\ &\leq |x(t) - x(t_0)| + \sum_{i=0}^N (i+1)|M_i(t) - M_i(t_0)| + 2\varepsilon. \end{aligned}$$

But then, by continuity of the real valued functions x and M_i , $i = 0, 1, 2, \dots$,

$$0 \leq \liminf_{t \rightarrow t_0} \|y(t) - y(t_0)\| \leq \limsup_{t \rightarrow t_0} \|y(t) - y(t_0)\| \leq 2\varepsilon.$$

From this and the arbitrariness of ε we conclude that $\lim_{t \rightarrow t_0} \|y(t) - y(t_0)\| = 0$. \square

Only from the definition of solution and without further assumptions we know, by conditions (iii) in Definition 2.2, that $\sum_{i=1}^{\infty} iq_i M_i(t) < \infty$, for a.e. t in $(0, T)$. This means, in particular, that if $y(0) \in X_+$ but the previous series is divergent in $t = 0$ then, generically, for $t > 0$, it is convergent and thus the sequences $(y_n(t))_{n \in \mathbb{N}}$ will decay faster than $y(0)$ as $n \rightarrow \infty$. However, the next two propositions show that with additional hypothesis we can be more specific about this behaviour.

Proposition 5.4. *If $(k_i)_{i \in \mathbb{N}_0}$, $(p_i)_{i \in \mathbb{N}_0}$ and $(q_i)_{i \in \mathbb{N}_0}$ are nonnegative and furthermore, $k_i = O(i)$, and $(q_i)_{i \in \mathbb{N}_0}$ is bounded, then, for each solution $y = (x, M_0, M_1, \dots)$ on $[0, T)$, the function $Q(\cdot) := \sum_{i=1}^{\infty} i q_i M_i(\cdot)$, is absolutely continuous on each compact subinterval of $[0, T)$. Furthermore, if $(p_i)_{i \in \mathbb{N}_0}$ is bounded, then the function $P(\cdot) := \sum_{i=1}^{\infty} i p_i M_i(\cdot)$ is also absolutely continuous on each compact subinterval of $[0, T)$.*

Proof. Let the first set of conditions be fulfilled. Then, the sequence $g_i = i q_i$ satisfies (38) and hypothesis (A) of Theorem 5.1, so that we consider the corresponding version of (39) with $m = 1$. Take any $T' \in (0, T)$. Then, that expression shows that, for each $t \in [0, T')$, $Q(t) - Q(0)$ can be written as a sum of integrals of Lebesgue integrable functions on $[0, T')$. Therefore, it is an absolutely continuous function of t on $[0, T')$. The same applies to $P(\cdot)$ if we consider the more strict hypothesis on the p_i coefficients. \square

The previous Proposition imposes too stringent restrictions to the sequences (p_i) and (q_i) that might not be satisfied in some applications. Having this in mind we can still obtain the following result without those restrictive hypothesis at the expense of not guaranteeing the absolute continuity of $Q(\cdot)$ down to $t = 0$.

Proposition 5.5. *Let $(k_i)_{i \in \mathbb{N}_0}$ and $(q_i)_{i \in \mathbb{N}_0}$ satisfy, $(i+1)q_{i+1} \geq i q_i$, for sufficiently large i , and $((i+1)q_{i+1} - i q_i)k_i = O(i q_i)$. Consider a solution y on some interval $[0, T)$. Then, the function $Q(\cdot)$ is absolutely continuous on each compact subinterval of $(0, T)$.*

Proof. Take any pair (t_1, t_2) such that, $0 < t_1 < t_2 < T'$, for some fixed $T' \in (0, T)$, and $\sum_{i=1}^{\infty} i q_i M_i(t_p)$ is finite for $p = 1, 2$. We claim that $g_i = i q_i$ satisfies (38) and hypothesis (B) of Theorem 5.1. Since $((i+1)q_{i+1} - i q_i)k_i = O(i q_i)$ and by (ii) and (iii) of Definition 2.2, we can conclude that (38) is satisfied. On the other hand, the hypothesis also states that, for our choice of the sequence $(g_i)_{i \in \mathbb{N}_0}$, we have $g_{i+1} \geq g_i$, for sufficiently large i . This, together with the way t_1 and t_2 were chosen, proves our claim. This establishes the validity of (39) with $m = 1$ for this choice of t_1, t_2 and $g_i = i q_i$, $i = 0, 1, 2, \dots$. But the first integral on the right-hand side of (39) is bounded by a constant depending only on T' , and, by the hypothesis, for some $C > 0$ independent of the particular choice of t_1, t_2 , we have

$$\int_{t_1}^{t_2} \sum_{i=1}^{\infty} (g_{i+1} - g_i) x(s) k_i M_i(s) ds \leq C \int_{t_1}^{t_2} \sum_{i=1}^{\infty} g_i M_i(s) ds.$$

Therefore, by (39), there is another positive constant C depending only of T' such that,

$$Q(t_2) - Q(t_1) \leq C \left(1 + \int_{t_1}^{t_2} Q(s) ds \right),$$

so that, by fixing t_0 for which $Q(t_0)$ is finite, we have, by Gronwall inequality,

$$0 \leq Q(t) \leq (C + Q(t_0))(1 + CT' e^{CT'}),$$

for t a.e. in $(0, T')$. Thus, $Q \in L^\infty(0, T')$. Using this fact again in (39) together with our hypothesis, we conclude that Q is, in fact, in $C((0, T'))$ and it is absolutely continuous on this interval. This completes the proof. \square

6. DIFFERENTIABILITY

We state the first order differentiability properties of solutions for the silicosis system under two different sets of assumptions. In the first one, we can draw a conclusion slightly stronger than in the second one.

Proposition 6.1. *Let $(k_i)_{i \in \mathbb{N}_0}$ be a nonnegative sequence such that $k_i = O(i)$. Let $(q_i)_{i \in \mathbb{N}_0}$ be bounded. Let y be a solution of (1) in some interval $[0, T)$, $0 < T \leq \infty$. Then, $x \equiv y_1$ and $M_i \equiv y_{i+1}$, $i \in \mathbb{N}_0$, are C^1 functions on $[0, T)$ and satisfy the equations (1) in the classical sense that is, in their differential form, for all $t \in [0, T)$.*

Proof. That M_i , $i \in \mathbb{N}_0$, are C^1 in $[0, T)$ is clear from the continuity of x and M_i stated in (ii) of Definition 2.2 and the equations (2), (3) from (iv) in the same definition. It remains to prove the regularity of x . The continuity of $x \sum_{i=1}^{\infty} k_i M_i$ is a consequence of the continuity of x , the assumptions, and Corollary 5.3. The continuity of $Q = \sum_{i=1}^{\infty} i q_i M_i$, under these hypotheses was stated in Proposition 5.4. By (4) in Definition 2.2 we conclude that $x \in C^1([0, T))$. \square

Proposition 6.2. *Let $(k_i)_{i \in \mathbb{N}_0}$ and $(q_i)_{i \in \mathbb{N}_0}$ satisfy the hypothesis of Proposition 5.5 and $k_i = O(i)$. Let y be a solution of (1) in some interval $[0, T)$, $0 < T \leq \infty$. Then, $x \equiv y_1$ and $M_i \equiv y_{i+1}$, $i \in \mathbb{N}_0$, are C^1 on $(0, T)$ and they satisfy equations (1) in the classical sense that is, in their differential form, for all $t \in (0, T)$.*

Proof. This is just a consequence of the same arguments used in the proof of the previous proposition together with Proposition 5.5. \square

Corollary 6.3. *Let $(k_i)_{i \in \mathbb{N}_0}$, $(p_i)_{i \in \mathbb{N}_0}$, and $(q_i)_{i \in \mathbb{N}_0}$ be nonnegative sequences such that, $k_i = O(i)$, and, for sufficiently large i , $q_i = ai^\eta$, for some $a, \eta > 0$. Then, the same conclusions in Proposition 6.2 hold.*

Proof. From the fact that, as $i \rightarrow \infty$,

$$\frac{(i+1)^{\eta+1} - i^{\eta+1}}{i^{\eta+1}} i \rightarrow \eta + 1,$$

and from Proposition 6.2, the result immediately follows. \square

7. UNIQUENESS

Theorem 7.1. *Let $(k_i)_{i \in \mathbb{N}_0}$ and $(g_i)_{i \in \mathbb{N}_0}$ be as in the hypothesis of the existence Theorem 4.1. Assume furthermore that $k_i = O(i)$ and $ik_i = O(g_i)$. Let also $(p_i), (q_i)$ be nonnegative sequences such that, $(p_i)_{i \in \mathbb{N}_0}$ is bounded and $q_i = O(i)$. Then, for each $y_0 = (x_0, M_{00}, M_{01}, \dots) \geq 0$, satisfying $\sum_{i=0}^{\infty} g_i y_{0i} < \infty$, there is exactly one solution $y = (x, M_0, M_1, \dots)$ of (1) on $[0, \infty)$ satisfying $y(0) = y_0$.*

Proof. Let y be a solution as stated above, proved to exist in Theorem 4.1, and let \tilde{y} be another solution with $\tilde{y}(0) = y_0$. Define $z(t) = y(t) - \tilde{y}(t)$. Fix any finite $T > 0$. By Definition 2.2 all the coordinates of y, \tilde{y} and, therefore, of z are absolutely continuous functions on $[0, T]$. Hence, $t \mapsto |z(t)|$ has the same property and furthermore, for $t \in [0, T]$ a.e.,

$$\frac{d}{dt}|z(t)| = (\operatorname{sgn} z(t)) \frac{dz}{dt}(t).$$

We then know that the differential versions of equations (2) and (3) from the definition of solution, Definition 2.2, are satisfied for a.e. $t \in [0, T]$. Fix any finite

$N \geq 2$. Having in mind that $y_1 = x$ and $y_i = M_{i-2}$, $i = 2, 3, \dots$, and the same for \tilde{y} , rewriting each one of those equations using y_i, \tilde{y}_i notation, we can obtain, for a.e. $t \in [0, T]$,

$$\begin{aligned}
\frac{d}{dt} \sum_{i=0}^{N-2} (i+1) |M_i - \tilde{M}_i| &= \frac{d}{dt} \sum_{i=2}^N (i-1) |z_i| \\
&= \sum_{i=2}^N (i-1) \frac{dz_i}{dt} \operatorname{sgn}(z_i) \\
&= \sum_{i=2}^N k_{i-2} (y_1 y_i - \tilde{y}_1 \tilde{y}_i) [i \operatorname{sgn}(z_{i+1}) - (i-1) \operatorname{sgn}(z_i)] \\
&\quad - N k_{N-2} (y_1 y_N - \tilde{y}_1 \tilde{y}_N) \operatorname{sgn}(z_{N+1}) \\
&\quad - \sum_{i=2}^N (i-1) (p_{i-2} + q_{i-2}) |z_i|. \tag{51}
\end{aligned}$$

Now for the first sum in the right-hand side of the last equation we observe that, for $i = 2, 3, \dots$,

$$\begin{aligned}
(y_1 y_i - \tilde{y}_1 \tilde{y}_i) [i \operatorname{sgn}(z_{i+1}) - (i-1) \operatorname{sgn}(z_i)] &= \\
&= (\tilde{y}_1 z_i + y_i z_1) [i \operatorname{sgn}(z_{i+1}) - (i-1) \operatorname{sgn}(z_i)] \\
&= (\tilde{y}_1 |z_i| \operatorname{sgn}(z_i) + y_i z_1) [i \operatorname{sgn}(z_{i+1}) - (i-1) \operatorname{sgn}(z_i)] \\
&= \tilde{y}_1 |z_i| [i \operatorname{sgn}(z_{i+1}) - (i-1)] + y_i z_1 [i \operatorname{sgn}(z_{i+1}) - (i-1) \operatorname{sgn}(z_i)] \\
&\leq \tilde{y}_1 |z_i| + y_i |z_1| (2i-1),
\end{aligned}$$

and using this in (51) we obtain, after integrating in $[0, t]$, for every $t \in [0, T]$,

$$\begin{aligned}
\sum_{i=2}^N (i-1) |z_i(t)| &\leq \int_0^t \left[\tilde{y}_1 \sum_{i=2}^N k_{i-2} |z_i| + |z_1| \sum_{i=2}^N k_{i-2} (2i-1) y_i \right] ds \\
&\quad - \int_0^t N k_{N-2} (y_1 y_N - \tilde{y}_1 \tilde{y}_N) \operatorname{sgn}(z_{N+1}) ds. \tag{52}
\end{aligned}$$

Since, by hypothesis, $ik_i = O(g_i)$, and y satisfies (8), we know that

$$\sup_{t \in [0, T]} \sum_{i=2}^{\infty} k_{i-2} (2i-1) y_i < \infty. \tag{53}$$

By (46),

$$z_1(t) + \sum_{i=2}^{\infty} (i-1) z_i(t) = - \int_0^t \sum_{i=2}^{\infty} ((i-1) p_{i-2} + q_{i-2}) z_i(s) ds,$$

and using the hypothesis on (p_i) and (q_i) , we obtain, for all $t \in [0, T]$,

$$|z_1(t)| \leq \sum_{i=2}^{\infty} (i-1) |z_i(t)| + C_0 \int_0^t \sum_{i=2}^{\infty} (i-1) |z_i(s)| ds, \tag{54}$$

for some positive constant C_0 . Therefore, for $t \in [0, T]$,

$$\begin{aligned} \int_0^t |z_1(s)| ds &\leq \int_0^t \sum_{i=2}^{\infty} (i-1) |z_i(s)| ds + C_0 \int_0^t \int_0^s \sum_{i=2}^{\infty} (i-1) |z_i(\tau)| d\tau ds \\ &= \int_0^t (1 + C_0(t-s)) \sum_{i=2}^{\infty} (i-1) |z_i(s)| ds \\ &\leq C_1 \int_0^t \sum_{i=2}^{\infty} (i-1) |z_i(s)| ds, \end{aligned} \quad (55)$$

for some positive (T -dependent) constant, C_1 .

For the last term in (52), we use (48) to prove that it converges to zero as $N \rightarrow \infty$. In fact, from the definition of solution and the bounded convergence theorem we have that

$$\begin{aligned} \sum_{i=m}^{\infty} i M_i(t) - \sum_{i=m}^{\infty} i M_i(0) + \int_0^t \sum_{i=m}^{\infty} i(p_i + q_i) M_i(s) ds \\ - \int_0^t \sum_{i=m}^{\infty} x(s) k_i M_i(s) ds \rightarrow 0, \quad \text{as } m \rightarrow \infty, \end{aligned}$$

and thus,

$$\int_0^t m x(s) k_{m-1} M_{m-1}(s) ds \rightarrow 0, \quad \text{as } m \rightarrow \infty,$$

and this is equivalent to

$$\int_0^t N k_{N-2} y_1(s) y_N(s) ds \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

Proceeding similarly with respect to \tilde{y} , and since

$$\left| \int_0^t N k_{N-2} (y_1 y_N - \tilde{y}_1 \tilde{y}_N) \operatorname{sgn}(z_{N+1}) ds \right| \leq \int_0^t N k_{N-2} y_1 y_N ds + \int_0^t N k_{N-2} \tilde{y}_1 \tilde{y}_N ds,$$

we conclude that

$$\int_0^t N k_{N-2} (y_1 y_N - \tilde{y}_1 \tilde{y}_N) \operatorname{sgn}(z_{N+1}) ds \rightarrow 0, \quad \text{as } N \rightarrow \infty. \quad (56)$$

Using in (52) the boundedness of \tilde{y} and the hypothesis on k_i , together with (53), (55), and (56), we deduce that there exists some constant $C > 0$ such that, for each $t \in [0, T]$,

$$\sum_{i=2}^{\infty} (i-1) |z_i(t)| \leq C \int_0^t \sum_{i=2}^{\infty} (i-1) |z_i(s)| ds,$$

and thus, since by the definition of z , $z_i(0) = 0$, we conclude that

$$\sum_{i=2}^{\infty} (i-1) |z_i(t)| = 0,$$

for all $t \in [0, T]$. Hence, for $i = 2, 3, \dots$, $z_i(t) = 0$, which means that $y_i(t) = \tilde{y}_i(t)$. But then, by (54) we also have $z_1(t) = 0$, which means that $y_1(t) = \tilde{y}_1(t)$. Since $T > 0$ can be chosen arbitrarily large, the proof is complete. \square

Remark 7.2. Observe that the uniqueness theorem in [2] (Theorem 3.6) for the Becker-Döring equations, which corresponds to the uniqueness result, Theorem 7.1, in our present setting, does not require any type of restrictions on the fragmentation coefficients b_r , in contrast with our restrictions on the coefficients p_i and q_i . The reason for this is that, to estimate y_1 in terms of y_i , $i \geq 2$, that is, to obtain (54), we used the mass balance equation (46) thus requiring some hypothesis on the coefficients p_i, q_i , while in [2] to estimate c_1 in terms of the other coordinates c_r , the authors were able to use the conservation of mass property which did not required restrictions on the growth rate of the b_r coefficients.

Corollary 7.3. Let $(p_i), (q_i), (k_i)$ be nonnegative sequences satisfying $p_i = O(1)$, $q_i = O(i)$, and $k_i = O(i^\gamma)$ for some $\gamma \in [0, 1]$. Then, for each $y_0 = (y_{0i})_{i \in \mathbb{N}} \in X_+$ such that

$$\sum_{i=2}^{\infty} (i-1)^{1+\gamma} y_{0i} < \infty, \quad (57)$$

there is a unique local solution y of the silicosis system (1) satisfying $y(0) = y_0$ and, moreover, it is extendable to $[0, \infty)$. In particular, if (k_i) is bounded (i.e., if $\gamma = 0$) then this uniqueness result holds true for solutions with initial conditions with finite mass, that is, for nonnegative y_0 with

$$y_{01} + \sum_{i=2}^{\infty} (i-1) y_{0i} < \infty.$$

The following result is also a consequence of the uniqueness theorem:

Corollary 7.4. Let $y_0 \in X_+$ and suppose that the hypothesis of Theorem 7.1 are satisfied. Let y be the unique solution of the silicosis system (1) such that $y(0) = y_0$. If $(y^n)_{n \in \mathbb{N}}$ is the sequence of solutions of the truncated systems (5) considered in the proof of Theorem 4.1, then, as $n \rightarrow \infty$, $y^n(t) \rightarrow y(t)$ uniformly on compact intervals of $[0, \infty)$.

Proof. In order to reach a contradiction, suppose that there is a subsequence of (y^n) , (y^{n_j}) say, for which there is $T > 0$ and $\eta > 0$, such that, for all $j \in \mathbb{N}$, $\sup_{t \in [0, T]} |y^{n_j}(t) - y(t)| \geq \eta$. We can take this subsequence in place of the whole approximating sequence in the proof of Theorem 4.1 and conclude by that proof and the uniqueness theorem, that there is another subsequence $(y^{n_{j'}})$ such that, for each $i \in \mathbb{N}$, as $j' \rightarrow \infty$, $y_i^{n_{j'}}(t) \rightarrow y_i(t)$, on $[0, T]$. \square

8. SEMIGROUP PROPERTY

The last result we shall consider is a property that, together with the existence and the uniqueness results, is crucial for the study of the dynamics of solutions: the semigroup property.

Due to the fact that we could only prove the uniqueness result stated in Theorem 7.1 by imposing a more restrictive set of conditions on the coefficients than those required for the proof of existence in Theorem 4.1, and, in particular, as written above in Corollary 7.3, we can only proof uniqueness of solution in the full space X_+ when (k_i) is a bounded sequence, we are going to prove the semigroup property for initial conditions y_0 satisfying an extra higher moment condition like (57). In order to do this we need to start by proving that, under the assumptions

of Corollary 7.3, if we take any initial condition $y_0 \in X_+^\mu := X^\mu \cap X_+$, with $\mu \geq 1$, where

$$X^\mu := \left\{ y \in X : \|y\|_\mu := |y_1| + \sum_{i=2}^{\infty} (i-1)^\mu |y_i| < \infty \right\}, \quad (58)$$

then the solution stays in X_+^μ . Using this invariance property we then prove that, under those conditions, the set of solutions in these subspaces form a C_0 -semigroup.

Proposition 8.1. *Let $(p_i), (q_i), (k_i)$ be nonnegative sequences satisfying $p_i = O(1)$, $q_i = O(i)$, and $k_i = O(i^\gamma)$ for some $\gamma \in [0, 1]$. Take any $y_0 \in X_+^{1+\gamma}$ and let $y(\cdot)$ be the unique solution of (1) with $y(0) = y_0$. Then $y(t) \in X_+^{1+\gamma}$ for all $t > 0$.*

Proof. With the coefficients satisfying these assumptions and taking the sequence (g_i) defined by $g_1 = 1$ and $g_i = (i-1)^{1+\gamma}$ if $i \geq 2$, all conditions of Theorem 4.1 (existence) and Theorem 7.1 (uniqueness) are satisfied. Truncating the initial condition and working with the solutions to the n_j -truncated systems we can repeat the argument from (18) until (20) with the (g_i) above, concluding from (20) that the solution y of the infinite dimensional system obtained by making $n_j \rightarrow \infty$ satisfies $y_1(t) + \sum_{i=2}^{\infty} (i-1)^{1+\gamma} y_i(t) < C_1 e^{C_2 t}$, for all $t \in [0, T]$ and with C_1 and C_2 the same as before. Since under these conditions solutions are unique, through y_0 there is only one solution y and the argument above proves that $y(t) \in X_+^{1+\gamma}$, for all t . \square

Theorem 8.2. *Let $(p_i), (q_i), (k_i)$ be nonnegative sequences satisfying $p_i = O(1)$, $q_i = O(i)$, and $k_i = O(i^\gamma)$ for some $\gamma \in [0, 1]$. Let $y_0 \in X_+^{1+\gamma}$. Denote by $T(\cdot)y_0$ the unique solution $y(\cdot; y_0)$ of (1) satisfying the initial condition $y(0; y_0) = y_0$. Then, $\{T(t) : X_+^{1+\gamma} \rightarrow X_+^{1+\gamma} \mid t \geq 0\}$ is a C_0 -semigroup, i.e.,*

- (i) $T(0) = id$, the identity operator;
- (ii) $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$;
- (iii) $(t, y_0) \mapsto T(t)y_0$ is a continuous mapping from $[0, \infty) \times X_+^{1+\gamma}$ into $X_+^{1+\gamma}$.

Proof. Conditions (i) and (ii) are obvious from the fact that equations (2)–(4) are autonomous. To prove the continuity property in (iii) we first observe that, due to the uniform convergence in $t \in [0, T]$ of $x^n(t) \rightarrow x(t)$ and $M_i^n(t) \rightarrow M_i(t)$ as $n \rightarrow \infty$, it is sufficient to prove the continuity separately in t and in the initial condition y_0 .

Let us start by the continuity in t . The continuity of each map $y_i(\cdot; y_0)$, $i \in \mathbb{N}$, is a consequence of the definition of solution itself. For the continuity in the $X_+^{1+\gamma}$ norm we use the integrated version of the moments' equation (39). In fact, by our hypothesis on the coefficients k_i , by taking $g_i = i^{1+\gamma}$, the hypothesis set (B) of Theorem 5.1 is satisfied so that (39) holds for any $0 \leq t_1 < t_2$ and for any integer $m \geq 1$. Fix some $T > 0$, take $t_1 = 0$, $t_2 = T$, and isolate the integral term of the left-hand side. Then, by using on the other terms, the uniform boundedness of $\|y(t; y_0)\|_{1+\gamma}$ for $t \in [0, T]$, and the dominated convergence theorem, we conclude that

$$\phi_m(T, y_0) := \int_0^T \sum_{i=m}^{\infty} i^{1+\gamma} (p_i + q_i) M_i(s) ds \longrightarrow 0, \quad \text{as } m \rightarrow \infty.$$

Let $\varepsilon > 0$ be given. Then, again by the uniform boundedness of $\|y(t; y_0)\|_{1+\gamma}$ for $t \in [0, T]$, there is $\delta_0 > 0$, depending only on T and y_0 , such that the right-hand

side of (39) lies in $[0, \varepsilon/3)$, if $0 \leq t_1 < t_2 \leq T$ and $t_2 - t_1 < \delta_0$. Fix the integer $m > 2$ so that $\phi_m < \varepsilon/3$. Then, there is $\delta_1 \in (0, \delta_0)$, depending only on T and y_0 , such that, if $0 \leq t_1 < t_2 \leq T$, and $t_2 - t_1 < \delta_1$,

$$\begin{aligned} 0 \leq \int_{t_1}^{t_2} \sum_{i=1}^{\infty} i^{1+\gamma} (p_i + q_i) M_i(s) ds &\leq \sum_{i=1}^{m-1} i^{1+\gamma} (p_i + q_i) \int_{t_1}^{t_2} M_i(s) ds + \phi_m(T; y_0) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3}, \end{aligned}$$

and hence, from (39) we conclude that,

$$\left| \sum_{i=1}^{\infty} i^{1+\gamma} M_i(t_2) - \sum_{i=1}^{\infty} i^{1+\gamma} M_i(t_1) \right| < \varepsilon,$$

thus, together with the continuity of $x(\cdot)$ and $M_0(\cdot)$, allowing us to conclude that, for each y_0 , $\|T(\cdot)y_0\|_{1+\gamma} = \|y(\cdot; y_0)\|_{1+\gamma}$ is continuous in $[0, \infty)$. This, together with the continuity of each coordinate $y_i(\cdot; y_0)$ implies the continuity in the $X_+^{1+\gamma}$ norm as a result of standard weak* convergence results (see Lemma 3.3 in [2], for example).

It remains to prove in (iii) the continuity with respect to y_0 in $X^{1+\gamma}$. Consider a sequence $(y_0^n)_{n \in \mathbb{N}_0}$ and assume that, for some $y_0 \in X_+^{1+\gamma}$, we have $y_0^n \rightarrow y_0$ strongly in $X_+^{1+\gamma}$ as $n \rightarrow \infty$. By our present assumptions, for each y_0^n there is a unique globally defined solution in X_+^γ , $y^n(\cdot) := T(\cdot)y_0^n$, and also a unique global solution in X_+^γ , $y(\cdot) := T(\cdot)y_0$. We now need to prove that, for each t , $y^n(t) \rightarrow y(t)$ strongly in X_+^γ as $n \rightarrow \infty$.

To this goal we start by repeating the proof of the existence result Theorem 4.1 with y^n taking the place of the solutions of the n -truncated systems. Actually, the only difference between what we do now and what was done in the proof of Theorem 4.1 is that now one must take $\chi_{i,j} \equiv 1$ and, instead of using Lemma 3.1 for estimations of the moment of y^n , we need to use Theorem 5.1. This allows us to conclude that, for the above functions $y^n = (x^n, M_0^n, M_1^n, \dots)$, we have, when $n \rightarrow \infty$, $y^n \xrightarrow{*} y$ in X_+ which, in particular, implies that for each i , $M_i^n(s) \rightarrow M_i(s)$ as $n \rightarrow \infty$.

Now take a solution $y^n = (x^n, M_0^n, M_1^n, \dots)$ and consider the moment equation (39) satisfied by it with $m = 1$, $t_1 = 0$, $t_2 = t$ and $g_i = i^{1+\gamma}$. We want to prove that, when we pass to the limit $n \rightarrow \infty$, the same equation is valid for the limit solution $y = (x, M_0, M_1, \dots)$. This clearly implies without additional effort that $\|y^n(t)\|_{1+\gamma} \rightarrow \|y(t)\|_{1+\gamma}$ which, together with the corresponding version of Lemma 3.3 in [2], implies the result. We shall make use of the bound (9) (which is valid for every solution) to get uniform bounds on the components $x^n(s)$ and $M_i^n(s)$ of the solution y^n , namely

$$x^n(t), M_i^n(t) \leq \|y^n(t)\| \leq \|y_0^n\| + (\alpha + r)t \leq 2\|y_0\| + (\alpha + r)t \quad (59)$$

where the last inequality holds for all sufficiently large n (because $y_0^n \rightarrow y_0$ in $X_+^{1+\gamma} \subset X_+$). The same holds for the components of the limit solution y .

Let $t \in [0, T]$ for any fixed $T < \infty$, and write (39) as follows:

$$\begin{aligned} \sum_{i=1}^{\infty} i^{1+\gamma} M_i^n(t) &= \sum_{i=1}^{\infty} i^{1+\gamma} M_i^n(0) - \int_0^t \sum_{i=1}^{\infty} i^{1+\gamma} (p_i + q_i) M_i^n(s) ds + \\ &+ \int_0^t k_0 x^n(s) M_0^n(s) ds + \int_0^t x^n(s) \sum_{i=1}^{\infty} ((i+1)^{1+\gamma} - i^{1+\gamma}) k_i M_i^n(s) ds. \end{aligned} \quad (60)$$

By the assumption on the initial data the first term in the right-hand side converges to $\sum_{i=1}^{\infty} i^{1+\gamma} M_i(0)$ as $n \rightarrow \infty$.

Using the bounds provided by (59), the dominated convergence theorem applied to the second integral in the right-hand side of (60) allows for the convergence of that integral to $\int_0^t k_0 x(s) M_0(s) ds$ as $n \rightarrow \infty$.

To deal with the third integral in the right-hand side of (60) we take a fixed $m \in \mathbb{N}$ and separate the sum into a sum with $i \leq m-1$ and another with $i \geq m$. For this “tail sum”, observe that, with the assumption that $k_i = O(i^\gamma)$, applying Lagrange’s mean value theorem gets us

$$((i+1)^{1+\gamma} - i^{1+\gamma}) k_i \sim C_\gamma i^{2\gamma} = C_\gamma \frac{1}{i^{1-\gamma}} i^{1+\gamma} \leq \frac{C_\gamma}{m^{1-\gamma}} i^{1+\gamma}.$$

Also, from (20) and the fact that C_1 and C_2 in that bound can be chosen independent of n (because $y_0^n \rightarrow y_0$ in $X_+^{1+\gamma}$) we have the estimate

$$\sum_{i=m}^{\infty} ((i+1)^{1+\gamma} - i^{1+\gamma}) k_i M_i^n(s) \leq \frac{C_\gamma}{m^{1+\gamma}} \sum_{i=m}^{\infty} i^{1+\gamma} M_i^n(s) \leq \frac{1}{m^{1+\gamma}} C_\gamma C_1 e^{C_2 s}$$

and thus, by (59) and the dominated convergence theorem, we conclude that, for every $\varepsilon > 0$ and $s \in [0, T]$, there is a m_0 such that, for all $m \geq m_0$ the tail contribution satisfies

$$\int_0^t x^n(s) \sum_{i=m}^{\infty} ((i+1)^{1+\gamma} - i^{1+\gamma}) k_i M_i^n(s) ds \leq \frac{1}{m^{1+\gamma}} C(T, \|y_0\|, \|y_0\|_{1+\gamma}) < \frac{1}{3} \varepsilon, \quad (61)$$

where the constant $C(T, \|y_0\|, \|y_0\|_{1+\gamma})$ does not depend on n . Clearly (61) is also valid with x^n and M_i^n changed to x and M_i , respectively. For the sum with $1 \leq i \leq m-1$ the pointwise convergence of $M_i^n(s)$ to $M_i(t)$ and the bound (59) allow us to apply again the dominated convergence theorem to conclude that, for every $\varepsilon > 0$, there exists a n_0 such that, for all $n > n_0$,

$$\begin{aligned} &\left| \int_0^t x^n(s) \sum_{i=1}^{m-1} ((i+1)^{1+\gamma} - i^{1+\gamma}) k_i M_i^n(s) ds - \right. \\ &\quad \left. - \int_0^t x(s) \sum_{i=1}^{m-1} ((i+1)^{1+\gamma} - i^{1+\gamma}) k_i M_i(s) ds \right| < \frac{1}{3} \varepsilon \end{aligned} \quad (62)$$

Plugging together (62) and (61) proves that the last integral in the right-hand side of (60) converges to the integral with x and M_i in place of x^n and M_i^n , respectively.

We finally consider the first integral in the right-hand side of (60). Again we separate the sum into a part with $1 \leq i \leq m-1$ and a tail contribution with $i \geq m$. For the tail term we can use (39), Proposition 8.1, and what we proved above for

the remaining integral terms to conclude that, for every $\varepsilon > 0$ there exists m and n_0 such that, for all $n > n_0$,

$$\int_0^t \sum_{i=m}^{\infty} i^{1+\gamma} (p_i + q_i) M_i^n(s) ds < \frac{1}{3}\varepsilon, \quad (63)$$

and the same holds with M_i^n substituted by M_i . For the sum with $1 \leq i \leq m-1$ it suffices to note that, for all $s \in [0, T]$ and for $n > n_0$ (augmenting n_0 if needed)

$$\sum_{i=1}^{m-1} i^{1+\gamma} (p_i + q_i) M_i^n(s) \leq (m-1)^\gamma \max_{1 \leq i \leq m-1} (p_i + q_i) (2\|y_0\| + (\alpha + r)T), \quad (64)$$

and so, applying the dominated convergence theorem one last time, we conclude that, for every $\varepsilon > 0$ there exists m and n_0 such that, for all $n > n_0$,

$$\left| \int_0^t \sum_{i=1}^{m-1} i^{1+\gamma} (p_i + q_i) M_i^n(s) ds - \int_0^t \sum_{i=1}^{m-1} i^{1+\gamma} (p_i + q_i) M_i(s) ds \right| < \frac{1}{3}\varepsilon. \quad (65)$$

Hence, by (63) and (65) we conclude that, when $n \rightarrow \infty$, the first integral in the right-hand side of (60) converges to a similar term with M_i in place of M_i^n .

We now have all the ingredients to pass to the limit $n \rightarrow \infty$ in both sides of (60) and obtain, for all $t \in [0, T]$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} i^{1+\gamma} M_i^n(t) &= \sum_{i=1}^{\infty} i^{1+\gamma} M_i(0) - \int_0^t \sum_{i=1}^{\infty} i^{1+\gamma} (p_i + q_i) M_i(s) ds + \\ &+ \int_0^t k_0 x(s) M_0(s) ds + \int_0^t x(s) \sum_{i=1}^{\infty} ((i+1)^{1+\gamma} - i^{1+\gamma}) k_i M_i(s) ds. \end{aligned} \quad (66)$$

Being $y = (x, M_0, M_1, \dots)$ the unique solution of (1) with initial condition $y_0 \in X_+^{1+\gamma}$, from Theorem 5.1 we conclude that the right-hand side of (66) is equal to $\sum_{i=1}^{\infty} i^{1+\gamma} M_i(t)$. That is, given the arbitrariness of T ,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} i^{1+\gamma} M_i^n(t) = \sum_{i=1}^{\infty} i^{1+\gamma} M_i(t), \quad \forall t \geq 0.$$

This and the expression for the norm of $X^{1+\gamma}$ in (58) allow us to conclude without further effort that, with the present assumptions, solutions $y(\cdot; y_0)$ of (1) depend on the initial condition y_0 continuously in the norm topology of $X^{1+\gamma}$. This completes the proof. \square

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