

TWO GENERALIZATIONS OF HOMOGENEITY IN GROUPS WITH APPLICATIONS TO REGULAR SEMIGROUPS

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ABSTRACT. Let X be a finite set such that $|X| = n$ and let $i \leq j \leq n$. A group $G \leq \mathcal{S}_n$ is said to be (i, j) -homogeneous if for every $I, J \subseteq X$, such that $|I| = i$ and $|J| = j$, there exists $g \in G$ such that $Ig \subseteq J$. (Clearly (i, i) -homogeneity is i -homogeneity in the usual sense.)

A group $G \leq \mathcal{S}_n$ is said to have the k -universal transversal property if given any set $I \subseteq X$ (with $|I| = k$) and any partition P of X into k blocks, there exists $g \in G$ such that Ig is a section for P . (That is, the orbit of each k -subset of X contains a section for each k -partition of X .)

In this paper we classify the groups with the k -universal transversal property (with the exception of two classes of 2-homogeneous groups) and the $(k-1, k)$ -homogeneous groups (for $2 < k \leq \lfloor \frac{n+1}{2} \rfloor$). As a corollary of the classification we prove that a $(k-1, k)$ -homogeneous group is also $(k-2, k-1)$ -homogeneous, with two exceptions; and similarly, but with no exceptions, groups having the k -universal transversal property have the $(k-1)$ -universal transversal property.

A corollary of all the previous results is a classification of the groups that together with any rank k transformation on X generate a regular semigroup (for $1 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$).

The paper ends with a number of challenges for experts in number theory, group and/or semigroup theory, linear algebra and matrix theory.

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1. INTRODUCTION AND PRELIMINARIES

One of the fundamental trends in semigroup theory has been the study of how idempotents shape the structure of the semigroup. Howie's book [19] can be seen as an excellent survey of the results obtained from the 40s to the 90s on this general problem. Evidently, there is the analogous question for the group of units, namely, to what extent the group of units shapes the structure of the semigroup. (And a similar question can be asked about the normalizer of the semigroup; more on this below.) However, unlike the idempotents case, the group of units approach quickly leads to problems that could not be tackled with the tools available 30 years ago, let alone 70 years ago. Fortunately now the situation is totally different, since the enormous progress made in the last decades in the theory of permutation groups provides the necessary tools to develop semigroup theory from this different point of view.

A particular instance of the general problem of investigating how the group of units shapes the whole semigroup might be described as follows: *classify the pairs (a, G) , where a is a map on X and G is a group of permutations of X , such that the semigroup $\langle a, G \rangle$, generated by a and G , has a given property.* (Observe that whenever S is a semigroup with group of units G we have $S = \cup_{a \in S} \langle a, G \rangle$.)

A very important class of groups that falls under this general scheme is that of synchronizing groups, groups of permutations on a set that together with any non-invertible map on the same set generate a constant (see [6], [10], [34]). These groups are very interesting from a group theoretic point of view and are linked to the Černý conjecture, a longstanding open problem in automata theory.

Another instance of the general problem described above is the following: *classify the permutation groups on a set that together with any map on that set generate a regular semigroup.* (An element a in a semigroup S is said to be regular if there exists $b \in S$ such that $a = aba$. The semigroup is said to be regular if all its elements are regular.) This question has been answered in [3] as follows. (From now on \mathcal{S}_n will denote the symmetric group on the set $[n] := \{1, \dots, n\}$; by \mathcal{T}_n we will denote the full transformation monoid on $[n]$. We use the notation $D(2 * n)$ for the dihedral group of order $2n$, called either D_n or D_{2n} in the literature; other notation for finite groups is standard.)

Theorem 1.1. *If $n \geq 1$ and G is a subgroup of \mathcal{S}_n , then the following are equivalent:*

- (i) *The semigroup $\langle G, a \rangle$ is regular for all $a \in \mathcal{T}_n$.*
- (ii) *One of the following is valid for G and n :*
 - (a) *$n = 5$ and $G \cong C_5$, $D(2 * 5)$, or $\text{AGL}(1, 5)$;*
 - (b) *$n = 6$ and $G \cong \text{PSL}(2, 5)$ or $\text{PGL}(2, 5)$;*
 - (c) *$n = 7$ and $G \cong \text{AGL}(1, 7)$;*
 - (d) *$n = 8$ and $G \cong \text{PGL}(2, 7)$;*
 - (e) *$n = 9$ and $G \cong \text{PSL}(2, 8)$ or $\text{PFL}(2, 8)$;*
 - (f) *$G = \mathcal{A}_n$ or \mathcal{S}_n .*

The critical observation that led to the proof of Theorem 1.1 is that for $n \geq 12$, if $G (\leq \mathcal{S}_n)$ satisfies the property that any rank $\lfloor \frac{n}{2} \rfloor$ map $a \in \mathcal{T}_n$ is regular in the semigroup $\langle G, a \rangle$, then G contains the alternating group. Therefore Theorem 1.1 can be seen as an (almost) immediate corollary of the following result.

Theorem 1.2. *If $n \geq 12$ and G is a subgroup of \mathcal{S}_n , then the following are equivalent:*

- (i) The element a is regular in $\langle G, a \rangle$, for all $a \in \mathcal{T}_n$ such that $\text{rank}(a) = \lfloor \frac{n}{2} \rfloor$.
- (ii) $G = \mathcal{A}_n$ or \mathcal{S}_n .

Below a sharper version of this result is going to be stated.

The aim of this paper is to carry out a deeper group theoretic analysis in order to prove a lemma on the groups G that satisfy the following property: for every rank k map a (with fixed k such that $1 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$), a is regular in $\langle G, a \rangle$; and then extract many important consequences from this result.

In order to understand these groups we need some observations. Suppose $G \leq \mathcal{S}_n$ and all rank k maps $a \in \mathcal{T}_n$ are regular in $\langle G, a \rangle$. Then there exists $b \in \langle G, a \rangle$ such that $a = aba$ and hence ab is an idempotent in $\langle G, a \rangle$ having the same rank as a , where $b = a$ or $b \in G$ or $b = g_1 a g_2 a \dots a g_m$, for $m \geq 2$. Suppose $b \in G$. Then there exists $g \in G$ (namely $g = b$) such that ag is idempotent and hence $[n]ab$ is a section for the partition of $[n]$ induced by the kernel of a . Suppose $b = g_1 a g_2 a \dots a g_m$ with $m \geq 2$. Since ab is idempotent and $ab = a g_1 a g_2 a \dots a g_m$ it follows that there exists $g \in G$ (namely $g = g_1$) such that a and aga have the same rank. In both cases there exists $g \in G$ such that $[n]ag$ is a section for the partition of $[n]$ induced by the kernel of a . As this property must hold for all rank k maps $a \in \mathcal{T}_n$, it follows that G must satisfy the following *k-universal transversal property*: for every k -set $I \subseteq [n]$ and every partition P of $[n]$ into k blocks, there exists $g \in G$ such that Ig is a section for P . The next six results (almost) provide the classification of the groups possessing the *k-universal transversal property*.

The first theorem handles the permutation groups of small degree.

Theorem 1.3. *For $n < 11$ and $2 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$, a group $G \leq \mathcal{S}_n$ with the *k-universal transversal property* is *k-homogeneous*, with the following exceptions:*

- (1) $n = 5$, $G \cong C_5$ or $D(2 * 5)$ and $k = 2$;
- (2) $n = 6$, $G \cong \text{PSL}(2, 5)$ and $k = 3$;
- (3) $n = 7$, $G \cong C_7$ or $G \cong D(2 * 7)$, and $k = 2$; or $G \cong \text{AGL}(1, 7)$ and $k = 3$;
- (4) $n = 8$, $G \cong \text{PGL}(2, 7)$ and $k = 4$;
- (5) $n = 9$, $G \cong 3^2 : 4$ or $G \cong 3^2 : D(2 * 4)$ and $k = 2$;
- (6) $n = 10$, $G \cong \mathcal{A}_5$ or $G \cong \mathcal{S}_5$ and $k = 2$; or $G \cong \text{PSL}(2, 9)$ or $G \cong \mathcal{S}_6$ and $k = 3$.

The next results deal with the groups of degree larger than 10. We start by the case of groups possessing the *k-universal transversal property*, for large values of k .

Theorem 1.4. *Let $n \geq 11$, $G \leq \mathcal{S}_n$. If $6 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$, then the following are equivalent:*

- (1) G has the *k-universal transversal property*;
- (2) $\mathcal{A}_n \leq G$.

The four next results deal with groups possessing the *k-universal transversal property*, when $k \in \{2, \dots, 5\}$.

Theorem 1.5. *Let $n \geq 11$, $G \leq \mathcal{S}_n$ and let $2 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$. The following are equivalent:*

- (1) G has the *5-universal transversal property*;
- (2) G is *5-homogeneous*, or $n = 33$ and $G = \text{P}\Gamma\text{L}(2, 32)$.

Unlike the previous cases, the classification of groups possessing the *4-universal transversal property* was not possible. So far we have the following results, and we believe the remaining cases require very delicate considerations.

Theorem 1.6. *Let $n \geq 11$, $G \leq \mathcal{S}_n$ and let $2 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$. If G is 4-homogeneous, or $n = 12$ and $G = M_{11}$, then G has the 4-universal transversal property.*

If there are more groups possessing the 4-universal transversal property, then they must be groups G such that $\text{PSL}(2, q) \leq G \leq \text{P}\Gamma\text{L}(2, q)$, with either q prime or $q = 2^p$ for p prime. Note, however, that the groups $\text{PSL}(2, q)$ for $q \equiv 1 \pmod{4}$ cannot possess the 4-universal transversal property since they fail to satisfy the necessary condition of 3-homogeneity.

Similarly to the previous case, a full classification of the groups possessing the 3-universal transversal property was not possible.

Theorem 1.7. *Let $n \geq 11$, $G \leq \mathcal{S}_n$ and let $2 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$. G has the 3-universal transversal property if G is 3-homogeneous, or one of the following groups*

- (1) $\text{PSL}(2, q) \leq G \leq \text{P}\Sigma\text{L}(2, q)$, where $q \equiv 1 \pmod{4}$;
- (2) $\text{Sp}(2d, 2)$ with $d \geq 3$, in either of its 2-transitive representations;
- (3) $2^{2d} : \text{Sp}(2d, 2)$;
- (4) HS ;
- (5) Co_3 ;
- (6) $2^6 : G_2(2)$ and its subgroup of index 2;
- (7) $\text{AGL}(1, p)$ where, for all $c \in \text{GF}(p) \setminus \{0, 1\}$, $|\langle -1, c, c-1 \rangle| = p-1$.

If there are more groups possessing the 3-universal transversal property, then they must be Suzuki groups $\text{Sz}(q)$, possibly with field automorphisms adjoined, and/or subgroups of index 2 in $\text{AGL}(1, p)$ for $p \equiv 11 \pmod{12}$.

Finally, possessing the 2-universal transversal property is just another way of saying primitive.

Theorem 1.8. *A permutation group has the 2-universal transversal property if and only if the group is primitive.*

These theorems immediately imply an analogue of the Livingstone–Wagner [30] result on k -homogeneous groups (for $2 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$).

Corollary 1.9. *Let $n \geq 5$, let $2 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$, and let $G \leq \mathcal{S}_n$ be a group having the k -universal transversal property. Then G has the $(k-1)$ -universal transversal property.*

Now consider partitions of type $(\overbrace{1, \dots, 1}^{k-1}, n-k+1)$ (that is, $k-1$ classes of size 1 and one class of size $n-k+1$). Any group G satisfying the k -universal transversal property must also satisfy the following property: for every k -partition P of $[n]$ of type $(1, \dots, 1, n-k+1)$ and for every k -set I there exists $g \in G$ such that Ig is a section for P . In particular this implies that the union of all the singleton blocks is contained in Ig ; as the union of the singleton blocks can be any $(k-1)$ -subset of $[n]$, it follows that any group G possessing the k -universal transversal property must be $(k-1, k)$ -homogeneous, that is, for every $(k-1)$ -set I and for every k -set J there exists $g \in G$ such that $Ig \subseteq J$. In this new setting we can state the sharper version of Theorem 1.2 announced above.

Theorem 1.10. *If $n \geq 12$ and G is a subgroup of \mathcal{S}_n , then the following are equivalent:*

- (i) G is $(\lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor)$ -homogeneous.
- (ii) The map a is regular in $\langle G, a \rangle$ for all $a \in \mathcal{T}_n$ such that $\text{rank}(a) = \lfloor \frac{n}{2} \rfloor$ and a has kernel type $(1, \dots, 1, \lceil \frac{n}{2} \rceil + 1)$.

- (iii) The map a is regular in $\langle G, a \rangle$ for all $a \in \mathcal{T}_n$ such that $\text{rank}(a) = \lfloor \frac{n}{2} \rfloor$.
- (iv) $G = \mathcal{A}_n$ or \mathcal{S}_n .

Our second main theorem on groups provides the following classification of the $(k-1, k)$ -homogeneous groups.

Theorem 1.11. *If $n \geq 1$ and $2 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$, then the following are equivalent:*

- (i) G is a $(k-1, k)$ -homogeneous subgroup of \mathcal{S}_n ;
- (ii) G is $(k-1)$ -homogeneous or G is one of the following groups
 - (a) $n = 5$ and $G \cong C_5, D(2 * 5)$, $k = 3$;
 - (b) $n = 7$ and $G \cong \text{AGL}(1, 7)$, with $k = 4$;
 - (c) $n = 9$ and $G \cong \text{ASL}(2, 3)$ or $\text{AGL}(2, 3)$, with $k = 5$.

Once again an analogue of the Livingstone–Wagner [30] result is immediate.

Corollary 1.12. *Let $n \geq 1$, let $3 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$, and let $G \leq \mathcal{S}_n$ be a $(k-1, k)$ -homogeneous group. Then G is a $(k-2, k-1)$ -homogeneous group, except when $n = 9$ and $G \cong \text{ASL}(2, 3)$ or $\text{AGL}(2, 3)$, with $k = 5$.*

By [24, Theorem 2.3] we know that every rank k map $a \in \mathcal{T}_n$ is regular in $\langle G, a \rangle$ if and only if G has the k -universal transversal property, that is, in the orbit of every k -set there exists a transversal (or section) for every k -partition. Therefore we can state our main results about semigroups.

A quasi-permutation is a transformation in which all kernel classes but one are singletons.

Theorem 1.13. *Let $G \leq \mathcal{S}_n$ and let $1 < k \leq \lfloor \frac{n+1}{2} \rfloor$. Then the following are equivalent*

- (1) for every quasi-permutation a , such that $\text{rank}(a) = k$, the semigroup $\langle G, a \rangle$ is regular;
- (2) G is $(k-1)$ -homogeneous or G is one of the following groups
 - (a) $n = 5$ and $G \cong C_5, D(2 * 5)$, with $k = 3$;
 - (b) $n = 7$ and $G \cong \text{AGL}(1, 7)$, with $k = 4$;

We are now ready to state the result that dramatically generalizes Theorem 1.1 (and also [24, Theorem 2.3] taking advantage of the fact that if a group has the k -universal transversal property, then it also has the $(k-1)$ -universal transversal property, by Corollary 1.9).

Theorem 1.14. *Let $n \geq 5$, $G \leq \mathcal{S}_n$ and let $1 < k \leq \lfloor \frac{n+1}{2} \rfloor$. Then the following are equivalent:*

- (1) all rank k transformations $a \in \mathcal{T}_n$ are regular in $\langle a, G \rangle$;
- (2) for all rank k transformations $a \in \mathcal{T}_n$, the semigroup $\langle a, G \rangle$ is regular;
- (3) G has the k -universal transversal property (and hence is one of the groups listed in the classification).

One last word about the normalizer. It is well known that not every semigroup has a group of units, and hence the approach proposed in this paper might seem limited. Therefore two observations should be made here. The first is that it is commonly believed that the majority of finite semigroups have only one idempotent (which is a zero), but that did not prevent experts in semigroup theory to investigate how idempotents shape the structure of a semigroup; and the second observation is that by [24, Theorem 2.3, (iii) \Leftrightarrow (iv) \Leftrightarrow (v)], Theorem 1.13 and Theorem 1.14 admit versions in terms of conjugates. As a sample result we have the following immediate (from [24, Theorem 2.3]) version of the previous theorem.

Theorem 1.15. *Let $n \geq 5$, $G \leq \mathcal{S}_n$ and let $1 < k \leq \lfloor \frac{n+1}{2} \rfloor$. Then the following are equivalent:*

(1) *all rank k transformations $a \in \mathcal{T}_n$ are regular in*

$$\langle g^{-1}ag \mid g \in G \rangle;$$

(2) *for all rank k transformations $a \in \mathcal{T}_n$ the semigroup $\langle g^{-1}ag \mid g \in G \rangle$ is regular;*

(3) *G has the k -universal transversal property.*

This observation is important because the transformation semigroup $S \leq \mathcal{T}_n$ might contain no group of units, but every transformation semigroup S has a normalizer and hence the results of this paper can be used to extract information about the structure of S from its normalizer. For example, if $S = \langle t_1, \dots, t_m \rangle$ is a semigroup generated by m rank 3 maps (for example in \mathcal{T}_{176}) and it turns out that the normalizer $N(S) := \{g \in \mathcal{S}_n \mid g^{-1}Sg = S\}$ contains the Higman–Sims group, then we know that the semigroup S is regular.

An even more striking consequence of Corollary 1.9 and of the fact that possessing the k -universal transversal property is closed upwards (that is, if $G \leq H \leq \mathcal{S}_n$ and G has the k -universal transversal property, then H also has it), is the following result.

Theorem 1.16. *Let $n \geq 5$, $G \leq \mathcal{S}_n$ and let $1 < k < \lfloor \frac{n+1}{2} \rfloor$. Then the following are equivalent:*

(i) *G has the k -universal transversal property;*

(ii) *G has the l -universal transversal property for all l such that $1 \leq l \leq k$;*

(iii) *H has the k -universal transversal property for all H such that $G \leq H \leq \mathcal{S}_n$;*

(iv) *H has the l -universal transversal property for all H such that $G \leq H \leq \mathcal{S}_n$ and for all l such that $1 \leq l \leq k$;*

As a consequence, the following are equivalent.

(1) *all rank k transformations $a \in \mathcal{T}_n$ are regular in $\langle a, G \rangle$;*

(2) *all rank k transformations $a \in \mathcal{T}_n$ are regular in $\langle g^{-1}ag \mid g \in G \rangle$;*

(3) *for all rank k transformations $a \in \mathcal{T}_n$ and for all groups H such that $G \leq H \leq \mathcal{S}_n$ we have that $\langle h^{-1}ah \mid h \in H \rangle$ is a regular semigroup.*

(4) *for all rank k transformations $a \in \mathcal{T}_n$ and for all groups H such that $G \leq H \leq \mathcal{S}_n$ we have that $\langle a, H \rangle$ is a regular semigroup.*

Finally we summarize what this paper brings to groups and to semigroups:

(1) We have generalized the notion of $(k-1)$ -homogeneity in permutation groups; we have extended it first to the obvious notion of $(k-1, k)$ -homogeneous groups and then extended this to the notion of groups having the k -universal transversal property (for $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$).

(2) The $(k-1, k)$ -homogenous groups were fully classified, and the groups having the k -universal transversal property have been classified, with the exception of a class of groups (for $k = 3$) and another class (for $k = 4$). These two classes left undecided are surely very interesting problems for group theorists and combinatorialists.

(3) As a corollary of the classification it follows that $(k-1, k)$ -homogenous groups are $(k-2, k-1)$ -homogenous with two exceptions; and groups having the k -universal transversal property have the $(k-1)$ -universal transversal property. And this fact is extremely important for the impact of these results on the theory of semigroups.

- (4) Regarding semigroups, we take deep results out of the classification of finite simple groups and show that it is possible to follow the promising path of investigating how the group of units (or other groups associated to the semigroup such as the normalizer) shape the structure of the semigroup. This mimics what has been done in semigroup theory for the last 70 years with the set of idempotents.
- (5) The paper ends with a number of challenges for experts in number theory, group and/or semigroup theory, linear algebra and matrix theory.

2. THE CLASSIFICATION OF $(k - 1, k)$ -HOMOGENEOUS GROUPS

A permutation group G of degree n is k -homogeneous if it acts transitively on the set of k -subsets of its domain. Since k -homogeneity is clearly equivalent to $(n - k)$ -homogeneity, it is usually assumed that $k \leq (n - 1)/2$. With this assumption, Livingstone and Wagner, in an elegant paper [30], proved that a k -homogeneous group is $(k - 1)$ -homogeneous, and is k -transitive if $k \geq 5$. Kantor [20, 21] determined all k -homogeneous groups which are not k -transitive for $2 \leq k \leq 4$. The 2-transitive groups have been determined as a consequence of the Classification of Finite Simple Groups; lists of them can be found in [8] and [12].

For $k \leq l$, the permutation group G is (k, l) -homogeneous if, given subsets K, L of the domain with $|K| = k$ and $|L| = l$, there is an element of G which maps K to a subset of L . Note that

- (k, k) -homogeneity is equivalent to k -homogeneity, and for fixed k the concept of (k, l) -homogeneity becomes formally weaker as l increases;
- (k, l) -homogeneity is equivalent to the “dual” concept of (l, k) -homogeneity, requiring that for given K and L as before, there is an element of G mapping L to a superset of K ;
- (k, l) -homogeneity is equivalent to $(n - l, n - k)$ -homogeneity.

In this section we are concerned with $(k, k + 1)$ -homogeneity. Because of the third property above, we lose no generality in assuming, as Livingstone and Wagner did, that $k \leq (n - 1)/2$; indeed this condition will be used in our proofs several times. We had hoped to find arguments as elegant as those of Livingstone and Wagner, but have not succeeded. We prove the following theorem:

Theorem 2.1. *Let G be a $(k, k + 1)$ -homogeneous permutation group of degree $n \geq 2k + 1$. Then either G is k -homogeneous, or G is one of the following groups:*

- (1) $n = 5$, $k = 2$, $G \cong C_5$ or $D(2 * 5)$;
- (2) $n = 7$, $k = 3$, $G = \text{AGL}(1, 7)$;
- (3) $n = 9$, $k = 4$, $G = \text{ASL}(2, 3)$ or $\text{AGL}(2, 3)$.

2.1. General observations. Let G be $(k, k + 1)$ -homogeneous of degree $n \geq 2k + 1$. We begin with a few general observations.

Proposition 2.2. *The number of G -orbits on k -sets is at most k .*

Proof. Since a fixed $(k + 1)$ -set contains a representative of every orbit on k -sets, there are at most $k + 1$ orbits on k -sets. This bound can be reduced to k . For suppose there are $k + 1$ orbits; then each $(k + 1)$ -set contains exactly one k -set from each orbit. Let V_k be the

\mathbb{Q} -vector space of functions from k -sets to \mathbb{Q} , and let $T : V_k \rightarrow V_{k+1}$ be defined by

$$(Tf)(L) = \sum_{x \in L} f(L \setminus \{x\})$$

for $f \in V_k$ and $|L| = k + 1$. Since $n \geq 2k + 1$, it is known that T is injective (Kantor [22]). However, if f is the characteristic function of any G -orbit, then Tf is the all-1 function. \square

This gives a lower bound for $|G|$, namely $|G| \geq \binom{n}{k}/k$. We refer to this as the *order bound*. The right-hand side of this bound is a monotonic function of k for $k < n/2$; so, whenever we rule out a group G on the basis of this inequality for a certain value of k , then it cannot occur for any larger value of k either.

The *Ramsey number* $R(k, l, r)$, for positive integers k, l, r with $k \leq l$ and $r \geq 1$, is the smallest number n such that, if the k -element subsets of an n -set are coloured with r colours, there exists a l -element set all of whose k -element subsets have the same colour.

Proposition 2.3. *If G is $(k, k + 1)$ -homogeneous but not k -homogeneous of degree n , then $n < R(k, k + 1, 2)$.*

Proof. Colour the k -sets in one G -orbit red and the others blue. Each $(k + 1)$ -set contains k -sets of each colour. \square

It happens that the Ramsey numbers $R(2, 3, 2) = 6$ and $R(3, 4, 2) = 13$ are two of the very few which are known exactly. The first is the well-known “party problem”; the second was computed by McKay and Radzizowski [32] in 1991 (see [37] for a survey). The number $R(4, 5, 2)$ is not known, and the known upper bounds are too large for our purpose.

In the case $k = 2$, we have $n \leq 5$, and it is easy to see that the cyclic and dihedral groups are examples and hence we have (1) of Theorem 2.1. So we may assume that $k \geq 3$.

Our general results allowed us to decide which groups are $(k, k + 1)$ -homogeneous, except for a number of groups of small degrees. To decide those cases we used GAP [13] and include here a word about those computations. For $n \leq 20$, the simplest method is to compute the orbits of a given group on k -sets and $(k + 1)$ -sets, and for each orbit representative on $(k + 1)$ -sets, test whether it contains representatives of all the k -set orbits.

For larger n , the memory requirements of this method are too heavy, so we proceed a little differently. First, as we will prove below, any candidate group must be 2-transitive; so we reject groups which either fail to be 2-transitive or are k -homogeneous. We also reject groups which fail the order bound $|G| \geq \binom{n}{k}/k$. Then, if G is t -transitive, we loop over all pairs (K', L') , where K' and L' are subsets of $\{t + 1, \dots, n\}$ of cardinality $k - t$ and $k + 1 - t$ respectively, and check whether there is an element of G mapping $\{1, \dots, t\} \cup K'$ to a subset of $\{1, \dots, t\} \cup L'$. If this fails for any pair (K', L') , we can terminate the computation and report that G is not $(k, k + 1)$ -homogeneous.

2.2. Transitivity. From now on, G will be a $(k, k + 1)$ -homogeneous but not k -homogeneous group of degree $n \geq 2k + 1$, which is not one of the exceptions listed in the statement of Theorem 2.1.

Proposition 2.4. *G is transitive.*

Proof. Let O be an orbit of G . There exists a k -set containing at least one point of O . Hence every $(k + 1)$ -set contains a k -set containing at least one point of O , and thus intersects O . So $|O| \geq n - k > n/2$. Since O was arbitrary, there is only one orbit. \square

2.3. Primitivity.

Proposition 2.5. *G is primitive.*

Proof. Suppose that G is imprimitive, with r blocks of size s .

If $k \leq s$, then there is a k -set contained in a block. But, since $k + 1 \geq 4$, there is a $(k + 1)$ -set and all instances of an k -set or an $(k + 1)$ containing at least two points of each of two blocks; such a set cannot contain a k -set of the type just described.

So $k > s$, and $r = n/s > n/k > 2$, so $r \geq 3$. There is a $(k + 1)$ -set which contains either $\lfloor (k + 1)/r \rfloor$ or $\lceil (k + 1)/r \rceil$ points from each block. On the other hand, there is a k -set containing all the points of a block. So $(k + 1)/r \geq \lceil (k + 1)/r \rceil - 1 \geq s - 1$, whence $k + 1 \geq (r - 1)s \geq 2n/3$, a contradiction. \square

Since lists of primitive groups are conveniently available in computer algebra systems such as GAP, we have checked all primitive groups of degree at most 20, and find no counterexamples for the statement of Theorem 2.1. In view of our remarks about Ramsey numbers earlier, we may from now on assume that $k \geq 4$.

2.4. 2-homogeneity.

Proposition 2.6. *G is 2-homogeneous.*

Proof. Assume that G is not 2-homogeneous; let it have r orbits on 2-element subsets. Each is the edge set of one of the symmetrised orbital graphs for G ; each of these graphs is vertex-primitive and edge-transitive. First we show:

- each symmetrised orbital graph has valency at least k ;
- there are at most two such graphs (that is, $r \leq 2$).

For the first point, suppose that there is a graph whose valency $v - 1$ is smaller than k , so that $v \leq k$. Then some k -set contains a vertex and all its neighbours in this graph, and hence every $(k + 1)$ -set does so. The number of $(k + 1)$ -sets is $\binom{n}{k+1}$, whereas the number of ways of choosing the closed neighbourhood of a vertex in the graph, and then adjoining $k + 1 - v$ more points to make a $(k + 1)$ -set is $n \binom{n-v}{k+1-v}$. Since the second method overcounts, we have $\binom{n}{k+1} \leq n \binom{n-v}{k+1-v}$. A short calculation yields $n \leq k + 2$, a contradiction.

For the second, suppose that $r \geq 3$, and let $\Gamma_1, \Gamma_2, \Gamma_3$ be three of the orbital graphs. By the first point, we can find a $(k + 1)$ -set consisting of a vertex x and k of its neighbours in the graph Γ_1 . This set must contain a k -set consisting of a point y and $k - 1$ of its neighbours in Γ_2 , and a k -set consisting of a point z and $k - 1$ of its neighbours in Γ_3 . Now it is clear that x, y, z are distinct; but then the pair $\{y, z\}$ must be an edge in both Γ_2 and Γ_3 , a contradiction.

Now we conclude the proof. Suppose that $r = 2$ and let Γ be one of the two complementary orbital graphs. Suppose first that the valency of Γ is at least $k + 1$. Then we can choose a $(k + 2)$ -set consisting of a vertex x and a set X of $k + 1$ of its neighbours. Now one point of X , say y , must be joined to at least $k - 1$ further points of X , say those in a subset Y . Now the induced subgraph on the $k + 1$ points $\{x, y\} \cup Y$ has minimum valency at least 2, and so cannot contain a vertex which is nonadjacent to all but one point of this set, a contradiction.

So Γ has valency k , as does its complement, and $n = 2k + 1$.

There are $\binom{n}{k+1}$ choices of a $k + 1$ -set; each contains a vertex joined to all or all but one of the remaining vertices. But each vertex lies in just one $k + 1$ -set in which it is joined to all

other vertices, and to k^2 in which it is joined to all but one (choose one neighbour to omit, and one non-neighbour to include). So $\binom{n}{k} \leq nk^2$.

This inequality fails for $k \geq 5$; if $k = 4$ then $n = 9$, which has already been disposed of by computation. \square

2.5. 2-transitivity.

Proposition 2.7. *G is 2-transitive.*

Proof. According to Kantor's classification, a 2-homogenous, but non 2-transitive group G is contained in a one-dimensional affine group, and has order at most $q(q-1)/2 \cdot \log_p q$, where $n = q$ is a power of p and is congruent to 3 (mod 4); so p is congruent to 3 (mod 4) and $\log_p q$ is odd.

If $k = 3$, then we have $q(q-1)/2 \cdot \log_p q \geq q(q-1)(q-2)/18$, so $q-2 \leq 9 \log_p q$. It is easy to check that this inequality is satisfied only for $q = 7, 11, 27$. The first two cases are covered by computation.

Suppose that $q = 27$, so that $|G| \leq 27 \cdot 13 \cdot 3$. If $k = 4$, then our inequality $|G| \geq \binom{27}{4}/4$ is violated. As remarked earlier, this settles larger values of k also. \square

2.6. Completion of the proof. We have a list of 2-transitive groups. It is now a case of going through the list.

The condition of $(k, k+1)$ -homogeneity is closed upwards; so we can usually assume that the groups we are considering are maximal subgroups of the symmetric or alternating group. The only exception is when we are testing the $(k, k+1)$ -homogeneity of a group which has a k -homogeneous overgroup. Since $k \geq 4$ and we may assume that $n \geq 20$, the only cases which need to be considered are $\text{PSL}(2, 23) \leq M_{24}$ (with $n = 24$, $k = 4$ or 5) and $\text{PGL}(2, 32) \leq \text{P}\Gamma\text{L}(2, 32)$ (with $n = 33$, $k = 4$). Computation shows that neither group is $(k, k+1)$ -homogeneous.

According to Burnside, the 2-transitive groups are of two types: *affine groups*, whose minimal normal subgroup is elementary abelian; and *almost simple groups*, whose minimal normal subgroup is simple. For the affine groups, the maximal groups are $\text{AGL}(d, p)$ for p prime.

Case $G = \text{AGL}(d, p)$, with p prime.

Subcase $d = 1$. We have $p(p-1) \geq \binom{p}{4}/4$, so $p \leq 11$; these cases are excluded by computation.

So we may assume that $d \geq 2$. Below, $\mathbf{0}$ and $\mathbf{1}$ denote the all-zero and all-one vectors.

Subcase $k \leq d$. There is an affine independent $(k+1)$ -set. But since $k \geq 4$, there is a k -set containing three or four affine dependent points ($(\mathbf{0}, \mathbf{0})$, $(\mathbf{1}, \mathbf{0})$ and $(2, \mathbf{0})$ if $p > 2$, and $(\mathbf{0}, \mathbf{0}, \mathbf{0})$, $(\mathbf{1}, \mathbf{0}, \mathbf{0})$, $(\mathbf{0}, \mathbf{1}, \mathbf{0})$ and $(\mathbf{1}, \mathbf{1}, \mathbf{0})$ if $p = 2$) which cannot be contained in such a $(k+1)$ -set.

Subcase $d+1 \leq k \leq p^{d-1}$, excluding $k = d+1$, $p = 2$, d odd. If p is odd, or if $p = 2$ and d is odd, there exist $d+2$ points such that every hyperplane omits at least two, namely $\mathbf{0}$, $\mathbf{1}$, and the points with a single coordinate 1 and all others zero. (This construction needs to be modified if p is even and divides $d-1$: then replace the all-1 vector by $(2, \mathbf{1})$.) A $(k+1)$ -set containing it can contain no k points contained in a hyperplane, a contradiction.

If $p = 2$ and d is even, then $d \geq 5$, so we can add one more point and find a set of size $d+3$ with the claimed property: any non-zero vector with an even number of 1s will do.

Subcase $k = d+1$, d odd, $p = 2$. Since $d+2 \leq 2^{d-2}$, we can take a $(k+1)$ -set contained in a $(d-2)$ -flat and an affine independent k -set.

Subcase $p^{d-1} - 1 \leq k \leq p^d - d(p-1) - 2$. There is a set of $1 + d(p-1)$ points meeting every hyperplane, namely those with at most one non-zero coordinate. Its complement contains a $(k+1)$ -set omitting a point of every hyperplane. But there is a k -set containing a hyperplane. Subcase $k \geq p^d - d(p-1) - 1$. Since $2k + 1 \leq p^d$, we have $p^d \leq 2d(p-1) + 1$, which is satisfied only for $p = 3$, $d = 2$, giving the known examples.

For groups with simple socle, there are more cases.

Case: $G = \text{P}\Gamma\text{L}(2, q)$, $q = p^e$.

Subcase $k \geq 5$. We have $|G| = (q+1)q(q-1)e$. If $k \geq 5$, then $|G| \geq \binom{q+1}{5}/5$, so $(q-2)(q-3) \leq 600e$, so $q \leq 27$ or $q = 32$, handled by computation. For $k \geq 6$ the inequality gives $q \leq 17$, which is covered by computation.

Subcase $k = 4$. The orbits of $\text{PGL}(2, q)$ on the 4-tuples of distinct points are parametrised by cross-ratio, of which there are $q-2$ distinct values. A typical 4-set has six distinct cross-ratios; depending on the congruence of q , there may be a set with only two cross-ratios, and one with only three. So the group $\text{PGL}(2, q)$ has at least $(q+5)/6$ orbits on 4-sets. Adding field automorphisms at worst divides the number of orbits by e . So $q+5 \leq 24e$. If $e = 1$ then $q \leq 19$, covered by our computation. For $e > 1$, the remaining values to be checked are $q = 25, 27, 32, 64$ and 128 ; again computation shows there are no examples.

Case: G is a unitary, Suzuki or Ree group. These groups are smaller than $\text{PSL}(2, q)$ of the same degree; all are ruled out by the order test except for $\text{PFU}(3, q)$ with $q = 3, 4$. Now $\text{PFU}(3, 3)$ (with degree 28) is a subgroup of $\text{Sp}(6, 2)$, considered below. $\text{PFU}(3, 4)$ is handled by computation.

Case $G = \text{P}\Gamma\text{L}(d, q)$, $d \geq 3$. Here $n = (q^d - 1)/(q - 1)$. We follow similar arguments to the affine case.

Subcase: $k \leq d$. There exist $d+1$ points, no three collinear. On the other hand, there is a k -set containing three collinear points.

Subcase: $d < k < (q^{d-1})/(q-1)$. There is a k -set containing a basis for the vector space, and a $(k+1)$ -set contained in a hyperplane.

Subcase: $(q^{d-1} - 1)/(q - 1) \leq k < n - (q + 1)$. Since a line contains $q + 1$ points and meets every hyperplane, there is a $(k+1)$ -set containing no hyperplane; but there is a k -set which contains a hyperplane.

Subcase: $k \geq n - (q + 1)$. In this case, $k > n/2$, contrary to assumption.

Case: $G = \text{Sp}(2d, 2)$, with $n = 2^{2d-1} \pm 2^{d-1}$. We start with a brief description of these groups. Let V be a vector space of dimension $2d$ over the field of two elements, and B a fixed nondegenerate alternating bilinear form on V . Let \mathcal{Q} be the set of all quadratic forms on V which polarize to B . These fall into two orbits under the action of the symplectic group $\text{Sp}(d, 2)$, of sizes $2^{2d-1} \pm 2^{d-1}$, corresponding to the two types of quadratic form, distinguished by the dimension of their maximal totally singular subspaces. The two types are designated $+$ and $-$, and the corresponding dimensions are d and $d-1$ respectively. Let \mathcal{Q}_ϵ be the set of forms of type ϵ . The symplectic group is 2-transitive on each orbit. We may assume that $d \geq 3$, since otherwise the degrees are smaller than 20.

It is readily checked from the order bound that the values of k which need to be considered satisfy $k+1 \leq |W|$, where W is a maximal totally singular subspace of the relevant quadratic form, except for $d = 3$ and type $-$ (acting on 28 points). This exceptional case can be handled by computation.

There is a ternary relation on \mathcal{Q}_ϵ preserved by the group. If Q_1, Q_2, Q_3 are three quadratic forms of the same type, then $Q_1 + Q_2 + Q_3$ is a quadratic form, which may be of the same or opposite type. Let \mathcal{T} be the set of all triples for which the sum is of the same type. It is easily checked that, for a fixed form Q , and a maximal totally singular subspace W for Q , the set $\{Q_w : w \in W\}$, where $Q_w(x) = Q(x) + B(x, w)$, is a set of $|W|$ forms, all triples of which belong to \mathcal{T} . Since $k + 1 \leq |W|$, and there exists a triple not belonging to \mathcal{T} , we see that G cannot be $(k, k + 1)$ -homogeneous.

Case: G is sporadic. The sporadic 2-transitive groups of degree greater than 20 are M_{22} and its automorphism group ($n = 22$), M_{23} ($n = 23$), M_{24} ($n = 24$), the Higman–Sims group ($n = 176$) and the Conway group Co_3 ($n = 276$). Computation handles all of these except the Conway group, which is a bit on the large side. However, the order test shows that we only need consider $k \leq 6$; the case $k = 4$ yields to computation, and the other cases cannot arise since inspection of the combinatorial object preserved by the group (a so-called “regular two-graph” see [41]) shows that there are seven substructures on five points, and so at least seven orbits on 5-sets and on 6-sets.

3. THE ANALOGUE OF THE LIVINGSTONE–WAGNER RESULT

Livingstone and Wagner [30] proved that if a group $G \leq S_n$ is k -homogeneous (for $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$), it is also $(k - 1)$ -homogeneous.

Corollary 3.1. *Let $n \geq 1$, let $3 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$, and let $G \leq S_n$ be a $(k - 1, k)$ -homogeneous group. Then G is a $(k - 2, k - 1)$ -homogeneous group, except when $n = 9$ and $G \cong \text{ASL}(2, 3)$ or $\text{AGL}(2, 3)$, with $k = 5$.*

Proof. We know that G either is $(k - 1)$ -homogeneous or is one of the five exceptions listed in Theorem 2.1. If the group is $(k - 1)$ -homogeneous, then it certainly is $(k - 2, k - 1)$ -homogeneous.

Regarding the three exceptions of degree 5 and 7, GAP shows they satisfy the corollary. Regarding the two groups of degree 9, GAP shows that they are not $(3, 4)$ -homogeneous as for both groups the orbit of $\{1, 2, 3\}$ does not contain a subset of $\{1, 2, 4, 5\}$. \square

Recall that if G is a permutation group of degree n , for $k \leq n$, we say that G possesses the k -universal transversal property if the orbit of any k -set contains a section for every k partition of $[n]$. It is clear that the class of groups the k -universal transversal property (for some $2 \leq k \leq \lfloor n/2 \rfloor$) is contained in the class of $(k - 1, k)$ -homogeneous groups (consider a k -partition with $k - 1$ singleton blocks); therefore the groups possessing the k -universal transversal property are $(k - 1)$ -homogeneous, with the exceptions listed in Theorem 2.1, and hence they have the $(k - 1)$ -universal transversal property, with the possible exception of the five exceptional groups listed in Theorem 2.1. Inspection of these groups leads to the following result.

Corollary 3.2. *Let $n \geq 5$, let $2 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$, and let $G \leq S_n$ be a group having the k -universal transversal property. Then G has the $(k - 1)$ -universal transversal property.*

If $k > \lfloor \frac{n+1}{2} \rfloor$, then no analogue of the Livingstone–Wagner Theorem can hold. But the situation is actually very simple.

Theorem 3.3. *Let G be a subgroup of S_n , and let k be an integer satisfying $\lfloor \frac{n+1}{2} \rfloor < k < n$. Then the following are equivalent:*

- (1) G has the k -universal transversal property;
- (2) G is $(k-1, k)$ -homogeneous;
- (3) G is k -homogeneous.

In particular, if these conditions hold with $k < n-5$, then G is \mathcal{S}_n or \mathcal{A}_n .

Proof. (1) implies (2): The proof of this given earlier does not depend on the value of k .

(2) implies (3): Let G be $(k-1, k)$ -homogeneous. Then G is $(n-k, n-k+1)$ -homogeneous. Because of the inequality on k , the exceptional groups in Theorem 1.11 do not occur; so G is $(n-k)$ -homogeneous, and hence k -homogeneous.

(3) implies (1): Clear. □

4. THE CLASSIFICATION OF THE GROUPS POSSESSING THE k -UNIVERSAL TRANSVERSAL PROPERTY

Let G be a permutation group of degree n . For $k \leq n$, we say that G possesses the k -universal transversal property if the orbit of any k -set contains a section for every k partition of $[n]$. A group is said to have the universal transversal property if it has the k -universal transversal property for all $k \leq n$. In [3] the following theorem is proved.

Theorem 4.1. *A subgroup G of \mathcal{S}_n has the universal transversal property if and only if one of the following is valid:*

- (i) $n = 5$ and $G \cong C_5, D_5$, or $\text{AGL}(1, 5)$;
- (ii) $n = 6$ and $G \cong \text{PSL}(2, 5)$ or $\text{PGL}(2, 5)$;
- (iii) $n = 7$ and $G \cong \text{AGL}(1, 7)$;
- (iv) $n = 8$ and $G \cong \text{PGL}(2, 7)$;
- (v) $n = 9$ and $G \cong \text{PSL}(2, 8)$ or $\text{PFL}(2, 8)$;
- (vi) $G = \mathcal{A}_n$ or \mathcal{S}_n .

The goal of the following two sections is to prove that with some exceptions, the groups possessing the k -universal transversal property are k -homogeneous (for $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$). We abbreviate “ k -universal transversal property” to k -ut property.

Our main results are Theorems 1.3–1.8 stated in the introduction, and that we now state in a single theorem.

Theorem 4.2. *Let $n \geq 11$, $G \leq \mathcal{S}_n$ and let $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$.*

- (1) *If $6 \leq k \leq \lfloor \frac{n}{2} \rfloor$, then the following are equivalent:*
 - (a) G has the k -ut property;
 - (b) $\mathcal{A}_n \leq G$.
- (2) *The following are equivalent:*
 - (a) G has the 5-ut property;
 - (b) G is 5-homogeneous, or $n = 33$ and $G = \text{PFL}(2, 32)$.
- (3) *(a) if G is 4-homogenous, or $n = 12$ and $G = \text{M}_{11}$, then G has the 4-ut property.*
 - (b) *Apart from the possible exception of some G with $\text{PSL}(2, q) \leq G \leq \text{PFL}(2, q)$, the groups listed above are the only ones having the 4-ut property.*
- (4) *(a) if G is 3-homogenous, or one of the following groups*
 - (i) $\text{PSL}(2, q) \leq G \leq \text{P}\Sigma\text{L}(2, q)$, where $q \equiv 1 \pmod{4}$;

- (ii) $\text{Sp}(2d, 2)$ with $d \geq 3$, in either of its 2-transitive representations;
- (iii) $2^{2d} : \text{Sp}(2d, 2)$;
- (iv) Higman–Sims;
- (v) Co_3 ;
- (vi) $2^6 : \text{G}_2(2)$ and its subgroup of index 2;
- (vii) $\text{AGL}(1, p)$ where, for all $c \in \text{GF}(p)^*$, $|\langle -1, c, c - 1 \rangle| = p - 1$;

then G has the 3-ut property.

- (b) Apart from the possible exception of the Suzuki groups $\text{Sz}(q)$, the groups listed above are the only ones having the 3-ut property.

(5) The following are equivalent.

- (a) G has the 2-ut property;
- (b) G is primitive.

For $n < 11$, a group $G \leq \mathcal{S}_n$ with the k -universal transversal property is k -homogeneous, with the following exceptions:

- (i) $n = 5$, C_5 or $D(2 * 5)$ and $k = 2$;
- (ii) $n = 6$, $\text{PSL}(2, 5)$ and $k = 3$;
- (iii) $n = 7$, C_7 or $D(2 * 7)$, and $k = 2$; or $\text{AGL}(1, 7)$ and $k = 3$;
- (iv) $n = 8$, $\text{PGL}(2, 7)$ and $k = 4$;
- (v) $n = 9$, $3^2 : 4$ or $3^2 : D(2 * 4)$ and $k = 2$;
- (vi) $n = 10$, \mathcal{A}_5 or \mathcal{S}_5 and $k = 2$; or $\text{PSL}(2, 9)$ or \mathcal{S}_6 and $k = 3$.

Proof. In Sections 4 and 5 all the claims for $n \geq 11$ are proved. Regarding $n < 11$, all the claims can be easily checked with GAP.

For $n = 5, 6$ all the possible groups appear in the statement of the theorem.

For $n = 7$ the group $7 : 3$ is 2-homogenous, but does not have the 3-ut property as the orbit of $\{1, 2, 7\}$ has no transversal for $\{\{1\}, \{2, 7\}, \{3, 6, 4, 5\}\}$. The 2-homogeneous group $L(3, 2)$ also does not have the 3-ut property as the orbit of $\{1, 2, 4\}$ does not contain a transversal for $\{\{1\}, \{2, 4\}, \{3, 7, 6, 5\}\}$.

For $n = 8$, the 3-homogenous groups $\text{AGL}(1, 8)$, $\text{AFL}(1, 8)$, $\text{ASL}(3, 2)$ do not have the 4-ut property. There is no section for the partition $\{\{1\}, \{2\}, \{3, 4\}, \{5, 6, 7, 8\}\}$ in the orbits of the set $\{1, 2, 3, 4\}$. The 3-homogenous group $\text{PSL}(2, 7)$ does not have the 4-ut property as the orbit of $\{1, 2, 3, 5\}$ has no transversal for the partition $\{\{1, 5, 7, 8\}, \{2, 6\}, \{3\}, \{4\}\}$.

For $n = 9$, the 2-homogenous groups M_9 , $\text{AGL}(1, 9)$, $\text{AFL}(1, 9)$, $\text{ASL}(2, 3)$, $\text{AGL}(2, 3)$ do not have the 3-ut property. Their orbits on the set $\{1, 2, 3\}$ contain no section for the partition $\{\{1\}, \{2, 3\}, \{4, 5, 6, 7, 8, 9\}\}$.

For $n = 10$ the 3-homogeneous groups $\text{PGL}(2, 9)$, M_{10} , $\text{PFL}(2, 9)$ do not have the 4-ut property as the partition $\{\{1\}, \{2\}, \{3, 10\}, \{4, 9, 8, 6, 7, 5\}\}$ has no transversal in the orbit of $\{1, 2, 3, 10\}$. \square

Proposition 4.3. (1) A k -homogeneous group has the k -ut property.

(2) A group with the k -ut property is $(k - 1, k)$ -homogeneous, and hence is $(k - 1)$ -homogeneous or one of the five exceptions to Theorem 1.11.

(3) If $k > n/2$, then a group has the k -ut property if and only if it is k -homogeneous or one of the exceptions to Theorem 1.11.

The first part is trivial; the second is contained in the preamble to Theorem 1.10 and the result of Theorem 1.11; the third is contained in the preamble to Theorem 2.1.

Our aim is to determine, as completely as possible, the groups with the k -ut property which are not k -homogeneous. For $k = 2$, there are many such groups and no hope of a determination:

Proposition 4.4. *A subgroup of \mathcal{S}_n has the 2-ut property if and only if it is primitive.*

Proof. By Higman's Theorem [18], G is primitive if and only if all the non-diagonal orbital graphs are connected. (These are just the graphs whose edge sets are the orbits on 2-sets.) But a graph is connected if and only if, for every 2-partition of the vertices, there is an edge which is a section for the partition. \square

However, for $k > 2$, we are in a stronger position, due to the following pair of results, one negative, one positive:

Proposition 4.5. *If G is a group of automorphisms of a Steiner system $S(k-1, l, n)$ with $k-1 < l < n$, then G does not have the k -ut property.*

(A Steiner system is a collection of blocks or subsets of size l of the n -set so that each $(k-1)$ -set is contained in a unique block.)

Proof. Take the partition with $k-2$ singleton parts, one part of size $l-k+2$ consisting of the remaining points in some block containing these points, and one part consisting of everything else. A k -set which is contained in a block cannot be a section for this partition. \square

This shows that the following $(k-1)$ -homogeneous groups do not have the k -ut property:

- (1) Subgroups of $\text{AGL}(d, p)$, if $d > 1$ and $p > 2$, with $k = 3$ (these groups preserve the geometry of affine points and lines).
- (2) 3-transitive subgroups of $\text{AGL}(d, 2)$ for $d > 2$, with $k = 4$ (these preserve geometry of affine points and planes).
- (3) Subgroups of $\text{P}\Gamma\text{L}(d, q)$ with $d > 2$, for $k = 3$ (these preserve the geometry of projective points and lines).
- (4) The unitary and Ree groups, with $k = 3$ (these preserve unitals).
- (5) Subgroups of $\text{P}\Gamma\text{L}(2, q)$ containing $\text{PSL}(2, q)$, where q is a proper power of an odd prime or $q = 2^e$ where e is not prime, with $k = 4$ (these preserve circle geometries). Note that we can exclude subgroups of $\text{P}\Sigma\text{L}(2, q)$ containing $\text{PSL}(2, q)$ for $q \equiv 1 \pmod{4}$, since these groups are not 3-homogeneous.
- (6) The Mathieu groups M_{11} , M_{12} , M_{22} (and $\text{Aut}(M_{22})$), M_{23} and M_{24} in their usual representations, with $k = 5, 6, 4, 5, 6$, respectively, as these preserve famous Steiner systems.

A permutation group G is k -primitive if it is $(k-1)$ -transitive and the pointwise stabiliser of $k-1$ points acts primitively on the remaining points. It is *generously k -transitive* if the setwise stabiliser of $k+1$ points induces the symmetric group on these points, and is *almost generously k -transitive* if the setwise stabiliser of $k+1$ points induces the symmetric or alternating group on them (Neumann [33]).

Lemma 4.6. *If a permutation group G is k -primitive and almost generously k -transitive, then every orbit of G on $(k+1)$ -sets contains a section for any $(k+1)$ -partition of which $k-1$ of the parts are singletons.*

Proof. In an almost generously k -transitive group, each orbit on $(k+1)$ -sets corresponds in a natural way to an orbit of the $(k-1)$ -point stabiliser on pairs of points outside the given $k-1$ points. Now the proof concludes as in Proposition 4.4. \square

Proposition 4.7. *Each of the following 2-transitive groups G has the 3-ut property:*

- (1) $\text{PSL}(2, q) \leq G \leq \text{P}\Sigma\text{L}(2, q)$, where $q \equiv 1 \pmod{4}$;
- (2) $\text{Sp}(2d, 2)$ with $d \geq 3$, in either of its 2-transitive representations;
- (3) $2^{2d} : \text{Sp}(2d, 2)$;
- (4) Co_3 .

Proof. Each of these groups is 2-primitive and generously 2-transitive, so by the previous lemma it is enough to consider partitions $\{X, Y, Z\}$ in which no part is a singleton. We assume that X is the smallest part.

Also, each of these groups has just two orbits on 3-sets, and each orbit T is a *regular two-graph* (Taylor [41]), that is,

- any two points lie in exactly λ members of T , for some λ ;
- any four points contain an even number of members of T .

Now suppose that the orbit T contains no section for the partition $\{X, Y, Z\}$. For any $x_1, x_2 \in X$, $y \in Y$ and $z \in Z$, we have $x_1yz, x_2yz \notin T$, and so both or neither of x_1x_2y and x_1x_2z belong to T . Suppose that $x_1x_2y \in T$, for some $y \in Y$. Then $x_1x_2z \in T$ for all $z \in Z$, and $x_1x_2y \in T$ for all $y \in Y$. Hence we have $\lambda \geq |Y| + |Z| \geq 2n/3$. In the contrary case, neither of these triples belong to T , and $\lambda \leq |X| - 2 \leq n/3 - 2$.

However, for these groups, it is easily checked that these inequalities fail in all cases:

- for $\text{PSL}(2, q)$, $n = q + 1$, $\lambda = (q - 1)/2$;
- for $\text{Sp}(2d, 2)$, $n = 2^{2d-1} \pm 2^{d-1}$, $\lambda = 2^{2d-2}$ or $\lambda = 2^{2d-2} \pm 2^{d-1} - 2$;
- for $2^{2d} : \text{Sp}(2d, 2)$, $n = 2^{2d}$, $\lambda = 2^{2d-1}$ or $\lambda = 2^{2d-1} - 2$;
- for Co_3 , $n = 276$, $\lambda = 112$ or $\lambda = 162$.

\square

These results, together with computation for $n \leq 11$, resolve the question of the k -ut property for all groups G which are $(k-1)$ -homogeneous but not k -homogeneous, but for the following exceptional cases:

- (1) $k = 3$:
 - (a) $\text{AGL}(1, p)$, p prime, or its subgroup of index 2 if $p \equiv 3 \pmod{4}$;
 - (b) $2^6 : \text{G}_2(2)$ and its subgroup of index 2;
 - (c) $\text{Sz}(q)$;
 - (d) The Higman–Sims group.
- (2) $k = 4$:
 - (a) $\text{PSL}(2, q) \leq G \leq \text{P}\Gamma\text{L}(2, q)$, with either q prime (except $\text{PSL}(2, q)$ for $q \equiv 1 \pmod{4}$, which is not 3-homogeneous), or $q = 2^p$ for p prime;
 - (b) M_{11} , degree 12.
- (3) $k = 5$:
 - (a) $\text{P}\Gamma\text{L}(2, 32)$, degree 33.

5. THE EXCEPTIONAL CASES

In this section we are going to look at the exceptional cases listed at the end of the previous section.

5.1. The group M_{11} has the 4-ut property. The group M_{11} of degree 12 can easily be handled with GAP. This group has two orbits on 4-sets (in GAP are the orbits of $\{1, 2, 3, 4\}$ and of $\{1, 2, 3, 7\}$). GAP checks in less than one minute that the orbit of each one of these sets contains a section for all possible 4-partitions of $\{1, \dots, 12\}$.

5.2. Some general results. In this subsection we are going to prove a number of auxiliary results. We start by associating a graph to a t -homogeneous group $G \leq \mathcal{S}_n$ as follows. Let $B \subseteq [n]$ and $c \in [n]$ such that $|B| = t - 1$. Then we define the following graph on the points in $[n] \setminus B$:

$$G(B, c) = \{\{x, y\} \mid \{x, y\} \cup B \in (\{1, \dots, t, c\}G)\}.$$

In the particular case of $B = \{b\}$, we will write $G(b, c)$ rather than $G(\{b\}, c)$.

Observe that for every $g \in G$ we have $G(B, c) \cong G(Bg, c)$. Therefore, as t is larger than the order of B , it follows that $G(B, c)$ is connected if and only if $G(B', c)$ is connected, for all B' such that $|B'| = |B| = t - 1$.

Proposition 5.1. *If a t -homogeneous group $G \leq \mathcal{S}_n$ has the $(t + 1)$ -ut property, then $G(B, c)$ is connected, for all $B \subset [n]$ (with $|B| = t - 1$) and all $c \in [n]$.*

Proof. If $G(B, c)$ is not connected, then there exists a connected component D contained in the graph. Now consider the partition $P = (\{b_1\}, \dots, \{b_{t-1}\}, D, R)$, where R contains the remaining elements, that is, $R = [n] \setminus (B \cup D)$, and $B = \{b_1, \dots, b_{t-1}\}$.

Any section for the partition P must contain B . Therefore any set containing B and in the orbit of $\{1, \dots, t, c\}$ must be of the form $\{x, y\} \cup B$ and hence x and y are connected in $G(B, c)$. Thus, either $x \in D$ and hence $y \in D$ (because D is a connected component of $G(B, c)$) so that $\{x, y\} \cup B$ is not a section for P ; or $x \notin D$ and hence $y \notin D$ thus implying $x, y \in R$. Again $\{x, y\} \cup B$ is not a section for P . The result follows. \square

This proposition immediately implies that, for example, the 2-homogeneous group $G = \text{AGL}(1, 17)$ does not have the 3-ut property.

In fact, according to GAP, the graph $G(\{17\}, 4)$ has the following two connected components $\{\{1, 2, 3, 4, 6, 9, 10, 15\}, \{5, 7, 8, 11, 12, 13, 14, 16\}\}$. And it can be checked, in fact, that the orbit of $\{1, 2, 4\}$ under G has no section for the partition

$$P = (\{1, 2, 3, 4, 6, 9, 10, 15\}, \{5, 7, 8, 11, 12, 13, 14, 16\}, \{17\}).$$

(More on these groups below.)

For the particular case of the 3-ut property, another important graph is the following: for a set $C \subseteq [n]$ and $c \in [n]$, we have

$$\Gamma(C, c) = \bigcup_{b \in C} (G(b, c) \cap ([n] \setminus C) \times ([n] \setminus C)).$$

Let G be a group admitting a bad 3-partition, that is, $P = (A, C, A')$ such that no set in the orbit of $\{1, 2, c\}$ is a section for P . This means that the distance from A to A' in the graph $\Gamma(C, c)$ must be infinite. In fact, if it is not infinite, it must be one as every vertex in this graph either is on A or in A' . That means that there exist $a \in A$ and $a' \in A'$ that are

connected in $\Gamma(C, c)$. But, by definition, Γ is a union of subgraphs of $G(b, c)$ and hence it follows that for some $b \in C$ we have that $\{a, a'\}$ is an edge in $G(b, c)$. Thus $b \in C$, $a \in A$, $a' \in A'$ and hence $\{a, b, a'\}$ is a section for P that belongs to the orbit of $\{1, 2, c\}$, by the definition of $G(b, c)$. It is proved that bad partitions $P = (A, C, A')$ induce graphs $\Gamma(C, c)$ in which the distance from A to A' is infinite.

This observation leads to the following procedure that (if it ends) allows to check that a group G has the 3-ut property. We already know that if $G(n, c)$ is disconnected, then the group does not have the 3-ut property. The question is whether there exists a group G with connected graph $G(n, c)$, but that does not have the k -ut property. Therefore we assume that $G(n, c)$ is connected and start with three sets (A, C, A') , all contained in $[n]$, such that (without loss of generality because we only consider 2-homogeneous transitive groups) $\{1\} = A$ and $\{n\} = C$, where n is the degree of the group G . We are going to try to build a bad partition and hence include in $A \cup A'$ and C all the elements that must necessarily be in each one of this sets provided that we want $\Gamma(C, c)$ to be disconnected on $A \cup A'$. So we proceed as follows (denote the distance from a to b in graph Gr by $D_{Gr}(a, b)$): for a fixed $d \in [n]$ that will be the distance in $G(n, c)$ from A to A' ,

- (1) put in A' an element $y \in [n]$ such that $D_{G(n, c)}(1, y) = d$; add to $A \cup A'$ the set $\{x \in [n] \mid \{1, y\} \in G(x, c)\}$. (Observe that these x must be in $A \cup A'$ because if one of them is in C , then $\{1, y, x\}$ would be a section for (A, C, A') and hence any oversets of A, A', C yielding a partition of $[n]$ would have a section in the orbit of $\{1, 2, c\}$.)
- (2) add to C the set $\{x \in [n] \mid 1 - \dots - x - \dots - y\}$, where $a - b$ means that $\{a, b\}$ is an edge in $G(n, c)$. (Observe that if one of these x is not in C , then $D_{G(n, c)}(A, B) < d$, contrary to our assumption.)
- (3) check if $A \cup A'$ is connected under $\Gamma(C, c)$; if it is connected, then all the overpartitions of (A, C, A') are good; if it is not connected, then
- (4) add to $A \cup A'$ the set $\{x \in [n] \mid A \cup A' \text{ is connected in } \Gamma(C \cup \{x\}, n)\}$ (as such x cannot go to C).
- (5) $A \cup C \supseteq \{x \mid D_{G(n, c)}(A, x) < d\}$ and $A' \cup C \supseteq \{x \mid D_{G(n, c)}(A', x) < d\}$;
- (6) add to A the set $(A \cup C) \cap (A \cup A')$.
- (7) add to A' the set $(A' \cup C) \cap (A \cup A')$.
- (8) add to C the set $(A \cup C) \cap (C \cup A')$.
- (9) go to (3).

5.3. The two exceptional groups of degree 64. The group $H = 2^6 : G_2(2)$ and its subgroup G , of index 2, have $G(64, c)$ connected. Therefore the guess is that they have the 3-ut property. To test that conjecture we are going to apply to G the procedure outlined at the end of the previous subsection, as if G has the 3-ut property, then the overgroup H also has. On the set $[64]$ the group G has three orbits on 3-sets, namely (in GAP) $\{1, 2, 3\}$, $\{1, 2, 5\}$ and $\{1, 2, 29\}$.

The points y such that $D_{G(64, 3)}(1, y) = 2$ are

$$\{2, 3, 4, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 27, 28, 37, 38, 39, 40, 41, 42, 43, 44, 49, 50, 51, 52, 61, 62, 63\}.$$

To all of them the procedure ends yielding the result that at a certain point A and A' are connected under $\Gamma(C, 3)$. And there are no y such that $D_{G(64, 3)}(1, y) > 2$.

The points y such that $D_{G(64,5)}(1, y) = 2$ are

$$\{5, 6, 7, 8, 9, 10, 11, 12, 15, 17, 18, 19, 20, 23, 25, 29, 30, 31, 32, 33, 34, \\ 35, 36, 40, 42, 45, 46, 47, 48, 50, 53, 54, 55, 56, 57, 58, 59, 60\}.$$

To all of them the procedure ends yielding the result that at a certain point A and A' are connected under $\Gamma(C, 5)$. And there are no y such that $D_{G(64,5)}(1, y) > 2$.

Regarding $\{1, 2, 29\}$ (the 3-set with the smallest orbit), the points y such that $D_{G(64,29)}(1, y) = 2$ are

$$\{2, 3, 4, 13, 14, 16, 21, 22, 24, 26, 27, 28, 37, 38, 39, 41, 43, 44, 49, 51, 52, 61, 62, 63\}.$$

To all of them the procedure ends yielding the result that at a certain point A and A' are connected under $\Gamma(C, 29)$. In this case there are also some y such that $D_{G(64,29)}(1, y) = 3$:

$$\{5, 6, 7, 8, 9, 10, 11, 12, 17, 18, 19, 20, 29, 30, 31, 32, 33, 34, 35, 36, 45, 46, 47, 48, 53, 54, 55, 56, 57, 58, 59, 60\}.$$

To all of them the procedure ends yielding the result that at a certain point A and A' are connected under $\Gamma(C, 29)$. And there are no y such that $D_{G(64,29)}(1, y) > 3$.

It is checked that G has the 3-ut property and hence the same holds for H .

5.4. The group $\text{P}\Gamma\text{L}(2, 32)$. The graph $G(\{1, 2, 3\}, c)$ is connected and hence the guess is that this group has the 5-ut property. It is too big to be tested directly and the algorithm used in subsection 5.3 does not work here. Therefore we used the following algorithm (in GAP) to prove that indeed this group has the 5-ut property.

G has three orbits on 5-sets: $\{1, \dots, 5\}$, $\{1, \dots, 4, 6\}$ and $\{1, \dots, 4, 10\}$ are representatives. Suppose we want to prove that $\{1, \dots, 4, 10\}G$ contains a section for all partitions. To do that we are going to try to build a bad partition (one that has no section in $\{1, \dots, 4, 10\}G$).

- (1) We start with the subpartition $p := \{\{1\}, \{2\}, \{3\}, \{4\}, \{6\}\}$ and add the number 5 (the smallest not in $\{1, 2, 3, 4, 6\}$) to the blocks of p in the five possible ways. We get 5 subpartitions and remove all the partitions that have a section in $\{1, \dots, 4, 10\}G$. According to GAP all the 5 are left.
- (2) Repeat the previous step with 7 being included in each one of the previous five subpartitions, and then removing the ones that have a section in $\{1, \dots, 4, 10\}G$. According to GAP all the 25 are left.
- (3) Repeating with 8, we end up with 115 subpartitions.
- (4) With 9, we get 337 subpartitions.
- (5) Repeating with 10, \dots , 19, we get, respectively, the following number of subpartitions

$$719, 1052, 1065, 1193, 912, 229, 211, 137, 50, 2.$$

And it does not matter where we put 20, we end always with a partition admitting a section in $\{1, \dots, 4, 10\}G$. This proves that $\{1, \dots, 4, 10\}G$ contains a section for all the partitions in which 1, 2, 3, 4, 6 are all in different blocks.

- (6) Then we start with a subpartition $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$ and the orbit $\{1, \dots, 4, 10\}G$, and follow the previous algorithm. Again we get that $\{1, \dots, 4, 10\}G$ contains a section for all 5-partitions (in which 1, 2, 3, 4, 5 are in different blocks). This proves that $\{1, \dots, 4, 10\}G$ contains a section for all the 5-partitions since there is only another orbit of partitions: those in which 1, 2, 3, 4, 10 are in different blocks and for those $\{1, \dots, 4, 10\}G$ trivially contains a section.
- (7) Finally we repeat the same algorithm with $\{1, \dots, 5\}G$ and $\{1, \dots, 4, 6\}G$. The worse case is when $P := \{\{1\}, \{2\}, \{3\}, \{4\}, \{6\}\}$ and we have the orbit of $\{1, \dots, 4, 10\}$.

5.5. The Higman–Sims group. Let G be the Higman–Sims group, a group of order $2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$ and degree 176. Then G has three orbits O_1, O_2, O_3 on ordered triples, with cardinalities $176 \cdot 175 \cdot t$, for $t = 12, 72, 90$; representatives of the orbits are $(1, 2, 16)$, $(1, 2, 3)$ and $(1, 2, 6)$ respectively.

According to Taylor [41], O_2 is a regular two-graph with $\lambda = 72$, and exactly the same argument as in Proposition 4.7 shows that every 3-partition has a section belonging to the orbit O_2 . So we only have to deal with the orbits O_1 and O_3 .

Moreover, every 3-partition is equivalent under G to one with 1, 2, 3 in different parts; so we started with the subpartition $\{\{1\}, \{2\}, \{3\}\}$ and applied the same algorithm used in the preceding subsection, concluding that this group has the 3-ut property.

5.6. The groups $\text{AGL}(1, p)$ for p prime. We proved above that if the graph $G(B, c)$ is not connected for some B and c , then G does not have the k -ut property, for $k = |B| + 2$. Unfortunately, the groups $\text{AGL}(1, p)$ have disconnected graphs for some p , and connected graphs for other p . Therefore we need sharper results. The aim of this section is to prove them.

We start by providing a characterization of connectedness in this setting. We denote by $\text{GF}(p)$ the field with p elements and by $\text{GF}(p)^*$ its non-zero elements.

Proposition 5.2. *Let $G = \text{AGL}(1, p)$, with p prime. If $G(0, c)$ is not connected then $H := \langle -1, c, c-1 \rangle \subset \text{GF}(p)^*$. In such a case, the orbit of $\{0, 1, c\}$ under $\text{AGL}(1, p)$ has no section for the partition $\{\{0\}, H, \text{rest}\}$.*

Proof. Observe that $\{b, x, y\}$ is in the orbit of $\{0, 1, c\}$ if and only if

$$\{x, y\} \in \{\{\alpha + b, c\alpha + b\}, \{b - \alpha, (c-1)\alpha + b\}, \{(1-c)\alpha + b, b - c\alpha\} \mid \alpha \in \text{GF}(p)^*\}.$$

In particular, for $b = 0$, we have that

$$\{x, y\} \in \{\{\alpha, c\alpha\}, \{-\alpha, (c-1)\alpha\}, \{(1-c)\alpha, -c\alpha\} \mid \alpha \in \text{GF}(p)^*\}.$$

Now if $G(0, c)$ is not connected, then there exists a set $A \subset \text{GF}(p)^*$ such that for every $\{x, y\} \in G(0, c)$ we have

$$x \in A \Leftrightarrow y \in A.$$

In particular, for $\{x, y\} = \{\alpha, c\alpha\}$ we have $\alpha \in A \Leftrightarrow c\alpha \in A$, that is, $A = Ac$. In the same way we get the conditions $A = -A(c-1)$ and $Ac = A(c-1)$. Collecting these three conditions we get that:

$$A = Ac = A(c-1) = -A.$$

Now, for $c_i \in \{-1, c, c-1\}$, we have $A = Ac_1 = (Ac_2)c_1 = \dots = Ac_n \dots c_1$ and hence $A = A\langle -1, c, c-1 \rangle$. Clearly, saying that *there exists a proper subset $A \subset \text{GF}(p)^*$ such that $A = -A = Ac = A(c-1)$* is equivalent to saying that *the group $\langle -1, c, c-1 \rangle$ is strictly contained in $\text{GF}(p)^*$* . The first claim follows.

Regarding the second claim, any section for the partition must have the form $\{0, c_1, r\}$, with $c_1 \in H$, and hence we must have $\{c_1, r\} \in G(0, c)$. As we saw above, this means that

$$\{c_1, r\} \in \{\{\alpha, c\alpha\}, \{-\alpha, (c-1)\alpha\}, \{(1-c)\alpha, -c\alpha\} \mid \alpha \in \text{GF}(p)^*\}.$$

Checking all the possibilities always leads to the conclusion that $r \in H$. □

The next result is our main result regarding the groups $\text{AGL}(1, p)$.

Theorem 5.3. *Let p be a prime and let $c \in \text{GF}(p) \setminus \{0, 1\}$. Then the following are equivalent:*

- (1) *the orbit of $\{0, 1, c\}$ under $\text{AGL}(1, p)$ contains a section for all the 3-partitions of $\{0, \dots, p\}$;*
- (2) $|\langle -1, c, c-1 \rangle| = p-1$.

Proof. We already proved that if $|\langle -1, c, c-1 \rangle| < p-1$, then there exists a 3-partition P such that no set in the orbit of $\{0, 1, c\}$ (under $\text{AGL}(1, p)$) is a section for P .

Conversely, suppose that $|\langle -1, c, c-1 \rangle| = p-1$ and suppose that there exists a bad partition $P = (A, C, A')$. This implies that $\Gamma(C, c)$ is not connected, that is, for all $\{x, y\} \in \Gamma(C, c)$, either $x, y \in A$ or $x, y \in A'$. As $\Gamma(C, c)$ is a union of graphs, this means that for all $b \in C$, if $\{x, y\} \in G(b, c)$, then

$$x \in A \Leftrightarrow y \in A.$$

Now, repeating the arguments in the previous result we observe that this last condition is equivalent to saying that

$$\begin{aligned} (\forall b \in C, \alpha \in \text{GF}(p)^*) \quad \alpha \in A - b &\Leftrightarrow c\alpha \in A - b; \\ (\forall b \in C, \alpha \in \text{GF}(p)^*) \quad -\alpha \in A - b &\Leftrightarrow (c-1)\alpha \in A - b; \\ (\forall b \in C, \alpha \in \text{GF}(p)^*) \quad (1-c)\alpha \in A - b &\Leftrightarrow -c\alpha \in A - b. \end{aligned}$$

The first equivalence implies that $(A - b) = (A - b)c$, that is, $B = Bc$, for $B = A - b$; the second implies $B = B(c-1)$, and the last implies $B = -Bc^{-1}(1-c)^{-1}$. All these three together imply $B = Bc = B(c-1) = -B$. We already proved that this implies $|B| \geq |\langle -1, c, c-1 \rangle| = p-1$, and clearly $|B| = |A|$. A contradiction since in $P = (A, C, A')$ the set A cannot have $p-1$ elements. \square

If $p \equiv 1 \pmod{3}$ and $p > 7$, then we can take c to be a primitive 6th root of the unity; then $c^2 = c-1$, so $\langle c, c-1, -1 \rangle$ is a subgroup of order 6. Thus $\text{AGL}(1, p)$ does not have the 3-ut property if $p \equiv 1 \pmod{3}$ and $p > 7$.

Also, if $p \equiv 1 \pmod{4}$ and $p > 5$, then there are consecutive quadratic residues in $\{1, \dots, p-1\}$; if c is the larger of such a pair, then $\langle c, c-1, -1 \rangle$ is contained in the subgroup of squares and again $\text{AGL}(1, p)$ does not have the 3-ut property. (If no two consecutive residues exist, then as 1 and $p-1$ are residues, we see that residues and non-residues must alternate, apart from one pair of non-consecutive non-residues. But consecutive integer squares in $\{1, \dots, p-1\}$ are an odd distance apart, and so there must be consecutive non-residues between them. So there are only two such squares, namely 1 and 4 and so $p = 5$.)

Thus, only for primes $p \equiv 11 \pmod{12}$ is the question undecided.

We have not so far considered the subgroup of index 2 in $\text{AGL}(1, p)$, which is 2-homogeneous for $p \equiv 3 \pmod{4}$. But if $p \equiv 7 \pmod{12}$, then $\text{AGL}(1, p)$ does not have the 3-ut property, and neither does its subgroup. So these are also undecided only for $p \equiv 11 \pmod{12}$.

5.7. The groups $\text{PSL}(2, q)$. Regarding the groups $\text{PSL}(2, q) \leq G \leq \text{P}\Gamma\text{L}(2, q)$, with either q prime (with the exception of $\text{PSL}(2, q)$ for $q \equiv 1 \pmod{4}$, which are not 3-homogeneous), or $q = 2^p$ for p prime, we have the following:

- (1) Suppose p is such that for some $c \in \text{GF}(p)^*$ we have that $\langle -1, c, c-1 \rangle$ is a proper subgroup of $\text{GF}(p)^*$. Then there exists a 3-partition $P = (A, B, C)$ such that the orbit of $\{0, 1, c\}$ under $\text{AGL}(1, p)$ has no section for P . Therefore the partition $(\{\infty\}, A, B, C)$ has no section in the orbit of $\{\infty, 0, 1, c\}$ under $\text{PGL}(2, p)$. This

follows from the fact that any set in this orbit containing ∞ is of the form $\{\infty\} \cup D$ where D is a 3-set in the orbit of $\{0, 1, c\}$ under $\text{AGL}(1, p)$.

- (2) By the previous observation, it follows that, for the same p , G does not have the 4-ut property for all $G \leq \text{PGL}(2, p)$.
- (3) for the case $\text{PSL}(2, q)$, where $q = 2^p$, we have $\text{PSL}(2, q) = \text{SL}(2, q)$.
- (4) Below we have all the edges of $G(\{0, 1\}, c)$.

$$\begin{aligned} & \left\{ \frac{c\alpha^2}{c\alpha^2-c+1}, 2 \frac{\alpha^2}{2\alpha^2-1} \right\} \left\{ 2 \frac{\alpha^2}{2\alpha^2+1}, 2 \frac{c\alpha^2}{2c\alpha^2-c+2} \right\} \left\{ \frac{c\alpha^2}{c\alpha^2-1+c}, 2 \frac{c\alpha^2}{2c\alpha^2-2+c} \right\} \\ & \left\{ \frac{\alpha^2(c^2+2-3c)}{-3c\alpha^2+2\alpha^2+c^2\alpha^2-1}, \frac{c\alpha^2(c-1)}{1-c\alpha^2+c^2\alpha^2} \right\} \left\{ \frac{\alpha^2(c^2+2-3c)}{-3c\alpha^2+2\alpha^2+c^2\alpha^2+1}, \frac{c\alpha^2(c-2)}{2-2c\alpha^2+c^2\alpha^2} \right\} \\ & \left\{ \frac{c\alpha^2(c-1)}{-c\alpha^2-1+c^2\alpha^2}, \frac{c\alpha^2(c-2)}{c^2\alpha^2-2c\alpha^2-2} \right\} \left\{ \frac{\beta^2(c-1)}{c\beta^2-\beta^2-2+c}, \frac{\beta^2(c-1)}{c\beta^2-\beta^2-c} \right\} \\ & \left\{ \frac{\beta^2(c-1)}{c\beta^2-c-\beta^2+2}, \frac{\beta^2}{\beta^2-2} \right\} \left\{ \frac{\beta^2(c-1)}{c\beta^2+c-\beta^2}, \frac{\beta^2}{\beta^2+2} \right\} \left\{ \frac{\beta^2(c-2)}{c\beta^2+4c-2\beta^2-4}, \frac{\beta^2}{\beta^2+2} \right\} \\ & \left\{ \frac{\beta^2(c-2)}{c\beta^2-2\beta^2+4-4c}, \frac{\beta^2(c-2)}{c\beta^2-2\beta^2-2c} \right\} \left\{ \frac{\beta^2}{\beta^2-2}, \frac{\beta^2(c-2)}{c\beta^2+2c-2\beta^2} \right\} \end{aligned}$$

But it does not seem clear how an approach similar to the one used in $\text{AGL}(1, p)$ can be carried out here...

5.8. The state of the art. Regarding our list of exceptional cases the situation is the following:

- (1) $k = 3$:
 - (a) regarding $\text{AGL}(1, p)$, with p prime, or its subgroup of index 2 if $p \equiv 3 \pmod{4}$, we have:
 - (i) $\text{AGL}(1, p)$, or (if $p \equiv 3 \pmod{4}$) its subgroup of index 2, does not have the 3-ut property, unless possibly when $p \equiv 11 \pmod{12}$;
 - (ii) in general $\text{AGL}(1, p)$ does not have the 3-ut property if and only if there exists $c \in \text{GF}(p) \setminus \{0, 1\}$ such that $|\langle c, c-1, -1 \rangle| < p-1$.
 - (b) The group $2^6 : \text{G}_2(2)$ has the 3-ut property; the same happens to its subgroup of index 2.
 - (c) The groups $\text{Sz}(q)$ appear to have connected $G(b, c)$ and hence, probably, each one of them has the 3-ut property.
 - (d) The Higman–Sims group has the 3-ut property.
- (2) $k = 4$:
 - (a) For the groups $\text{PSL}(2, q) \leq G \leq \text{PTL}(2, q)$, with either q prime, or $q = 2^p$ for p prime, the situation is this:
 - (i) if q is prime and there exists $c \in \text{GF}(p)^*$ such that $|\langle c, c-1, -1 \rangle| < q$, then $G \leq \text{PGL}(2, q)$ does not have the 4-ut property.
 - (ii) for $q \equiv 1 \pmod{4}$ the group $\text{PSL}(2, q)$ is not 3-homogeneous.
 - (iii) what happens in the other groups is undecided.
 - (b) M_{11} , degree 12, has the 4-ut property.
- (3) $k = 5$:
 - (a) $\text{PTL}(2, 32)$, degree 33, has the 5-ut property.

6. REGULAR SEMIGROUPS

Arguably, three of the most important classes of semigroups are groups, inverse semigroups and regular semigroups, defined as follows: for a semigroup S we have that

- S is a group if for all $a \in S$ there exists a unique $b \in S$ such that $a = aba$;
- S is inverse if for all $a \in S$ there exists a unique $b \in S$ such that $a = aba$ and $b = bab$;
- S is regular if for all $a \in S$ there exists $b \in S$ such that $a = aba$.

Recall from the introduction that to a large extent semigroup structure theory is (almost) all about trying to show how the idempotents shape the structure of the semigroup. Therefore it is no surprise that groups and inverse semigroups can be characterized by their idempotents:

- a semigroup is inverse if and only if it is regular and the idempotents commute (see [19, Theorem 5.5.1]);
- a semigroup is a group if and only if it is regular and contains exactly one idempotent (see [19, Ex. 3.11]).

Inverse semigroups, apart from being the class of (non-group) semigroups with the largest number of books dedicated to them, were introduced by geometers and they keep being very important to them [35].

The full transformation semigroup $T(X)$ is regular, and all regular semigroups embed in some $T(X)$; every group embeds in some $T(X)$ as a group of permutations; and every inverse semigroup embeds in some $T(X)$ as an inverse semigroup of quasi-permutations, that is, transformations in which all but one of the kernel classes are singletons. (This follows from the Vagner–Preston representation [19, Theorem 5.1.7] that maps every inverse semigroup into an isomorphic semigroup of partial bijections on a set; and every partial bijection f on X can be extended to a quasi-permutation \bar{f} on $X \cup \{\infty\}$, defining $x\bar{f} = \infty$, for all x not in the domain of f , and $x\bar{f} = xf$ elsewhere, yielding a semigroup of full quasi-permutations isomorphic to the original one.)

In the introduction we provided the classification of the groups $G \leq \mathcal{S}_n$ such that the semigroup generated by G and any map $a \in \mathcal{T}_n$ is regular. Our aim now is to dramatically improve that result by extending it to quasi-permutations and transformations of a given rank. The main observation is the following straightforward lemma.

Lemma 6.1. *Let $a \in \mathcal{T}_n$ and let $G \leq \mathcal{S}_n$. Then a is regular in $\langle a, G \rangle$ if and only if there exists $g \in G$ such that $\text{rank}(aga) = \text{rank}(a)$.*

Proof. Suppose that a is regular in $\langle a, G \rangle$. Then there exists $b \in \langle a, G \rangle$ such that $a = aba$. As $\text{rank}(uv) \leq \min\{\text{rank}(u), \text{rank}(v)\}$ it follows that $\text{rank}(b) \geq \text{rank}(a)$. Now, either $b \in G$ and the result follows, or $b = g_1ag_2 \dots g_mag_{m+1}$ and $\text{rank}(b) \leq \text{rank}(a)$, that is, $\text{rank}(a) = \text{rank}(b)$. Therefore, for every $g_i \in \{g_2, \dots, g_m\}$ we have $\text{rank}(ag_ia) = \text{rank}(a)$. It is proved that there exists $g \in G$ such that $\text{rank}(aga) = \text{rank}(a)$.

Conversely, if $\text{rank}(aga) = \text{rank}(a)$, then $[n]ag$ is a transversal of $\text{Ker}(a)$ and hence ga permutes $[n]a$. Therefore, for some natural m , $(ga)^m$ acts on $[n]a$ as the identity and hence $a(ga)^m = a$. The lemma follows. \square

In [24] a stronger version of the previous result is proved.

Theorem 6.2. ([24, Theorem 2.3 and Corollary 2.4]) *Let $G \leq \mathcal{S}_n$ and let $a \in \mathcal{T}_n$. Then the following are equivalent:*

- (1) *there exists $g \in G$ such that $\text{rank}(aga) = \text{rank}(a)$;*
- (2) *a is regular in $\langle G, a \rangle$;*
- (3) *every $b \in \langle G, a \rangle$, such that $\text{rank}(b) = \text{rank}(a)$, is regular in $\langle G, a \rangle$.*

With the new tools developed in the previous sections we can now prove our first main theorem regarding regularity of semigroups generated by a group and a quasi-permutation. Recall that $a \in \mathcal{T}_n$ is a quasi-permutation if all, but one, of the $\text{Ker}(a)$ -classes have one element.

Theorem 6.3. *Let $G \leq \mathcal{S}_n$ and let $1 < k < n$. Then the following are equivalent*

- (1) *every quasi-permutation a , such that $\text{rank}(a) = k$, is regular in $\langle G, a \rangle$;*
- (2) *G is $(k-1)$ -homogeneous or G is one of the following groups*
 - (a) *$n = 5$ and $G \cong C_5, D_5$, with $k = 2$;*
 - (b) *$n = 7$ and $G \cong \text{AGL}(1, 7)$, with $k = 3$;*
 - (c) *$n = 9$ and $G \cong \text{ASL}(2, 3)$ or $\text{AGL}(2, 3)$, with $k = 4$.*

Proof. Every quasi-permutation a of rank k has a kernel of the form

$$(\{a_1\}, \dots, \{a_{k-1}\}, \{a_k, \dots, a_n\})$$

and has image $\{b_1, \dots, b_k\}$. By the previous lemma we know that a is going to be regular in $\langle a, G \rangle$ if and only if there exists $g \in G$ such that $\text{rank}(a) = \text{rank}(aga)$. But this is equivalent to saying that there exists $g \in G$ such that $([n] \setminus \{b_1, \dots, b_k\})g \subseteq \{a_k, \dots, a_n\}$. As these sets are arbitrary, it follows that G satisfies the property that each quasi-permutation a is regular in $\langle a, G \rangle$ if and only if G is $(n-k, n-k+1)$ -homogeneous. By Theorem 2.1 this last condition is equivalent to (2). It is proved that (1) and (2) are equivalent. \square

Now we can state and prove our second main result about quasi-permutations.

Theorem 6.4. *Let $G \leq \mathcal{S}_n$ and let $1 < k \leq \lfloor \frac{n}{2} \rfloor$. Then the following are equivalent*

- (1) *for every quasi-permutation a , such that $\text{rank}(a) = k$, the semigroup $\langle G, a \rangle$ is regular;*
- (2) *G is $(k-1)$ -homogeneous or G is one of the following groups*
 - (a) *$n = 5$ and $G \cong C_5, D_5$, with $k = 2$;*
 - (b) *$n = 7$ and $G \cong \text{AGL}(1, 7)$, with $k = 3$;*

Proof. Clearly (1) implies that every quasi-permutation a is regular in $\langle a, G \rangle$ and hence, by the previous result, it follows that G must be one of the groups listed in the statement of the theorem, or $n = 9$ and $G \cong \text{ASL}(2, 3)$ or $\text{AGL}(2, 3)$. However, for $a := \begin{pmatrix} \{1\} & \{2\} & \{3\} & \{4, \dots, 9\} \\ 1 & 4 & 5 & 2 \end{pmatrix}$, the semigroup $\langle a, G \rangle$ is not regular (when G is $\text{ASL}(2, 3)$ or $\text{AGL}(2, 3)$). It is proved that (1) implies (2).

Conversely, let a be a rank k quasi-permutation and let $G \leq \mathcal{S}_n$ be a $(k-1)$ -homogeneous group. By the previous theorem ((2) implies (1)) we know that a is regular in $\langle a, G \rangle$ and hence, by Theorem 6.2, every $b \in \langle a, G \rangle$ such that $\text{rank}(b) = k$ is regular in $\langle a, G \rangle$. Now suppose that $b \in \langle a, G \rangle$ and $\text{rank}(b) = l < k$. Then G is l -homogeneous (because $k \leq \lfloor \frac{n}{2} \rfloor$) and hence there exists $g \in G$ such that bgb has rank l . Thus, once again by Lemma 6.1, b is regular in $\langle b, G \rangle$. As $\langle b, G \rangle \subseteq \langle a, G \rangle$, it follows that b is regular in $\langle a, G \rangle$.

That the groups listed in (2) (a) and (b) satisfy the condition (1) follows from Theorem 1.1. \square

Now we turn to the case of transformations of a given rank. Our main theorem is the following.

Theorem 6.5. *Let $n \geq 5$, $G \leq \mathcal{S}_n$ and let $1 < k \leq \lfloor \frac{n+1}{2} \rfloor$. Then the following are equivalent:*

- (1) for all rank k transformations $a \in \mathcal{T}_n$, we have that a is regular in $\langle a, G \rangle$;
- (2) for all rank k transformations $a \in \mathcal{T}_n$, the semigroup $\langle a, G \rangle$ is regular;
- (3) G has the k -ut property (and hence is one of the groups listed in Theorem 4.2).

Proof. It follows from Lemma 6.1 that (1) and (3) are equivalent, and (2) implies (1) trivially. In addition, (1) (together with Theorem 6.2) implies that b is regular in $\langle a, G \rangle$, for all $b \in \langle a, G \rangle$, with $\text{rank}(b) = \text{rank}(a)$. It also follows from Proposition 4.3 (2) that if G has the k -ut property, then G is $(k-1)$ -homogenous, or it is one of the exceptions in Theorem 1.11. If G is $(k-1)$ -homogenous, then for every $c \in \langle a, G \rangle$, with $\text{rank}(c) < \text{rank}(a)$, we have that G is $\text{rank}(c)$ -homogenous and hence c is regular in $\langle c, G \rangle \subseteq \langle a, G \rangle$. Thus c is regular in $\langle a, G \rangle$.

If G is one of the three exceptions of degree 5 and 7, then (by Theorem 1.13) $\langle a, G \rangle$ is regular, for all rank(k) maps a . Thus it is proved that (1) and (3) imply (2). \square

When $n = 9$ and $G = \text{ASL}(2, 3)$ or $G = \text{AGL}(2, 3)$, (by Theorem 6.4) there exist rank-4 maps $a \in \mathcal{T}_n$ such that $\langle a, G \rangle$ is not regular. Also these groups do not have the 4-ut property.

7. PROBLEMS

We start by proposing a problem to experts in number theory. If this problem can be solved, the results on $\text{AGL}(1, p)$, in Section 5.6, will be dramatically sharpened.

Problem 1. *Classify the prime numbers p congruent to 11 (mod 12) such that for some $c \in \text{GF}(p)^*$ we have $|\langle -1, c, c-1 \rangle| < p-1$.*

Problem 2. *Do the Suzuki groups $\text{Sz}(q)$ have the 3-ut property?*

Classify the groups G that have the 4-ut property, when $\text{PSL}(2, q) \leq G \leq \text{PTL}(2, q)$, with either q prime (except $\text{PSL}(2, q)$ for $q \equiv 1 \pmod{4}$, which is not 3-homogeneous), or $q = 2^p$ for p prime.

Problem 3. *Prove results analogous to Theorem 6.4 and to Theorem 6.5, but for the transformations of rank $k > \lfloor \frac{n+1}{2} \rfloor$.*

The difficulty here (when $\text{rank } k > \lfloor \frac{n+1}{2} \rfloor$) is that a k -homogenous group is not necessarily $(k-1)$ -homogenous. Therefore a rank k map $a \in \mathcal{T}_n$ might be regular in $\langle a, G \rangle$, but we are not sure that there exists $g \in G$ such that $\text{rank}(bgb) = \text{rank}(b)$, for $b \in \langle a, G \rangle$ such that $\text{rank}(b) < \text{rank}(a)$.

Problem 4. *In what concerns this paper, the most general problem that has to be handled is the classification of pairs (a, G) , where $a \in \mathcal{T}_n$ and $G \leq \mathcal{S}_n$, such that $\langle a, G \rangle$ is a regular semigroup.*

When investigating $(k-1)$ -homogenous groups without the k -ut property, it was common that some of the orbits on the k -sets have transversals for all the partitions. Therefore the following definition is natural. A group $G \leq \mathcal{S}_n$ is said to have the weak k -ut property if there exists a k -set $S \subseteq [n]$ such that the orbit of S under G contains a section for all k -partitions. And such a set is called a G -universal transversal set.

Problem 5. *Classify the groups with the weak k -ut property; in addition, for each one of them, classify their G -universal transversal sets.*

In this paper we considered groups such that the orbit of every k -set contains a section for every k -partition. And this is of course a very strong requirement. In order to attack Problem 4, it seems the next step (in addition to Problem 5) is to consider groups such that the orbit of every k -set contains sections for some (not all) partitions.

Problem 6. *Let π be a partition of n . A map $a \in \mathcal{T}_n$ has kernel type π if the partition of n induced by the cardinalities of the kernel blocks is equal to π . Classify the groups $G \leq \mathcal{S}_n$ such that for all maps $a \in \mathcal{T}_n$ of a given kernel type π , the semigroup $\langle G, a \rangle$ is regular.*

In McAlister's celebrated paper [31] it is proved that if $e^2 = e \in \mathcal{T}_n$ is a rank $n - 1$ idempotent, then $\langle G, e \rangle$ is regular for all groups $G \leq \mathcal{S}_n$. In addition, assuming that $\{\alpha, \beta\}$ is the non-singleton kernel class of e and $\alpha e = \beta$, if α and β are not in the same orbit under G , then $\langle e, G \rangle$ is an orthodox semigroup (that is, the idempotents form a subsemigroup); and $\langle G, e \rangle$ is inverse if and only if α and β are not in the same orbit under G and the stabilizer of α is contained in the stabilizer of β .

Problem 7. *Classify the groups $G \leq \mathcal{S}_n$ that together with any idempotent [rank k idempotent] generate a regular [orthodox, inverse] semigroup.*

Classify the pairs (G, a) , with $a \in \mathcal{T}_n$ and $G \leq \mathcal{S}_n$, such that $\langle G, a \rangle$ is inverse [orthodox]. (Recall that by [39] every element $a \in \mathcal{T}_n$ is contained in an inverse subsemigroup of \mathcal{T}_n ; in addition it is a longstanding open problem to describe the maximal inverse subsemigroups of \mathcal{T}_n .)

A group $G \leq \mathcal{S}_n$ has the $(n - 1)$ -universal transversal property if and only if it is transitive. And $\langle G, a \rangle$ contains all the rank $n - 1$ maps of \mathcal{T}_n if and only if G is 2-homogeneous. In this last case $\langle G, a \rangle$ is regular for all $a \in \mathcal{T}_n$ because $\langle G, a \rangle = \{b \in \mathcal{T}_n \mid |[n]b| \leq n - 1\} \cup G$, and this semigroup is well known to be regular.

Problem 8. *Classify the groups $G \leq \mathcal{S}_n$ such that G together with any rank $n - k$ map, where $k \leq 5$, generate a regular semigroup. We already know that such G must be k -homogeneous and so are classified.*

The majority of the previous problems (and theorems) admit an obvious analogous with regular replaced everywhere by *idempotent generated*.

Problem 9. *Classify all the pairs (a, G) , where $a \in \mathcal{T}_n$ and $G \leq \mathcal{S}_n$, such that $\langle a, G \rangle \setminus G$ is idempotent generated (that is, $\langle a, G \rangle \setminus G$ is generated by its own idempotents).*

Solve particular instances of this general problem analogous to the list of problems above.

The theorems and problems in this paper admit linear versions that are interesting for experts in groups and semigroups, but also to experts in linear algebra and matrix theory. However, for the linear case, not even an analogue of Theorem 1.1 exists. All we know is that any singular matrix with any group containing the special linear group generate a regular semigroup [4, 5] (see also the related papers [15, 36, 38]).

Problem 10. *Prove (or disprove) that if $G \leq GL(n, q)$ such that for all singular matrix a there exists $g \in G$ with $\text{rank}(a) = \text{rank}(aga)$, then G contains the special linear group.*

It is clear that such a group must satisfy the following property. If V is a vector space (over a finite field) with $\dim(V) = n$, and $U, T \leq V$ are two non-null subspaces such that $\dim(U) + \dim(T) = n$, then there exists $g \in G$ such that $V = Ug \oplus T$.

For $n = 2$ and for $n = 3$, this condition is equivalent to irreducibility of G . But we conjecture that, for sufficiently large n , it implies that G contains the special linear group.

Problem 11. *Classify the groups $G \leq GL(n, q)$ such that for all rank k (for a given k) singular matrix a we have that a is regular in $\langle G, a \rangle$ [the semigroup $\langle G, a \rangle$ is regular].*

To handle this problem it is useful to keep in mind the following results. Kantor [22] proved that if a subgroup of $PGL(d, q)$ acts transitively on k -dimensional subspaces, then it acts transitively on l -dimensional subspaces for all $l \leq k$ such that $k + l \leq n$; in [23], he showed that subgroups transitive on 2-dimensional subspaces are 2-transitive on the 1-dimensional subspaces with the single exception of a subgroup of $PGL(5, 2)$ of order $31 \cdot 5$; and, with the second author [9], he showed that such groups must contain $PSL(d, q)$ with the single exception of the alternating group A_7 inside $PGL(4, 2) \cong A_8$. Also Hering [16, 17] and Liebeck [27] classified the subgroups of $PGL(d, p)$ which are transitive on 1-spaces.

Problem 12. *Solve the analogue of Problem 11 for independence algebras (for definitions and fundamental results see [1, 2, 11, 14]).*

Recall from Subsection 5.2 the graph $G(B, c)$, where G is a t -homogeneous group and $|B| = t - 1$.

Problem 13. *Is it true that the group G has the $(t + 1)$ -ut property if and only if $G(B, c)$ is connected? If so, is it possible to find an elementary proof of that (without using the classification of finite simple groups)?*

Problem 14. *Prove Corollary 1.9 and Corollary 1.12 without using the classification of finite simple groups.*

Regarding this problem, observe that the proof that k -ut implies $(k - 1)$ -ut in fact follows from the classification of the $(k - 1, k)$ -homogeneous groups (and hence, for that purpose, we can bypass the classification of groups with the k -ut property). In fact, if the group possesses the k -ut property (for $k \leq \lfloor \frac{n}{2} \rfloor$), then it is $(k - 1, k)$ -homogeneous and hence, with few exceptions, it is $(k - 1)$ -homogeneous so that it has the $(k - 1)$ -ut property. So in the previous theorem what really is at stake is to find an elementary proof to Corollary 1.12.

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